Risk-Averse Bargaining in a Stochastic Optimization Context

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Problem definition: Bargaining situations are ubiquitous in economics and management. We consider the problem of bargaining for a fair ex-ante distribution of random profits arising from a cooperative effort of a fixed set of risk-averse agents. Our approach integrates optimal managerial decision making into bargaining situations with random outcomes and explicitly models the impact of risk aversion. The proposed solution rests on a firm axiomatic foundation and yet allows to compute concrete bargaining solutions for a wide range of practically relevant problems. **Methodology/results**: We model risk preferences using coherent acceptability functionals and base our bargaining solution on a set of axioms that can be considered a natural extension of Nash bargaining to our setting. We show that the proposed axioms fully characterize a bargaining solution, which can be efficiently computed by solving a stochastic optimization problem. We characterize special cases where random payoffs of players are simple functions of overall project profit. In particular, we show that for players with distortion risk functionals, the optimal bargaining solution can be represented by an exchange of standard options contracts with the project profit as the underlying asset. We illustrate the concepts in the paper with a detailed example of risk-averse households that jointly invest into a solar plant. **Managerial implications**: We demonstrate that there is no conflict of interest between players about management decisions and that risk aversion facilitates cooperation. Furthermore, our results on the structure of optimal contracts as a basket of option contracts provides valuable guidance for negotiators.

Key words: Stochastic bargaining games; coherent risk measures; stochastic programming; photovoltaics

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1. Introduction

In this paper, we study situations where several economic agents jointly undertake a project involving management decisions under uncertainty about some random parameters that determine project outcomes. In this context the questions on how to optimally manage the project and how to distribute its profits arise. These two fundamental questions, though intimately related, are usually treated in two separate strands of literature.

The first question falls into the domain of stochastic optimization, which was successfully applied to a wide range of decision problems under uncertainty in the last decades (Shapiro et al. 2009, Birge and Louveaux 2011). However, stochastic optimization typically is concerned with optimal decisions for single agents as opposed to strategic interactions between several agents. The second question of distributing the rewards of the project is the subject of cooperative game theory (e.g. Maschler et al. 2013) where solution approaches include bargaining games and coalitional games. As opposed to stochastic optimization, cooperative game theory usually treats project outcomes as parameters of the problem and completely abstracts from the underlying decision situation faced by the cooperating agents.

In this paper, we propose an integrated view on stochastic optimization and cooperative game theory, focusing on bargaining games, i.e., situations where agents bargain for shares of the profits earned by the grand coalition and cannot form sub-coalitions. We are interested in real-world decision problems where agents whose risk aversion can be described by coherent acceptability functionals bargain for the uncertain outcomes of a project. The paper proposes a framework on how to optimize and distribute profits that depend on both decisions and stochastic risk factors. In particular, we aim for an ex-ante split of profits that can be contractually fixed before random outcomes are observed thus enabling a transfer of risk between the agents.

Based on a set of axioms and mild assumptions on the nature of decisions, we characterize a unique distribution of acceptability values which can be efficiently computed by solving a stochastic optimization problem. Furthermore, we show that if all agents use distortion functionals, the split-up of profits can be represented by an exchange of standard call and put options contracts, simplifying the real world implementation of the optimal allocation. Finally, we demonstrate the applicability of our approach in an example considering the joint investment in solar panels mounted on an apartment house owned by several parties.

There is a rich literature on theoretical properties of bargaining games (Maschler et al. 2013) and bargaining situations are ubiquitous in economics and management. Examples include, but are not limited to, bankruptcy problems where debtors bargain for shares of the collateral (e.g., Bulow and Rogoff 1989, Stutzer 2018), individual and collective bargaining for wages (e.g., Haller and Holden 1990, Malcomson 1997), the theory of joint ventures (e.g., Lee 2004, Lukas and Welling 2012), optimal ownership and the theory of the firm (e.g., De Meza and Lockwood 1998, Schmitz 2013), bargaining within a family and implications for the marriage market (e.g., Lundberg and Pollak 1993, Pollak 2019), applications in politics such as negotiations for fishery quotas as in Kampas (2015), bargaining for a monetary union as in Chang (1995), or bargaining over water resources as in Degefu et al. (2016), and applications in pre-trial bargaining and litigation (e.g., Bebchuk 1984, Anderlini et al. 2019).

Apart from bargaining there are also different approaches in cooperative game theory such as envy free allocations (Fragnelli and Marina 2003), maximal overall utility (Melese et al. 2017), or, most prominently, membership in the core of a coalitional game (Habis and Herings 2011, Karsten et al. 2015, Toriello and Uhan 2017, Németh and Pintér 2017, Asimit and Boonen 2018, Leng et al. 2020). In contrast to our approach, most papers focus on theoretical properties such as existence and stability of solutions, but do not derive an explicit split up of random profits (e.g., Habis and Herings 2011, Xu and Veinott 2013, Parilina and Tampieri 2018).

Another difference of our approach to the extant literature is that almost all papers in stochastic cooperative game theory assume risk-neutral decision making, neglecting the impact of risk aversion on the outcome of the game (e.g. Charnes and Granot 1976, 1977, Xu and Veinott 2013, Parilina and Tampieri 2018). Among the authors that consider risk preferences, most use expected utility (e.g. Suijs et al. 1999, Fragnelli and Marina 2003, Timmer et al. 2005, Melese et al. 2017, Németh and Pintér 2017) to model risk attitudes and so far there are only few papers that use coherent risk measures in cooperative games (Uhan 2015, Boonen et al. 2016, Toriello and Uhan 2017, Asimit and Boonen 2018).

Although unrealistic when it comes to the design of contracts, ex-post bargaining after the random outcome is known is the mainstream in the stochastic bargaining literature. Exceptions are early works on two player Nash-bargaining over simple lotteries, which are first considered in Roth and Rothblum (1982) with the aim to analyze the impact of risk-aversion on the bargaining outcome. However, since the description of randomness is restricted to lotteries between two outcomes, the approach is not suitable for the analysis of most real-world situations. Another exception is Riddell (1981), where two players bargain based on expected utility on finite state spaces. In this simplified framework it is possible to characterize Pareto efficient allocations by elementary calculation. However, such an approach is not possible in our multi-player setup with coherent acceptability functionals on general probability spaces.

The papers Uhan (2015) and Toriello and Uhan (2017) study a special class of linear programming coalitional games, i.e., games where the payoff of players derives from the value that they have in sub-coalitions backing threats to break away from the grand coalition. The papers are close to our approach as the authors model risk aversion using coherent risk measures and derive an ex-ante split-up of random profits. The authors define profits for all sub-games and show that the core of the considered game is non-empty. Apart from the fact that Uhan (2015) and Toriello and Uhan (2017) study coalitional games instead of bargaining games, these papers differ from the present paper in that they impose a rather special separable, linear structure to ensure that the game can be *decomposed* into games played by sub-coalitions. Furthermore, the authors make limiting assumptions on the employed risk measures and probability spaces. In contrast, our approach makes virtually none of these assumptions and therefore is more widely applicable.

This paper thus contributes to the literature in the following ways:

1. We propose a novel solution approach for bargaining games defined on general probability spaces where joint management decisions with stochastic payoffs are explicitly modeled. The players at the same time bargain *ex-ante* for optimal management decisions and a fair allocation of the resulting (random) outcome using coherent acceptability measures to describe their risk preferences.

   Using theoretical results on the theory of sup-convolution of risk measures in Acciaio and Svindland (2009), Barrieu and El Karoui (2005), Rüschendorf (2013), we are able to show that the resulting optimization problems fully integrate with two-stage and multi-stage stochastic programming. Our approach thus combines stochastic optimization with bargaining theory. Furthermore, our results exhibit interesting structural similarity to recent work on competitive equilibria between risk-averse agents as studied in Heath and Ku (2004), Ralph and Smeers (2015), Philpott et al. (2016).

2. Our solution rests on an axiomatic framework that can be seen as a natural modification of the classical Nash bargaining axioms. In particular, we take into account that for risky projects the amount of money a player is willing to invest (and therefore to put at risk) influences the outcomes of the bargaining process. As a consequence, we deviate from the classical invariance axiom in Nash bargaining theory and replace it by a "strategy proofness" axiom, which is similar to some proposals in the literature on deterministic allocation rules (e.g. Hougaard 2009). We show that under mild assumptions our axioms yield a unique allocation of acceptability values which can be efficiently computed from the solution of a stochastic optimization problem.

   In particular, our axiomatic setup leads to a reformulation in terms of stochastic bargaining with bargaining powers. The bargaining powers of the players follow canonically from the axioms and the risk preferences of the agents and do not have to be determined by additional arguments *outside* the mathematical specification of the bargaining problem.

   In summary, while our results are based on rather technical arguments, the practical implementation of the resulting allocations is rather simple: Once an arbitrary Pareto efficient solution is found, there is no additional computational effort required to obtain the correct allocations as these follow from simple formulas. Furthermore, in most cases of practical relevance, the allocations are simple enough to be part of real-world contracts and lend themselves to implementation by non-specialists.

3. Our results have several real-world implications for negotiations between risk-averse agents, which are of interest not only for the computation of optimal outcomes but also as guidance for negotiators involved in bargaining processes.
   (a) We show that there is no conflict of interest between players on how to manage the project. Thus managerial decisions are not *strategic*, implying that project management can be conceptually, temporally, and organizationally decoupled from the process of bargaining for the allocation of profits.
   (b) Interestingly, our findings imply that risk aversion facilitates cooperation. In particular, we can show that there is a *diversification gain* from cooperating, which allows players to reduce their risk by pooling their endowments, much like in the case of a portfolio whose risk is smaller than the risk of the individual assets it consists of. This effect is absent in deterministic bargaining and risk-neutral bargaining and adds an independent motivation to cooperate, which increases the attractiveness of large coalitions.
   (c) The proposed bargaining outcomes are fair and are likely to be accepted by participants. In particular, we show that the random shares of participants are comonotone, i.e., that for optimal allocations all individual profits increase with project profits and there are no

situations where the gain of one player is the loss of another. Furthermore, we derive a simple and comprehensible rule for the distribution of overall acceptability of the project which takes into account the size of the contributions and therefore leads to transparent and fair allocations.

(d) Our results show that a simple ex-post proportional split up of project profits, as advocated in large parts of the literature and often used in practice, can only be optimal in degenerate cases. However, we show that under mild conditions on risk preferences, optimal allocations take the form of an exchange of standard option contracts between the players. This structure is conducive to implementing the calculated *ex-ante* allocation of profits in contractual agreements and thereby facilitates the real world implementation of the bargaining outcome and allows negotiators to focus on agreements of this type, which helps to structure and simplify negotiations.

The paper is structured as follows: In Section 2, we introduce stochastic bargaining games with a fixed set of players in which agents agree on an allocation of payoffs before the randomness realizes. The solution thus entails a distribution of profits in every possible future scenario.

We impose the axioms of Pareto optimality, symmetry, and strategy proofness for a solution. While the symmetry axiom encapsulates a notion of fairness that makes our approach similar to Nash bargaining, strategy proofness avoids the problem of strategic splitting of agents and leads to endogenously defined unique bargaining powers. We argue that these axioms are natural in our context and show that they uniquely characterize a bargaining solution.

In Section 3, we analyze the resulting bargaining problems using results about the sup-convolution of coherent risk measures. In particular, we obtain necessary and sufficient conditions for the existence of a solution in terms of the subgradient sets of the players' coherent risk functionals.

We show that affine and linear allocation rules as used, e.g., in Deprez and Gerber (1985), Suijs et al. (1999), Barrieu and El Karoui (2005), Timmer et al. (2005), Baeyens et al. (2013) can only be optimal, if there is a *least risk-averse* player. Furthermore, we prove that in the much more realistic case where the risk preferences of all agents can be described by distortion functionals, the exchange of finitely many standard option contracts achieves the optimal allocation of profits.

In Section 4, we analyze a detailed example of a bargaining game between the owners of an apartment building who install solar panels on the roof of their jointly owned house. The question of stimulating the build-up of renewable capacity is central in the combat of climate change, and the complicated ownership structures of apartment buildings are a significant impediment to an extension of privately owned solar power beyond single-family houses. The results of the case study, in particular the findings on optimal tilt angles under risk aversion, are therefore of independent interest.

Section 5 concludes the paper and discusses avenues for further research.

All proofs are relegated to the appendix.

2. The Bargaining Problem

In this section, we introduce an axiomatic framework for a class of bargaining problems under uncertainty. Section 2.1 outlines the general setting, introduces notation, and discusses our choice of risk preferences. In Section 2.2, we introduce and motivate three axioms for risk-averse bargaining under uncertainty and show that our axiomatic framework uniquely characterizes a bargaining solution. Lastly, we show that the bargaining solution can be represented as the solution to a stochastic optimization problem, which generalizes classical Nash bargaining with bargaining power to stochastic games.

2.1. Motivation & Setting

We consider a project that yields random profits and is jointly undertaken by n risk-averse players. All random quantities are defined on a common probability space $\mathcal{Y} = (\Omega, \mathcal{F}, \mathbb{P})$.

Each participant $i \in N := \{1, \dots, n\}$ obtains a profit $R_i : \Omega \to \mathbb{R}$ when playing alone and not as a member of the group, while the whole group of n players that cooperates receives a joint random profit of $M : \mathcal{X} \times \Omega \to \mathbb{R}$. Note that M may depend on joint managerial decisions $x \in \mathcal{X}$, where $\mathcal{X}$ is the set of feasible decisions. For a fixed $x \in \mathcal{X}$, we denote by $M_x : \Omega \to \mathbb{R}$ the random variable $\omega \mapsto M(x, \omega)$. Positive M and R_i indicate profits, whereas negative values model losses.

We assume that projects can only be undertaken jointly by all players and forming sub-coalitions is either legally, organizationally, or practically impossible. In particular, we consider a bargaining setup, where players cannot obtain a higher share of M based on the threat of forming sub-coalitions.

When cooperating, the group decides on $x \in \mathcal{X}$ and aims at distributing the joint outcome M_x among the participants in a "fair way", which leads to individual (random) payoffs $L(x, \omega) = (L_1(x, \omega), \dots, L_n(x, \omega))$ such that

$$M(x, \omega) = \sum_{i=1}^{n} L_i(x, \omega), \quad a.s. \tag{1}$$

It is worthwhile to point out that in the above setting, the allocation of profits cannot be decided *ex-post*, i.e., after the uncertainty realizes. This is made clear by the following example.

Example 1. Let there be two risk-averse players and two equally likely future states of the world, i.e., $N = \{1, 2\}$, $\Omega = \{\omega_1, \omega_2\}$, and $\mathbb{P}(\{\omega_1\}) = \mathbb{P}(\{\omega_2\}) = 0.5$. Assume further that $R_1(\omega_j) = -R_2(\omega_j)$ and $R_j(\omega_1) = -R_j(\omega_2)$ for $j \in \{1, 2\}$. If the players are risk-averse, they would prefer a certain payment of 0 to their endowments and thus have an incentive to agree to *pool their risks* by defining a payout rule

$$L_i(\omega) = \frac{1}{2}(R_1(\omega) + R_2(\omega)), \quad \forall i \in N, \tag{2}$$

which yields a certain profit of 0 for both players in both states of the world.

Note that, in agreeing to (2), the players shift wealth between future states to achieve a better risk profile *ex-ante*, much like when entering an insurance contract. They are, however, not necessarily better off ex-post, since in each of the two states, there is one player who regrets having pooled her risks. For this reason, an ex-post allocation which is based on the allocation of wealth after the realization of randomness cannot be ex-ante optimal, as players would not agree to shift wealth between the scenarios after the randomness realized. Hence, in the above example an ex-post allocation would result in the same payoffs as a solution without any cooperation and therefore in ex-ante welfare losses.

Consequently, we have to consider the risk preferences of agents in deciding about random ex-ante payoffs L. We assume that the risk preferences of player i can be expressed by a coherent acceptability functional $\mathcal{A}_i$ defined on the Lebesgue space $\mathcal{L}^p(\mathcal{Y})$ of p-integrable random variables. Throughout this paper we will assume that $p \geq 1$.

Definition 1. $\mathcal{A} : \mathcal{L}^p(\mathcal{Y}) \to \mathbb{R}$ is a coherent acceptability functional if for all $X,\ Y \in \mathcal{L}^p(\mathcal{Y})$, $\lambda \in \mathbb{R}_0^+$, and $c \in \mathbb{R}$ the following holds:

1. Monotonicity: If $X \leq Y$ almost surely, then $\mathcal{A}(X) \leq \mathcal{A}(Y)$.
2. Positive homogeneity: $\mathcal{A}(\lambda X) = \lambda \mathcal{A}(X)$.
3. Super-additivity: $\mathcal{A}(X+Y) \geq \mathcal{A}(X) + \mathcal{A}(Y)$.
4. Translation invariance: $\mathcal{A}(X+c) = \mathcal{A}(X) + c$.

Economists traditionally think about acceptability in terms of expected utility. However, since the seminal paper by Artzner et al. (1999), coherent risk and acceptability functionals have gained wide popularity as an alternative in finance and management science because of their ease of interpretation and analytical tractability. In particular, coherent acceptability measures can be interpreted in terms of monetary units, which is why they are sometimes also called *monetary utility functions* (Föllmer and Schied 2004).

The most important and widely used acceptability functional is the *Average Value-at-Risk*, which is defined as

$$\mathrm{AVaR}_\alpha(X) = \frac{1}{\alpha} \int_0^\alpha F_X^{-1}(t)\, dt \tag{3}$$

with α being the risk aversion parameter α and F_X the cumulative distribution function of a random variable X. For continuous distributions, AVaR can be interpreted as the expectation of X, conditional that it falls below its α-quantile. The family of AVaR acceptability measures contains the essential infimum ($\alpha = 0$) and the expectation ($\alpha = 1$) as limiting special cases.

A generalization of the AVaR is the class of distortion functionals (see, e.g., Föllmer and Schied 2004, Pflug and Römisch 2007) which are defined as

$$\mathcal{A}_i(X) = \int_0^1 \mathrm{AVaR}_\alpha(X)\, dm_i(\alpha), \tag{4}$$

where m_i are arbitrary probability measures on $[0,1]$, which models the relative importance of different quantiles for valuing the acceptability of X. Distortion functionals form an important subset of coherent acceptability functionals containing most functionals relevant for practical applications.

2.2. An Axiomatic Approach

This subsection develops an axiomatic theory of bargaining solutions, which is close to the Nash bargaining approach, but, as opposed to classical theory, allows profits L_i and opportunity losses R_i to be random and players to be risk-averse.

DEFINITION 2. An *instance* of the bargaining problem is a triple $(R, M, \mathcal{A})$ with $R = (R_1, \dots, R_n)$, $R_i \in \mathcal{L}^p(\mathcal{Y})$ $(i \in N)$, $M : \mathcal{X} \times \Omega \to \mathbb{R}$, and $\mathcal{A} = (\mathcal{A}_1, \dots, \mathcal{A}_n) : \bigtimes_{i=1}^n \mathcal{L}^p(\mathcal{Y}) \to \mathbb{R}^n$ where $\mathcal{A}_i : \mathcal{L}^p(\mathcal{Y}) \to \mathbb{R}$ are coherent acceptability measures for all $i \in N$.

For a given instance $(R, M, \mathcal{A})$, we define the set of feasible *acceptability allocations* that exceed the acceptability of the endowment for each player as

$$\mathcal{U}(R, M, \mathcal{A}) = \{u \in \mathbb{R}^n \mid \exists x \in \mathcal{X}\ \exists L : \sum_{i=1}^n L_i \underset{a.s.}{=} M_x, L_i \in \mathcal{L}^p(\mathcal{Y}),\ \mathcal{A}_i(R_i) \leq u_i \leq \mathcal{A}_i(L_i),\ i \in N\}. \quad (5)$$

We assume that the values $v_i := \mathcal{A}_i(R_i)$, representing the personal evaluations of the opportunity losses, are always strictly positive. The instance $(R, M, \mathcal{A})$ is called *feasible* if the set $\mathcal{U}(R, M, \mathcal{A})$ is non-empty. A sufficient condition for feasibility of an instance $(R, M, \mathcal{A})$ is $\sum_{i=1}^n R_i \leq M_x$ almost surely for some x (see Corollary 1, Equation (17) for a weaker condition that is also necessary). We restrict ourselves to feasible instances, i.e., to cases where cooperation has a non-negative value for the players.

DEFINITION 3. A *bargaining solution* is a mapping F that assigns a vector $u = (u_1, \dots, u_n) \in \mathcal{U}(R, M, \mathcal{A})$ of acceptability values to each feasible instance $(R, M, \mathcal{A})$.
In the following, we discuss three axioms and argue why they should be fulfilled in the context of the bargaining games outlined in this paper. The first axiom is Pareto optimality, which is a basic efficiency requirement also used in the classical Nash bargaining approach.

AXIOM 1 (PAR). The acceptability allocation $u^* = F(R, M, \mathcal{A})$ prescribed by the bargaining solution is *Pareto optimal.* In particular, if $u \in \mathcal{U}(R, M, \mathcal{A})$ is another acceptability allocation, then

$$\exists i \in N : u_i > u_i^* \ \Rightarrow\ \exists j \in N : u_j < u_j^*.$$

Any allocation that does not fulfill PAR could be improved, at least for some players, without making anybody else worse off. Choosing such an allocation is obviously wasteful and therefore undesirable.

The next axiom is symmetry, which captures the essence of the notion of *fairness* employed by the Nash bargaining approach. The axiom requires that if all players are indistinguishable in every relevant aspect of the game, they should obtain the same acceptability value. More formally, we can state the symmetry axiom as follows.

AXIOM 2 (SYM). If $\mathcal{A}_i(R_i) = \mathcal{A}_j(R_j)$ for all $i, j \in N$, and if for every permutation σ of N,

$$(u_1, \dots, u_n) \in \mathcal{U}(R, M, \mathcal{A}) \ \Rightarrow\ (u_{\sigma(1)}, \dots, u_{\sigma(n)}) \in \mathcal{U}(R, M, \mathcal{A}),$$

then $u_i^* = u_j^*$ for all $i, j \in N$.

Lastly, we add an axiom that ensures what is sometimes called *strategy proofness* of the allocation rule. The idea is that players should not benefit from strategically splitting up or merging. This is made sure by requiring that larger investments (larger opportunity losses) should entail larger shares of the profit. Similar ideas have been applied in different forms in the literature, e.g., in the concept of the proportional Shapley value (e.g. Feldman et al. 1999, Béal et al. 2018), but also in the general context of (deterministic) allocation rules (see Hougaard 2009, Chapter 2).

More formally, we require the following to hold.

Axiom 3 (STR). If $(R', M, \mathcal{A}')$ is a game associated with $(R, M, \mathcal{A})$ by $N' = N \setminus \{i\} \cup \{(i,k) \,|\, k = 1, \ldots, \ell\}$ and

$$R'_j = R_j \quad \forall j \neq i,\; R'_{ik} = R_i/\ell \quad (k = 1, \ldots, \ell),$$
$$\mathcal{A}'_j = \mathcal{A}_j \quad \forall j \neq i,\; \mathcal{A}'_{ik} = \mathcal{A}_i \quad (k = 1, \ldots, \ell),$$

then the acceptability allocations $u = F(R, M, \mathcal{A})$ and $u' = F(R', M, \mathcal{A}')$ are interrelated by

$$u'_j = u_j \quad \forall j \neq i, \quad u'_{ik} = u_i/\ell \quad (k = 1, \ldots, \ell). \tag{6}$$

Thus, in the new game, where player i "splits herself" into ℓ subplayers, splitting also her opportunity losses in equal parts, the benefits should be split as well.

Remark 1. In a purely deterministic context, (Hougaard 2009) uses a related axiom called *No Advantageous Splitting* (NAS), which is stronger than STR as it considers the split up of players into parts of arbitrary size. Our results extend the framework in (Hougaard 2009) to the case of coherent acceptability functionals and consequently require a technically more involved analysis to make use of STR in the axiomatic characterization of allocations.

In the following, we give a characterization of those bargaining solutions that satisfy Axioms 1 – 3. To this end, we will require the following assumption, which we assume to hold for the rest of the paper.

Assumption 1. *The set $\mathcal{X}$ is a compact topological space and $x \mapsto M_x$ is a continuous mapping from $\mathcal{X}$ to $\mathcal{L}^p(\mathcal{Y})$ equipped with the weak topology σ.*

Using this assumption, we are able to show the following result, which ensures that there is always a solution to every feasible bargaining game. The proof is relegated to the appendix.

Proposition 1. *Under Assumption 1, the set $\mathcal{U}(R, M, \mathcal{A})$ is compact.*

Building on Proposition 1, we prove that $\mathcal{U}$ is a convex polyhedron in the next result.

Proposition 2. *The set $\mathcal{U}(R, M, \mathcal{A})$ is the polytope*

$$\{u \in \mathbb{R}^n \,|\, \forall i \in N : u_i \geq \mathcal{A}_i(R_i), \sum_{i=1}^n u_i \leq z\} \tag{7}$$

with

$$z = z(R, M, \mathcal{A}) = \max_{u,L,x} \left\{ \sum_{i=1}^n u_i \,|\, \sum_{i=1}^n L_i = M_x,\; \mathcal{A}_i(R_i) \leq u_i \leq \mathcal{A}_i(L_i),\, \forall i \in N,\, x \in \mathcal{X} \right\}. \tag{8}$$

We are now in the position to show that Axioms 1 – 3 characterize a unique acceptability allocation.

Theorem 1. *A bargaining solution F satisfies Axioms 1 – 3 if and only if for every instance $(R,M,\mathcal{A})$ the acceptability allocations $u=F(R,M,\mathcal{A})$ are given by*

$$u_i = \frac{v_i}{\sum_{j=1}^n v_j} \cdot z(R,M,\mathcal{A}), \quad \forall i \in N, \tag{9}$$

where z is defined by (8) and $v_i = \mathcal{A}_i(R_i)$ for all $i \in N$.

We call a bargaining solution F satisfying Axioms 1 – 3 a *Nash-type bargaining solution* or Nash bargaining solution, for short. The next theorem provides the justification for the terminology by showing that the bargaining solution (u,L,x) can be obtained by solving a stochastic optimization problem with a product-form objective function similar to classical Nash bargaining without going through the stepwise procedure of first solving (8) and then applying Theorem 1 to find optimal u_i.

Theorem 2. *If the bargaining solution F fulfills Axioms 1 – 3, $u=F(R,M,\mathcal{A})$ is the optimal solution to the stochastic optimization problem*

$$\begin{array}{ll} \max\limits_{u,L,x} & \prod_{i=1}^n (u_i - v_i)^{v_i} \\ s.t. & u_i \geq v_i, \quad \forall i \in N \\ & u_i \leq \mathcal{A}_i(L_i), \quad \forall i \in N \\ & \sum_{i=1}^n L_i \leq M_x \quad \textit{almost surely,} \\ & x \in \mathcal{X}. \end{array} \tag{10}$$

Remark 2. In Theorem 2, the parameter x is one of the decision variables the players have to decide on. However, as Theorem 1 shows, the case where x can be negotiated by the players does not add any complexity to the bargaining problem itself: Since the dependence on x only impacts the factor $z(R,M,\mathcal{A})$, the optimization with respect to x can be fully decoupled from the bargaining process. In particular, there is no conflict of interest between the players over managerial decisions, since x, loosely speaking, only influences the absolute size of the pie, i.e., the size of the set $\mathcal{U}$, but not the relative shares. This parallels Suijs and Borm (1999), Uhan (2015), Toriello and Uhan (2017) who show a similar result for allocations in the core of stochastic coalitional games between risk-averse players.

Remark 3. Note that the solution to (10) need not be unique in L or x, i.e., it can happen that different choices of L and/or x produce the same optimal solution u. Hence, uniqueness of the bargaining solution can only be guaranteed on the level of the acceptability values u.

Problem (10) can be interpreted as Nash bargaining with bargaining powers for stochastic games played by risk-averse agents.

The v_i that enter the formulation as exponents in the objective ensure that Axiom 3 is respected, which implies that the opportunity loss of agent i can be interpreted as her *bargaining power*. If agent i splits into several agents, the bargaining power decreases and the resulting new agents collectively get the same as agent i gets in the original game.

Beginning with Roth (1979) and Binmore (1980) there is a large literature on Nash bargaining with bargaining power. Bargaining power in this literature is represented by nonnegative numbers γ_i which are used as exponents in the objective function $\prod_{i=1}^n (u_i - v_i)^{\gamma_i}$. In many papers on bargaining power, the values of γ_i are chosen more or less arbitrarily. A notable exception is Binmore et al. (1986), where bargaining power is related to the bargainer's time preferences and to the risk of a breakdown of negotiations in dynamic bargaining models. In our setup, bargaining power (as well as the opportunity loss) is related to the acceptability $\mathcal{A}_i(R_i)$ or lost opportunities, which is a natural point of view for investment problems. Furthermore, this choice has the advantage that bargaining powers are endogenous to the axiomatic framework and do not have to be imposed by additional reasoning.

We conclude with a discussion of the two axioms *invariance with respect to affine transformations* (INV) and *invariance with respect to irrelevant alternatives* (IIA), which are fulfilled in classical Nash bargaining but are absent from our approach.

We do not impose INV, since strategy proofness cannot be maintained simultaneously with INV: As seen from (9), in the presence of PAR and SYM, the axiom STR entails a nonlinear dependence of the values u_i on the parameters v_i, whereas INV prescribes a linear dependence (Muthoo 1999).

Dropping the assumption INV may seem as a gratuitous deviation from traditional Nash bargaining. It should be noted, however, that INV consists of two parts: invariance with respect to multiplication by a positive scalar (i.e., a rescaling of benefits by expressing them in a different unit), and invariance with respect to translation (i.e., a rescaling by shifting the zero point of benefit evaluation). The first type of invariance is fully preserved in our model, but it does not need an explicit axiom, as it follows in our context from the positive homogeneity of coherent acceptability measures: if we express payoffs, e.g., in euros instead of dollars, positive homogeneity of the $\mathcal{A}_i$ entails that the resulting bargaining solution will be the same. The second part of INV, however, cannot be preserved in our framework as we base our model on a particular strand of the cooperative game theory literature (e.g., Feldman et al. 1999, Béal et al. 2018, Hougaard 2009, Besner 2019) where the amount of investment into a joint project *counts*: if player 1 invests a million dollars into the project while player 2 only invests 10 dollars, then the extra profit resulting from the synergy effect should not be divided in shares 50 : 50, as, by virtue of INV, classical (deterministic) Nash bargaining prescribes it for the simplest case of the feasible set. We think that the classical consideration does not make too much sense anymore in the case of risky projects where investments are *at stake*, and therefore replace it by the STR axiom.

We also note that IIA – the most controversial axiom in Nash bargaining (e.g., Luce and Raiffa 1957, Tversky 1972, Kalai and Smorodinsky 1975) – is not needed in our setup: The role of IIA in the classical theory is to deal with nonlinear boundaries of the bargaining set but the set of feasible acceptability values $\mathcal{U}(\mathcal{R}, \mathcal{M}, \mathcal{A})$ has a linear border (see Proposition 2).

3. Characterizing the Optimal Allocation

The main result so far, Theorem 1, characterizes the fair and efficient allocation of the acceptability values u_i to the participants. However, the allocation L of the project profit M as well as a complete characterization of the optimal decision x remain open questions at this point. In particular, solutions of (8) are not unique and in general lead to allocations L that need not be consistent with the prescribed shares in Theorem 1, even if they yield the optimal value z.

This section is devoted to these questions. Section 3.1 characterizes optimal allocations L based on the sup-convolution of coherent acceptability functionals and its properties. In addition, the optimization with respect to x is analyzed in more detail and we discuss the structural differences of risk-averse and risk-neutral bargaining. In Section 3.2, we discuss two important special cases. In particular, it turns out that the presence of a dominating acceptability functional leads to affine allocations. Moreover, we establish that if all functionals $\mathcal{A}_i$ are distortion functionals, then the allocations L_i can be expressed in terms of standard options on the joint success variable M. Section 3.3 is devoted to an algorithmic description that details how a bargaining problem can be solved numerically for concrete practically relevant cases and a discussion of the computational complexity of our approach.

3.1. Characterizing the optimal allocation via sup-convolutions

We start our analysis of problem (8) by introducing the following additional running assumption.

Assumption 2. *The coherent acceptability functionals $\mathcal{A}_i$ are proper and upper semicontinuous.*

From this rather weak assumption it follows that the $\mathcal{A}_i : \mathcal{L}^p(\mathcal{Y}) \to \mathbb{R}$ have dual representations (e.g. Pflug and Römisch 2007, Theorem 2.30)

$$\mathcal{A}_i(X) = \inf\{\mathbb{E}[\zeta_i X] : \zeta_i \in \mathcal{Z}_i\}, \tag{11}$$

where $\mathcal{Z}_i$ are convex sets in $\mathcal{L}^q(\mathcal{Y})$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $p \geq 1$ and

$$\mathcal{Z}_i \subseteq \{\zeta_i \in \mathcal{L}^q(\mathcal{Y}) : \zeta_i \geq 0 \wedge \mathbb{E}[\zeta_i] = 1\}. \tag{12}$$

Assumption 2 also guarantees that the infimum in (11) is attained. It is fulfilled for typical acceptability functionals treated, e.g., in Pflug and Römisch (2007), Föllmer and Schied (2004). As a consequence, $\mathcal{A}_i$ is the support function $\sigma_{\mathcal{Z}_i}$ of $\mathcal{Z}_i$ defined by the right hand side of (11) and is therefore completely characterized by the set $\mathcal{Z}_i$.

For the further development it is important to observe that the maximum sum of acceptability values in (8) can be expressed as the so-called *sup-convolution* $(\square_{i=1}^n \mathcal{A}_i)(M)$ of the individual functionals $\mathcal{A}_i$. In particular, for a given $M \in \mathcal{L}^p(\mathcal{Y})$, we write

$$\mathcal{A}^{\max}(M) := \left(\mathop{\square}_{i=1}^{n} \mathcal{A}_i\right)(M) = \sup\left\{\sum_{i=1}^n \mathcal{A}_i(X_i) \in \mathcal{L}^p(\mathcal{Y}) | \sum_{i=1}^n X_i = M\right\}. \tag{13}$$

Optimal risk sharing problems like (13) were studied by many authors in the mathematical finance and insurance literature starting in the context of classical insurance risk theory with Borch (1962), Bühlmann (1970), Gerber (1979). Subsequently, these methods were generalized using advanced methods of convex analysis such as the inf-convolution and sup-convolution (e.g., Barrieu and El Karoui 2005, Jouini et al. 2008, Ludkovski and Rüschendorf 2008, Acciaio 2009, Acciaio and Svindland 2009). An excellent account of this topic can be found in Rüschendorf (2013).

The following theorem states some important facts about the sup-convolution of coherent acceptability functionals.

Theorem 3 **(Sup-Convolution of Acceptability Functionals)**. *If*

$$\mathcal{Z}^{\max} = \bigcap_{i=1}^{n} \mathcal{Z}_i \neq \emptyset, \tag{14}$$

then:

1. *The domain of $\mathcal{A}^{\max}$ is not empty and we have the following dual representation*

$$\mathcal{A}^{\max}(M) = \inf_{\zeta \in \mathcal{L}^q} \{\mathbb{E}[\zeta M] : \zeta \in \mathcal{Z}^{\max}\}, \quad \forall M \in \mathcal{L}^p(\mathcal{Y}). \tag{15}$$

2. *$\mathcal{A}^{\max}$ is a proper coherent upper semi-continuous acceptability functional.*
3. *An allocation $X = (X_1, \ldots, X_n)$ is optimal for (13) if and only if there is a $\Lambda \in \mathcal{L}^q(\mathcal{Y})$ with*

$$\Lambda \in \partial \mathcal{A}_i(X_i) = \arg\min\{\mathbb{E}[\zeta_i X_i] : \zeta_i \in \mathcal{Z}_i\}, \quad \forall i \in N. \tag{16}$$

   Conversely, for any $\Lambda \in \arg\min_{\zeta \in \mathcal{L}^q}\{\mathbb{E}[\zeta X] : \zeta \in \mathcal{Z}^{\max}\}$ obtained from (15) an optimal allocation X can be found such that (16) is fulfilled.
4. *If $p > 1$, the supremum in (13) is attained and there exists an optimal allocation X. Under the stronger condition (and after suitable reordering) $\mathcal{Z}_1 \cap \bigcap_{i=2}^{n} \operatorname{int} \mathcal{Z}_i \neq \emptyset$, this is also true for $p = 1$.*
5. *The elements of an optimal allocation $X = (X_1, \ldots, X_n)$ are comonotone random variables, i.e.,*

$$(X_i - \mathbb{E}[X_i])(X_j - \mathbb{E}[X_j]) \geq 0, \text{ almost surely}, \forall i, j \in N.$$

Note that comonotonicity of the allocation implies that the profits of all players increase as a function of the optimal project profit M, i.e., it cannot be that the gain of one player is the loss of another. This property is arguably desirable because it enhances the perceived fairness of the solution and therefore its acceptance amongst players. Comonotonicity of optimal allocations resulting from the sup-convolution of coherent risk measures has been derived earlier, based on arguments on comonotone improvement in convex-order arguments (Ludkovski and Rüschendorf 2008). However, since our proof is more direct and much less involved, we decided to include it in the appendix.

Condition (16) clearly resembles standard optimality conditions often found in economics that require the marginal utilities of all players to coincide for an optimal allocation of goods. It is also similar to the conditions found for competitive Nash equilibria in markets where agents have coherent risk preferences (e.g., Heath and Ku 2004, Ralph and Smeers 2015, Philpott et al. 2016). Furthermore, Toriello and Uhan (2017) find a similar condition for the solution of a linear stochastic coalitional game.

To relate the results in Theorem 3 to the bargaining games introduced in Section 2, we first consider the case where M does not depend on auxiliary decisions x.

Based on Theorem 1 and Theorem 3, it turns out that the optimal z in (8) may be obtained by solving the (simpler) sup-convolution problem and that optimal allocations L for the bargaining problem can be easily constructed from any optimal allocation X in (13) by appropriate deterministic side payments.

Corollary 1. *Let $M:\Omega\to\mathbb{R}$ be the profit of the grand coalition and $X=(X_1,\dots,X_n)$ an optimal allocation for the sup-convolution problem* (13). *If*

$$\mathcal{A}^{\max}(M)\geq\sum_{i=1}^{n}\mathcal{A}_i(R_i), \tag{17}$$

then

$$z(R,M,\mathcal{A})=\mathcal{A}^{\max}(M) \tag{18}$$

and the allocation

$$L_i=X_i+\frac{v_i}{\sum_{j=1}^{n}v_j}\mathcal{A}^{\max}(M)-\mathcal{A}_i(X_i) \tag{19}$$

is both optimal for (13) *and a solution of the bargaining game.*

In particular,

$$\Lambda\in\partial\mathcal{A}_i(L_i)=\arg\min\{\mathbb{E}\left[\zeta_i X_i\right]:\ \zeta_i\in\mathcal{Z}_i\},\quad \forall i\in N \tag{20}$$

holds for any

$$\Lambda\in\partial\mathcal{A}^{\max}(M)=\arg\min\{\mathbb{E}\left[\zeta_i M\right]:\ \zeta_i\in\mathcal{Z}^{\max}\} \tag{21}$$

and $L_1,\dots,L_n,M$ are comonotone random variables.

If (17) *is violated, the bargaining problem has no solution.*

Remark 4. The value z can be rewritten as

$$z=\mathcal{A}^{\max}(M)=\mathbb{E}[\Lambda M], \tag{22}$$

where Λ is optimal for (15). This implies that the dual variable Λ can be interpreted as the (stochastic) shadow price of a change in M.

Next, we consider the general case, where M depends on management decisions x. As pointed out in Remark 2, the bargaining process about acceptability values can be completely separated from the maximization with respect to x. Based on the above results on the sup-convolution, it is possible to make this more concrete.

Corollary 2. *Let*

$$A=\arg\max_{y\in\mathcal{X}}\{\mathcal{A}^{\max}(M_y)\}\neq\emptyset \tag{23}$$

and

$$z=\max_{y\in\mathcal{X}}\{\mathcal{A}^{\max}(M_y)\}. \tag{24}$$

Under the assumption that an optimal allocation $X=(X_1,\dots,X_n)$ for (24) *exists, any $x\in A$ is an optimal management decision of the bargaining game, provided that the condition of Corollary 1 holds for $M=M_x$. The optimal allocation of the bargaining problem can be obtained by applying* (19) *and the related acceptability allocation is given by* (9) *with $z(R,M_x,A)=z$.*

Separation of the managerial decision x from the decision on the split-up of M_x simplifies the decision process. In particular, the decision x can be taken in order to maximize overall acceptability without any interference of strategic considerations and ensures that bargaining does not lead to suboptimal outcomes and therefore welfare losses. This is clearly a desirable property of the proposed bargaining approach.

We devote the last part of this section to a discussion of the impact of the risk aversion on the bargaining outcome. To this end, let us contrast bargaining among risk-neutral players ($\mathcal{A}_i = \mathbb{E}$) with the general situation where $\mathcal{A}_i$ describe truly risk-averse preferences. In the case of risk-neutral bargaining, the project is managed such that the *expected* profit of the grand coalition is maximized and the players share the outcomes based on (9) using the expected value of their endowments R_i. Note that in this case, the pointwise distribution of profits can be done more or less arbitrarily as is demonstrated in Example 3, for example, by assigning the whole uncertain project outcome M to any agent i who compensates the other agents by deterministic payments according to their shares u_j.

In the risk-averse case, the situation changes in three important ways:

1. The optimal group decision is made under risk aversion. Note that the risk functional $\mathcal{A}^{\max}$ is a amalgamation of the risk preferences of all the agents and therefore each $\mathcal{A}_i$ potentially affects optimal project management decisions x and consequently changes the distribution M of joint profits.
2. The risk aversion of an agent determines the allocation of acceptability values and therefore the share of M allocated to the agent in every scenario. In particular, for a given fixed value of the group decision x, the share of agent i increases in its evaluation of the endowment R_i, i.e., if the agent would be more risk-averse she would get a smaller share of the overall acceptability. However, at the same time the agent would receive a *more secure* share of M, increasingly insuring her against negative outcomes for higher levels of risk aversion.
3. The fact that agents are risk-averse provides a reason to cooperate, which is not present in conventional deterministic and risk-neutral bargaining. To see this point, consider the situation where there is no management decision x and agents can cooperate by pooling their random endowments R_i such that
$$M = \sum_{i=1}^{n} R_i.$$
In this case, there is no gain from cooperation in the classical sense, since the payoff of the coalition is just the sum of the payoffs of the individual players in the conflict point. Consequently, in deterministic bargaining, agents would not have an incentive to cooperate. This is obviously still true if $\mathcal{A}_i = \mathbb{E}$ for all agents, since in this case
$$\mathcal{A}^{\max}(M) = \mathbb{E}(M) = \sum_{i=1}^{n} \mathbb{E}(R_i),$$
which implies that cooperation does not yield a gain for a coalition of risk-neutral players.

However, note that in the risk-averse case the situation is different, since by the super-additivity of acceptability functionals, we get

$$\mathcal{A}^{\max}(M) \geq \sum_{i=1}^{n} \mathcal{A}^{\max}(R_i) \geq \sum_{i=1}^{n} \mathcal{A}_i(R_i),$$

i.e., the acceptability of the grand coalition is always greater or equal to the sum acceptabilities that can be generated by the single players alone. Furthermore, if the above inequality is strict, then it is easy to see that every player individually profits from cooperation.

This *diversification effect* in the risk-averse setting is absent in deterministic and risk-neutral bargaining and provides an additional incentive to cooperate which transcends the usual motivations to do so.

3.2. Two Important Special Cases: Affine Allocations & Options Contracts

Up to now, we analyzed general optimality and fairness conditions. In this section, we discuss two special cases of risk functionals $\mathcal{A}_i$, which allow for simple functional characterizations of the individual profits L_i in terms of M. The simple forms of L are conducive to a real world implementation of the bargaining solution. Such an implementation would otherwise be complicated by the requirement to agree on a contract that specifies payoffs $L_i(\omega)$ for all agents i in all future states ω of the world, possibly without any further structure to it. We discuss cases where optimal allocations turn out to be either affine functions of the overall profit or can be represented by options contracts with the project profits as the underlying asset.

We start with cases where a dominating acceptability functional is present, i.e. there exists an i_{max} such that $\mathcal{Z}_{i_{\max}} \subseteq \mathcal{Z}_i$ for all $i \in N$ and consequently $i_{\max}$ is *the least risk-averse* player. Such a situation can, for example, occur, if the $\mathcal{A}_i$ are from the same single-parameter family of acceptability functionals as is demonstrated in the following.

Example 2. Consider the situation where the preferences are described by $\mathcal{A}_i = \mathrm{AVaR}_{\alpha_i}$ and assume that $\alpha_n > \alpha_i$ for all $i < n$. Then the conjugate representation

$$\mathrm{AVaR}_{\alpha_i}(X) = \inf\{\mathbb{E}\,[\zeta X] : \mathbb{E}[\zeta] = 1,\, 0 \leq \zeta \leq \alpha_i^{-1}\} \tag{25}$$

implies $\mathcal{Z}_n \subset \mathcal{Z}_i$ for all $i < n$ and AVaR_{α_n} is the dominating acceptability functional.
If this condition is met, the optimal L_i are affine functions of M as the next result shows.

Proposition 3. *If there is an* $i_{\max} \in \{1, \ldots, n\}$ *such that*

$$\mathcal{Z}_{i_{\max}} \subseteq \mathcal{Z}_i \text{ for all } i \in \{1, \ldots, n\}, \tag{26}$$

then $z(R, M, A) = \mathcal{A}_{i_{\max}}(M)$ *and* $L_i = a_i + b_i M$, *where*

$$a_i = \left(\frac{v_i}{\sum_{j=1}^{n} v_j} - b_i \right) \mathcal{A}_{i_{\max}}(M) \text{ and } b_i = \begin{cases} \beta_i, & i \in \mathcal{I}^{\max} = \{j : \mathcal{Z}_j = \mathcal{Z}_{i_{\max}}\} \\ 0, & \text{else}, \end{cases} \tag{27}$$

solves the bargaining game if the condition in Corollary 1 is fulfilled and $\beta_j \geq 0$, $\sum_{j \in \mathcal{I}^{\max}} \beta_j = 1$.

Hence, in the case where there are least risk-averse players, these players take all the risk, while the other players receive deterministic side payments and bear no risk in the optimal allocation. An analogous result was proven by Toriello and Uhan (2017) in the context of coalitional games.

If (26) does not hold, then $\mathcal{A}^{\max}(M) > \max_{i\in N} \mathcal{A}_i(M)$ and it follows that no affine allocation can be a solution of the bargaining problem. To see this, assume that an affine allocation $L_i = a_i + b_i M$ solves the bargaining game. Note that since $\sum_i L_i = M$ almost surely, it follows that $\sum_i a_i = 0$ and $\sum_i b_i = 1$ and consequently

$$\sum_{i=1}^n \mathcal{A}_i(L_i) = \sum_{i=1}^n \mathcal{A}_i(b_i M) = \sum_{i=1}^n b_i \mathcal{A}_i(M) \leq \max\{\mathcal{A}_i(M)\} < \mathcal{A}^{\max}(M), \tag{28}$$

which contradicts the optimality of the allocation L in (13).

The assumption (26) of a dominating acceptability functional is a simplification and will be relaxed below. Nevertheless, it can be useful in practical situations where only a rough assessment of risk aversion in terms of a hierarchy of acceptability functionals is available. In some special cases, very simple managerial implications can be derived. The following examples discuss such situations.

Example 3 (Identical risk aversion). Assume $\mathcal{A}_i = \mathcal{A}$ for all $i \in N$ for a given acceptability functional $\mathcal{A}$. Then all $\mathcal{Z}_i$ are identical and $\mathcal{Z}_i = \mathcal{Z}_{i_{\max}}$ for all i and therefore $\mathcal{A}_{i_{\max}} = \mathcal{A}$ and the optimal acceptability allocation is given by

$$u_i = \frac{v_i}{\sum_{j=1}^n v_j} \mathcal{A}(M).$$

According to Proposition 3, the allocation is affine and highly ambiguous, since it does not matter which agent is assigned how much *risky* payoff M as long as she is compensated by a deterministic side payment. In particular, for each i the extreme solution $b_i = 1$ is possible, which results in player i bearing all the risk. In this case, the side payments have to be chosen as

$$a_i = -\frac{\sum_{j\neq i}^n v_j}{\sum_{j=1}^n v_j} \mathcal{A}(M),\ a_k = \frac{v_k}{\sum_{j=1}^n v_j} \mathcal{A}(M), \quad \forall k \in N \setminus \{i\}\,.$$

Another possible solution is the linear allocation with

$$a_i = 0 \text{ and } b_i = \frac{v_i}{\sum_{j=1}^n v_j}, \quad \forall i \in N, \tag{29}$$

which treats all players according to their relative bargaining power and leads to the pointwise allocation

$$L_i = \frac{v_i}{\sum_{j=1}^n v_j} M, \quad \forall i \in N. \tag{30}$$

Linear allocations play a prominent role in the discussion of fair allocations. In particular, a naive and often applied solution assigns fixed shares of the random profit according to the players' share of invested capital. In the scientific literature, linear rules have, e.g., been derived in Borch (1962) and Gerber (1979) in the context of classical insurance risk theory and have later been generalized to dilated convex risk measures and expected risk functionals, e.g., Deprez and Gerber (1985), Barrieu and El Karoui (2005), Timmer et al. (2005), Baeyens et al. (2013).

Note that in the present context purely linear rules are never possible unless all players use the same acceptability functional, because otherwise there would have to be at least one player with $b_i = 0$ in contradiction to (29).

EXAMPLE 4 (IDENTICAL DISAGREEMENT POINTS). If $v_i = v$ for all i, then $u_i = \frac{1}{n}\mathcal{A}_{i_{\max}}(M)$, i.e., the agents receive equal shares of the overall acceptability, which is an intuitive consequence of the fairness of the allocation. However, note that this does *not* imply that the payoffs are shared equally pointwise. In general, those players who are less risk-averse will bear higher losses in the *unfavorable scenarios* than more risk-averse players but receive higher profits in the *favorable scenarios*.

In particular, in the extreme case, where there is a player i that is least risk-averse such that $\mathcal{A}_i = \mathcal{A}^{\max}$ and $\mathcal{A}_j < \mathcal{A}^{\max}$ for all $j \neq i$, it follows by a similar reasoning as in Example 3 that

$$L_i = M - \frac{n-1}{n}\mathcal{A}^{\max}(M),\ L_k = \frac{1}{n}\mathcal{A}^{\max}(M), \quad \forall k \in N \setminus \{i\}\,.$$

In the case that there are several players with $\mathcal{A}_i = \mathcal{A}^{\max}$, it is optimal to choose $b_i = m^{-1}$, where $m = |\mathcal{I}^{\max}|$ is the number of least risk-averse players. This leads to

$$L_i = \frac{M}{m} - \left(\frac{1}{m} - \frac{1}{n}\right)\mathcal{A}_{i_{\max}}(M).$$

The players $j \notin \mathcal{I}^{\max}$ only get a deterministic side payment of

$$a_j = L_j = \frac{1}{n}\mathcal{A}_{i_{\max}}(M)$$

and correspondingly $b_j = 0$.

Lastly, in case of identical acceptability functionals as in Example 3, (30) simplifies even more to

$$L_i = \frac{1}{n}M. \tag{31}$$

This can be seen as a direct consequence of the axiom (SYM). However, note that in order for the players to be fully symmetric such that their payoffs agree pointwise as in (31), the endowment v_i *as well as* the risk preferences have to agree.

Next, we consider a more general setting in which allocations take the form of standard options contracts. To that end, we make the assumption that the agents' acceptability functionals are so called *distortion functionals*, and the project profit is bounded below as stated in the following assumption.

ASSUMPTION 3. $\operatorname{ess\,inf} M_x = \sup\{t \in \mathbb{R} : \mathbb{P}(M_x(\omega) \leq t) = 0\} = C > -\infty$ *for all* $x \in \mathcal{X}$*, and all* $\mathcal{A}_i$ *are distortion functionals, i.e., of the form* (4).

In the following, we show that, under Assumption 2 and Assumption 3, any optimal allocation $X = (X_1, \ldots, X_n)$ of the sup-convolution (13) has an explicit representation that can be used to write the X_i as baskets of standard options. The following results therefore hold for bargaining games, but are not restricted to this setting. In particular, they extend to any Pareto optimal allocation of profits (see Rüschendorf 2013, Chapter 8, for numerous examples). However, to maintain consistency we will still talk about the agents $i = 1, \ldots, n$ as *players* throughout this section. For bargaining games, it follows as a corollary that every bargaining solution L can as well be represented by option contracts between the players plus deterministic side payments.

Using Assumption 3, we can dissect the random variables X_i into *slices* and then show that the acceptability of X_i can be built up from the acceptability of the slices. In particular, we establish the following lemma, which expresses the random variable X_i by means of an integral over the space $\mathbb{R}$ of possible acceptability values t, and derives from that an integral representation of the acceptability $\mathcal{A}_i(X_i)$.

LEMMA 1. *Under Assumption 2 and Assumption 3, the following holds:*

1. *The optimal X_i in (13) can be written as*

$$X_i(\omega) = \xi_i(C)C + \int_{-\infty}^{\infty} \mathbb{1}_{\{x \in \mathbb{R}:\, x \geq t\}}(M(\omega))\xi_i(t)\, dt, \quad \forall \omega \in \Omega \tag{32}$$

   for some functions $\xi_i : \mathbb{R} \to [0,1]$ with $\sum_{i=1}^n \xi_i(t) = 1$ for $t \geq C$.

2. *If $\mathcal{A}_i$ is a distortion functional, then*

$$\mathcal{A}_i(X_i) = \int_{-\infty}^{\infty} \mathcal{A}_i(\mathbb{1}_{\{x \in \mathbb{R}:\, x \geq t\}}(M))\xi_i(t)\, dt.$$

Note that the integrand in (32) is $\xi_i(t)$ if $t \leq M(\omega)$, and 0 otherwise, such that, loosely speaking, the ξ_i determine how to distribute the t-th dollar of profit amongst the players.

Using this representation, we can show the following theorem.

THEOREM 4. *If Assumption 2 and Assumption 3 are fulfilled, then there is always an optimal solution X of (13) for which the ξ_i in the representation (32) are of the form*

$$\xi_i(t) = \begin{cases} \delta_i(t), & \mathcal{A}_i(\mathbb{1}_{\{x \in \mathbb{R}:\, x \geq t\}}(M)) = \max_j \mathcal{A}_j(\mathbb{1}_{\{x \in \mathbb{R}:\, x \geq t\}}(M)) \\ 0, & \textit{otherwise,} \end{cases} \tag{33}$$

for almost all $t > C$, with $\delta_i(t) \in \{0,1\}$ such that $\sum_{i=1}^n \delta_i = 1$.

The functions $M \mapsto X_i$ are thus monotonically increasing and piecewise affine with interchanging constant pieces and pieces with slope 1. The payoff can therefore be written as a partition of the worst case payoff C plus a portfolio of standard call and put options on M, which are either received by or written to the community of the other players.

The next theorem establishes that in many practically relevant cases, finitely many options are enough to represent the split of M into $X = (X_1, \ldots, X_n)$.

THEOREM 5. *Under Assumption 2 and Assumption 3, the players can reach an optimal allocation of payoffs X by exchanging a finite number of options on M as an underlying asset if*

1. *the measures m_i in (4) are discrete with finitely many atoms or*
2. *the distribution of M is discrete and has only finitely many atoms.*

Finally, the following corollary transfers these results for the sup-convolution to allocations obtained by Nash-bargaining and thus facilitates a real world implementation of optimal allocations of the bargaining process. In fact, Corollary 1 shows that every bargaining solution is also a solution of the sup-convolution problem. Therefore, bargaining solutions *inherit* all properties shown for general solutions of (13), which proves the following result.

Corollary 3. *The results shown in Lemma 1, Theorem 4, and Theorem 5 for the solutions X of the sup-convolution* (13) *also hold for solutions L of bargaining games that fulfill Assumption 1 – 3 and Axioms 1 – 3.*

3.3. Numerical Calculation of Optimal Allocations

In this section, we detail how optimal allocations can be computed based on our previous results. Clearly, the sup-convolution and Corollary 1 can be used to find some optimal allocation. Corollary 3 delivers a simple solution when there is a dominating functional. Theorem 4 does the same for the more general case of distortion functionals, although the resulting calculations are slightly more involved and will be explained in detail below.

To this end, we assume that stochastic optimization problems are two-stage and the involved random variables are discrete with finitely many atoms, such that the problems can be solved numerically. This condition may either be fulfilled from the beginning or, if this is not the case and the random variables are continuously distributed, a discrete approximation can be constructed, either as sample average approximation (see, e.g., Shapiro et al. 2009) or by scenario generation (e.g. Heitsch and Römisch 2009, Pflug and Pichler 2014).

We denote by $S \in \mathbb{N}$ the number of scenarios which have probabilities $p_s > 0$ and by R_{is}, L_{is} and $M(s,x)$ the value of the random variables in the problem formulation in scenario s. We furthermore assume that the conditions in Theorem 5 are fulfilled and the solution can be written as a combination of option contracts. Under these circumstances, the bargaining solution can be found using Algorithm 1.

In lines 1 – 4 the joint stochastic optimization problem is solved (one may also use problem (23) instead of (8)) and the optimal shares of the overall acceptability are computed according to Theorem 1. Lines 5 – 8 find the options that define the structure of the payoffs according to Theorem 4, where H_s is a random variable defined on the scenarios $\Omega^S = \{1, \dots, S\}$ with probabilities p_s such that

$$H_s(\omega) = \mathbb{1}_{\{x \in \mathbb{R}:\, x \geq s\}}(\omega),\ \forall \omega \in \Omega^S$$

is the shifted heaviside function. Note that in case Assumption 3 is not fulfilled, this computation is not required and the L_i from the optimal solution of $z(R,M,A)$ can be used instead after applying Corollary 1. Finally, lines 9 – 11 correct the level of payoffs by adding a deterministic payment so that the acceptabilities coincide with the prescribed levels u_i.

We note that our approach to bargaining is not restricted to problems that fulfill the assumptions above but extends to multi-stage stochastic optimization problems and problems with continuous

Data: R, M, $\mathcal{A}$

1 Compute $z(R,M,\mathcal{A})$ and x by solving problem (8)

2 $v_i \leftarrow \mathcal{A}_i(R_i), \quad \forall i \in N$

3 $u_i \leftarrow \frac{v_i}{\sum_{j=1}^{n} v_j} \cdot z(R,M,\mathcal{A}), \quad \forall i \in N$

4 Renumber scenarios such that $M(s_1,x) \leq M(s_2,x)$ for $s_1 < s_2$

`/* Distribute gains in scenarios to players */`

5 **for** $s \leftarrow 2$ **to** S **do**

6 | $i^* \leftarrow \arg\max_{i \in N} \mathcal{A}_i(H_s)$

7 | $L_{i^*} \leftarrow L_{i^*} + H_s(M(s,x) - M(s-1,x))$

8 **end**

`/* Correct level of payments to match` u_i `*/`

9 **for** $i \leftarrow 1$ **to** n **do**

10 | $L_i \leftarrow L_i + (u_i - \mathcal{A}_i(L_i))$

11 **end**

Algorithm 1: Algorithm to compute a bargaining solution

randomness. In these cases, Algorithm 1 can be easily be adapted by employing more sophisticated approaches to solve the stochastic optimization problem in line 1 of the algorithm.

Finally, we would like to point out that one of the advantages of our approach is that it does not introduce any essential computational effort in addition to the solution of the basic stochastic optimization problem of maximizing the joint benefit (the sum of the acceptability values) of the entire set of players. Hence, if the problem is hard to start with it remains hard in our approach, but if it is tractable, then a bargaining solution can be easily computed. In particular, the effort for determining the maximally possible joint benefit is unavoidable anyway as long as we strive for *efficient* bargaining solutions in line with Axiom PAR: It is easy to see that an acceptability allocation $(u_1, \ldots, u_n)$ that is not optimal for the maximization problem in (8) is dominated by another feasible acceptability allocation and is therefore not Pareto optimal.

As soon, however, as the stochastic optimization problem (8) has been solved, the optimal allocations of the players can be found by simply inserting into (19), which is computationally cheap.

In the cases where the solution can be represented as a basket of standard options, our approach is especially attractive in so far as the underlying general and quite complex theory leads to simple allocation rules that can be understood and executed also by non-specialists. This is different from approaches that distribute joint profits *ex-post* according to some complicated game theoretic allocation rule (such as, e.g., the Shapley value or deterministic Nash bargaining), which is practically impossible to specify in real-world contracts that are signed *ex-ante* by the parties.

4. An Application Example: Construction of a Solar Roof

In this section, we illustrate the use of the proposed theory in a concrete real-world application example. As we argued in the introduction, the applications of bargaining theory where deterministic

Nash bargaining is routinely applied are abundant in the literature. However, we choose a novel application from the field of energy for our example.

In the course of the global transition to clean, renewable energy, there are many situations where several *small players* cooperate in projects that would be too costly or otherwise too demanding for a single entity. Examples are households that jointly invest in solar power plants or community storage (Chakraborty et al. 2019a,b) or the case of companies that own renewable generation assets and pool their production in a virtual power plant for cost efficient market access as well as for diversification of market and production risks (Baeyens et al. 2013, Nguyen and Le 2018, Kovacevic et al. 2018, Gersema and Wozabal 2018, Han et al. 2019).

Here, we analyze the problem of joint investment into solar power. In particular, we consider an investment in 7.5 kW peak (kWp) of photovoltaic (PV) panels on a duplex house near Munich that is jointly owned by a young couple and a family consisting of two kids and two parents, one of whom is working. We describe the setup, the assumptions, and the models for stochastic variables in Section 4.1 and discuss the results of the bargaining game in Section 4.2.

4.1. Parameters and Modeling

In the following, we distinguish variables between the two households by the subscripts f (family) and c (couple). For our calculation we assume that the lifetime of the panels is 20 years and that investment costs are €1300 per kWp (Kost et al. 2018). For simplicity, we assume zero maintenance cost.

The two owners of the duplex house jointly invest in the project and can both use electricity generated from the panels. Electricity that is not directly consumed is sold to the grid for a feed-in tariff F that is fixed at 10 cents/kWh for the lifetime of the plant, which roughly corresponds to the current subsidy regime in Germany (Fraunhofer ISE 2020).

Since major investments and alterations of a duplex house can only be decided unanimously by all owners, we assume that it is not possible for any of the two households to install solar panels on the roof of the building without the participation of the respective other household. Hence, if no bargaining solution is reached, no panels will be installed.

The joint profit from installing the PV panels is thus the sum of profits generated from selling to the grid for the feed-in tariff and the avoided cost of self-consumption. Since we assume identical electricity prices for both households, the overall savings in the electricity bill are not affected by which of the two households consumes the electricity. Therefore, we assume that electric current obeys physical laws and flows to the households dependent solely on consumption patterns. Since the consumption of the two households is random, the savings in the electricity bills of the households are random as well and deviate from the bargaining solution. The households therefore agree to make the necessary financial transactions that ensure the agreed upon allocation of profits.

The reward of the project is random as well, since electricity production P of the PV panels, average household power prices G, hourly demand C_c and C_f, as well as the returns of the alternative investments R_c and R_f are random. We assume these factors to be independent and, for our calculations, represent them by $S = 1000$ equally probable scenarios, which are sampled from the models outlined below.

For simplicity, we assume that the investments are split up proportionately to average electricity consumption of the two households (see below), which can be seen as a proxy for apartment size. This results in investments of €4505 by the family and €5245 by the couple. The opportunity losses are defined by the 20 year random profits R_c and R_f from investing in a diversified portfolio of German stocks. We model the value of the portfolio as a geometric Brownian motion with a yearly drift of 6.93% and a volatility of 21.83%, which we estimate based on daily closing prices from 01/1988 to 12/2019 (ignoring missing values) for the DAX performance index ˆGDAXI, freely available from *Yahoo! Finance*.

To generate samples for average household electricity prices, we fit a GBM process to average yearly household prices from 2000 – 2018 obtained from BDEW (2019) and simulate 20 years of yearly price changes starting from a price of 30.43 cents per kWh, which was the average price in 2018.

We model a yearly profile of solar irradiation for one square meter of ground in Munich in hourly resolution following Twidell and Weir (2015), Chapter 4. In particular, given values for the total radiation G_t and the diffuse radiation G_d, the produced energy per square meter is given by

$$\eta \left[(G_t - G_d) \cos\theta + G_d \right]. \tag{34}$$

Here, η denotes the efficiency of the panel and θ is the angle of incidence, i.e., the angle between the sun beam and the tilted surface of the solar panel. The angle of incidence varies with time and depends on several components:

1. The angle β between the panel and the horizontal (tilt) where $0° \leq \beta \leq 90°$ if the surface is directed towards the equator and $90° < \theta \leq 180°$ otherwise.
2. The angle γ between the normal to the solar panel surface, projected to the horizontal, and the local longitude meridian (azimuth). If $\gamma = 0°$ the panel faces south, if $\gamma = 90°$ the panel faces west, and if $\gamma = -90°$ the panel faces east.
3. The latitude ϕ and the longitude ψ of the location of the panel.
4. The declination (northern hemisphere)
$$\delta = 23.45° \sin \frac{360°(284 + d)}{365}$$
where d denotes the day of the year with $d = 1, \ldots, 365$.
5. The rotation angle w (hour angle) since the last solar noon,
$$w = (15°)(t_{zone} - 12) + (\psi - \psi_{zone}).$$
Here t_{zone} is the civil time in hours of the time zone containing longitude ψ, and ψ_{zone} is the longitude where civil time t_{zone} and the solar time coincide. For simplicity, we neglect the so called equation of time, which is a small correction term.

Based on these components, the angle of incidence, respectively its cosine used in (34), fulfills

$$\cos\theta = (A - B)\sin\delta + [D\sin w + (E + F)\cos w]\cos\delta,$$

where

$$A = \sin\phi\cos\beta,\ B = \cos\phi\sin\beta\cos\gamma,\ D = \sin\beta\sin\gamma,\ E = \cos\phi\cos\beta,\ F = \sin\phi\sin\beta\cos\gamma.$$

In order to calculate usable electricity from (34) for a location in Munich, we use the relevant geographic information $\phi = 48.137154°$, $\psi = 11.576124°$, $\psi_{zone} = 15°$. Moreover, we derive the local radiation values G_t and G_d from the global radiation maps published by the German Weather Service (DWD 2020). Finally we assume that 8m^2 of area are required per kWp capacity and that the efficiency of the panels is $\eta = 0.1$.

For a combination of angles $x = (\beta, \gamma)$ we calculate the nominal production $P(x)$ by (34) and the production in scenario $s = 1, \ldots, S$ as

$$P_s(x) = P(x)\epsilon_s$$

where ϵ_s are independent normally distributed errors with mean 1 and standard deviation 0.2, which model multiplicative deviations from the long-term mean.

To generate hourly electricity demands, we use the free tool *LoadProfileGenerator*[1] using default profiles CHR33 and CHR44 for 2018 for the couple and the family, respectively. We let devices be generated randomly and assume that the couple owns an electric car which is charged at home and used for a 30km commute every day. To obtain random consumption profiles, we randomize the generated profiles by resampling days. More specifically, for a given day, we sample from the days in the generated profile that are in the same month and on the same weekday.

We assume that the risk preferences of the two households are given by

$$\mathcal{A}_c(X) = 0.6\,\mathbb{E}(X) + 0.4\ \text{AVaR}_{0.1}(X), \quad \mathcal{A}_f(X) = 0.7\,\mathbb{E}(X) + 0.3\ \text{AVaR}_{0.05}(X).$$

Note that the above risk preferences are distortion functionals with $m_f(1) = 0.7$, $m_f(0.05) = 0.3$, $m_c(1) = 0.6$, and $m_c(0.1) = 0.4$.[2]

4.2. Results

For simplicity, we solve the hourly planning problem, which is the basis for the allocation of profits and the alignment of the PV panels, only for one year of operation and multiply the result by 20 to obtain an estimate for the profits over the whole lifetime of the installation. Hence, we implicitly assume that the considered year is typical.

[1] See `https://www.loadprofilegenerator.de/`

[2] The specification of preferences in the example is purely illustrative. In an actual application, acceptability functionals and their parameters would have to be elicited from the households. In particular, there is an extensive literature on the estimation of utility functions using questionnaires (e.g., Mosteller and Nogee 1951, Halter and Mason 1978, Evans and Viscusi 1991). In the same way, parameters of acceptability measures may be inferred as is evidenced in the literature about inferring risk functionals from portfolio decisions (see, e.g., Grechuk and Zabarankin 2014, 2016).

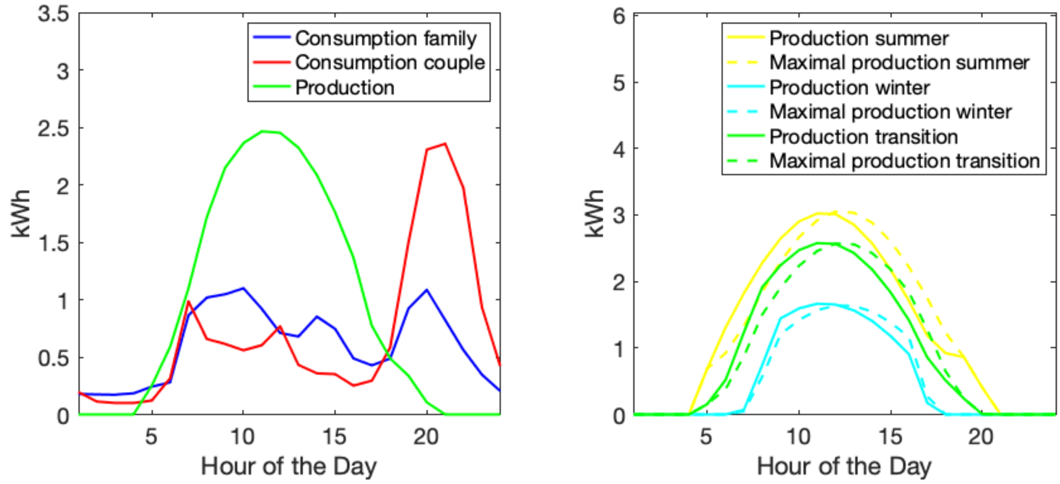

Figure 1 **Consumption of the two investors against production of PV panels (left), optimal consumption versus configuration maximizing overall production.**

To obtain scenarios G_s for household electricity prices, we simulate trajectories G_s^t for $t = 1, \ldots, 20$, $s = 1, \ldots, S$ for the 20 year lifetime of the plant from the yearly electricity price process discussed above. We then obtain scenarios for average average household prices $G_s = 20^{-1} \sum_{t=1}^{20} G_s^t$, which we use in our calculation.

Apart from the decision on the split up of project profits, the investors have to decide on the azimuth γ and the panel tilt β for the solar panels, which together constitute the decision $x = (\beta, \gamma)$ in our framework. We therefore use $\mathcal{X} = [-180, 180] \times [0, 90]$ as the feasible set of our problem.

Since the production depends in a non-convex way on x, we discretize $\mathcal{X}$ using a grid $\mathcal{X}^g$ with mesh size 1° in both dimensions and solve the problems

$$\Pi(x) = \begin{cases} \max\limits_{L,u,M} \; \mathcal{A}_c(L_c) + \mathcal{A}_f(L_f) \\ \text{s.t.} \quad M_s = \min(P_s(x), C_{cs} + C_{fs})G + \max(P_s(x) - C_{cs} - C_{fs}, 0)F, \; \forall s = 1, \ldots, S \\ \quad\quad M_s = L_{cs} + L_{fs}, \; \forall s = 1, \ldots, S \\ \quad\quad u_c = \frac{v_c}{v_f + v_c}(\mathcal{A}_c(L_c) + \mathcal{A}_f(L_f)) \\ \quad\quad u_f = \frac{v_f}{v_f + v_c}(\mathcal{A}_c(L_c) + \mathcal{A}_f(L_f)) \\ \quad\quad \mathcal{A}_c(L_c) \geq v_c \\ \quad\quad \mathcal{A}_f(L_f) \geq v_f \end{cases} \tag{35}$$

for all $x \in \mathcal{X}^g$ and then find

$$x^* = \arg\min_{x \in \mathcal{X}^g} \Pi(x).$$

The optimal M, L, and u can then be found from the optimal solution of the problem $\Pi(x^*)$. Note that, for a fixed x, the above problem can be efficiently solved as a linear optimization problem.

The optimal angles are $x^* = (-18°, 45°)$ which deviates significantly from the angles $x^+ = (4°, 41°)$ maximizing overall electricity production, which is an interesting result in its own right. The production curves in summer, winter, and the transition periods (spring and autumn) are depicted in Figure 1. Relative to x^+, the panels are facing eastward to receive more sunlight in the morning to match the consumption pattern of the households better. Furthermore, the higher tilt of 45° ensures increased production in winter as compared to x^+. Although this choice leads to less overall

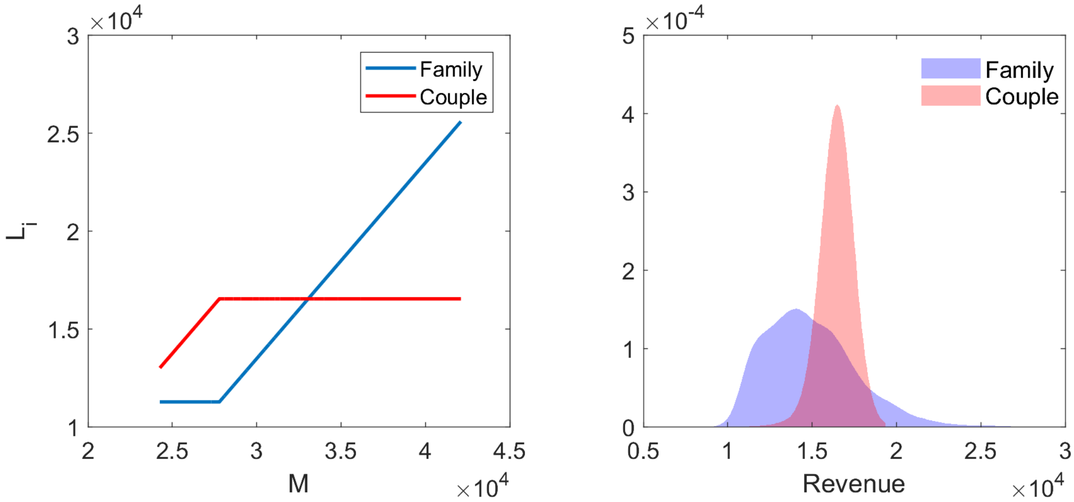

Figure 2 Allocation of profits as a function of M (left) and as density plots (right).

production, it maximizes acceptability, since self-consumption is strictly preferred to feeding into the grid.

The split up of the profits is plotted in the left panel of Figure 2. In accordance with the theory in Section 3.2, the payoffs are split up into simple options on the underlying M: the family receives a call option while the couple is short a put option on top of the distribution of the minimal value of M. Clearly, a contract specifying these payoffs as a function of M is legally feasible and easily understood by the parties.

Inspecting the density plot of the profits in the right panel of Figure 2, we see that the distribution ensures that the risk-averse couple gets a moderate reward for sure which is capped once M reaches the strike price of the put option. Consequently, the distribution of the rewards of the couple is concentrated around its mean with less downside risk but also limited upside potential. Contrary to that, the family bears most of the downside risk of the project but also has a more pronounced upside potential due to the payout structure of the call option.

The differences in the investments and consequently in the conflict points lead to a split up of acceptability values of $u_c = 16,145$ and $u_f = 13,788$ for the couple and the family, respectively. Due to the low prices of the panels and the rather high prices of grid electricity in Germany, these acceptability values clearly exceed the acceptability values of the alternative investments which are $v_c = 6,030$ and $v_f = 5,150$.

5. Conclusions

We formulate a bargaining game for risk-averse players who face uncertain profits from cooperation and whose risk preferences can be described by coherent acceptability functionals. Besides the allocation of acceptability values, we put special emphasis on finding ex-ante agreements on the allocation of profits. Additionally, the players also have to agree on a usage of their resources by taking optimal managerial decisions.

We impose three axioms for a “fair solution” and show that these uniquely characterize a distribution of acceptability values. We show that bargaining solutions can be found by solving a stochastic optimization problem that can be interpreted as a stochastic version of Nash bargaining with bargaining powers. In the optimum, the individual acceptability values are fractions of the optimal overall acceptability plus side-payments between the players. As an important consequence, it follows that managerial decisions can be separated from the question of a fair allocation.

We show that in the case where agents' risk preferences can be described by distortion risk functionals, the optimal allocation of profits can be characterized as an exchange of standard options contracts between the agents, which makes the approach practically feasible for real world applications.

In order to demonstrate the practical applicability of our approach, we present an illustrative case study of a joint investment in a solar roof on a duplex house. Two households with different consumption patters, risk preferences, and investment size optimize the alignment of the solar panels and search for a fair allocation of the profits from selling electricity to the grid and from the reduction of their energy bills by self-consumption. In order to maximize self-consumption, households choose an alignment that deviates substantially from the alignment with maximal energy production. The optimal allocation of profits can be reached by fixed payments and a long position in a call option on the project profit for the player with smaller risk aversion and a short position in a put option for the more risk-averse player.

Interesting topics for further research include profit sharing problems in energy applications such as jointly controlling a virtual power plant or managing a community storage, bargaining between nations for the reductions in climate gas emission, as well as problems in economics and politics where deterministic bargaining approaches are routinely applied.

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Appendix A: Online Appendix (Proofs)

A.1. Proof of Proposition 1

In order to prove Proposition 1, we require three technical results.

LEMMA 2. *Let X and Y be two topological spaces. If Y is compact, then the projection*

$$\mathrm{proj}_1 : X \times Y \to X, \text{ with } \mathrm{proj}_1(x,y) = x$$

is a closed mapping.

LEMMA 3. *For every X, there is a $\alpha_0(X)$ such that*

$$\mathcal{A}_i(X) = \mathrm{AVaR}_{\alpha_0(X)}(X).$$

If $\mathcal{A}_i \neq \mathbb{E}$ and X is not constant, then $\alpha_0(X) < 1$ is unique.

Proof. Note that $\alpha \mapsto \mathrm{AVaR}_\alpha(X)$ is continuous and strictly monotonically increasing if X is not almost surely constant. If X is almost surely constant, then $\mathrm{AVaR}_\alpha(X)$ is constant as well and α_0 can be chosen arbitrarily and in particular $\mathrm{AVaR}_1(X) = \mathbb{E}(X) = \mathcal{A}_i(X)$.

For the non-constant case, clearly,

$$\mathrm{AVaR}_0(X) = \operatorname{ess\,inf} X \le \mathcal{A}_i(X) \le \mathbb{E}(X) = \mathrm{AVaR}_1(X)$$

and strict monotonicity yields a unique $\alpha_0(X)$ by the intermediate value theorem. □

LEMMA 4. *Let $\mathcal{A}_i \neq \mathbb{E}$ be coherent acceptability measures, then the set*

$$K = \left\{ (L_1, \ldots, L_n) : \exists x \in \mathcal{X} \text{ with } \sum_i L_i \le M_x,\ \mathcal{A}_i(L_i) \ge v_i > -\infty \right\}$$

is relatively weakly compact in $\bigtimes_{i=1}^n \mathcal{L}^p$.

Proof. By the theorem of Banach-Alaoglu, K is relatively weakly compact if it is norm-bounded, where we use

$$||(L_1, \ldots, L_n)|| = \sum_i ||L_i||_p$$

as the norm in $\bigtimes_{i=1}^n \mathcal{L}^p$.

Suppose there is a sequence $(L^k)_{k\in\mathbb{N}} \subseteq K$ with corresponding $(x^k)_{k\in\mathbb{N}} \subseteq \mathcal{X}$, such that the former is unbounded. Possibly by selecting a subsequence, we can find a j such that $||L_j^k||_1 \to \infty$. Similarly, we can assume without loss of generality that there is a player i for whom there are sets $A_i^k \subseteq \Omega$ with

$$\int_{A_i^k} L_i^k \, d\mathbb{P} \xrightarrow{n\to\infty} -\infty. \tag{36}$$

Let $\alpha^k = \alpha_0(L_i^k, \mathcal{A}_i)$. If $\alpha_0^k \not\to 1$, then there exists an accumulation point $\bar{\alpha}_0 < 1$ and sets $B_i^k \subseteq \Omega$ such that $\mathbb{P}(A_i^k \cup B_i^k) = \alpha^k < 1$ and $L_i^k(\omega') \ge L_i^k(\omega)$ for all $\omega \in A_i^k \cup B_i^k$, $\omega' \in \Omega \setminus (A_i^k \cup B_i^k)$ and

$$\frac{1}{\alpha^k} \int_{A_i^k \cup B_i^k} L_i^k \, d\mathbb{P} \ge v_i,$$

implying that $\int_{B_i^k} L_i^k \, d\mathbb{P} \to \infty$ and therefore for any other $j \neq i$

$$\begin{aligned} \mathcal{A}_j(L_j^k) &\le \int L_j^k \, d\mathbb{P} \le \int_{A_i^k \cup B_i^k} (M_{x^k} - L_i^k) \, d\mathbb{P} + \int_{\Omega\setminus(A_i^k \cup B_i^k)} (M_{x^k} - L_i^k) \, d\mathbb{P} \\ &\le \sup_{x\in\mathcal{X}} \mathbb{E}(|M_x|) - v_i - \int_{\Omega\setminus(A_i^k \cup B_i^k)} L_i^k \, d\mathbb{P} \xrightarrow{n\to\infty} -\infty, \end{aligned}$$

since $\sup_{x\in\mathcal{X}} \mathbb{E}(|M_x|) = \sup_{x\in\mathcal{X}} |||M_x||_1 < \infty$ because of Assumption 1. This violates the assumption that $\mathcal{A}_j(L_j^k) \geq v_j > -\infty$ and thus shows that $L^k \notin K$ eventually.

What is left, is the case $\alpha^k \to 1$. If $\mathbb{E}(L_i^k) \to \infty$, then clearly for $j \neq i$

$$\mathcal{A}_j(L_j^k) \leq \mathbb{E}(L_j^k) \leq \mathbb{E}(M_{x^k} - L_i^k) = \mathbb{E}(M_{x^k}) - \mathbb{E}(L_i^k) \xrightarrow{n\to\infty} -\infty < v_j.$$

Hence, we can assume that $\mathbb{E}(L_i^k)$ remains bounded above and we therefore can find a finite $C \in \mathbb{R}$, such that for every $0 < \alpha \leq 1$ and $0 \leq \gamma < \alpha$

$$\int_\gamma^\alpha F_{L_i^k}^{-1}(t)\, dt \leq \int_{L_i^k \geq 0} L_i^k \, d\mathbb{P} \leq C, \quad \forall n \in \mathbb{N}. \tag{37}$$

Now for $0 < \alpha \leq 1$

$$\text{AVaR}_\alpha(L_i^k) = \frac{1}{\alpha}\left(\int_0^{\mathbb{P}(A_i^k)} F_{L_1^k}^{-1}(t)\, dt + \int_{(\mathbb{P}(A_i^k),\alpha]} F_{L_1^k}^{-1}(t)\, dt\right) \xrightarrow{n\to\infty} -\infty,$$

since the first term in the brackets diverges to $-\infty$ because of (36), while the second term is bounded above by C due to (37). However, this in particular, implies that

$$\lim_{n\to\infty} \mathcal{A}_i(L_i^k) \leq \lim_{n\to\infty} \mathbb{E}(L_i^k) = \lim_{n\to\infty} \text{AVaR}_1(L_i^k) = -\infty$$

and therefore $\mathcal{A}_i(L_i^k) \geq v_i$ is eventually violated, which leads to a contradiction to $L^k \in K$, proving the claim. □

Proof of Proposition 1 Since $\mathcal{A}_i(M) \leq \mathbb{E}(M)$, the set is clearly bounded.

To show that $\mathcal{U}(v, M, \mathcal{A})$ is closed, we write $\mathcal{U}(v, M, \mathcal{A}) = \text{proj}_1(\mathcal{V})$ with

$$\begin{aligned}\mathcal{V} = \text{hypo}(\mathcal{A}) \cap \left(\{u : u_i \geq v_i\} \times \bigtimes_{i=1}^n \mathcal{L}^p\right) \cap \left(\mathbb{R}^n \times \left\{L \in \bigtimes_{i=1}^n \mathcal{L}^p : \exists x \in \mathcal{X} \text{ with } \sum_i L_i = M_x\right\}\right)\\ \subseteq \mathbb{R}^n \times \bar{K}^\sigma.\end{aligned}$$

The first set is the hypograph of $\mathcal{A}$ which is closed in $\bigtimes_{i=1}^n \mathcal{L}^p$ since $\mathcal{A}$ is upper semi-continuous. Since the set is also convex, it is also closed in $\mathbb{R}^n \times (\bigtimes_{i=1}^n \mathcal{L}^p, \sigma)$, where σ is the weak topology in $\bigtimes_{i=1}^n \mathcal{L}^p$. Clearly, the second set is also closed in the same topology. To analyze the third set note that by Assumption 1 the set $M(\mathcal{X})$ is weakly compact in $\mathcal{L}^p$. Define the linear function $f(L) = \sum_{i=1}^n L_i$ from $\bigtimes_{i=1}^n \mathcal{L}^p$ to $\mathcal{L}^p$. Since f is bounded, it is weakly continuous and consequently $f^{-1}(M(\mathcal{X}))$ is closed in the weak topology in $\bigtimes_{i=1}^n \mathcal{L}^p$.

It follows that $\mathcal{V}$ is closed in $\mathbb{R}^n \times \bar{K}^\sigma$ and because $\bar{K}^\sigma$ is weakly compact due to Lemma 4, Lemma 2 shows that $\mathcal{U}(v, M, \mathcal{A})$ is closed. □

A.2. Proof of Proposition 2

The set $\mathcal{U}(R, M, \mathcal{A})$ defined by (5) is obviously just the u-projection of the feasible set $\mathcal{U}^+(R, M, \mathcal{A})$ of the maximization problem (8). Since the objective function of (8) only depends on u, it is immediately seen that

$$\sup_{u,L,x} \{\sum_{i=1}^n u_i \mid (u, L, x) \in \mathcal{U}^+(R, M, \mathcal{A})\} = \sup_u \{\sum_{i=1}^n u_i \mid u \in \mathcal{U}(R, M, \mathcal{A})\}.$$

$\mathcal{U}(R, M, \mathcal{A})$ is compact due to Proposition 1, therefore (8) attains its maximum.

We show that if (u, L, x) is feasible for (8), $u_i' \geq v_i \ \forall i$, and $\sum_{i=1}^n u_i' \leq \sum_{i=1}^n u_i$, then there is an L' and an x' such that also (u', L', x') is feasible. To see this, set $u_i'' = u_i' + \frac{1}{n}\left(\sum_{j=1}^n u_j - \sum_{j=1}^n u_j'\right) \geq u_i'$ for all $i \in N$. Then, clearly, $\sum_{i=1}^n u_i'' = \sum_{i=1}^n u_i$ and setting $L_i' = L_i - u_i + u_i''$ and $x' = x$, we have $\sum_{i=1}^n L_i' = M_{x'}$. Using translation invariance of $\mathcal{A}_i$, we get

$$\mathcal{A}_i(L_i') = \mathcal{A}_i(L_i + (u_i'' - u_i)) = \mathcal{A}_i(L_i) + (u_i'' - u_i) \geq u_i + u_i'' - u_i = u_i'' \geq u_i',$$

which establishes feasibility of (u', L', x').

Let now (u^*, L^*, x^*) be an optimal solution of (8) with optimal value z. Since (u^*, L^*, x^*) is a feasible solution, it follows from the above that every $u \in \mathbb{R}^n$ with $u_i \geq v_i \; \forall i$ and $\sum_{i=1}^n u_i \leq \sum_{i=1}^n u_i^* = z$ can be extended by a suitable L and a suitable x to a feasible (u, L, x). Therefore, the set (7) is a subset of the u-projection of $\mathcal{U}^+(R, M, \mathcal{A})$. Conversely, suppose that for some feasible (u, L, x), the component u does *not* lie in the set (7). This could only be the case if $\sum_{i=1}^n u_i > z$, but then there would be a better solution to the optimization problem (8) than (u^*, L^*, x^*). Thus, there cannot be a feasible u lying outside of (7). □

A.3. Proof of Theorem 1

LEMMA 5. *With $v_i = \mathcal{A}_i(R_i)$ for all $i \in N$ and $v = (v_1, \ldots, v_n)$, the function $v \mapsto z(R, M, \mathcal{A})$ is continuous.*

Proof. Starting from the definition

$$z = \max_{L,x} \{\sum_j \mathcal{A}_j(L_j) \mid \sum_j L_j = M_x, \; \mathcal{A}_j(L_j) \geq v_j \; (j \in N)\} \tag{38}$$

for some chosen $i \in N$ and $\epsilon > 0$, we define a *perturbed problem*

$$z' = \max_{L,x} \{\sum_j \mathcal{A}_j(L_j) \mid \sum_j L_j = M_x, \; \mathcal{A}_j(L_j) \geq v_j \; (j \neq i), \; \mathcal{A}_i(L_i) \geq v_i - \epsilon\} \tag{39}$$

Clearly, problem (39) is a relaxation of problem (38), so in particular $z' \geq z$.

First, we show the following auxiliary result: Let L' be the optimal solution of (39). Then there is a feasible solution L of (38) with $|\mathcal{A}_i(L_i') - \mathcal{A}_i(L_i)| \leq \epsilon$, $\forall i \in N$.

The assertion is obviously valid if L' is a feasible solution of (38). So let us assume now that this is not the case. Then, for suitable x,

$$\sum_j L_j' = M_x, \quad \mathcal{A}_j(L_j') \geq v_j \;\; (j \neq i), \quad \mathcal{A}_i(L_i') \geq v_i - \epsilon, \text{ and } \mathcal{A}_i(L_i') < v_i.$$

Setting $u_j' = \mathcal{A}_j(L_j')$ $(j \in N)$ and $0 \leq \delta = v_i - u_i' \leq \epsilon$, we have

$$\sum_{j \neq i} u_j' \geq \sum_{j \neq i} v_j + \delta, \tag{40}$$

since otherwise

$$z' = \sum_j u_j' = v_i - \delta + \sum_{j \neq i} u_j' < v_i - \delta + \sum_{j \neq i} v_j + \delta = \sum_j v_j,$$

i.e., z' would be smaller than z, in contradiction to the fact that (39) is a relaxation of (38).

Setting $\delta_j = u_j' - v_j \geq 0$ $(j \neq i)$, it follows from (39) that

$$\sum_{j \neq i} \delta_j \geq \delta. \tag{41}$$

We define $L_i = L_i' + \delta$, $L_j = L_j' - \delta_j'$ for $j \neq i$ with $0 \leq \delta_j' \leq \delta_j$ $(j \neq i)$ and $\sum_{j \neq i} \delta_j' = \delta$, which is possible because of (41). Then

$$\sum_j L_j = (L_i' + \delta) + \sum_{j \neq i} L_j' - \sum_{j \neq i} \delta_j' = \sum_j L_j' + \delta - \sum_{j \neq i} \delta_j' = \sum_j L_j' = M_x.$$

Furthermore, using translation invariance of the functionals $\mathcal{A}_i$, we get:

$$\mathcal{A}_i(L_i) = \mathcal{A}_i(L_i' + \delta) = \mathcal{A}_i(L_i') + \delta = u_i' + \delta = v_i,$$

as well as

$$\mathcal{A}_j(L_j) = \mathcal{A}_j(L_j' - \delta_j') = \mathcal{A}_j(L_j') - \delta_j' = u_j' - \delta_j' \geq u_j' - \delta_j = v_j \tag{42}$$

for each $j \neq i$. Hence, L is feasible solution of (38), with the value for x a part of the optimal solution (39). Finally,

$$|\mathcal{A}_i(L_i') - \mathcal{A}_i(L_i)| = |\mathcal{A}_i(L_i') - \mathcal{A}_i(L_i' + \delta)| = |-\delta| \leq \epsilon,$$

and because of (42), for $j \neq i$,

$$|\mathcal{A}_j(L_j') - \mathcal{A}_j(L_j)| = |-\delta_j'| = \delta_j' \leq \sum_{k \neq i} \delta_k' = \delta \leq \epsilon.$$

This proves the auxiliary statement.

Now suppose that some $\epsilon > 0$ is given, and consider the perturbed problem (39) for the given ϵ and some component i. By the auxiliary result, we can associate to the optimal solution of (39) a feasible solution of the basic problem (38) for which the objective value $\sum_i \mathcal{A}_i(L_i)$ is at most worse by $n \cdot \epsilon$ compared to the solution value of (39). As problem (39) relaxes problem (38), this means that the solution values z and z' can differ by not more than $n \cdot \epsilon$.

Let us write $\zeta(v)$ for $z(R, M, \mathcal{A})$ with $\mathcal{A}_i(R_i) = v_i$ $(i \in R)$. Then what has been shown above is that for two vectors v and v',

$$v_j = v_j' \quad \forall j \neq i \text{ and } |v_i - v_i'| \leq \epsilon \Rightarrow |\zeta(v) - \zeta(v')| \leq n\epsilon.$$

Consider now, to given v, a vector $\bar{v}$ with $|v_j - \bar{v}_j| \leq \epsilon$ for all j. Then

$$\begin{aligned}|\zeta(v) - \zeta(\bar{v})| \leq |\zeta(v) - \zeta(\bar{v}_1, v_2, \ldots, v_n)| + |\zeta(\bar{v}_1, v_2, \ldots, v_n) - \zeta(\bar{v}_1, \bar{v}_2, v_3, \ldots, v_n)| \\ + |\zeta(\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_{n-1}, v_n) - \zeta(\bar{v})| \leq n^2\epsilon,\end{aligned}$$

showing the continuity of $\zeta(v)$ as claimed. □

Equipped with the technical lemma above, we can now prove Theorem 1.

Proof of Theorem 1. Let F be a bargaining solution that satisfies Axioms 1 – 3. For $u^* = F(R, M, \mathcal{A}) \in \mathcal{U}$, we must have $\sum_{i=1}^n u_i^* = z$ with $z = z(R, M, \mathcal{A})$ from (8), since $\sum_{i=1}^n u_i^* < z$ would be in contradiction to PAR. This shows that $u^* = F(R, M, \mathcal{A})$ must be an element of

$$\mathcal{U}_0 = \{u \in \mathbb{R}^n \mid u_i \geq v_i,\ \forall i \in N,\ \sum_{i=1}^n u_i = z\}.$$

To show that (9) holds first assume $v_i \in \mathbb{Q}$ for all i, i.e., $v_i = p_i/q$ with $q \in \mathbb{N}$ and $p_i \in \mathbb{N}$ for all $i \in N$. Define an instance $(R', M, \mathcal{A}')$ by splitting each player i into p_i players (i, k) with $k = 1, \ldots, p_i$, $i \in N$, as in Axiom STR. Then all players have identical values $v_{ik}' = \mathcal{A}_{ik}'(R_{ik}')$, since, by the positive homogeneity of the $\mathcal{A}_i$,

$$\mathcal{A}_{ik}'(R_{ik}') = \mathcal{A}_i(R_i/p_i) = \frac{1}{p_i}\mathcal{A}_i(R_i) = \frac{v_i}{p_i} = \frac{1}{q}.$$

Successive application of Axiom STR for $i = 1, \ldots, n$ yields

$$u_{ik}' = \frac{u_i^*}{p_i} \quad \forall k = 1, \ldots, p_i,\ \forall i \in N.$$

Because of the identical values v_{ik}', the feasible set $\mathcal{U}'$ of $(R', M, \mathcal{A}')$ is invariant under permutations of the coordinates. Thus both conditions of Axiom SYM are satisfied, and we can conclude that all players get the same acceptability $u_{ik}' = z \cdot \left(\sum_{j=1}^n p_j\right)^{-1} = \tilde{z}$ and hence

$$u_i^* = p_i \cdot \tilde{z} = \frac{p_i}{\sum_{j=1}^n p_j} \cdot z = \frac{v_i}{\sum_{j=1}^n v_j} \cdot z$$

as required.

For $v_i \notin \mathbb{Q}$, the assertion follows because of the continuity of the functions on the right hand side of (9) by a straightforward convergence argument, representing $v_i \in \mathbb{R}$ as limits of rational numbers. The continuity of the quotient $v_i / \sum_j v_j$ is clear, while the continuity of $z(R, M\mathcal{A})$ is proved in Lemma 5.

To show the converse, we assume that the bargaining solution F satisfies (9) and show that Axioms (1) – (3) are satisfied.

Axiom PAR: Suppose that, contrary to PAR, the acceptability allocation $u^* = F(R, M, \mathcal{A}) \in \mathcal{U}(R, M, \mathcal{A})$ is dominated by some other $u \in \mathcal{U}(R, M, \mathcal{A})$, i.e., $u_i > u_i^*$ for some i and $u_j \geq u_j^*$ $\forall j \in N$. Then

$$\sum_{j=1}^{n} u_j > \sum_{j=1}^{n} u_j^* = z(R, M, \mathcal{A}),$$

which contradicts $u \in \mathcal{U}(R, M, \mathcal{A})$ because of (7).

Axiom SYM: In view of (7), the set $\mathcal{U}(R, M, \mathcal{A})$ can only be invariant under permutations of coordinates, if $v_i = v_j$ $\forall i, j \in N$. In this case, (9) implies identical utilities $u_i = z/n$ $\forall i \in N$.

Axiom STR: Let the conditions of STR be satisfied for the two instances $(R, M, \mathcal{A})$ and $(R', M, \mathcal{A}')$. Using positive homogeneity and translation invariance, it is easy to see that $z = z'$ for the corresponding solution values of (8). Then,

$$v_j' = v_j \quad (j \neq i), \qquad v_{ik}' = \frac{v_i}{\ell} \quad (k = 1, \ldots, \ell),$$

where the last equality follows by means of the positive homogeneity of the $\mathcal{A}_i$. Therefore (9) implies

$$u_j' = \frac{v_j}{\sum_{s \neq i} v_s + \sum_{s=1}^{\ell} \frac{v_s}{\ell}} \cdot z = u_j \quad \forall j : j \neq i$$

and

$$u_{ik}' = \frac{\frac{v_i}{\ell}}{\sum_{s \neq i} v_s + \sum_{s=1}^{\ell} \frac{v_s}{\ell}} \cdot z = \frac{u_i}{\ell} \quad \forall k : k = 1, \ldots, \ell,$$

as claimed by Axiom STR. □

A.4. Proof of Theorem 2

By monotonicity of the objective function and Proposition 2, the feasible set in (10) can be restricted to those u that satisfy $\sum_{i=1}^{n} u_i = z$ where $z = z(R, M, \mathcal{A})$. Except in the trivial boundary case where $z = \sum_j v_j$, a feasible solution with $u_i > v_i$ for all i exists. Taking the logarithm of the objective function in (10), we arrive at the following problem

$$\begin{array}{ll} \max\limits_{u} & \sum_{i=1}^{n} v_i \log(u_i - v_i) \\ \text{s.t.} & u_i \geq v_i, \quad \forall i \in N \\ & \sum_{i=1}^{n} u_i = z. \end{array} \tag{43}$$

Obviously, this is a convex optimization problem, as the objective function to be maximized is concave and the constraints are linear. The Lagrange function of (43) without the inequality constraint reads

$$\mathcal{L} = \sum_{i=1}^{n} v_i \log(u_i - v_i) - \lambda \left(\sum_{i=1}^{n} u_i - z \right).$$

The first order condition with respect to u_i therefore yields

$$u_i = \frac{1 + \lambda}{\lambda} \cdot v_i, \quad \forall i \in N. \tag{44}$$

Because $\sum_i u_i = z$, it follows that

$$\frac{1 + \lambda}{\lambda} = \frac{z}{\sum_{i=1}^{n} v_i} \geq 1 \tag{45}$$

and therefore $u_i \geq v_i$ is fulfilled. Plugging (45) into (44) yields

$$u_i = \frac{v_i}{\sum_j v_j} \cdot z,$$

which is the solution identified by Theorem 1. □

A.5. Proof of Theorem 3

To show some properties of sup-convolutions relevant in the later discussion, we require the following Lemma, about dual cones $\mathcal{Z}^*$ for sets $\mathcal{Z}$ defined as

$$\mathcal{Z}^*(\Lambda) = \{X \in \mathcal{L}^p(\mathcal{Y}) : \mathbb{E}[X(Z-\Lambda)] \geq 0 \text{ for all } Z \in \mathcal{Z}_i\}. \tag{46}$$

.

LEMMA 6. *Let $\mathcal{A}_i : \mathcal{L}^p(\mathcal{Y}) \to \mathbb{R}$ with $1 < p < \infty$, then*

$$(\mathcal{Z}^{\max})^*(\zeta) = \left(\bigcap_{i=1}^n \mathcal{Z}_i\right)^*(\zeta) = \sum_{i=1}^n \mathcal{Z}_i^*(\zeta), \tag{47}$$

for any $\zeta \in \mathcal{Z}^{\max} \neq \emptyset$.

Proof. By Assumption 2, the functionals $\mathcal{A}_i$ are upper semicontinuous and concave. Hence, their hypographs are (norm) closed and convex. By the Hahn-Banach separation theorem, it follows that they are weakly-closed as well. Since $1 < p < \infty$, $\mathcal{L}^p(\mathcal{Y})$ is reflexive and the weak topology is equivalent to the weak* topology. Therefore the hypographs are weak*-closed.

The (positive homogeneous) functionals $\mathcal{A}_i$ are by definition identical with the support functions $\sigma_{\mathcal{Z}_i}$ of the defining sets $\mathcal{Z}_i$. Because the hypographs of the individual support functions are weak*-closed, the sum

$$\sum_{i=1}^n \operatorname{hypo} \sigma_{\mathcal{Z}_i}, \text{ is weak*-closed.} \tag{48}$$

Finally, by Theorem 3.1 in Burachik and Jeyakumar (2005), (47) follows from (48). □

Proof of Theorem 3 The first point follows directly from Rüschendorf (2013), Proposition 11.1 (for the nonempty domain) and Acciaio (2009), Theorem 2.1 (for the dual representation).

To prove the second point, note that the fact that $\mathcal{A}^{\max}$ is a coherent acceptability measure directly follows from the dual representation (15). Upper semi-continuity follows by applying Theorem 2 and Theorem 4 of Section 6.2 in Chapter IV of Bourbaki (1989) to the family of upper semicontinuous (in fact even continuous) functionals $X \mapsto \mathbb{E}[\zeta X]$ for $\zeta \in \mathcal{Z}^{\max}$. Finally, $\mathcal{A}^{\max}$ is proper, since all $\mathcal{A}_i$ are proper by Rüschendorf (2013), Proposition 11.1.

The third statement directly follows from Proposition 11.4. and Theorem 11.5 in Rüschendorf (2013).

In order to prove the fourth point for $p > 1$, note first that because of the properties of $\mathcal{A}$ in the second point, by Kaina and Rüschendorf (2009) Theorem 2.9, there exist $\Lambda \in \mathcal{Z}^{\max} \neq \emptyset$ such that

$$\Lambda \in \underset{\zeta \in \mathcal{L}^q}{\arg\min} \{\mathbb{E}[\zeta M] : \zeta \in \mathcal{Z}^{\max}\}. \tag{49}$$

Assume now $1 < p < \infty$, $\mathcal{Z}^{\max} \neq \emptyset$ and consider any Λ, fulfilling (49), which can be rewritten as

$$M \in \mathcal{Z}^{\max *}(\Lambda). \tag{50}$$

Applying Lemma 6 then yields

$$M \in \sum_{i=1}^n \mathcal{Z}_i^*(\Lambda), \tag{51}$$

which shows that for any Λ that fulfills (49) there exist $X_i \in \mathcal{Z}_i^*(\Lambda)$, which sum up to M. Hence, there exist Λ such that (49) holds for these X_i, which are consequently optimal for the sup-convolution problem (13). If $p = \infty$, we apply the above argument for an arbitrary $1 < p' < \infty$. Since $M \in \mathcal{L}^{p'}(\mathcal{Y})$ and (51) ensures that the resulting L_i are in $\mathcal{L}^\infty(\mathcal{Y})$, the argument still goes through.

Existence of an allocation for $p = 1$ (in fact, for $1 \leq p \leq \infty$) under the stronger condition was shown in Rüschendorf (2013), Theorem 11.3.

Finally, to prove the last statement, we show that for any pair of optimal assignments X_i, X_j the set

$$A(i,j) = \{\omega \in \Omega : \exists \omega' \in \Omega : \; X_j(\omega) \geq X_j(\omega') \wedge X_i(\omega) < X_i(\omega')\} \tag{52}$$

has probability zero.

To this end, assume that $\mathbb{P}(A(i,j)) > 0$ and consider optimal X_i, X_j, an arbitrary $\omega \in A(i,j)$, and a related ω' fulfilling the defining condition of $A(i,j)$, i.e.,

$$X_j(\omega) \geq X_j(\omega') \wedge X_i(\omega) < X_i(\omega'). \tag{53}$$

Consider the conjugate representations (11) and the related optimal dual variables ζ_i and ζ_j for X_i and X_j. From (12) we know that $\mathbb{E}(\zeta_i) = \mathbb{E}(\zeta_j) = 1$ and $\zeta_i \geq 0$ and $\zeta_j \geq 0$ must hold almost surely. It follows from the optimality of ζ_i, ζ_j and (53) that $\zeta_j(\omega) \geq \zeta_j(\omega')$ and $\zeta_i(\omega) < \zeta_i(\omega')$. This holds for almost all $\omega \in A(i,j)$, which has positive probability as assumed above. But this violates (16) which states that $\zeta_i = \zeta_j$ almost surely, implying that X_i, X_j cannot be optimal. It follows that $A(i,j)$ must have probability 0 and because i, j were arbitrary, the same is true for the set $A(j,i)$. Together this yields that X_i and X_j are comonotone. □

A.6. Proof of Corollary 1 and Corollary 2

Proof of Corollary 1 Recall that the $\mathcal{A}_i$ are translation equivariant. Hence, if the allocation X_i is optimal for (13) then any allocation $X_i + a_i$ such that $\sum_{i=1}^n a_i = 0$ is also optimal because $\sum_{i=1}^n (X_i + a_i) = M$ and $\sum_{i=1}^n \mathcal{A}_i(X_i + a_i) = \sum_{i=1}^n \mathcal{A}_i(X_i)$. In particular L_i, defined in (19) is therefore optimal for (13) and the allocation L_i leads to the acceptability allocation $u_i = \frac{v_i}{\sum_{j=1}^n v_j} \mathcal{A}^{\max}(M)$.

If (17) holds then the u_i are automatically feasible for (8), $z = \mathcal{A}^{\max}(M)$, and the u_i fulfill (1) such that L, u is a solution of the bargaining game.

The allocation L_i is also optimal for the sup-convolution problem (13), because the L_i sum up to M and the objective value is not changed. Therefore the fourth statement of Theorem 3 implies (20) and the fifth statement implies comonotonicity.

If on the other hand (17) does not hold, then clearly $\mathcal{A}^{\max}(M)$ cannot be allocated such that $u_i \geq \mathcal{A}_i(R_i)$ for all i. □

Proof of 2. This follows directly from Proposition 2 and Corollary 1 because x is optimal for z and cannot be improved by any other management decision. □

A.7. Proof of Proposition 3

Condition (26) implies $\mathcal{Z}^{\max} = \mathcal{Z}_{i_{\max}}$, hence $\sum_{j=1}^n \mathcal{A}_j(b_j M) = \mathcal{A}_{i_{\max}}(M)$ and the allocation $X_j = b_j M$ must be optimal for the sup-convolution $\mathcal{A}^{\max}(M)$. Applying Corollary 1 directly shows that $L_i = a_i + b_i M$ solves the bargaining game. □

A.8. Proof of Lemma 1

By the comonotonicity of the X_i shown in Theorem 3.5 and by Proposition 5.16 in McNeil et al. (2005), X_i can be written as a monotonous function of M, i.e., $X_i = g_i(M)$ with $g_i : \mathbb{R} \to \mathbb{R}$ monotonously increasing. Since $M = \sum_{i=1}^n X_i$, the mappings g_i have to be absolutely continuous.

Therefore, it follows from the fundamental theorem of calculus that

$$X_i(\omega) = g_i(M(\omega)) = \int_C^{M(\omega)} g_i'(t)\, dt + g_i(C) = \int_{-\infty}^{\infty} \mathbb{1}_{\{x \in \mathbb{R} :\, x \geq t\}}(M(\omega)) \xi_i(t)\, dt + g_i(C)$$

with

$$\xi_i(t) = \begin{cases} g_i'(t), & \text{if } t > C \\ 0, & \text{if } t < C. \end{cases}$$

Also note that by the condition $M = \sum_{i=1}^n X_i$ it follows that for $t \geq C$, $\xi_i(t) \geq 0$ and $\sum_{i=1}^n \xi_i(t) = 1$ almost surely, establishing the first part of the theorem.

To prove the second part, note the random variables $\mathbb{1}_{\{x\in\mathbb{R}:\, x\geq t\}}(M)$ are comonotone for all $t \in \mathbb{R}$. Now define a set of points $(t_i)_{i\in\mathbb{N}}$ such that

$$\sum_{i=1}^N \mathbb{1}_{\{x\in\mathbb{R}: x\geq t_i\}}(M)\xi_i(t_i) \xrightarrow[a.s.]{N\to\infty} \int_{-\infty}^{\infty} \mathbb{1}_{\{x\in\mathbb{R}:\, x\geq t\}}(M)\xi_i(t)\, dt.$$

We then get

$$\begin{aligned} \mathcal{A}_i(X_i) &= \mathcal{A}_i\left(\lim_{N\to\infty}\sum_{i=1}^N \mathbb{1}_{\{x:\in\mathbb{R}:\, x\geq t_i\}}(M)\xi_i(t_i) + \xi_i(C)C\right) = \lim_{N\to\infty} \mathcal{A}_i\left(\sum_{i=1}^N \mathbb{1}_{\{x:\in\mathbb{R}:\, x\geq t_i\}}(M)\xi_i(t_i)\right) + \xi_i(C)C \\ &= \lim_{N\to\infty}\sum_{i=1}^N \mathcal{A}_i\left(\mathbb{1}_{\{x\in\mathbb{R}:\, x\geq t_i\}}(M)\xi_i(t_i)\right) + \xi_i(C)C = \int_{-\infty}^{\infty} \mathcal{A}_i(\mathbb{1}_{\{x\in\mathbb{R}:\, x\geq t\}}(M))\xi_i(t)\, dt + \xi_i(C)C \end{aligned}$$

where the second equality follows from Theorem 3.2 in Wozabal and Wozabal (2009) and the third one by comonotone additivity of $\mathcal{A}_i$ (e.g., Pflug and Römisch 2007, Proposition 2.49). □

A.9. Proof of Theorem 4

We first prove the statement for $\delta_i \in [0,1]$. Suppose that the statement is false and there exist optimal X_i, which violate (33). Then for a set $S \subseteq \mathbb{R}$ with positive probability there are players i and j such that

$$\mathcal{A}_i(\mathbb{1}_{\{x\in\mathbb{R}:\, x\geq t\}}(M)) > \mathcal{A}_j(\mathbb{1}_{\{x\in\mathbb{R}:\, x\geq t\}}(M))$$

for $t \in S$ and $\xi_i(t) < \xi_j(t)$.

Defining $\xi_i'(t) = \xi_i(t) + \mathbb{1}_S(t)\xi_j(t)$ and $\xi_j'(t) = \xi_j(t) - \mathbb{1}_S(t)\xi_j(t)$, we obtain a feasible $X' = (X_1, \dots, X_i', \dots, X_j', \dots, X_n)$ with

$$\begin{aligned} \sum_{i=1}^n \mathcal{A}_i(X_i') - \sum_{i=1}^n \mathcal{A}_i(X_i) &= \mathcal{A}_i(X_i') - \mathcal{A}_i(X_i) + \mathcal{A}_j(X_j') - \mathcal{A}_j(X_j) \\ &= \int_S \xi_j(t)\left(\mathcal{A}_i(\mathbb{1}_{\{x\in\mathbb{R}:\, x\geq t\}}(M)) - \mathcal{A}_j(\mathbb{1}_{\{x\in\mathbb{R}:\, x\geq t\}}(M))\right) dt > 0, \end{aligned}$$

by Lemma 1, contradicting the assumption of optimality of X.

It follows that if for some $t > C$ the set $\mathcal{I}(t) = \arg\max_j \mathcal{A}_j(\mathbb{1}_{\{x\in\mathbb{R}:\, x\geq t\}}(M))$ has only one element, i.e., $\mathcal{I}(t) = \{i\}$, then $\delta_i = 1$ and $\delta_j = 0$ for $j \neq i$. If $|\mathcal{I}(t)| > 1$, any ξ_i' with $\sum_{i\in\mathcal{I}} \delta_i'(t) = 1$ and $\delta_i'(t) \in [0,1]$ for $t > C$ is optimal for (13). We thus can randomly pick an $i \in \mathcal{I}$ and set $\delta_i' = 1$ to obtain an optimal solution as described in the statement of the theorem. □

A.10. Proof of Theorem 5

To prove the first point note that for any α

$$\mathrm{AVaR}_\alpha(\mathbb{1}_{\{x\in\mathbb{R}:\, x\geq t\}}(M)) = \frac{1}{\alpha}\int_0^\alpha F_{\mathbb{1}_{\{x\in\mathbb{R}:\, x\geq t\}}}(M)^{-1}(y)\, dy = \max\left(\frac{F^{-1}(\alpha) - t}{\alpha}, 0\right). \tag{54}$$

It follows that if m_i has only finitely many atoms $\alpha_1, \dots, \alpha_k$ with probabilities $m_1, \dots, m_k$, then

$$\mathcal{A}_i(\mathbb{1}_{\{x\in\mathbb{R}: x\geq t\}}(M)) = \sum_{i=1}^k m_i\, \mathrm{AVaR}_{\alpha_i}(\mathbb{1}_{\{x\in\mathbb{R}:\, x\geq t\}}(M)),$$

which together with (54) implies that $t \mapsto \mathcal{A}_i(\mathbb{1}_{\{x \in \mathbb{R}:\, x \geq t\}}(M))$ are piecewise affine functions with finitely many pieces. Since these functions can cross only finitely often, $\mathbb{R}$ can be divided into finitely many intervals where $i^* = \arg\max_j \mathcal{A}_j(\mathbb{1}_{\{x \in \mathbb{R}:\, x \geq t\}}(M))$ remains constant. In these intervals player i^* receives all the additional payments from the project, i.e., $M \mapsto X_{i^*}(M)$ has slope 1 while $X_i(M)$ has slope 0 for all $i \neq i^*$. This structure of payments can be achieved by combining finitely many standard call and put options.

Similarly, to prove the second point, note that if M has finite support there are only finitely many distinct $\mathbb{1}_{\{x \in \mathbb{R}:\, x \geq t\}}$, i.e., the sets $\arg\max_j \mathcal{A}_j(\mathbb{1}_{\{x \in \mathbb{R}:\, x \geq t\}}(M))$ can only change finitely often, leading to L which can be expressed by finitely many options. $\square$