Sum theorems for maximal monotone operators under weak compactness conditions

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Abstract

This note presents a summary of our most recent results concerning the maximal monotonicity of the sum of two maximal monotone operators defined in a locally convex space under the classical interiority qualification condition when one of their domains or ranges has a weak relative compactness property.

Keywords maximal monotone operator, Minkowski sum, weak compact set

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1 Preliminaries

The main objective of this article is to provide conditions under which the sum of two maximal monotone operators defined in a locally convex space remains maximal monotone when one of the domains or ranges of those operators has a weak relative compactness property. Our main source of inspiration is the extrapolation of the argument in the new proof given in [6] for the sum theorem for maximal monotone operators defined in reflexive Banach spaces.

All the locally convex spaces (LCS for short) considered in this paper are Hausdorff separated. Given (X, τ) a LCS we denote by X^* its topological dual, by "w" the weak topology on X, by "w^*" the weak-star topology on X^*, and by "τ_b^*" the strong topology on X^* which is the topology of the uniform convergence on $\tau-$bounded sets of X. The Mackey topology on $X(X^*)$ relative to the duality (X, X^*) $((X^*, X))$ is denoted by τ_M (τ_M^*).

To a multifunction $T : X \rightrightarrows X^*$ we associate its

- *graph:* $\operatorname{Graph} T = \{(x, x^*) \in X \times X^* \mid x^* \in T(x)\}$,
- *inverse:* $T^{-1} : X^* \rightrightarrows X$, $\operatorname{Graph} T^{-1} = \{(x^*, x) \mid (x, x^*) \in \operatorname{Graph} T\}$,
- *domain*: $D(T) := \{x \in X \mid T(x) \neq \emptyset\} = \operatorname{Pr}_X(\operatorname{Graph} T)$, and
- *range*: $R(T) := \{x^* \in X^* \mid x^* \in T(x) \text{ for some } x \in X\} = \operatorname{Pr}_{X^*}(\operatorname{Graph} T)$,

where Pr_X and Pr_{X^*} are the projections of $X \times X^*$ onto X and X^*, respectively. When no confusion can occur, T will be identified with $\operatorname{Graph} T$.

A multi-valued operator $T : X \rightrightarrows X^*$ is *monotone* if, for every $x_1^* \in T(x_1)$, $x_2^* \in T(x_2)$, $\langle x_1 - x_2, x_1^* - x_2^* \rangle \geq 0$. Here $\langle x, x^* \rangle := c(x, x^*) := x^*(x)$, $x \in X$, $x^* \in X^*$ is the *duality product* or *coupling* of $X \times X^*$.

An element $z = (x, x^*) \in X \times X^*$ is *monotonically related* (m.r. for short) *to* T if, for every $(a, a^*) \in \operatorname{Graph} T$, $\langle x - a, x^* - a^* \rangle \geq 0$.

An operator $T : X \rightrightarrows X^*$ is *maximal monotone* if it is monotone and every m.r. to T element $z = (x, x^*) \in X \times X^*$ belongs to $\operatorname{Graph} T$.

In general $T : X \rightrightarrows X^*$ is $\mu-$*representable* in $X \times X^*$, where μ is a topology on $X \times X^*$, if there exists $f : X \times X^* \to \overline{\mathbb{R}}$ proper convex $\mu-$lower semicontinuous such that $f \geq c$ and

$$\operatorname{Graph} T = [f = c] := \{(x, x^*) \in X \times X^* \mid f(x, x^*) = \langle x, x^* \rangle\}.$$

Such a function f is called a $\mu-$*representative* of T.

An operator is simply called *representable* if it is $\mu-$representable with respect to any topology μ on $X \times X^*$ compatible with the natural duality $(X \times X^*, X^* \times X)$.

For a subset A of a LCS (E, μ) we denote by: • "$\operatorname{co} A$" the *convex hull* of A, • "$\operatorname{cl}_\mu(A) = \overline{A}^\mu$" the $\mu-$*closure* of A, and by • "$\operatorname{int}_\mu A$" the topological interior of A.

When the topology μ is implicitly understood (such is the case when we deal with the initial topology of a LCS) the use of the $\mu-$notation is avoided.

To a proper function $f : X \to \overline{\mathbb{R}}$ and a topology τ on X we associate:

- the *convex hull* of f: $\operatorname{conv} f : X \to \overline{\mathbb{R}}$, which is the greatest convex function majorized by f, $(\operatorname{co} f)(x) := \inf\{t \in \mathbb{R} \mid (x, t) \in \operatorname{co}(\operatorname{epi} f)\}$, $x \in X$,

- the $\tau-$*lsc convex hull* of f: $\operatorname{cl}_\tau \operatorname{co} f : X \to \overline{\mathbb{R}}$, is the greatest τ–lsc convex function majorized by f, $(\operatorname{cl}_\tau \operatorname{co} f)(x) := \inf\{t \in \mathbb{R} \mid (x, t) \in \operatorname{cl}_\tau \operatorname{co} \operatorname{epi} f\}$, $x \in X$,

- the *convex conjugate* of $f : X \to \overline{\mathbb{R}}$ with respect to the dual system (X, X^*): $f^* : X^* \to \overline{\mathbb{R}}$, $f^*(x^*) := \sup\{\langle x, x^* \rangle - f(x) \mid x \in X\}$, $x^* \in X^*$.

- the *subdifferential* of f at $x \in X$: $\partial f(x) := \{x^* \in X^* \mid \langle x' - x, x^* \rangle + f(x) \leq f(x'),\ \forall x' \in X\}$ for $x \in \operatorname{dom} f$; $\partial f(x) := \emptyset$ for $x \notin \operatorname{dom} f$. Recall that $N_C = \partial I_C$ is the *normal cone* of $C \subset X$, where I_C is the *indicator function* of $C \subset X$ defined by $I_C(x) := 0$ for $x \in C$ and $I_C(x) := \infty$ for $x \in X \setminus C$.

Let (X, τ) be a LCS and let $Z := X \times X^*$. It is known that $(Z, \tau \times w^*)^* = Z$ via the coupling

$$z \cdot z' := \langle x, x'^* \rangle + \langle x', x^* \rangle, \quad \text{for } z = (x, x^*),\ z' = (x', x'^*) \in Z.$$

For a proper function $f : Z \to \overline{\mathbb{R}}$ all the above notions are defined similarly. The conjugate of f with respect to the natural dual system (Z, Z) induced by the previous coupling is given by

$$f^{\square} : Z \to \overline{\mathbb{R}}, \quad f^{\square}(z) = \sup\{z \cdot z' - f(z') \mid z' \in Z\},$$

and, by the biconjugate formula, $f^{\square\square} = \mathrm{cl}_{\tau\times w^*}\mathrm{co}f$ whenever $f^{\square}$ or $\mathrm{cl}_{\tau\times w^*}\,\mathrm{conv}\, f$ is proper.

Note that $f^{\square}(x, x^*) = f^*(x^*, x)$ where f^* is the convex conjugate relative to the dual system $(X \times X^*, X^* \times X^{**})$. Here $X^{**} := (X^*, \tau_b^*)^*$.

To a multi-valued operator $T : X \rightrightarrows X^*$ we associate:

- the *indicator function of* T; $\iota_T(z) = 0$, if $z \in \mathrm{Graph}\, T$, $\iota_T(z) = \infty$, otherwise;

- the *Fitzpatrick function* of T (introduced in [1]): $\varphi_T : Z \to \overline{\mathbb{R}}$, $\varphi_T := c_T^{\square}$, where $c_T : Z \to \overline{\mathbb{R}}$, $c_T := c + \iota_T$; or in extended form

$$\varphi_T(x, x^*) := \sup\{\langle x-a, a^*\rangle + \langle a, x^*\rangle \mid (a, a^*) \in \mathrm{Graph}\, T\},\ (x, x^*) \in X \times X^*, \tag{1}$$

- $\psi_T := \mathrm{cl}_{\,\tau\times w^*}\,\mathrm{conv}\, c_T$; $\psi_T = \varphi_T^{\square} = c_T^{\square\square}$ whenever φ_T or ψ_T is proper (for example when T is non-empty monotone (see e.g. [3, Proposition 3.2])).

Note that $[\varphi_T \le c]$ describes the set of all $z \in Z$ that are monotonically related to T.

2 Sum representability

Recall the following two results due to Zălinescu:

Proposition 1. *(Zălinescu [7, Proposition 1]) Let X_1, X_2 be LCS's and let $f_1, f_2 : X = X_1 \times X_2 \to \overline{\mathbb{R}}$ be proper convex functions. If there exists $(x_1, x_2) \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$ such that $f_1(\cdot, x_2)$ is continuous at x_1 and $f_2(x_1, \cdot)$ is continuous at x_2 then, for every $x^* \in X^* = X_1^* \times X_2^*$*

$$(f_1 + f_2)^*(x^*) = \min\{f_1^*(u^*) + f_2^*(x^* - u^*) \mid u^* \in X^*\}.$$

Here "min" stands for an infimum that is attained when finite.

Theorem 2. *Let E, F be LCS's and let $\phi_1, \phi_2 : E \times F \to \overline{\mathbb{R}}$ be proper convex functions. Consider $\rho : E \times F \to \overline{\mathbb{R}}$ defined by*

$$\rho(x, y) := \inf\{\phi_1(x, y_1) + \phi_2(x, y_2) \mid y_1 + y_2 = y\}.$$

Assume that there exists $(x_0, y_0) \in \operatorname{dom} \phi_2$ such that $x_0 \in \operatorname{Pr}_E(\operatorname{dom} \phi_1)$ and $\phi_2(\cdot, y_0)$ is continuous at x_0. Then, for every $x^* \in E^*$, $y^* \in F^*$

$$\rho^*(x^*, y^*) = \min\{\phi_1^*(x_1^*, y^*) + \phi_2^*(x_2^*, y^*) \mid x_1^* + x_2^* = x^*\}.$$

For a proof see [4, Theorem 34, p. 166].

Proposition 3. *under review*

Proposition 4. *Let (X, τ) be a LCS and let $A, B : X \rightrightarrows X^*$ be representable with representatives r_A, r_B. Assume that there exists $(x_0, x_0^*) \in \operatorname{dom} r_B^\square$ such that $x_0 \in \operatorname{Pr}_X(\operatorname{dom} r_A^\square)$ and $r_B^\square(\cdot, x_0^*)$ is τ_M-continuous at x_0. Then $A + B$ is representable and $r := r_A \square_2 r_B$ is exact and a representative of $A + B$. Here*

$$(r_A \square_2 r_B)(x, x^*) := \min\{r_A(x, x_1^*) + r_B(x, x_2^*) \mid x_1^* + x_2^* = x^*\},\ (x, x^*) \in X \times X^*,$$

and "min" stands for an infimum that is attained when finite.

Proposition 5. *Let (X, τ) be a LCS and let $A, B : X \rightrightarrows X^*$ be representable such that $\operatorname{co} R(B)$ is weak-star relatively compact in X^*. Then $A + B$ is representable.*

Proposition 6. *Let (X, τ) be a locally convex, let $A : X \rightrightarrows X^*$ be representable, and let $f : X \to \mathbb{R}$ be proper convex lower semicontinuous. If f is $\tau-$continuous at some $x_0 \in D(A)$ then $A + \partial f$ is representable.*

3 Two intermediary results

Theorem 7. *under review*

Remark 8.

Theorem 9. *Let (X, τ) be a LCS, let $T : X \rightrightarrows X^*$ be maximal monotone and let $C \subset X$ be $\tau-$closed convex. If $D(T) \cap \operatorname{int} C \neq \emptyset$ and $\operatorname{cl}\operatorname{co} D(T)$ is weakly compact then $T + N_C$ is maximal monotone.*

Remark 10. The previous result shows that every maximal monotone operator T with a weak relatively compact $\operatorname{co} D(T)$ is distinguished (see [5, Definition 3]) with respect to any topology compatible with the duality (X, X^*). As a result, with respect to the Mackey topology τ_M which is the strongest duality compatible topology, T is identifiable, $D(T)$ is thick, and $\operatorname{cl}_{\tau_M} D(T)$ is convex (see [4, 5] for details).

As a consequence of [5, Theorem 36] we have

Theorem 11. *Let (X, τ) be a barreled LCS. If $A, B : X \rightrightarrows X^*$ are maximal monotone with $\operatorname{cl}\operatorname{co} D(A)$ weak-compact, $D(A) \cap \operatorname{int} D(B) \neq \emptyset$, and $\operatorname{cl} D(B)$ convex then $A + B$ is maximal monotone.*

4 Sum theorems

Theorem 12. *Let (X, τ) be a LCS and let $A, B : X \rightrightarrows X^*$ be maximal monotone such that $\operatorname{co} D(A)$ is weak relatively compact in X and $\operatorname{co} R(B)$ is weak-star relatively compact in X^*. Then $A+B$ is and $A^{-1}+B^{-1}$ are maximal monotone.*

The next result is a generalization of Heisler's result (see [2, Theorem 40.4, p.156]).

Theorem 13. *Let (X, τ) be a LCS and let $A, B : X \rightrightarrows X^*$ be representable such that $D(A) = D(B) = X$ and $A + B$ is representable. Then A, B, $A + B$ are maximal monotone.*

In particular if X is a Banach space or A, B are maximal monotone with $\operatorname{co} R(A)$, $\operatorname{co} R(B)$ weak-star relatively compact in X then $A + B$ is maximal monotone.

Theorem 14. *Let (X, τ) be a LCS and let $A : X \rightrightarrows X^*$ be maximal monotone such that $\operatorname{co} D(A)$ is weak relatively compact in X. Let $f : X \to \mathbb{R}$ be proper convex lower semicontinuous such that f is $\tau-$continuous at some $x_0 \in D(A)$. Then $A + \partial f$ is maximal monotone.*

Theorem 15. *Let (X, τ) be a LCS and let $A : X \rightrightarrows X^*$ be maximal monotone such that $\operatorname{co} R(A)$ is weak-star relatively compact in X and let $f : X \to \mathbb{R}$ be proper convex lower semicontinuous. If f is $\tau-$continuous at some $x_0 \in \operatorname{int}\operatorname{dom} f$ then $A + \partial f$ is maximal monotone.*

Proposition 16. *Let (X, τ) be a LCS and let $f : X \to \mathbb{R}$ be proper convex lower semicontinuous. If f is $\tau-$continuous at some $x_0 \in \operatorname{dom} f$ then ∂f is maximal monotone. In this case $\overline{D(\partial f)}^{\tau} = \operatorname{cl}_\tau \operatorname{dom} f$.*

Proposition 17. *Let (X, τ) be a LCS and let $f : X \to \mathbb{R}$ be proper convex lower semicontinuous. If $\operatorname{dom} f$ is weak relatively compact then ∂f is maximal monotone. In this case, for every locally convex topology μ on X that is compatible with the duality (X, X^*), $\overline{D(\partial f)}^{\mu} = \operatorname{cl}_\mu \operatorname{dom} f$.*

Corollary 18. *Let (X, τ) be a LCS, let $A : X \rightrightarrows X^*$ be maximal monotone such that $\operatorname{co} D(A)$ is weak relatively compact in X or $\operatorname{co} R(A)$ is weak-star relatively compact in X, and let $f : X \to \mathbb{R}$ be proper convex lower semicontinuous. If f^* is τ_M^*-continuous at some $x_0^* \in \operatorname{int} \operatorname{dom} f^*$ then $A^{-1} + \partial f^*$ is maximal monotone.*

Theorem 19. *Let (X, τ) be a barreled LCS. If $A, B : X \rightrightarrows X^*$ are maximal monotone, $\operatorname{cl} \operatorname{co} R(A)$ is weak-star compact, $\operatorname{int} D(B) \neq \emptyset$, and $\operatorname{cl} D(B)$ is convex then $A + B$ is maximal monotone.*

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