Weakly homogeneous optimization problems

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ABSTRACT
This paper investigates a new class of optimization problems whose objective functions are weakly homogeneous relative to the constrain sets. Two sufficient conditions for nonemptiness and boundedness of solution sets are established. We also study linear parametric problems and upper semincontinuity of the solution map.

KEYWORDS
Optimization problem; weakly homogeneous function; asymptotic cone; pseudoconvex function; nonemptiness; boundedness; upper semincontinuity

1. Introduction

The class of weakly homogeneous functions, which contains the subclass of all polynomial functions, has been introduced and studied recently by Gowda and Sossa [1] in variational inequality problems. This paper introduces a new class of weakly homogeneous functions, which is stronger in some sense than that one of [1], and investigates weakly homogeneous optimization problems. Asymptotic analysis play an important role in this study; the normalization argument (see, e.g., [1,2]) is used almost in the proofs.

We establish two criteria for the nonemptiness and compactness of the solution sets. The first one is that the kernel of a weakly homogeneous optimization problem is trivial. When the kernel is non-trivial, the second one concerns the case that the objective function is pseudoconvex. These results are considered as extension versions of the Frank-Wolfe type theorem and the Eaves type theorem for polynomial optimization problems in [3].

The kernel, the domain, and the range of an affine variational inequality were introduced in [4,5]. Here, we developt these notions for a weakly homogeneous optimization problem. The two first ones are useful in the investigation of solution existence and stability of linear parametric optimization problems.

The organization of the paper is as follows. Section 2 gives a brief introduction to asymptotic cones, weakly homogeneous functions, pseudoconvexity, and optimization problems. Section 3 discusses on asymptotic problems. Two results on the nonemptiness and compactness of the solution sets are shown in Section 4. Sections 5 and 6 investigate the solution existence and stability of linear parametric optimization problems, respectively.

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2. Preliminaries

Let $C \subset \mathbb{R}^n$ be a closed cone, K be a nonempty closed subset of C, and $f : C \to \mathbb{R}$ be a continuous function. The *asymptotic cone* of K is defined by

$$K^\infty = \Big\{ v \in \mathbb{R}^n : \exists t_k \to +\infty, \exists x_k \in K \text{ with } \lim_{k\to\infty} \frac{x_k}{t_k} = v \Big\}.$$

The cone K^∞ is closed and $K^\infty \subset C$. Recall that if K is convex then K^∞ coincides with the recession cone of K, which is the set of vectors $v \in \mathbb{R}^n$ such that $x + tv \in K$ for any $x \in K$ and $t \geq 0$; i.e., $K = K + K^\infty$.

Definition 2.1. One says that the function f is *weakly homogeneous of degree α relative to K* if there exists a positively homogeneous function h of degree $\alpha > 0$ on C, i.e., $h(tx) = t^\alpha h(x)$ for all $x \in C$ and $t > 0$, $h(x)$ is nonzero on C, such that $f(x) - h(x) = o(\|x\|^\alpha)$ on K.

In Definition 2.1, the asymptotic homogeneous function h is not unique. We denote by $[f^\infty_\alpha]$ the class of all asymptotic homogeneous functions of degree $\alpha > 0$ of f on C. Clearly, if $g(x) = o(\|x\|^\alpha)$ on C then $f + g$ is also weakly homogeneous of degree α relative to K. The space of all continuous functions g on C, such that $g(x) = o(\|x\|^\alpha)$ on K, is denoted by $\mathcal{O}^\alpha_K$.

Remark 2.1. The notion in Definition 2.1 is different from the asymptotic function notion in the monograph of Auslender and Teboulle [2, Definition 2.5.1] and is stronger, in some sense (see Example 2.1 and Remark 4.1), than that one of [1]. Here, we emphasize the phase "*relative to K*" to find a positively homogeneous function h such that its degree is smallest among positively homogeneous functions on C.

Example 2.1. Consider the cone $C = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0\}$, the set $K = \{(x_1, x_2) : (x_1 - 2)^2 + (x_2 - 2)^2 \leq 1\} \cup \{(x_1, x_2) : x_1 \geq 1, x_2 = 0\}$, and the function $f(x_1, x_2) = x_1x_2 + \sqrt{x_1}$. Clearly, f is weakly homogeneous of degree $\alpha = \frac{1}{2}$ relative to K. There are two different asymptotic functions $h_1(x_1, x_2) = \sqrt{x_1}$ and $h_2(x_1, x_2) = \sqrt{x_1} + \sqrt{x_2}$ of f on $K^\infty = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 = 0\}$. Meanwhile, f also is an weakly homogeneous function of degree $\alpha = 2$ (the involved asymptotic function must be $h(x_1, x_2) = x_1x_2$) in sense of [1].

The minimization problem with the constraint set K and the objective function f is written formally as follows:

$$\mathrm{OP}(K, f) \qquad \text{minimize } f(x) \text{ subject to } x \in K.$$

The solution set of $\mathrm{OP}(K, f)$ is abbreviated to $\mathrm{Sol}(K, f)$. Clearly, if $\mathrm{Sol}(K, f)$ is nonempty then f is bounded from below on K.

Remark 2.2. Assume that K is a cone and f is positively homogeneous function of degree $\alpha > 0$ on K. If f is bounded from below on K then $\mathrm{Sol}(K, f)$ is nonempty. Indeed, take $x = 0 \in K$, there are some $y \in K \setminus \{0\}$ such that $f(y) < f(0)$. Since $f(0) = 0$, one has $f(y) < 0$. It follows that

$$\lim_{t\to+\infty} f(ty) = \lim_{t\to+\infty} t^\alpha f(y) = -\infty.$$

This contradicts to our assumption.

Remark 2.3. Assume that K is a cone and f is positively homogeneous function of degree $\alpha > 0$ on K. If $\mathrm{Sol}(K, f)$ is nonempty then this set is a closed cone. To prove this assertion, we suppose that $\mathrm{Sol}(K, f) \neq \emptyset$, $x \in \mathrm{Sol}(K, f)$, and $t > 0$. One has

$$f(y) \geq f(x), \ \forall y \in K. \tag{1}$$

Let t be an arbitrary positive real number. If $f(y) \geq f(x)$ then, by multiplying this inequality by t^α, we obtain $f(ty) \geq f(tx)$. Since $K = tK$, the condition (1) implies that $f(y) \geq f(tx)$, $\forall y \in K$. This shows that tx is a solution of $\mathrm{OP}(K, f)$. Hence, $\mathrm{Sol}(K, f)$ is a cone. The closedness of this set immediately follows from the closedness of K and the continuity of f.

Remark 2.4. Assume that K is a cone, f is positively homogeneous function of degree $\alpha > 0$ on K, and $\mathrm{Sol}(K, f)$ is nonempty. From Remark 2.3, one has $0 \in \mathrm{Sol}(K, f)$. Since $f(x) \geq f(0) = 0$ for all $x \in K$, f is non-negative on K. Furthermore, $\mathrm{Sol}(K, f)$ is the set of zero points of f in K, i.e. $\mathrm{Sol}(K, f) = \{x \in K : f(x) = 0\}$.

Assume that f is differentiable on $\operatorname{int} C$. The gradient of f is denoted by ∇f. The function f is *pseudoconvex* on $\operatorname{int} C$ if, for any $x, y \in \operatorname{int} C$ such that $\langle \nabla f(x), y - x\rangle \geq 0$, we have $f(y) \geq f(x)$. Recall that f is pseudoconvex on $\operatorname{int} C$ if and only if ∇f is pseudomonotone on $\operatorname{int} C$ (see, e.g., [6, Theorem 4.4]), i.e. if, for any $x, y \in \operatorname{int} C$ such that $\langle \nabla f(x), y - x\rangle \geq 0$, then $\langle \nabla f(y), y - x\rangle \geq 0$. If f is convex on $\operatorname{int} C$ then it is pseudoconvex on $\operatorname{int} C$.

Remark 2.5. Assume that K is convex, $K \subset \operatorname{int} C$, and f is pseudoconvex on $\operatorname{int} C$. If $x^0 \in \mathrm{Sol}(K, f)$ then $\langle \nabla f(x), x - x^0\rangle \geq 0$ for all $x \in K$. Indeed, since x^0 is a solution of $\mathrm{OP}(K, f)$, one has $\langle \nabla f(x^0), x - x^0\rangle \geq 0$ for all $x \in K$ (see, e.g., [6, Proposition 5.2]). The pseudomonotonicity of the gradient implies that $\langle \nabla f(x), x - x^0\rangle \geq 0$ for all $x \in K$. Conversely, if the point $x^0 \in K$ satisfied $\langle \nabla f(x^0), x - x^0\rangle \geq 0$ for all $x \in K$ then $x^0 \in \mathrm{Sol}(K, f)$ (see, e.g., [6, Proposition 5.3]).

The following assumptions will be needed throughout the paper: K is nonempty, unbouded, and closed; f is continuous on the cone C containing K; f is weakly homogeneous of degree $\alpha > 0$ relative to K.

3. Properties of asymptotic problems

The optimization problem is given by the asymptotic pair (K^∞, f^∞) plays a vital role in the investigation of behavior at infinity of $\mathrm{OP}(K, f)$.

Proposition 3.1. *If $h, h' \in [f^\infty_\alpha]$, then $\mathrm{Sol}(K^\infty, h) = \mathrm{Sol}(K^\infty, h')$.*

Proof. Let h, h' be two asymptotic homogeneous functions of degree α of f on C. Suppose that $\mathrm{Sol}(K^\infty, h)$ is empty. From Remark 2.2, h are not bounded from below on K^∞. There is $\bar{x} \in K^\infty \setminus \{0\}$ and $\|\bar{x}\| = 1$ such that $h(\bar{x}) < 0$. There exists a sequence $\{x_k\} \subset K$ with $\|x_k\|^{-1} x_k \to \bar{x}$. One has

$$\lim_{k \to +\infty} \frac{h'(x_k)}{\|x_k\|^\alpha} = \lim_{k \to +\infty} \frac{(h'(x_k) - f(x_k)) + (f(x_k) - h(x_k)) + h(x_k)}{\|x_k\|^\alpha} = h(\bar{x}) < 0.$$

Hence, the sequence $\{h'(x_k)\}$ is not bounded from below on K.

Suppose that $\mathrm{Sol}(K^\infty, h)$ is nonempty. From above argument, $\mathrm{Sol}(K^\infty, h')$ also is nonempty. Now we prove $\mathrm{Sol}(K^\infty, h) = \mathrm{Sol}(K^\infty, h')$. Suppose that there is x in $\mathrm{Sol}(K^\infty, h)$ but it does not belong to $\mathrm{Sol}(K^\infty, h')$. From Remark 2.4, we have $h(x) = 0$ and $h'(x) > 0$.

$$\begin{aligned} h'(x) = \lim_{t\to+\infty} \tfrac{t^\alpha h'(x)}{t^\alpha} &= \lim_{t\to+\infty} \tfrac{h'(tx)-f(tx)+f(tx)-0}{t^\alpha} \\ &= \lim_{t\to+\infty} \tfrac{h'(tx)-f(tx)}{t^\alpha} + \lim_{t\to+\infty} \tfrac{f(tx)-0}{t^\alpha} = 0. \end{aligned}$$

This is a contradiction. Thus, the two solution sets are equal. □

From Proposition 3.1, we can write a member of $[f^\infty_\alpha]$ simply by f^∞ when no confusion can arise. We denote the closed cone $\mathcal{K}(K, f) := \mathrm{Sol}(K^\infty, f^\infty)$. According to Propostion 3.1, $\mathcal{K}(K, f)$ is not depend on the choice of f^∞. Sometimes, we call $\mathcal{K}(K, f)$ is the *kernel* of the weakly homogeneous optimization problem $\mathrm{OP}(K, f)$. From Remark 2.4, one sees that the kernel is the set of zero points of f^∞ in K^∞, i.e., $\mathcal{K}(K, f) = \{x \in K^\infty : f^\infty(x) = 0\}$.

Proposition 3.2. *Assume that K is convex. One has the following inclusion*

$$\bigcup_{g\in\mathcal{O}^\alpha_K} (\mathrm{Sol}(K, f + g))^\infty \subset \mathcal{K}(K, f),$$

here M^∞ is the asymptotic cone of M. Furthermore, if K is a cone then the inverse inclusion holds.

Proof. Let $g \in \mathcal{O}^\alpha_K$ be given. Suppose that $\bar{x} \in (\mathrm{Sol}(K, f + g))^\infty$ and $\bar{x} \neq 0$. There exist a sequence $\{x^k\} \subset \mathrm{Sol}(K, f + g)$ and a sequence $\{t_k\} \subset \mathbb{R}_+ \setminus\{0\}, t_k \to +\infty$, such that $t_k^{-1}x^k \to \bar{x}$. By assumptions, for each x^k, one has

$$f(y) + g(y) \geq f(x^k) + g(x^k),\ \forall y \in K. \tag{2}$$

Let $u \in K$ be fixed. Since K is convex, for every $v \in K^\infty$, one has $u + t_kv \in K$ for any k. From (2), we conclude that

$$f(u + t_kv) + g(u + t_kv) \geq f(x^k) + g(x^k),\ \forall y \in K. \tag{3}$$

Dividing the inequality in (3) by t^α_k and letting $k \to +\infty$, we obtain $f^\infty(v) \geq f^\infty(\bar{x})$. The above assertion holds for every $v \in K^\infty$. We conclude that $\bar{x} \in \mathcal{K}(K, f)$.

Assume that K is a cone. Clearly, $g(x) := f(x) - f^\infty(x)$ belongs to $\mathcal{O}^\alpha_K$. Hence, $f - g = f^\infty$ and $\mathcal{K}(K, f) = \mathrm{Sol}(K, f - g)$. Then the inverse inclusion is proved. □

We show a basic property of the asymptotic problem $\mathrm{OP}(K^\infty, f^\infty)$.

Proposition 3.3. *If the kernel $\mathcal{K}(K, f)$ is empty, then* $\mathrm{Sol}(K, f)$ *so is. Hence, if* $\mathrm{Sol}(K, f)$ *is nonempty, then the kernel also is nonempty.*

Proof. Suppose that $\mathcal{K}(K, f)$ is empty. Clearly, the cone K^∞ is nontrivial. Remark 2.2 says that f^∞ is not bounded from below on K^∞. There exists $\bar{x} \in K^\infty \setminus \{0\}$ with

$\|\bar{x}\| = 1$, such that $f^\infty(\bar{x}) < 0$. There exists $\{x_k\} \subset K$ with $\|x_k\|^{-1}x_k \to \bar{x}$. We get

$$\lim_{k\to+\infty} \frac{f(x_k)}{\|x_k\|^\alpha} = \lim_{k\to+\infty} \frac{(f(x_k) - f^\infty(x_k)) + f^\infty(x_k)}{\|x_k\|^\alpha} = f^\infty(\bar{x}) < 0.$$

We conclude that f is not bounded from below on K. Hence, the emptyness of $\mathrm{Sol}(K, f)$ is proved. □

4. Nonemptiness and compactness of solution sets

According to Proposition 3.3, the nonemptiness of $\mathcal{K}(K, f)$ is a necessary condition for the existence of solutions of $\mathrm{OP}(K, f)$. We introduce two criteria for the nonemptiness and compactness of $\mathrm{Sol}(K, f)$. These proofs of the first one and the second one can be modified from the proofs of [3, Theorem 3.1] and [3, Theorem 3.2], respectively.

The first one is the case that the kernel is trivial.

Theorem 4.1. *If the kernel of* $\mathrm{OP}(K, f)$ *is a trivial cone, then* $\mathrm{Sol}(K, f)$ *is nonempty and bounded.*

Proof. Suppose that $\mathcal{K}(K, f) = \{0\}$. Given a point $x^0 \in K$ and set

$$M := \Big\{x \in K : f(x^0) \geq f(x)\Big\}. \tag{4}$$

It is clear that M is nonempty and closed. We now show $\mathrm{Sol}(K, f) = \mathrm{Sol}(M, f)$. Since $M \subset K$, $\mathrm{Sol}(K, f)$ is a subset of $\mathrm{Sol}(M, f)$. Suppose that $\bar{x} \in \mathrm{Sol}(M, f)$, one has

$$f(z) \geq f(\bar{x}),\ \forall z \in M. \tag{5}$$

Because of $x_0 \in M$, we conclude that $f(x_0) \geq f(\bar{x})$. From (4), we have

$$f(z) > f(x_0) \geq f(\bar{x}),\ \forall z \in K \setminus M. \tag{6}$$

From (5) and (6), we get $f(z) \geq f(\bar{x})$ for all $z \in K$; hence, $\bar{x}$ solves $\mathrm{OP}(K, f)$. Therefore $\mathrm{Sol}(K, f) \supset \mathrm{Sol}(M, f)$; the desired equation is proved.

If M is compact, by Weierstrass' Theorem we get the desired result. Thus, we need only to consider the case that M is unbounded.

On the contrary, we suppose that M is unbounded. Then there exists a sequence $\{x^k\} \subset M$ such that $\|x^k\| \to +\infty$. Without loss of generality we can assume that $\|x^k\|^{-1}x^k \to v$ and $v \in K^\infty \setminus \{0\}$. For each k, we have

$$f(x^0) \geq f(x^k). \tag{7}$$

Dividing both sides in (7) by $\|x^k\|^\alpha$ and letting $k \to +\infty$, we obtain the fact that $0 \geq f^\infty(v)$. This implies that $v \in \mathcal{K}(K, f)$. It is impossible by $\mathcal{K}(K, f) = \{0\}$. Hence, M is bounded. The proof is complete. □

Example 4.1. Consider the objective f and the constraint set K given in Example 2.1. It is easy to see that $\mathcal{K}(K, f) = \{0\}$. According to Theorem 4.1, the solution set is nonempty and bounded. Meanwhile, $\mathrm{Sol}(K, f) = \{(1, 0)\}$.

Remark 4.1. We mentioned that our weakly homogeneous function notation is stronger than that one of [1] in sense that our kernel maybe is smaller than $\mathrm{Sol}(K^\infty, h)$ in [1]. In Example 2.1, the asymptotic function in sense of [1] is $h(x_1, x_2) = x_1 x_2$; hence, one has $\mathrm{Sol}(K^\infty, h) = \{(x_1, 0) : x_1 \geq 0\}$. This cone is larger than the kernel $\mathcal{K}(K, f)$.

When the kernel is non-trivial, we have a criterion for the nonemptiness and compactness of $\mathrm{Sol}(K, f)$ provided that f is pseudoconvex.

Theorem 4.2. *Assume that K is convex, $K \subset \operatorname{int} C$, and f is pseudoconvex on $\operatorname{int} C$. Consider the two following statements:*

(a) *For each $v \in \mathcal{K}(K, f) \setminus \{0\}$, there exists $x \in K$ such that $\langle \nabla f(x), v\rangle > 0$;*
(b) $\mathrm{Sol}(K, f)$ *is nonempty and compact.*

One has (a) $\Rightarrow$ (b)*. In addition, if K is convex then* (b) $\Rightarrow$ (a)*.*

Proof. (a) $\Rightarrow$ (b) Suppose that (a) holds. For each $k = 1, 2, \dots$, we denote $K_k = K \cap \mathbb{B}(0, k)$. Clearly, K_k is compact. We can assume that K_k is nonempty. According to Weierstrass' Theorem, $\mathrm{Sol}(K_k, f)$ has a solution, denoted by x^k.

We claim that $\{x^k\}$ is bounded. Indeed, on the contrary, suppose that this sequence is unbounded, here $x^k \neq 0$ for all k, and $\|x^k\|^{-1}x^k \to v$, where $v \in K^\infty$ and $\|v\| = 1$. For each k, we have

$$f(x) \geq f(x^k), \ \forall x \in K_k. \tag{8}$$

By fixing $x \in K_1$, hence $x \in K_k$ for any k, dividing two sides of the inequality in (8) by $\|x^k\|^d$ and letting $k \to +\infty$, we get $0 \geq f^\infty(v)$. This leads to $v \in \mathcal{K}(K, f) \setminus \{0\}$.

For each k, since f is pseudoconvex, from Remark 2.5, we have

$$\langle \nabla f(x), x - x^k\rangle \geq 0, \ \forall x \in K_k. \tag{9}$$

Let $x \in K$ be given, then $x \in K_k$ for k large enough. Dividing both sides in (9) by $\|x^k\|$ and letting $k \to +\infty$, we obtain $0 \geq \langle \nabla f(x), v\rangle$. This holds for all $x \in K$. It contradicts (a). Hence, $\{x^k\}$ is bounded.

We can assume that $x^k \to \bar{x}$. From (8), by the continuity of f, it not difficult to prove that $\bar{x}$ solves $\mathrm{OP}(K, f)$, so $\mathrm{Sol}(K, f)$ is nonempty.

To prove the boundedness of $\mathrm{Sol}(K, f)$, on the contrary, we suppose that there is an unbounded solution sequence $\{x^k\}$, with $\|x^k\|^{-1}x^k \to v$, where $v \in K^\infty$ and $\|v\| = 1$. For each k, the inequalities in (8) and (9) hold for any $x \in K$. By repeating the previous argument, we can get a similar contradiction. The first assertion is proved.

Assume that K is convex. One has $K = K + K_\infty$. Suppose $\mathrm{Sol}(K, f)$ be nonempty and compact, but there exists $v \in \mathcal{K}(K, f) \setminus \{0\}$ such that $\langle \nabla f(x), v\rangle \leq 0$ for all $x \in K$. Let x^0 be a solution of $\mathrm{OP}(K, f)$. For any $t \geq 0$, one has $x^0 + tv \in K$ and $\langle \nabla f(x^0 + tv), v\rangle \leq 0$, so

$$\langle \nabla f(x^0 + tv), x^0 - (x^0 + tv)\rangle \geq 0.$$

The pseudoconvexity of f yields $f(x^0) \geq f(x^0 + tv)$. Hence, $x^0 + tv \in \mathrm{Sol}(K, f)$ for any $t \geq 0$. This shows that $\mathrm{Sol}(K, f)$ is unbounded which contradicts our assumption. Thus (a) holds. The proof is complete. $\square$

Example 4.2. Consider the objective function $f(x_1, x_2) = \sqrt{x_2^5} + \frac{1}{2}x_1^2 - x_1x_2$ and the constraint set $K = \{(x_1, x_2) \in \mathbb{R}^2 : x_1x_2 \geq 2, x_2 \geq 16\} \subset \mathbb{R}^2_{>0}$ with $C = \mathbb{R}^2_{\geq 0}$. The gradient and the Hessian matrix of f on K, respectively, given by

$$\nabla f = \begin{bmatrix} x_1 - x_2 \\ \frac{5}{2}\sqrt{x_2^3} - x_1 \end{bmatrix}, \quad H_f = \begin{bmatrix} 1 & -1 \\ -1 & \frac{15}{4}\sqrt{x_2} \end{bmatrix}.$$

It is easy to check that H_f is positive semidefinite on the open set $O = \{(x_1, x_2) \in \mathbb{R}^2 : x_1x_2 > 1, x_2 > 15\} \subset \operatorname{int} C$ (with $K \subset O \subset C$); hence f is convex on K. One has $K^\infty = \mathbb{R}^2_{\geq 0}$ and $f^\infty(x_1, x_2) = \sqrt{x_2^5}$. This yields

$$\mathcal{K}(K, f) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 = 0\}. \tag{10}$$

Take $v = (\alpha, 0)$ in $\mathcal{K}(K, f) \setminus \{0\}$, then one has $\alpha > 0$. Choose $(x_1, x_2) = (17, 16)$ in the constraint set, we have $\langle \nabla f(x), v\rangle = \alpha > 0$. According to Theorem 4.2, the solution set of $\mathrm{OP}(K, f)$ is nonempty and compact.

5. Linear parametric problems

In this and next sections, we assume that $\alpha > 1$. We consider the parametric weakly homogeneous programs $\mathrm{OP}(K, f_u)$, where $u \in \mathbb{R}^n$ and

$$f_u(x) = f(x) - \langle u, x\rangle.$$

Here, C^* stands for the polar cone [8] (or, the negative dual cone) of C, i.e., $C^* = \{u \in \mathbb{R}^n : \langle x, v\rangle \leq 0\}$. Recall that x belongs to the interior $\operatorname{int} C^*$ of C if and only if $\langle x, v\rangle < 0$ for all $v \in C$ and $v \neq 0$.

The *range* of the weakly homogeneous optimization problem $\mathrm{OP}(K, f)$ is defined by

$$\mathcal{R}(K, f) := \{u \in \mathbb{R}^n : \mathrm{Sol}(K, f_u) \neq \emptyset\}.$$

Remark 5.1. To understand reason we use the term "range" to describe this set, we consider the relation between $\mathcal{R}(K, f)$ and $\nabla f(K)$. By $\nabla f_u = \nabla f - u$ and Fermat's theorem, $\mathcal{R}(K, f)$ is a subset of $\nabla f(K)$. When K is convex and f is convex on K, the range of $\mathrm{OP}(K, f)$ coincides with the range of the map ∇f on K, i.e.,

$$\mathcal{R}(K, f) = \nabla f(K).$$

To prove $\mathcal{R}(K, f) \supset \nabla f(K)$, we first recall that $f_u(x) = f(x) - \langle u, x\rangle$ also is convex on K for any u. Let u be a vector in $\nabla f(K)$. Then there are some $z \in K$ such that $\nabla f(z) - u = 0$. Hence that $\nabla f_u(z) = 0$. By the convexity of f_u on K, z must be a solution of $\mathrm{OP}(K, f_u)$.

Theorem 5.1. *Assume that f is bounded from below on K. Then, one has*

$$\operatorname{int} \mathcal{K}(K, f)^* \subset \mathcal{R}(K, f). \tag{11}$$

Furthermore, if $u \in \operatorname{int} \mathcal{K}(K, f)^*$ then $\mathrm{Sol}(K, f_u)$ is compact.

Proof. Let u be a vector in $\operatorname{int}\mathcal{K}(K,f)^*$. We now prove the nonemptiness of $\mathrm{Sol}(K,f_u)$. For each $k=1,2,\ldots$, we define

$$K_k = \{x \in \mathbb{R}^n : x \in K, \|x\| \leq k\}.$$

Clearly, K_k is compact. We can assume that K_k is nonempty. Weierstrass' Theorem says that $\mathrm{OP}(K_K,f_u)$ has a solution, denoted by x_k. One has

$$f(y) - \langle u, y\rangle \geq f(x_k) - \langle u, x_k\rangle\,,\ \forall y \in K_k. \tag{12}$$

One claims that the sequence $\{x^k\}$ is bounded. On the contrary, suppose that the sequence is unbounded. We can assume that $x^k \neq 0$ for all k, $\|x^k\|^{-1}x^k \to \bar{x}$ with $\bar{x} \in K^\infty$ (here $\|\bar{x}\| = 1$).

Let y in K_1 be fixed. For each k, from (12) one has

$$f(y) - \langle u, y\rangle \geq f(x_k) - \langle u, x_k\rangle\,. \tag{13}$$

Dividing the inequality (13) by $\|x^k\|^\alpha$ and letting $k \to +\infty$, we have $0 \geq f^\infty(\bar{x})$. By the boundedness of f on K, one can show that $f^\infty(\bar{x}) \geq 0$. Hence, one has $f^\infty(\bar{x}) = 0$. It follows that $\bar{x} \in \mathcal{K}(K,f)$. Furthermore, since f is bounded from below on K by γ, from (13) we see that

$$\langle u, x_k\rangle \geq \gamma - f(y) + \langle u, y\rangle\,.$$

This leads to $\langle u, \bar{x}\rangle \geq 0$. It contradicts to our assumption $u \in \operatorname{int}\mathcal{K}(K,f)^*$. Thus, the sequence $\{x^k\}$ must be bounded.

We can suppose that $x^k \to z$. It is not difficult to prove that z solves $\mathrm{OP}(K,f_u)$. It follows the nonemptiness of $\mathrm{Sol}(K,f_u)$. The inclusion (11) is proved.

The boundedness of $\mathrm{Sol}(K,f_u)$ is proved by assuming that there exists an unbounded sequence of solutions $\{x_k\} \subset \mathrm{Sol}(K,f_u)$, $x^k \neq 0$ for all k and $\|x^k\|^{-1}x^k \to \bar{x}$. Repeating the previous argument, we also obtain the facts that $\bar{x} \in \mathcal{K}(K,f)$ and $u \notin \operatorname{int}\mathcal{K}(K,f)^*$. It contradicts to our assumption.

The proof is complete. □

Corollary 5.1. *Assume that K is a pointed cone and f is bounded from below on K. Then $\mathcal{R}(K,f)$ is nonempty. Furthermore, if $\mathcal{R}(K,f)$ is closed then*

$$\mathcal{K}(K,f)^* \subset \mathcal{R}(K,f). \tag{14}$$

Proof. Because of $\mathcal{K}(K,f) \subset K$, we have $K^* \subset \mathcal{K}(K,f)^*$. Since the cone K is pointed, K^* has a nonempty interior; then $\operatorname{int}\mathcal{K}(K,f)^*$ is nonempty. By (11), $\mathcal{R}(K,f)$ also is nonempty. Clearly, from the inclusion (11), the closedness of $\mathcal{R}(K,f)$ implies (14). □

Example 5.1. Consider the objective function f and the constraint set K given in Example 4.2. Clearly, f is bounded from below on K. As (10), the interior of the dual cone of the kernel is determined by

$$\operatorname{int}\mathcal{K}(K,f)^* = \{(x_1,x_2) \in \mathbb{R}^2 : x_1 < 0\}.$$

According to Theorem 5.1, the solution set of $\mathrm{OP}(K,f_u)$ is nonempty and compact for any $u = (u_1,u_2)$ such that $u_1 < 0$.

The *domain* of the weakly homogeneous optimization problem $\mathrm{OP}(K, f)$ is defined by

$$\mathcal{D}(K,f) := \nabla f(K) + (K^{\infty})^{*}.$$

Lemma 5.1. *Assume that $K \subset \operatorname{int} C$ and f is differentiable on $\operatorname{int} C$. Then,*

$$u \in \operatorname{int} \mathcal{D}(K,f) \tag{15}$$

iff for each $v \in K^{\infty} \setminus \{0\}$ there is $x \in K$ such that $\langle \nabla f(x) - u, v\rangle > 0$.

Proof. Suppose that $u \in \operatorname{int} \mathcal{D}(K,f)$. There exists an $\varepsilon > 0$ such that

$$\mathbb{B}(u,\varepsilon) \subset \mathcal{D}(K,f), \tag{16}$$

where $\mathbb{B}(u,\varepsilon)$ is the open ball of radius ε centered at u. On the contrary, there is $\bar{v} \in K^{\infty} \setminus \{0\}$ such that $\langle \nabla f(x) - u, \bar{v}\rangle \le 0$ for all $x \in K$. Let a be a point in $\mathbb{B}(u,\varepsilon)$. Since (16), there exist $\bar{x} \in K$ and $w \in (K^{\infty})^{*}$ such that $a = \nabla f(\bar{x}) - w$. This leads to

$$\langle u - a, \bar{v}\rangle = -\langle \nabla f(\bar{x}) - u, \bar{v}\rangle + \langle w, \bar{v}\rangle \le 0.$$

This inequality holds for any $a \in \mathbb{B}(u,\varepsilon)$. It follows that $\bar{v} = 0$, and one has a contradiction.

Conversely, suppose that for each $v \in K^{\infty} \setminus \{0\}$ there exists $x \in K$ such that $\langle f(x) - u, v\rangle > 0$, but (15) is false, i.e., u belongs to the closed set $\mathbb{R}^n \setminus \operatorname{int} \mathcal{D}(K,f)$. There exists a convergent sequence $\{u^k\} \subset \mathbb{R}^n$, such that $u^k \to u$ and $u^k \notin \operatorname{int} \mathcal{D}(K,f)$ for every k. This follows that

$$(u^k - \nabla f(K)) \cap (K^{\infty})^{*} = \emptyset, \tag{17}$$

for every k. From (17), let $v^k \in K^{\infty} \setminus \{0\}$ be given, for any x in K, one has $\left\langle u^k - \nabla f(x), v^k\right\rangle \ge 0$. Suppose that $\|v^k\|^{-1} v^k \to \bar{v}$ where $\bar{v} \in K^{\infty} \setminus \{0\}$. Hence, from the last inequality, we obtain $\langle \nabla f(x) - u, \bar{v}\rangle \le 0$. This contradicts to the assumption. $\square$

Theorem 5.2. *Assume that K is convex, $K \subset \operatorname{int} C$, and f is pseudoconvex on $\operatorname{int} C$. The set $\mathrm{Sol}(K, f_u)$ is nonempty and compact if and only if $u \in \operatorname{int} \mathcal{D}(K,f)$.*

Proof. Assume that $\mathrm{Sol}(K, f_u)$ is nonempty and compact, but $u \notin \operatorname{int} \mathcal{D}(K,f)$. According to Lemma 5.1, there exists $\bar{v} \in K^{\infty} \setminus \{0\}$ such that $\langle \nabla f(x) - u, \bar{v}\rangle \le 0$ for all $x \in K$. Let x_0 be a solution of $\mathrm{OP}(K, f_u)$ and $t > 0$. By the convexity of K, $x_0 + t\bar{v} \in K$. It follows that $\langle \nabla f(x_0 + t\bar{v}) - u, \bar{v}\rangle \le 0$. So, one has

$$\langle \nabla f(x_0 + t\bar{v}) - u, x_0 - (x_0 + t\bar{v})\rangle = -t \langle \nabla f(x_0) - u, \bar{v}\rangle \ge 0.$$

From the pseudoconvexity of f, we conclude that $f_u(x_0) \ge f_u(x_0 + t\bar{v})$, i.e., $x_0 + t\bar{v}$ belongs to $\mathrm{Sol}(K, f_u)$. Clearly, $\mathrm{Sol}(K, f_u)$ contains the ray $\{x_0 + t\bar{v} : t \ge 0\}$. Hence, $\mathrm{Sol}(K, f_u)$ is unbounded. This contradicts to the assumption.

Conversely, we suppose that $u \in \operatorname{int} \mathcal{D}(K,f)$. We will show that $\mathrm{Sol}(K, f_u)$ is nonempty. For each $k = 1, 2, \ldots,$ we define $K_k = K \cap \mathbb{B}(0,k)$. Without loss of generality we can assume that $K_k \neq \emptyset$ for each k. Clearly, every set K_k is compact and convex.

According to the Hartman-Stampacchia Theorem [7, Chapter 1, Lemma 3.1], there exists $x^k \in K_k$ such that $\left\langle \nabla f(x^k) - u, x - x^k \right\rangle \geq 0$ for all $x \in K_k$. Since $\nabla f(x) - u$ also is pseudomonotone, from Remark 2.5, we obtain

$$\left\langle \nabla f(x) - u, x - x^k \right\rangle \geq 0 \ \forall x \in K_k. \tag{18}$$

We claim that $\{x^k\}$ is bounded. Indeed, on the contrary, the sequence is unbounded. We can assume that $x^k \neq 0$ and $\|x^k\|^{-1}x^k \to \bar{v}$ with $\bar{v} \in K^\infty \setminus \{0\}$. Let $x \in K$ be arbitrary given. Then there exists k_x such that $x \in K_k$ for all $k > k_x$. Dividing (18) by $\|x^k\|$ and letting $k \to +\infty$, one has $\langle \nabla f(x) - u, \bar{v} \rangle \leq 0$, and then $u \notin \operatorname{int} \mathcal{D}(K, f)$ by Lemma 5.1. This is a contradiction. Hence, $\{x^k\}$ is bounded.

Now, we assume that $x^k \to \bar{x}$. By the closedness of K, $\bar{x}$ belongs to K. Let $x \in K$ be arbitrary given. From (18), taking $k \to +\infty$, we obtain

$$\langle \nabla f(\bar{x}) - u, x - \bar{x} \rangle \geq 0.$$

This inequality holds for every $x \in K$. From Remark 2.5, $\bar{x}$ solves $\mathrm{OP}(K, f_u)$, i.e., $\mathrm{Sol}(K, f_u)$ is nonempty.

Now, we need only to prove that $\mathrm{Sol}(K, f_u)$ is bounded. On the contrary, suppose that $\mathrm{Sol}(K, f_u)$ is unbounded. There is an unbounded sequence $\{x^k\} \subset \mathrm{Sol}(K, f_u)$. Without loss of generality we can assume that $x^k \neq 0$ and $\|x^k\|^{-1}x^k \to \bar{v}$ with $\bar{v} \in K^\infty \setminus \{0\}$. By definition, from Remark 2.5, one has

$$\left\langle \nabla f(x) - u, x - x^k \right\rangle \geq 0,$$

for all $x \in K$. Let $x \in K$ be given. Dividing this inequality by $\|x^k\|$ and letting $k \to +\infty$, we get $\langle \nabla f(x) - u, \bar{v} \rangle \leq 0$. Applying Lemma 5.1, we obtain $u \notin \operatorname{int} \mathcal{D}(K, f)$. This is a contradiction. □

Corollary 5.2. *Assume that K is convex, $K \subset \operatorname{int} C$, and f is pseudoconvex on $\operatorname{int} C$. If u belongs to $\mathcal{R}(K, f) \setminus \operatorname{int} \mathcal{D}(K, f)$ then $\mathrm{Sol}(K, f_u)$ is unbounded.*

Proof. The conclusion immediately follows from the definition of $\mathcal{R}(K, f)$ and Theorem 5.2. □

6. Upper semincontinuity of solution maps

This section focuses on the upper semicontinuity of the solution map $\mathcal{S} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ given by $\mathcal{S}(u) = \mathrm{Sol}(K, f_u)$. The kernel and the domain play an important role in this investigation.

Recall that the set-valued map $\Phi : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is *locally bounded* at $\bar{x}$ if there exists an open neighborhood U of $\bar{x}$ such that $\cup_{x \in U} \Phi(x)$ is bounded [8, Definition 5.14]. The map Φ is *upper semicontinuous* at $x \in T$ iff for any open set $V \subset \mathbb{R}^n$ such that $\Phi(x) \subset V$ there exists a neighborhood U of x such that $\Phi(x') \subset V$ for all $x' \in U$. Recall that if Φ is upper semicontinuous at every $x \in T \subset \mathbb{R}^m$ then Φ is said that to be upper semicontinuous on T.

Remark 6.1. If Φ is closed, namely, the graph

$$\mathrm{gph}(\Phi) := \big\{(u,v) \in \mathbb{R}^m \times \mathbb{R}^n : v \in \Phi(u)\big\}$$

is closed in $\mathbb{R}^m \times \mathbb{R}^n$, and locally bounded at x, then Φ is upper semicontinuous at x [8, Theorem 5.19].

Remark 6.2. The solution map $\mathcal{S}$ is closed. Indeed, we will prove that the graph $\mathrm{gph}(\mathcal{S})$ is closed in $\mathbb{R}^n \times \mathbb{R}^n$. Take a sequence $\{(u^k, x^k)\}$ in $\mathrm{gph}(\mathcal{S})$ with $(u^k, x^k) \to (u, \bar{x})$. It follows that $u^k \to u$ and $x^k \to \bar{x}$. Let $y \in K$ be arbitrary fixed. By definition, one has $f_{u^k}(y) \geq f_{u^k}(x^k)$. Taking $k \to +\infty$, we get $f_u(y) \geq f_u(\bar{x})$, i.e., $\bar{x} \in \mathrm{Sol}(K, g)$. Hence, the graph is closed.

Theorem 6.1. *Assume that K is convex. If the kernel of* $\mathrm{OP}(K, f)$ *is trivial, then $\mathcal{S}$ is upper semicontinuous on $\mathbb{R}^n$.*

Proof. Suppose that $\mathcal{K}(K, f) = \{0\}$. By Remarks 6.1 and 6.2, we need only to prove $\mathcal{S}$ is locally bounded at $u \in \mathbb{R}^n$.

Let $\varepsilon > 0$ be given. Let $\mathbb{B}(u, \varepsilon)$ and $\overline{\mathbb{B}}(u, \varepsilon)$ be the open ball and the closed ball, respectively, of radius ε centered at u. Consider the following sets:

$$M_\varepsilon := \bigcup_{u \in \mathbb{B}(u,\varepsilon)} \mathcal{S}(u) \subset \bigcup_{u \in \overline{\mathbb{B}}(u,\varepsilon)} \mathcal{S}(u) =: N_\varepsilon. \tag{19}$$

We conclude that N_ε is bounded. We suppose on the contrary that N_ε is unbounded. There is an unbounded sequence $\{x^k\}$ and a sequence $\{u^k\} \subset \overline{\mathbb{B}}(u, \varepsilon)$ such that x^k solves $\mathrm{OP}(K, f_{u^k})$ with $x^k \neq 0$ for every k, and $\|x^k\|^{-1} x^k \to \bar{x}$ with $\|\bar{x}\| = 1$. By the compactness of $\overline{\mathbb{B}}(u, \varepsilon)$, we can assume that $u^k \to u$ with $u \in \overline{\mathbb{B}}(u, \varepsilon)$.

By assumptions, for every k, one has

$$f(y) - \left\langle u^k, y \right\rangle \geq f(x^k) - \left\langle u^k, x^k \right\rangle, \ \forall y \in K. \tag{20}$$

Let $u \in K$ be fixed and $v \in K_\infty$ be arbitrary. By the convexity of K, one has $u + \|x^k\| v \in K$ for any k. From (20), we conclude that

$$f(u + \|x^k\| v) - \left\langle u^k, u + \|x^k\| v \right\rangle \geq f(x^k) - \left\langle u^k, x^k \right\rangle.$$

Dividing this inequality by $\|x^k\|^\alpha$ and taking $k \to +\infty$, by $\alpha > 1$, we obtain $f^\infty(v) \geq f^\infty(\bar{x})$. It follows that

$$\bar{x} \in \mathcal{K}(K, f). \tag{21}$$

This contradicts our assumption. Hence, N_ϵ must be bounded.

By (19), the boundedness of M_ϵ follows that of N_ϵ. Thus, $\mathcal{S}$ is locally bounded at $u \in \mathbb{R}^n$. □

Theorem 6.2. *Assume that f is bounded from below on K. Then the map $\mathcal{S}$ is upper semicontinuous on* $\mathrm{int}\, \mathcal{K}(K, f)^*$.

Proof. Let $u \in \operatorname{int} \mathcal{K}(K,f)^*$ be given. Like as the proof of Theorem 6.1, we prove that $\mathcal{S}$ is locally bounded at u. We retain the argument and the notion from the proof of Theorem 6.1, one has (21).

Since f is bounded from below on K by γ, from (20) we see that

$$\langle u, x_k \rangle \geq \gamma - f(y) + \langle u, y \rangle \,,$$

where y is fixed. This leads to $\langle u, \bar{x} \rangle \geq 0$. It contradicts to our assumption that $u \in \operatorname{int} \mathcal{K}(K,f)^*$. □

Theorem 6.3. *Assume that K is convex, $K \subset \operatorname{int} C$, and f is pseudoconvex on $\operatorname{int} C$. Then the map $\mathcal{S}$ is upper semicontinuous on $\operatorname{int} \mathcal{D}(K,f)$.*

Proof. Suppose that $u \in \operatorname{int} \mathcal{K}(K,f)^*$ is given. We need to prove that $\mathcal{S}$ is locally bounded at u. Repeat the argument from the proof of Theorem 6.1, we get (21).

By Remark 2.5, one has $\left\langle \nabla f(x) - u, x - x^k \right\rangle \geq 0$, for all $x \in K$. Dividing this one by $\|x^k\|$ and letting $k \to +\infty$, we get $\langle \nabla f(x) - u, \bar{x} \rangle \leq 0$. From Lemma 5.1, we obtain $u \notin \operatorname{int} \mathcal{D}(K,f)$. This is a contradiction. □

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