# A hybrid projection-proximal point algorithm for solving nonmonotone variational inequality problems *

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**Abstract**

A hybrid projection-proximal point algorithm is proposed for variational inequality problems. Though the usual proximal point method and its variants require that the mapping involved be monotone, at least pseudomonotone, we assume only that the so-called Minty variational inequality has a solution, in order to ensure the global convergence. This assumption is less stringent than pseudomonotonicity. In particular, it applies to quasimonotone variational inequality having a nontrivial solution.

**Key words.** Variational inequality, Proximal point algorithm, Global convergence.

## 1. Introduction

The classical variational inequality problem is to find an element $x^* \in C$ such that

$$\langle F(x^*), y - x^* \rangle \geq 0, \quad \forall y \in C, \tag{1.1}$$

where $C$ is a nonempty closed convex set in $\mathbb{R}^n$, $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^n$, and $F$ is a continuous mapping from $\mathbb{R}^n$ into $\mathbb{R}^n$. Throughout this paper, we denote the variational inequality by $\mathrm{VI}(F, C)$.

To solve the variational inequality, projection-type methods are effective algorithms, including extragradient method, double projection method, and proximal point method, etc. For ensuring the global convergence of these projection-type algorithms, monotonicity of a certain kind is usually required. Strong monotonicity was assumed in the earliest projection method. Two decades later, strong monotonicity was weakened into pseudomonotonicity, a very weak kind of generalized monotonicity. Certainly the design of the algorithm was rather improved. Under the assumption of pseudomonotonicity, [1] proved that an extragradient method is globally convergent, [2] proved the global convergence of a double projection method, and [3] obtained the global convergence of a proximal point method.

Recently, the second author and his coauthor devised a new double projection method in [8], which did not require that the mapping be pseudomonotone. The only assumption is that the so-called Minty variational inequality has a solution, which is true if the mapping is pseudomonotone, or if the mapping is quasimonotone and the variational inequality has a nontrivial solution. Thus the new method can apply to a class of quasimonotone variational inequalities without loss of global convergence. Extending the result in [8], [4] and [5] introduced new projection methods for nonmonotone equilibrium problem, and [7] proposed a projection method for nonmonotone variational inequality problem having a set-valued mapping.

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Proximal point algorithm (in short, PPA) was introduced by Martinet [9] and further developed by many researchers [10-16]. The original PPA is proposed for finding zeroes of a maximal monotone operator. As the exact version of PPA is generally not easy to implement, developments of effectively implementing PPA have been studied extensively. Inexact version is the main topic. To improve the performance of inexact PPA, a hybrid projection-proximal point method was suggested in [14] which adds a projection to generate the next iteration.

PPA for finding a solution of a variational inequality is also discussed in the literature; see [17-21]. The iterative scheme of this algorithm is given by

$$x^{k+1} = \Pi_C(x^k - \tau F(x^{k+1})), \tag{1.2}$$

where $\tau > 0$ and $\Pi_C$ denotes the projection onto $C$. The convergence of PPA for (1.1) was proved in [3] under the assumption that the mapping $F$ is pseudomonotone. Without assuming pseudomonotonicity, we attempt to improve the PPA and prove its global convergence. Actually, this is realized by improving an equivalent version of PPA. We prove that (1.2) is equivalent to the following scheme

$$x^{k+1} = \Pi_{C\cap H_k}(x^k), \tag{1.3}$$

where $H_k := \{x \in \mathbb{R}^n : \langle F(y^k), x - y^k\rangle \le 0\}$, with $y^k$ satisfying $y^k = \Pi_C(x^k - \tau F(y^k))$. Though this observation is motivated by [14], our purpose is to weaken the monotone-type assumption while [14] is to improve the performance from the numerical point. It should be mentioned that [6] proposed a proximal point method for nonmonotone variational inequality problems subject to box constraints and obtained only that there exists a subsequence converging to a solution of variational inequalities.

We propose a new method which replaces (1.3) by the following scheme

$$x^{k+1} = \Pi_{C\cap H_{i_k}}(x^k), \tag{1.4}$$

where

$$i_k := \arg\max_{0\le j\le k} \frac{\langle F(y^j), x^k - y^j\rangle}{\|F(y^j)\|}. \tag{1.5}$$

The global convergence of this new PPA is proved in this paper, assuming that the Minty variational inequalities have a solution. The latter assumption is strictly less restrictive than pseudomonotonicity of the mapping, and satisfied by a very large class of quasimonotone variational inequalities. To our best knowledge, this is the first PPA for quasimonotone variational inequalities. In fact, the Minty variational inequalities having a solution does not necessarily imply pseudomonotonicity, even quasimonotonicity. Therefore, this method applies to nonmonotone variational inequalities, which is an important distinguish from known PPA.

Just like many known approximate proximate point algorithms (in short, APPA), it is either impossible or impractical to solve the PPA exactly. The use of approximate solutions is essential for implementing algorithms.

We propose an inexact version

$$\begin{aligned} &\text{step 1. Compute } y^k : \langle y^k + c_k F(y^k) - x^k - \varepsilon_k, y - y^k\rangle \ge 0,\ \forall y \in C,\\ &\text{step 2. Compute next iteration point} : x^{k+1} = \Pi_{C\cap H_{i_k}}(x^k), \end{aligned} \tag{1.6}$$

where $\{c_k\}$ is a bounded sequence of positive scalars, $i_k$ is defined in (1.5), and prove the global convergence under the assumption that Minty variational inequality has a solution.

The organization of this paper is as follows. In the next section, we recall some definitions and lemmas. In the section 3, we shall propose a new exact PPA and a new APPA for variational inequality and establish their global convergence.

## 2. Preliminaries

**Definition 2.1.** $x \in C$ is called a solution of the Minty variational inequality iff

$$\langle F(y), y - x\rangle \geq 0, \quad \forall\, y \in C.$$

Throughout this paper, we denote the Minty variational inequality by MVI$(F, C)$.

**Definition 2.2.** The mapping $F$ is said to be pseudomonotone on $C$ iff for each pair $x, y \in C$,

$$\langle F(x), y - x\rangle \geq 0 \Rightarrow \langle F(y), y - x\rangle \geq 0.$$

**Definition 2.3.** The mapping $F$ is said to be quasimonotone on $C$ iff for each pair $x, y \in C$,

$$\langle F(x), y - x\rangle > 0 \Rightarrow \langle F(y), y - x\rangle \geq 0.$$

It can be seen that pseudomonotone mappings are quasimonotone, but not vice versa. For example, if $F(x) := x^2$ and $C := [-1, 1]$, then $F$ is quasimonotone but not pseudomonotone on $C$.

**Definition 2.4.** $x \in C$ is called a trivial solution iff $\langle F(x), y - x\rangle = 0$ for all $y \in C$. If a solution of VI$(F, C)$ is not a trivial solution, it is called a nontrivial solution.

**Definition 2.5.** K is a subset of $\mathbb{R}^n$, the set valued mapping $G : K \rightrightarrows \mathbb{R}^n$ is said to be a KKM map iff

$$co\{x_1, x_2, \cdots, x_m\} \subset \cup_{i=1}^{m} G(x_i),\ \forall \{x_1, x_2, \cdots, x_m\} \subset K,$$

where $coA$ denotes the convex hull of $A$, $A \subset \mathbb{R}^n$ is a subset.

**Proposition 2.1.** [8] If the mapping $F$ is pseudomonotone and VI$(F, C)$ has a solution, then MVI$(F, C)$ has a solution.

**Proposition 2.2.** [8, 22] Some other results for the solvability of Minty variational inequality can be stated that if one of the following assumptions is satisfied:

(a) $F$ is quasimonotone and VI$(F, C)$ has a nontrivial solution;

(b) $F$ is quasimonotone on $C, F \neq 0$ and $C$ is bounded;

(c) $F$ is the gradient of $G$, where $G$ is a differentiable quasiconvex function on an open set $K \supset C$ and attains its global minimum on $C$;

(d) $F$ is quasimonotone on $C, F \neq 0$ on $C$ and there exists a positive number $r$ such that, for every $x \in C$ with $\|x\| \geq r$, there exists $y \in C$ such that $\|y\| \leq r$ and $\langle F(x), y - x\rangle \leq 0$,

then the solution set of Minty variational inequality is nonempty.

**Remark 2.1.** In [23], when $K \subset \mathbb{R}$ is a compact set, $F : K \rightrightarrows \mathbb{R}$ is quasimonotone if and only if MVI$(K, F)$ has a solution.

Note that the mapping can be not quasimonotone even if MVI$(F, C)$ has a solution. This can be seen from the following example.

**Example 2.1.** [23] Let

$$K := [-1, 1] \times [-1, 1]$$

and

$$F(x_1, x_2) = (x_1^2, x_2^2), \quad \forall (x_1, x_2) \in K.$$

If we take $x = (-1, 0)$ and $y = (-\frac{1}{2}, -1)$, then $\langle F(x), y - x\rangle = \frac{1}{2} > 0$, but $\langle F(y), y - x\rangle = -\frac{7}{8} < 0$. Therefore $F$ is not quasimonotone on $K$. However, $(-1, -1)$ is a solution of MVI$(F, K)$.

To solve the variational inequality problem, one usually uses the following natural residual function

$$r(x) := x - \Pi_C(x - F(x)), \tag{2.1}$$

where $\Pi_C$ denotes the metric projection onto $C$.

**Lemma 2.1.** [24] Let K be a subset of $\mathbb{R}^n$ and $G : K \rightrightarrows \mathbb{R}^n$ be a closed-valued KKM map. If there exists a $x \in K$ such that $G(x)$ is compact, then $\cap_{y \in K}(G(y)) \neq \emptyset$.

One can refer to [25] for the following results.

**Lemma 2.2.** For any $x, y \in R^n$, then
$\langle \Pi_C(x) - \Pi(y), x - y\rangle \geq \|\Pi_C(x) - \Pi_C(y)\|^2$.

**Lemma 2.3.** Let $y \in C$ and $x \in R^n$. Then $\langle y - \Pi_C(x), x - \Pi_C(x)\rangle \leq 0$.

**Lemma 2.4.** $r(x) = 0$ iff $x$ is a solution of VI$(F, C)$.

# 3. Main Results

The following proposition is based on the references [1-2].

**Proposition 3.1.** Assume that $x$ and $y$ satisfy the following three conditions:

(C1) $x, y \in C$ and $x \neq y$;

(C2) $\alpha > 0$;

(C3) $y = \Pi_C(x - \alpha F(y))$;

Then we have

$$y = \Pi_{C \cap H}(x),$$

where $H := \{x' \in \mathbb{R}^n : \langle F(y), x' - y\rangle \leq 0\}$.

*Proof.* Let

$$x(\alpha) = \Pi_C(x - \alpha F(y))$$

and

$$H^b := \{x' \in \mathbb{R}^n : \langle F(y), x' - y\rangle = 0\}.$$

Since

$$y = \Pi_C(x - \alpha F(y)) \in H^b,$$

it follows that

$$x(\alpha) = \Pi_{C \cap H^b}(x - \alpha F(y)).$$

By (C3) and Lemma 2.2, we get

$$\begin{aligned}\langle F(y), x-y\rangle &= \langle F(y), \Pi_C(x) - \Pi_C(x-\alpha F(y))\rangle \\ &= \frac{1}{\alpha}\langle x-(x-\alpha F(y)), \Pi_C(x) - \Pi_C(x-\alpha F(y))\rangle \\ &\geq \frac{1}{\alpha}\|x-y\|^2 \\ &> 0.\end{aligned}$$

This implies that

$$\langle F(y), x-y\rangle > 0 \text{ and } \|F(y)\| \neq 0. \tag{3.1}$$

If we take

$$\alpha^1 = \frac{\langle F(y), x-y\rangle}{\|F(y)\|^2},$$

then

$$x - \alpha^1 F(y) \in H^b.$$

By the definition of $H^b$, we have

$$\langle (\alpha^1 - \alpha)F(y), z-w\rangle = 0, \text{ for all } z,\ w \in H^b.$$

Let

$$\bar{x}(\alpha^1) = \Pi_{C\cap H^b}(x - \alpha^1 F(y)).$$

Furthermore, we have

$$\begin{aligned}\|x-\alpha^1 F(y) - \bar{x}(\alpha^1)\|^2 + \|(\alpha-\alpha^1)F(y)\|^2 = \|x-\alpha F(y) - \bar{x}(\alpha^1)\|^2 \\ \|x-\alpha^1 F(y) - x(\alpha)\|^2 + \|(\alpha-\alpha^1)F(y)\|^2 = \|x-\alpha F(y) - x(\alpha)\|^2.\end{aligned} \tag{3.2}$$

This implies that

$$\begin{aligned}&\|x-\alpha^1 F(y) - \bar{x}(\alpha^1)\|^2 - \|x-\alpha^1 F(y) - x(\alpha)\|^2 \\ &= \|x-\alpha F(y) - \bar{x}(\alpha^1)\|^2 - \|x-\alpha F(y) - x(\alpha)\|^2.\end{aligned}$$

Therefore, we obtain

$$x(\alpha) = y = \bar{x}(\alpha^1) = \Pi_{C\cap H^b}(x-\alpha^1 F(y)).$$

If we let

$$\hat{x} = \Pi_{C\cap H^b}(x) \text{ and } \bar{x} = \Pi_{H^b}(x),$$

then we have

$$\begin{aligned}\|y'-\bar{x}\|^2 &= \|y'-x\|^2 - \|x-\bar{x}\|^2 \\ &\geq \|\hat{x}-x\|^2 - \|x-\bar{x}\|^2 \\ &= \|\hat{x}-\bar{x}\|^2,\ \forall y' \in C\cap H^b.\end{aligned}$$

This implies that

$$\Pi_{C\cap H^b}(x)=\Pi_{C\cap H^b}(\bar{x}).$$

By the (3.1), we know

$$x\notin H,$$

which implies

$$\Pi_{C\cap H^b}(x)=\Pi_{C\cap H}(x).$$

Therefore, we have

$$y=\Pi_C(x-\alpha F(y))=\Pi_{C\cap H}(x).$$

□

By Proposition 3.1, the classical exact proximal point algorithm $x^{k+1}=\Pi_C(x^k-\tau F(x^{k+1}))$ is equivalent to

$$\begin{aligned} y^k &= \Pi_C(x^k-\tau F(y^k)) \\ x^{k+1} &= \Pi_{C\cap H_k}(x^k), \end{aligned} \tag{3.3}$$

where $H_k:=\{x\in\mathbb{R}^n:\langle F(y^k),x-y^k\rangle\le 0\}$. Now we present a modified proximal point algorithm, which is based on (3.3).

**Algorithm 1.** (New exact proximal point algorithm)
Step 0. Select any $\beta\in(0,+\infty),x^0\in C,k=0$.
Step 1. For $x^k\in C$, compute

$$y^k=\Pi_C(x^k-\beta F(y^k)), \tag{3.4}$$

If $x^k=y^k$, stop; else compute

$$i_k:=\arg\max_{0\le j\le k}\frac{\left\langle F(y^j),x^k-y^j\right\rangle}{\|F(y^j)\|}, \tag{3.5}$$

that is, $i_k\in\{0,1,\dots,k\}$ and

$$\frac{\left\langle F(y^{i_k}),x^k-y^{i_k}\right\rangle}{\|F(y^{i_k})\|}\ge\frac{\left\langle F(y^j),x^k-y^j\right\rangle}{\|F(y^j)\|},\quad \forall\, j\in\{0,1,\dots,k\}.$$

Step 2. Compute

$$x^{k+1}=\Pi_{C\cap H_{i_k}}(x^k), \tag{3.6}$$

where $H_i:=\{x\in\mathbb{R}^n:\langle F(y^i),x-y^i\rangle\le 0\}$. Let $k=k+1$ and return to Step 1.

**Remark 3.1.** (1) If (3.4) is well-defined and $x^k\ne y^k$, by Lemma 2.3, we have

$$\begin{aligned} &\langle x^k-\beta F(y^k)-y^k,x^k-y^k\rangle \\ &=\|x^k-y^k\|^2-\beta\langle F(y^k),x^k-y^k\rangle \\ &\le 0. \end{aligned} \tag{3.7}$$

This implies

$$\beta\langle F(y^k), x^k - y^k\rangle \geq \|x^k - y^k\|^2 \\ > 0. \tag{3.8}$$

Therefore, $\|F(y^k)\| \neq 0$, and (3.5) is well-defined.
(2) If MVI$(F, C)$ has a solution, then $C \cap H_{i_k}$ is a nonempty closed convex subset and therefore (3.6) is well-defined.

[3] proved that (3.4) is well-defined if the mapping is pseudomonotone. Now we prove that (3.4) is well-defined under the weaker assumption that the MVI$(F, C)$ has a solution.

**Proposition 3.2.** If $F : C \to \mathbb{R}^n$ is a continuous mapping and MVI$(F, C)$ has a solution, then $y^k$ satisfying (3.4) exists.

*Proof.* If we let $T(u) = u + \beta F(u) - x^k$, then (3.4) is equivalent to solving the VI$(T, C)$. For given $y \in C$, then we let

$$G(y) = \{u \in C : \langle T(u), y - u\rangle \geq 0\}. \tag{3.9}$$

We claim that $G$ is a KKM map. If not, there exists $\{y_1, y_2, \cdots, y_m\} \subset C$ such that

$$co\{y_1, y_2, \cdots, y_m\} \not\subset \cup_{i=1}^m(G(y_i)).$$

Thus there exists nonnegative numbers $\{\alpha_1, \alpha_2, \cdots, \alpha_m\} \subset [0, 1]$ such that $\Sigma_{i=1}^m \alpha_i = 1$ and

$$\langle T(\Sigma_{i=1}^m \alpha_i y_i), y_j - (\Sigma_{i=1}^m \alpha_i y_i)\rangle < 0, j = 1, 2, \cdots, m.$$

This implies that

$$\langle T(\Sigma_{i=1}^m \alpha_i y_i), \Sigma_{j=1}^m \alpha_j y_j - (\Sigma_{i=1}^m \alpha_i y_i)\rangle < 0. \tag{3.10}$$

This is a contradiction. Therefore, we verify the claim.

Since $F$ is a continuous mapping from $C$ to $\mathbb{R}^n$, it follows that $T$ is also a continuous mapping. Thus $G$ is a closed-valued KKM map. If $x^*$ is a solution of MVI$(F, C)$, then

$$\langle F(y), y - x^*\rangle \geq 0, \ \forall y \in C. \tag{3.11}$$

For any $u \in G(x^*)$, by (3.9) and the definition of $T$, we have

$$\begin{aligned} 0 &\leq \langle T(u), x^* - u\rangle \\ &= \langle u + \beta F(u) - x^k, x^* - u\rangle \\ &= \langle u - x^k, x^* - u\rangle + \beta\langle F(u), x^* - u\rangle. \end{aligned} \tag{3.12}$$

It follows from (3.11) that

$$\langle u - x^k, x^* - u\rangle \geq 0.$$

This implies that

$$\|u\|^2 - (\|x^k\| + \|x^*\|)\|u\| \leq \|x^k\|\|x^*\|.$$

This shows that $G(x^*)$ is a bounded set. Furthermore, $G(x^*)$ is closed, this implies that $G(x^*)$ is a compact set. By Lemma 2.1, we have

$$\cap_{y \in C} G(y) \neq \emptyset.$$

This verifies that $y^k$ satisfying (3.4) exists.

□

Compared to other proximal point algorithms for variational inequality in the literature, we present a global convergence result of Algorithm 1 under the assumption that MVI$(F, C)$ has a solution, without assuming usual generalized monotonicity.

**Theorem 3.1.** If there exists $x^* \in C$ such that

$$\langle F(y), y - x^* \rangle \geq 0, \quad \forall\, y \in C, \tag{3.13}$$

then either the algorithm terminates in a finite number of iterations or generates a sequence $\{x^k\}$ converging to a solution of the variational inequality problem.

*Proof.* By Lemma 2.3, we have

$$\begin{aligned}\|x^k - x^*\|^2 &= \|x^k - x^{k+1} + x^{k+1} - x^*\|^2 \\ &= \|x^k - x^{k+1}\|^2 + 2\langle x^k - x^{k+1}, x^{k+1} - x^* \rangle + \|x^{k+1} - x^*\|^2 \\ &\geq \|x^k - x^{k+1}\|^2 + \|x^{k+1} - x^*\|^2,\end{aligned} \tag{3.14}$$

which implies

$$\begin{aligned}\|x^{k+1} - x^*\|^2 &\leq \|x^k - x^*\|^2 - \|x^k - x^{k+1}\|^2 \\ &= \|x^k - x^*\|^2 - d^2(x^k, C \cap H_{i_k}) \\ &\leq \|x^k - x^*\|^2 - d^2(x^k, H_{i_k}) \\ &= \|x^k - x^*\|^2 - (\frac{\langle F(y^{i_k}), x^k - y^{i_k} \rangle}{\|F(y^{i_k})\|})^2.\end{aligned} \tag{3.15}$$

It follows that $\{x^k\}$ is bounded and

$$\frac{\langle F(y^{i_k}), x^k - y^{i_k} \rangle}{\|F(y^{i_k})\|} \to 0, \text{ as } k \to +\infty. \tag{3.16}$$

Using (3.8) and (3.5), we have that

$$\frac{\langle F(y^{i_k}), x^k - y^{i_k} \rangle}{\|F(y^{i_k})\|} \geq \frac{\langle F(y^k), x^k - y^k \rangle}{\|F(y^k)\|} > 0.$$

This and (3.16) together imply that

$$\frac{\langle F(y^k), x^k - y^k \rangle}{\|F(y^k)\|} \to 0, \text{ as } k \to +\infty. \tag{3.17}$$

Since $y^k \in C$, (3.13) implies $\langle F(y^k), y^k - x^* \rangle \geq 0$. It follows from Lemma 2.3 and (3.4) that

$$\begin{aligned}\langle x^k - y^k, x^* - y^k \rangle &\leq \langle x^k - y^k, x^* - y^k \rangle + \beta \langle F(y^k), y^k - x^* \rangle \\ &= \langle x^k - \beta F(y^k) - y^k, x^* - y^k \rangle \\ &\leq 0,\end{aligned}$$

which implies

$$\|y^k\|^2 - \langle x^k + x^*, y^k \rangle \leq -\langle x^k, x^* \rangle. \tag{3.18}$$

Since $\{x^k\}$ is bounded, it follows that $\{y^k\}$ is bounded. Furthermore, since $F$ is a continuous mapping, there exists $M_1 > 0$ such that

$$\|F(y^k)\| \leq M_1, \text{ for all } k. \tag{3.19}$$

From (3.8) and (3.19), we obtain

$$\frac{\langle F(y^k), x^k - y^k\rangle}{\|F(y^k)\|} \geq \frac{1}{\beta M_1}\|x^k - y^k\|^2. \tag{3.20}$$

This and (3.17) together imply that

$$\lim_{k\to+\infty} \|x^k - y^k\| = 0. \tag{3.21}$$

Since $\{x^k\}$ is bounded, it follows that there exist subsequence $\{x^{k_l}\}$ and $x' \in C$ such that

$$x^{k_l} \to x', \text{ as } l \to +\infty. \tag{3.22}$$

From (3.4) and Lemma 2.3, we have

$$\langle x^{k_l} - \beta F(y^{k_l}) - y^{k_l}, y - y^{k_l}\rangle \leq 0, \ \forall y \in C. \tag{3.23}$$

Letting $l \to +\infty$, we obtain that

$$\langle F(x'), y - x'\rangle \geq 0, \ \forall y \in C. \tag{3.24}$$

Thus $x'$ is a solution of VI$(F, C)$.
By (3.5), we get

$$\frac{\left\langle F(y^{i_{k_l}}), x^{k_l} - y^{i_{k_l}}\right\rangle}{\|F(y^{i_{k_l}})\|} \geq \frac{\left\langle F(y^j), x^{k_l} - y^j\right\rangle}{\|F(y^j)\|}, \quad \forall\, j \in \{0, 1, \ldots, k_l\}. \tag{3.25}$$

Combining (3.25) with (3.16) and letting $l \to +\infty$, we have that

$$\langle F(y^j), x' - y^j\rangle \leq 0, \text{ for all } j. \tag{3.26}$$

If we replace $x^*$ by $x'$ in the (3.14), then we obtain that the sequence $\{\|x^k - x'\|\}$ is decreasing. Moreover, since $x'$ is an accumulation point of $\{x^k\}$, we have $x^k \to x'$ as $k \to +\infty$. □

We now propose an inexact version and establish its global convergence in the following.

**Algorithm 2.** (New inexact proximal point algorithm)
Step 0. Select any $M > 0, \gamma_0 \in (0, 1), c_0 \in (0, +\infty), x^0 \in C$.
Step 1. For $x^k \in C$, compute $y^k$:

$$\langle y^k + c_k F(y^k) - x^k - \varepsilon_k, y - y^k\rangle \geq 0, \ \forall y \in C, \tag{3.27}$$

where $\|\varepsilon_k\| \leq \min\{M, \gamma_k\|x^k - y^k\|\}, \gamma_k \in (0, 1), c_k \in (0, +\infty)$.
If $x^k = y^k$, stop; else compute

$$i_k := \arg\max_{0\leq j\leq k} \frac{\left\langle F(y^j), x^k - y^j\right\rangle}{\|F(y^j)\|}, \tag{3.28}$$

that is, $i_k \in \{0, 1, \ldots, k\}$ and

$$\frac{\left\langle F(y^{i_k}), x^k - y^{i_k}\right\rangle}{\|F(y^{i_k})\|} \geq \frac{\left\langle F(y^j), x^k - y^j\right\rangle}{\|F(y^j)\|}, \quad \forall\, j \in \{0, 1, \ldots, k\}.$$

Step 2. Compute

$$x^{k+1} = \Pi_{C\cap H_{i_k}}(x^k), \tag{3.29}$$

where $H_i := \{x \in \mathbb{R}^n : \langle F(y^i), x - y^i\rangle \leq 0\}$. Let $k = k + 1$ and return to Step 1.

**Remark 3.2.** (1) If (3.27) is well-defined and $x^k \neq y^k$, we have

$$\begin{aligned} &\langle x^k - c_k F(y^k) + \varepsilon_k - y^k, x^k - y^k \rangle \\ &= \|x^k - y^k\|^2 + \langle \varepsilon_k, x^k - y^k \rangle - c_k \langle F(y^k), x^k - y^k \rangle \\ &\leq 0. \end{aligned} \tag{3.30}$$

This implies

$$\begin{aligned} c_k \langle F(y^k), x^k - y^k \rangle &\geq \|x^k - y^k\|^2 + \langle \varepsilon_k, x^k - y^k \rangle \\ &\geq \|x^k - y^k\|^2 - \|\varepsilon_k\| \|x^k - y^k\| \\ &\geq \|x^k - y^k\|^2 - \gamma_k \|x^k - y^k\|^2 \\ &\geq (1 - \gamma_k) \|x^k - y^k\|^2 \\ &> 0. \end{aligned} \tag{3.31}$$

Therefore, $\|F(y^k)\| \neq 0$, and (3.28) is well-defined.
(2) If MVI$(F, C)$ has a solution, then $C \cap H_{i_k}$ is a nonempty closed convex subset and therefore (3.29) is well-defined.
(3) Since exact solution of PPA is always an approximate solution, it follows that we only need to show that the (3.4) in this paper is well-defined.

**Theorem 3.2.** If there exists $x^* \in C$ such that

$$\langle F(y), y - x^* \rangle \geq 0, \quad \forall\, y \in C, \tag{3.32}$$

and $0 < \inf_k c_k \leq \sup_k c_k < +\infty$, $\sup \gamma_k < 1$, then either the algorithm terminates in a finite number of iterations or generates a sequence $\{x^k\}$ converging to a solution of the variational inequality problem.

*Proof.* By Lemma 2.3, we get

$$\begin{aligned} \|x^k - x^*\|^2 &= \|x^k - x^{k+1} + x^{k+1} - x^*\|^2 \\ &= \|x^k - x^{k+1}\|^2 + 2\langle x^k - x^{k+1}, x^{k+1} - x^* \rangle + \|x^{k+1} - x^*\|^2 \\ &\geq \|x^k - x^{k+1}\|^2 + \|x^{k+1} - x^*\|^2. \end{aligned} \tag{3.33}$$

This implies

$$\begin{aligned} \|x^{k+1} - x^*\|^2 &\leq \|x^k - x^*\|^2 - \|x^k - x^{k+1}\|^2 \\ &= \|x^k - x^*\|^2 - d^2(x^k, C \cap H_{i_k}) \\ &\leq \|x^k - x^*\|^2 - d^2(x^k, H_{i_k}) \\ &= \|x^k - x^*\|^2 - \left(\frac{\langle F(y^{i_k}), x^k - y^{i_k} \rangle}{\|F(y^{i_k})\|}\right)^2. \end{aligned} \tag{3.34}$$

From (3.34), we know that $\{x^k\}$ is bounded and

$$\frac{\langle F(y^{i_k}), x^k - y^{i_k} \rangle}{\|F(y^{i_k})\|} \to 0, \text{ as } k \to +\infty. \tag{3.35}$$

Using (3.31) and (3.28), we have

$$\frac{\langle F(y^{i_k}), x^k - y^{i_k} \rangle}{\|F(y^{i_k})\|} \geq \frac{\langle F(y^k), x^k - y^k \rangle}{\|F(y^k)\|} > 0.$$

This and (3.35) together imply that

$$\frac{\langle F(y^k), x^k - y^k \rangle}{\|F(y^k)\|} \to 0, \text{ as } k \to +\infty. \tag{3.36}$$

By (3.27), we have

$$
\begin{aligned}
&\langle x^k - y^k, x^* - y^k\rangle + \langle \varepsilon_k, x^* - y^k\rangle \\
&\leq \langle x^k - c_k F(y^k) + \varepsilon_k - y^k, x^* - y^k\rangle \\
&\leq 0.
\end{aligned}
$$

This implies that

$$
\|y^k\|^2 - \langle x^k + x^*, y^k\rangle - \langle \varepsilon_k, y^k\rangle \leq -\langle x^k, x^*\rangle - \langle \varepsilon_k, x^*\rangle. \tag{3.37}
$$

Therefore, we obtain that

$$
\|y^k\|^2 - (\|x^k\| + M + \|x^*\|)\|y^k\| \leq (\|x^k\| + M)\|x^*\|. \tag{3.38}
$$

Since $\{x^k\}$ is bounded, it follows that $\{y^k\}$ is bounded. Furthermore, since $F$ is a continuous mapping, there exists $M_2 > 0$ such that

$$
\|F(y^k)\| \leq M_2, \text{ for all } k. \tag{3.39}
$$

Combining (3.31) with (3.39), we obtain that

$$
\frac{\langle F(y^k), x^k - y^k\rangle}{\|F(y^k)\|} \geq \frac{1-\gamma_k}{c_k M_2}\|x^k - y^k\|^2. \tag{3.40}
$$

This, (3.36), $0 < \inf_k c_k \leq \sup_k c_k < +\infty$, and $\sup \gamma_k < 1$ together imply that

$$
\lim_{k\to+\infty} \|x^k - y^k\| = 0. \tag{3.41}
$$

Since $\{x^k\}$ is bounded, it follows that there exists an index set $K \subset \{0, 1, 2, 3, \cdots\}$, a subsequence $\{x^k\}_{k\in K}$, and $x' \in C$ such that

$$
x^k \to x', \text{ as } k \to +\infty, k \in K. \tag{3.42}
$$

From (3.27), we have

$$
\langle x^k - c_k F(y^k) + \varepsilon_k - y^k, y - y^k\rangle \leq 0, \ \forall y \in C, k \in K. \tag{3.43}
$$

Since $0 < \inf_k c_k \leq \sup_k c_k < +\infty$, it follows that there exists an index subset $K_0 \subset K$ such that $c_k \to c_0 > 0$ as $k \to +\infty, k \in K_0$. Letting $k \to +\infty, k \in K_0$, we obtain that

$$
\langle F(x'), y - x'\rangle \geq 0, \ \forall y \in C. \tag{3.44}
$$

Thus this implies that $x'$ is a solution of VI$(F, C)$. By (3.28), we get

$$
\frac{\left\langle F(y^{i_k}), x^k - y^{i_k}\right\rangle}{\|F(y^{i_k})\|} \geq \frac{\left\langle F(y^j), x^k - y^j\right\rangle}{\|F(y^j)\|}, \quad \forall\, j \in \{0, 1, \ldots, k\}. \tag{3.45}
$$

Combining (3.45) with (3.35) and letting $k \to +\infty, k \in K_0$, we have that

$$
\langle F(y^j), x' - y^j\rangle \leq 0, \text{ for all } j. \tag{3.46}
$$

If we replace $x^*$ by $x'$ in the (3.33), then we obtain that the sequence $\{\|x^k - x'\|\}$ is decreasing. Moreover, since $x'$ is an accumulation point of $\{x^k\}$, we have $x^k \to x'$ as $k \to +\infty$. □

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