Some heuristic methods for the p-median problem with maximum distance constraints. Application to a bi-objective problem.

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Abstract

In this work we study the p-median problem with maximum distance constraints (**PMPDC**) which is a variant of the classical p-median problem (**PMP**). First of all, we provide some different formulations for (**PMPDC**) because the heuristics procedures for the (**PMPDC**) with a formulation based on the approach that modifies the distance matrix that leads to difficulties and frequently infeasible solutions. Different heuristic procedures for the (**PMPDC**) problem are developed. First, a new Lagrangian relaxation algorithm, which differs from those existing in the literature, is developed. Second, we apply a new approach, based on the GRASP methodology addapted to (**PMPDC**) from (**PMP**). In addition, we study in depth the relation between the feasibility of (**PMPDC**) and the parameters p and maximum distance limits, providing an analytic-geometric characterization. Finally, we develop a new resolution methodology to solve a bi-objective problem of interest in facility location and healthcare optimization, the p-median-center tradeoff problem, finding all the points on the Pareto front very quickly.

Keywords: Facility location, p-median problem, Lagrangian relaxation, GRASP, bi-objective problem, p-median-center tradeoff problem.

1 Introduction

The p-median problem, initially described by Hakimi (1965) and formulated mathematically later by ReVelle and Swain (1970), is an NP-hard problem, classic in Location Theory. Although the literature about it is quite broad, the problem of the p-median with maximum distance constraints is poorly treated. It is possible to consider two approaches when formulating the p-median problem with maximum distance constraints: the indirect approach, introducing a sufficiently large M value in the distance matrix if the distance (or time) between a demand point and a facility site point exceeds a maximum distance (or time); and the direct approach, introducing a series of maximum distance constraints (or time) (henceforth, unless otherwise specified, we will consider distance and time indistinctly). As Choi and Chaudry (1993) point out, "*the performance of the indirect approach can greatly be dependent on the choice of the large number for big-M*". The application of this problem to public services (location of bus stops) can be seen in Murray and Wu (2003).

The p-median problem with maximum distance constraints appeared for the first time in Toregas *et al.* (1971). In Khumawala (1973, 1975), a specific heuristic for this problem appears. Instead, Hillsman and Rushton (1975) apply the heuristic of Teitz and Bart (1968). Rahman and Smith (1991) point out "*With maximum distance constraints the performance of the Teitz and Bart method is even better*", referring to the heuristic of Ardalan (1988) (this one reflects the superiority against the heuristic of Khumawala (1973)). Choi and Chaudry (1993) apply a Lagrangian relaxation procedure, obtaining optimal solutions in a reasonable time. Finally, Chaudry *et al.* (1994) correct the heuristic solutions (applying Teitz and Bart (1968) heuristics) obtained by Rahman and Smith (1991), comparing with the optimal solutions obtained by Lagrangian relaxation in Choi and Chaudry (1993). Since the 1990s, the use of metaheuristics in Operations Research has been increasing over the years, and there are now many of them to solve complex problems in Optimization. This is why, since in the existing literature for this problem no metaheuristics have been applied, we consider it appropriate to attack the problem by applying this type of heuristics. Morever, we apply our results and methodology of the p-median problem with maximum distance constraints to a bi-objective problem of interest in practice: the p-median-center tradeoff problem.

According to Halpern (1978), "*In many real world problems the objective function is a mixture of the two different, possibly adverse objectives mentioned above*". This phrase of Halpern, is associated with the following: *in many real problems of Location, there are two opposing objectives:* equity *and* efficiency. These objectives are related to two classic problems: first, with the p-center problem (to minimize maximum distance or maximum service time), traditionally associated with the location of emergency services; and second, with the p-median problem (to minimize the total distance needed to cover all demand), usually associated with the location of points of service or the distribution of people or goods. However, since the p-median problem is based on averaging, it can discriminate remote areas and population density or low demand, against areas or areas centrally located and characterized by high population density, or a large amount of demand, implying no equity (although quite efficient) (see Ogryczak (1997)). On the other hand, the location of a certain service from the p-center problem can cause a high total distance, which means no efficiency in service.

However, these objectives are often combined. An example of this is the location of a number of commercial establishments (say p). Consideration should be given to the p-center problem, as it is desirable that establishments should be close to customers, and also the p-median problem, since a fast distribution of stocks is needed. The convex combination of the objective functions associated with p-center and p-median problems is often known as a cent-dian function, due to Halpern (1976), or also as a medi-center function, due to Handler (1976). In the existing literature, the predominant focus of study for the bi-objective problem of the p-center and p-median is by means of convex methods or ones that weigh both objectives by weight. The resolution of these problems, either does not lead to a calculation of all the efficient points, or they can be calculated, but the success of finding all the efficient points depends on the measurement of the computational weights and time. This work is part of the classical studies on the medi-center (or cent-dian) problem, Daskin (2010, 2013) and Daskin and Maass (2015).

The article is structured as follows. Section 2 states the formulation of the problem and establishes the notation used in the rest of the article. Section 3 develops a new Lagrangian relaxation procedure. Section 4 develops a new heuristic procedure based on the GRASP metaheuristic. Section 5 presents the computational results obtained when comparing the different heuristics procedures and we give an in-depth study of the feasibility of the p-median problem with maximum distance constraints. Section 6 states a bi-objective problem, the p-median-center tradeoff problem, and we give a fast algorithm to compute all the points of the Pareto front. Finally, section 7 gives the final conclusions.

2 Formulation

In this section, we will show two equivalent formulations of the problem that we are discussing in this work: *dense formulation* and *sparse formulation*.

Let $M = \{1, 2, \ldots, m\}$ be a set of demand points of a certain service, with demands h_i, $i = 1, \ldots, m$, and a set $N = \{1, 2, \ldots, n\}$ of points where it is possible to locate or open a facility (service points). We will assume that the distance matrix of each demand point i at each point of facility j, (d_{ij}) is known. Moreover, for every $i \in M$, let s_i be the maximum distance limit between the demand point i and any facility.

Given a total number of p facilities that are to be opened, the problem of the p-median with maximum distance constraints consists in deciding at which points the facilities should be opened and assigning to each demand point an open facility that complies with maximum distance constraints, so that the total distance (averaged with the population) covered to serve the total demand is minimal.

2.1 Dense formulation

The dense formulation explicitly adds maximum distance constraints to the formulation of the p-median problem. More specifically, the p-median problem with maximum distance constraints (**PMPDC2**) (with dense formulation) is stated as follows:

$$
\begin{aligned}
(\textbf{PMPDC1})\ \text{Minimize}\ & \sum_{i=1}^{m}\sum_{j=1}^{n} h_i d_{ij} y_{ij} && (1)\\
\text{s.t.} & \sum_{j=1}^{n} y_{ij} = 1,\ i = 1,\ldots,m && (2)\\
& \sum_{j=1}^{n} d_{ij} y_{ij} \leqslant s_i,\ i = 1,\ldots,m && (3)\\
& y_{ij} \leqslant x_j,\ i = 1,\ldots,m,\ j = 1,\ldots,n && (4)\\
& \sum_{j=1}^{n} x_j = p, && (5)\\
& x_j \in \{0,1\},\ j = 1,\ldots,n && (6)\\
& y_{ij} \in \{0,1\},\ i = 1,\ldots,m,\ j = 1,\ldots,n && (7)
\end{aligned}
$$

Here, m is the number of demand points, n is the number of candidate points of possible locations of the facilities, p is the number of facilities to be opened, h_i is the point demand i, d_{ij} is the distance between the demand point i to the facility candidate point j, s_i is the maximum distance limit between the demand point i and any facility, $x_j \in \{0,1\}$ is a location variable, defined for each $j = 1,\ldots,n$, with $x_j = 1$ if the point j is selected as facility, and 0 otherwise. Finally, $y_{ij} \in \{0,1\}$ is an allocation variable, defined for all $i = 1,\ldots,m$ $j = 1,\ldots,n$ $y_{ij} = 1$ if the demand point i is assigned to the facility j and 0 otherwise.

The objective function (1) minimizes the total distance to be traversed by the entire population. Constraints (2) ensure that all demand points are assigned to a facility; with constraints (3) the maximum distance conditions are imposed, and with the constraints (4), we ensure that if a facility is not open, it cannot serve any point of demand. With the constraint (5), we ensure that exactly p facilities will open. Finally, the constraints (6) and (7) refer to the binary character of the decision variables.

The dense formulation presents a greater number of constraints than the indirect approach, which is based on modifying the distance matrix by introducing a sufficiently large value M if the distance between a demand point and a facility site point exceeds a maximum distance and suppresses the constraint (3). More concretely:

$$
d'_{ij} = \begin{cases} d_{ij}, & \text{if } d_{ij} \leqslant s_i \\ M, & \text{if } d_{ij} > s_i \end{cases}
$$

The p-median problem with maximum distance constraints, using this indirect approach, may seem a priori an easy problem to solve, since it is reduced to a classical p-median problem with the modified distance matrix (which is perhaps one of the reasons why it is not much studied in the existing literature). However, surprisingly, heuristics bring difficulties to the procedure and frequently infeasible solutions, while the quality of the obtained solutions depends on the value of big-M (Choi and Chaudry (1993)).

2.2 Sparse formulation

Next, we present a second formulation, which is the most appropriate for the resolution procedures. In addition, to the notation discussed above, we add the following, defining N_i as the set of facility sites within s_i distance units from the demand point $i \in M$.

We use the following notation and definitions:

N is the index set for facility sites.
M is the index set for demand points.
h_i is the demand at demand point i.
p is the number of facilities to be opened.
d_{ij} is the distance from demand point i to facility site j.
s_i is the maximum distance limit between a demand point i and any facility site.
N_i is the set of facility sites within s_i distance units from demand point $i \in M$, with $N_i \subseteq N$.
For all $j = 1, \ldots, n$, x_j is the location variable, with $x_j = 1$, if the facility point j is to be opened, and $x_j = 0$ otherwise.
For all $i = 1, \ldots, m$, $j = 1, \ldots, n$, y_{ij} is the assignment variable, with $y_{ij} = 1$ if the demand point i is assigned to facility site j, and $y_{ij} = 0$ otherwise.

The p-median problem with maximum distance constraints (**PMPDC2**) (with sparse formulation) may be stated as follows:

$$(\mathbf{PMPDC2}) \quad \text{Minimize} \sum_{i=1}^{m} \sum_{j \in N_i} h_i d_{ij} y_{ij} \tag{8}$$

$$\text{s.t.} \sum_{j \in N_i} y_{ij} = 1, \; i = 1, \ldots, m \tag{9}$$

$$y_{ij} \leqslant x_j, \; i = 1, \ldots, m, \; j \in N_i \tag{10}$$

$$\sum_{j=1}^{n} x_j = p, \tag{11}$$

$$x_j \in \{0, 1\}, \; j = 1, \ldots, n \tag{12}$$

$$y_{ij} \in \{0, 1\}, \; i = 1, \ldots, m, \; j = 1, \ldots, n \tag{13}$$

In the objective function (8), the total distance to be crossed by the entire population is minimized. The constraints (9) ensure that all demand points will be assigned to a facility within a maximum distance limit s_i associated with them. With the constraints (10), we ensure that if a facility is not open, it cannot serve any demand point (taking into account the maximum distance constraints). With the constraint (11), we ensure that exactly p facilities are opened. Finally, the constraints (12) and (13) refer to the binary character of the decision variables.

It is important to note that, for some $(p, \mathbf{s})$, $\mathbf{s} = (s_1, \ldots, s_m)$, there may be no feasible solution. In addition, having fixed the value of p , if the values of the maximum distances limits are increased, there exists a limit value $\bar{\mathbf{s}}$ in which the problem solution (**PMPDC2**) will not differ from the solution of the p-median classic problem (**PMP**). In practice, it is usual to consider that $s_i = s$ for all $i = 1, \ldots, m$. This is why this hypothesis will be assumed later.

3 A Lagrangian relaxation procedure

Lagrangian relaxation is a well-known procedure for solving large-scale and complex combinatorial optimization problems. It is based on exploiting the inherent structure of each problem in order to obtain lower and upper bounds on the optimal value of the problem (Guignard (2003)). Moreover, we will relax (9) to obtain an easy problem to solve. Note that in the Lagrangian relaxation procedure, we need to be able to solve the subproblems in an easy way, a method to obtain feasible solutions (a Lagrangian heuristic, see Avella *et al.* (2007), Daskin (2013)) and a procedure to update the multipliers (a subgradient optimization procedure, see Frangioni *et al* (2017)).

The Lagrangian relaxation procedure we apply here is new, and differs from that in Choi and Chaudry (1993), since they fall into a knapsack subproblem and we fall into a trivial problem. The reason for this is that we have been able to adapt the Lagrangian relaxation procedure for the classic p-median problem (**PMP**) (Daskin (2013)) to the p-median problem with maximum distance constraints (**PMPDC2**).

$$(\mathbf{PMPDC2}\boldsymbol{\lambda}) \quad \text{Minimize} \quad \sum_{i=1}^{m} \sum_{j \in N_i} h_i d_{ij} y_{ij} + \sum_{i=1}^{m} \lambda_i \left[1 - \sum_{j \in N_i} y_{ij} \right] \tag{14}$$

$$= \sum_{i=1}^{m} \sum_{j \in N_i} \left(h_i d_{ij} - \lambda_i \right) y_{ij} + \sum_{i=1}^{m} \lambda_i \tag{15}$$

$$\text{s.t.} \quad y_{ij} \leqslant x_j, \ i = 1, \ldots, m, \ j \in N_i \tag{16}$$

$$\sum_{j=1}^{n} x_j = p, \tag{17}$$

$$x_j \in \{0, 1\}, \ j = 1, \ldots, n \tag{18}$$

$$y_{ij} \in \{0, 1\}, \ i = 1, \ldots, m, \ j = 1, \ldots, n \tag{19}$$

For fixed values of multipliers, λ_i, we want to minimize the objective function (14). Note that the term $\sum_{i=1}^{m} \lambda_i$ is constant. Next, the proposed Lagrangian relaxation algorithm is solved to solve the p-median problem with maximum distance constraints.

- Step 1 (Initialization): Initialize $UB := +\infty$ (upper bound), $LB := -\infty$ (lower bound), $k := 0$ and take initial values for the multipliers. We have chosen to establish the objective value of the optimal solution of the linear relaxation of the problem as the initial LB and the dual variables associated with the optimal solution of the linear relaxation as initial multipliers.
- Step 2 (Resolution of subproblems) : In the k-th iteration, having fixed λ^k, we compute

$$V_j = \sum_{i \in M / j \in N_i} \min \left\{ 0, h_i d_{ij} - \lambda_i \right\}, \quad j \in N.$$

Find the p lowest values of V_j and let $x_j(\lambda^k) = 1$ in the corresponding facility sites and set $x_j(\lambda^k) = 0$ otherwise. Let $y_{ij} = 1$ if $x_j(\lambda^k) = 1$ and $h_i d_{ij} - \lambda_i < 0$ and $y_{ij} = 0$ otherwise. Calculate

$$V(\lambda^k) = \sum_{i=1}^{m} \sum_{j \in N_i} (h_i d_{ij} - \lambda_i)\, y_{ij} + \sum_{i \in I} \lambda_i^k \tag{20}$$

If $V(\lambda^k) > LB$ then $LB := V(\lambda^k)$.

- **Step 3** (Obtaining feasible solutions: Lagrangian Heuristic): Having fixed the p values of the variables $x_j(\lambda^k)$, for all $i = 1, \ldots, m$ find

  $$\widehat{j}_i = \arg\min \left\{ d_{ij} \Big/ x_j = 1,\ j \in N \right\}$$

  Let $y_{ij}(\lambda^k) = 1$ if $j = \widehat{j}_i$ and $y_{ij}(\lambda^k) = 0$ otherwise.

  Calculate

  $$\overline{V}(\lambda^k) = \sum_{i=1}^{m} h_i d_{i\widehat{j}_i}$$

  If $\overline{V}(\lambda^k) < UB$, then $UB := \overline{V}(\lambda^k)$.

- **Step 4** (Subgradient iteration): Calculate the subgradient ξ_i using

  $$1 - \sum_{j \in N_i} y_{ij}, \quad i \in M.$$

  Calculate the step size

  $$T_k := \frac{\theta(UB - LB)}{\displaystyle\sum_{i=1}^{m} \left(1 - \sum_{j \in N_i} y_{ij}\right)^2}$$

  with $0 < \theta \leqslant 2$.
  Update the multipliers:

  $$\lambda^{k+1} := \max\left\{0,\ \lambda^k + T_k \xi^k\right\}$$

  Moreover, if more than some number of prefixed iterations of the sugradient algorithm are performed without an increment of LB, then halve the step length parameter by setting $\theta := \theta/2$ (see Held *et al.* (1974)). In our numerical experiments, we have established 18 iterations.

- **Step 5** (Stopping criterion): If the duality gap (GAP=$\frac{(UB-LB)}{UB}$) is less than a prefixed tolerance, or if the number of maximum iterations has been reached, or if the norm of the subgradient or θ are less than a prefixed tolerance, STOP and set UB as the objective function value. Set $x_j(\lambda^k)$ and $y_{ij}(\lambda^k)$ as the final solution obtained. If not, $k := k + 1$ and go to step 2.

4 A heuristic method based on the GRASP methodology

GRASP (*Greedy Randomized Adaptive Search Procedure*) is a metaheuristic that works quite well for the classic p-median problem. Its adaptation to our p-median problem with maximum distance constraints is not entirely immediate, since the maximum distance contraints in the GRASP construction phase have an influence and non-feasible solutions can be obtained. So it is necessary to implement two phases in the construction phase of GRASP. Note that the GRASP implemented here requires two parameters: KRCL (cardinal of the restricted candidate list) and `iter_max` (number of maximum iterations of the procedure) (see Resende and Ribeiro (2016) for other aspects of GRASP). In addition, we assume the rest of the parameters of the p-median problem with maximum distance constraints and we also assume that the maximum distance is $s_i = s$ for all $i = 1, \ldots, m$. The following two basic phases are explained in an iteration of GRASP:

- **Constructive phase:** We randomize (using a restricted candidate list of cardinal KRCL) the following greedy procedure decomposed into two clearly differentiated phases:

  Phase 1 : The greedy heuristic for the set covering is applied (see Murty (1995) or Chvatal (1979)), where the number of facilities is minimized, taking $dc = s$ as the coverage distance. At the end of this heuristic, we get a k number of facilities. The following cases may occur:

  - If $k > p$, predictably the problem is not feasible. In our numerical experiments, whenever this case has been given, the problem was really not feasible.
  - If $k = p$, a feasible solution is already obtained.
  - If $k < p$, go to phase 2.

  Phase 2 : Given $k < p$ facilities, we have p complete facilities, with the original objective (1). More specifically, in any iteration of this phase 2, in order to complete up to p facilities, let X_k be the set containing the locations assigned to the k facilities. Let $d(i, X_k)$ be the minimum distance between the demand point i and the nearest point belonging to the set X_k. Similarly, let $d(i, j \cup X_k)$ be the minimum distance between the demand point i and the nearest point, which belongs to the set X_k increased by the new location j. The best site to locate a single facility, since the first k facilities are located in sites of the set X_k, is the location j that minimizes $Z_j = \sum_i h_i d(i, j \cup X_k)$. This procedure is repeated until p facilities are completed, in which case STOP.

- **Local search:** In the case of the classical p-median problem, the well-known local search algorithm was introduced by Teitz and Bart (1968). It is also known as *swap heuristic*, because in each iteration it exchanges an element that is in the solution (opened facility site point) for one that is not in the solution (closed facility site point). The heuristic of Teitz and Bart (1968) was later improved by Whitaker (1983) and known in the literature as *Whitaker's fast heuristics*. Finally, Resende and Werneck (2003) proposed an implementation procedure that significantly improves Whitaker's fast heuristics (1983). Here, Whitaker's fast heuristics have been implemented with the Resende and Werneck implementation (see Resende and Werneck (2003)) to adapt it to our problem.

5 Computational results

In this section, we present the computational results of the numerical experiments performed. In order to compare the different resolution methods, we will consider two sets of data and different test problems:

- The first set of data consists of test problems whose parameters come from real data about a certain health service: aint11, aint12 and aint13 with $m = 134$, $n = 121; m = 266, n = 21$, and $m = 430, n = 22$ respectively, where m is the number of demand points and n is the number of service points. In addition, we will take $s_i = s$ for all $i = 1, \ldots, m$.
- The second set of data has been generated randomly, to obtain test problems of greater size than the previous ones. More specifically, they are data with $n = m$ of sizes 500, 800 and 1000. We will denote these test problems as s500_1, s800_1 and s1000_1. In addition, the demands have been randomly generated following a uniform distribution between 10 and 100 and the distances have been obtained by rounding the Euclidean distances calculated from the uniformly generated coordinates between 0 and 100. In addition, we take $s_i = s$ for all $i = 1, \ldots, m$.

5.1 Maximum Distance Feasibility Intervals

To analyze the feasibility of problems and its relation to the parameters p and s, we are interested in calculating, the minimum value of s that makes the problem, which we will call *maximum distance feasibility threshold* and by $\underline{s}$, feasible . In addition, we will call the maximum value that makes the solution (**PMPDC2**) match the solution of (**PMP**) the *maximum distance invariance threshold* and we will represent it by $\overline{s}$. We will call the interval whose extremes are the maximum distance feasibility threshold and the maximum distance invariance threshold the *maximum distance feasibility interval* and we will represent it by $[\underline{s}, \overline{s}]$.

In addition, given a fixed value $s \in [\underline{s}, \overline{s}]$, we will call the subinterval (which can degenerate to the point s) where the solution is constant and equal to the solution associated with *s the invariance subinterval (associated to s)* and represent it by $\mathsf{inv}(s)$. Note that the lower extreme of any invariance subinterval coincides with the maximum distance of any demand point to its assigned facility at the optimum (such a value will be represented by d^*_{max}, which depends on s). Note that the maximum minimum distance (depending on the distance data and not on the value of p) may not coincide with the maximum distance feasibility threshold (which depends on p).

With fixed p, we want to know how it changes the objective value (of the optimal solution which we will represent by z (is the total distance)) by varying the value of the maximum distance in the range of feasibility of the maximum distance. Figures 1, 2 show some of these concepts, corresponding to dataset `aint13` with $p = 7$ and $p = 10$. Another example can be seen in Figure 3, corresponding to data set `aint12` with $p = 7$. It can be seen that there are more subintervals of invariance here with respect to the examples in Figures 1 and 2.

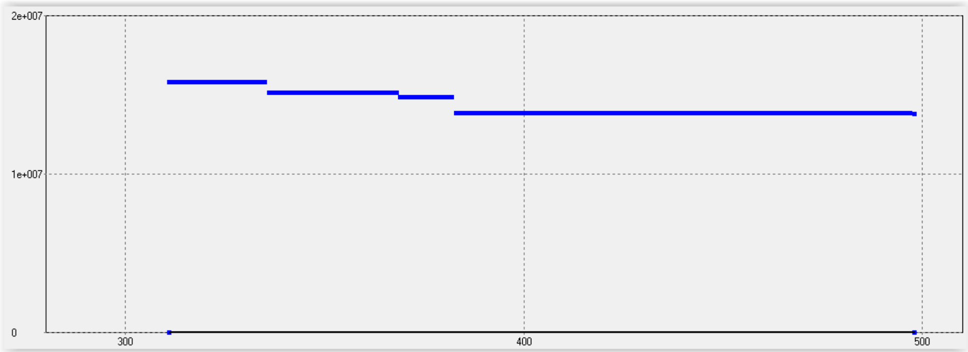

Figure 1: z (objective value, total distance) versus maximum distance, corresponding to the data set `aint13` with $p = 7$. The maximum distance feasibility interval is $[311, 498]$. For example, we note that by opening $p = 7$ facilities, reducing the maximum distance by $187 = 498 - 311$ units, the total distance increases by 14.48%.

- Note that as p increases, both the maximum distance feasibility threshold and the maximum distance invariance threshold are kept constant or decrease their value.
- The number of invariance subintervals decreases as p increases.
- If $p = n$ (that is, if all facility sites are opened) we have $s = \mathsf{inv}(s) = d^*_{max}$. At this value we will call it the *threshold of feasibility-invariance*.

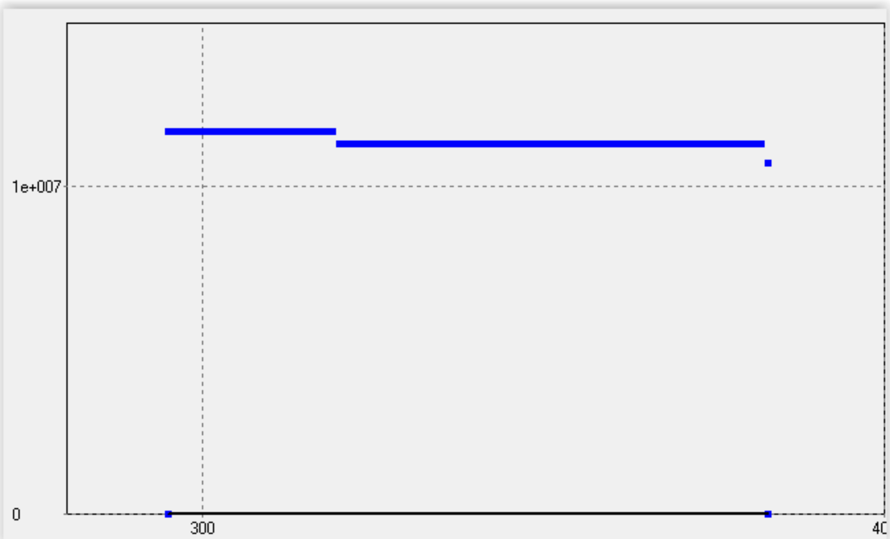

Figure 2: z (objective value, total distance) versus maximum distance, corresponding to the data set `aint13` with $p = 10$. The maximum distance feasibility interval is $[295, 383]$

- From a certain value of parameter p to some value $p^* \leqslant n$, the feasibility intervals coincide. In addition, as p increases, there is a point $p^* \leqslant n$ such that the maximum distance feasibility intervals degenerate at the threshold of feasibility-invariance.
- If the values of $\underline{s}$ coincide as p increases, then the threshold of feasibility-invariance is $\underline{s}$.

To see the variation of the objective function when p varies, Figure 4 shows for the data set aint11, the objective value plotted versus the variation of p fixed $s = 504$.

For each value of p and s that made the problem feasible (corresponding to dataset aint13), the average distance (optimal solution divided by total demand) has been drawn in Figure 5. Figure 6 shows the average p-distance plane associated with Figure 5.

The study of maximum distance feasibility intervals helps the decision maker to choose an appropriate p value, subject to a maximum distance (or service time) constraint (or vice versa) since, in practice, there are situations in which it has prefixed values of p, others in which maximum service or distance times, or even both, are prefixed. A balance between the values of p and the values of the maximum distance is neccesary, since, low service times are necessary, we can expect that a higher p number of facilities will be required (and conversely, if small values of p are needed, it is to be expected that high values of the maximum distance are required for the solutions to be feasible). In addition, we can calculate a value of p, as the cost increases if we decrease the maximum distance to the feasibility threshold.

5.2 Computational results analysis

The Table 1 shows the computational results of the different numerical experiments performed on an Intel (R) Core i7-4790 CPU @ 3.60 GHz PC with 16.0 GB of RAM and Windows 7 Professional operating system. The Lagrangian relaxation procedure has been implemented in Xpress-Mosel (version 8.0 of 64-bit program), as well as the optimal resolution of the model that has been

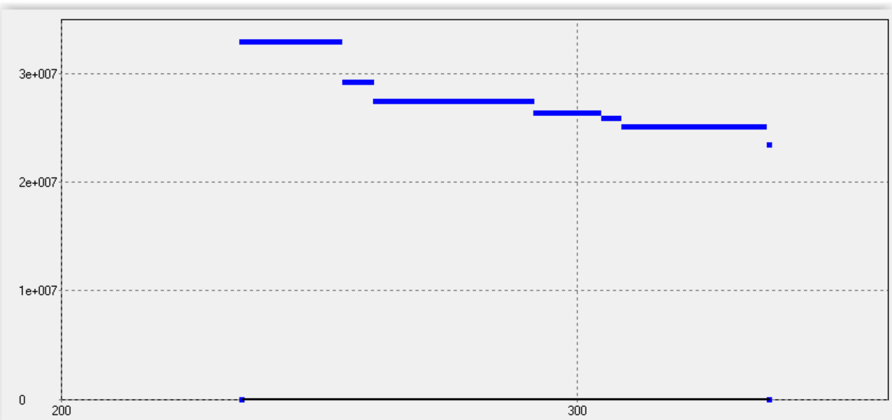

Figure 3: z (objective value, total distance) versus maximum distance, corresponding to the data set aint12 with $p = 7$. The maximum distance feasibility interval is $[235, 337]$

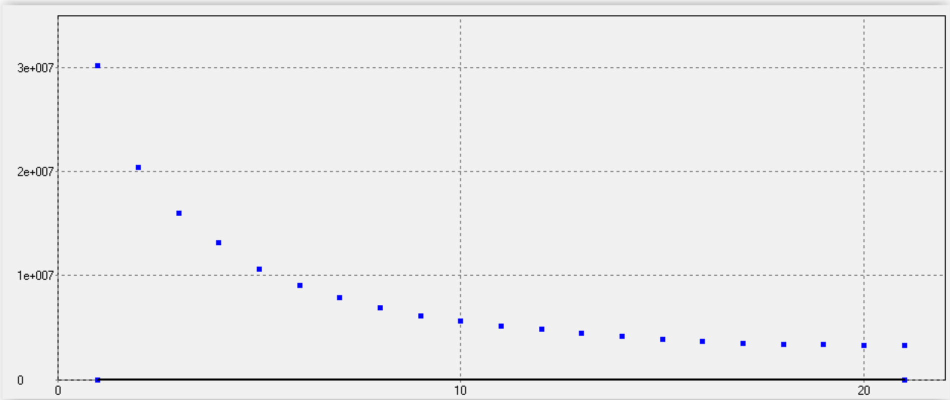

Figure 4: z (objective value, total distance) versus p fixed $s = 504$, corresponding to the data set aint11.

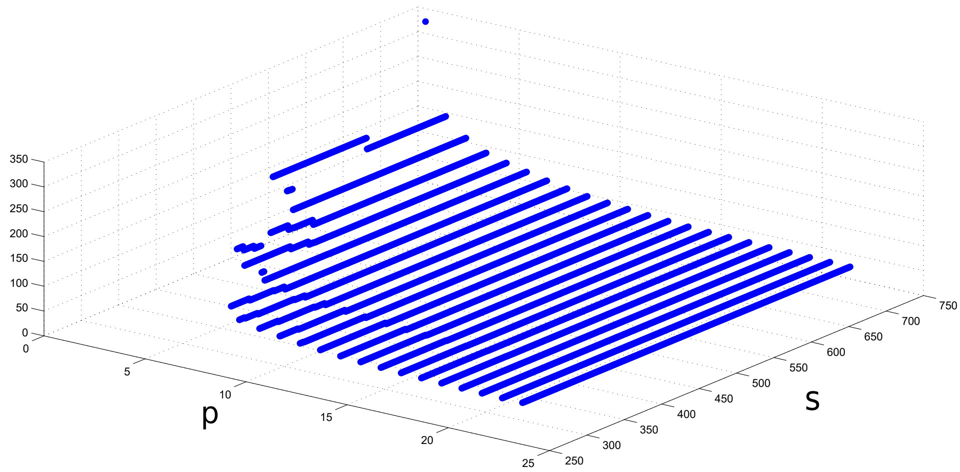

Figure 5:

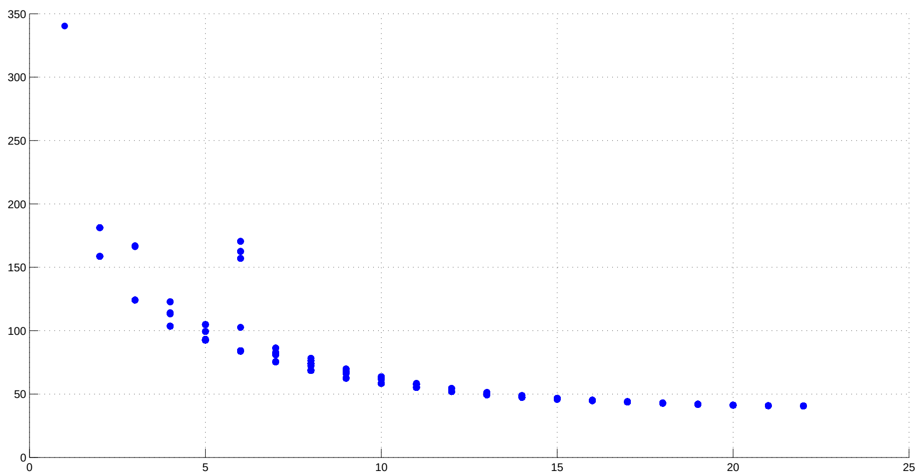

Figure 6: In this plane, we can see forall p, the number of the invariance subintervals

produced by XPRESS-Optimizer (which presents one of the most efficient implementations of the well-known Branch algorithm and includes, among other things, numerous cutting plane methods, heuristics and preprocessing methods). The GRASP has been implemented in C in order to exploit the power of this metaheuristic. In addition, for the same reason, we have established 50 iterations and 8 as the cardinal of restricted candidate list.

The parameters (p, s) appear in the first column, leading to the column of the set of data associated with the numerical experiments. The columns labeled z, z^{LR} and z^G show the average distance obtained by XPRESS, Lagrangian relaxation and GRASP respectively. The columns labeled t, t^{LR} and t^G show the computation time in seconds by XPRESS, Lagrangian relaxation and GRASP respectively. With XPRESS, we have calculated the maximum distance at the optimum, d^*_{max}, and for the Lagrangian and GRASP relaxation we have calculated the average distance deviations obtained by Lagrangian relaxation and GRASP respectively, as we have done with d^{LR} and d^G.

In addition, by way of example, Figure 7 shows a graph with the coordinates of the open candidate points and the demand points assigned to them, corresponding to the solution of the test problem s1000_1, with $p = 15$ and $s = 21$.

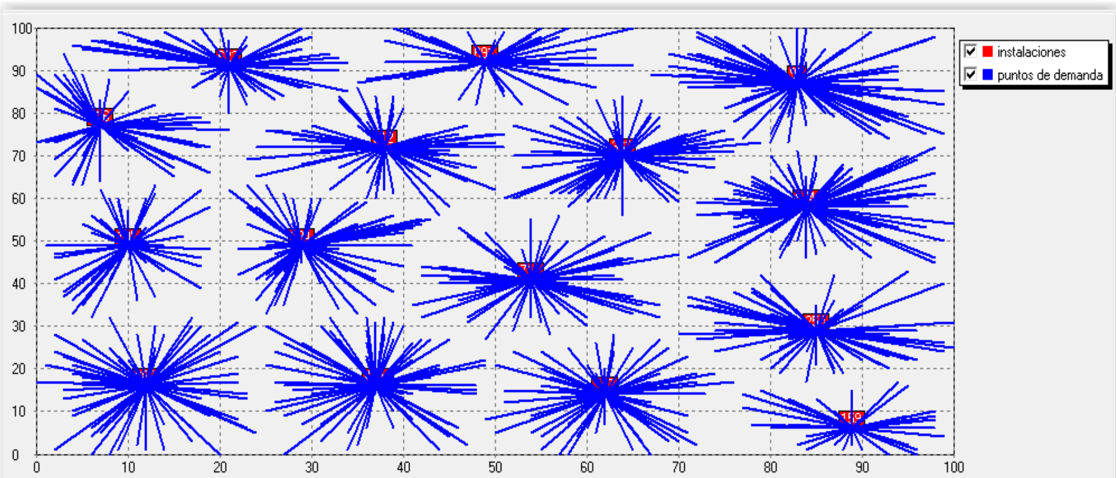

Figure 7:

We can observe that when the size of the problem grows, the computation time of XPRESS increases. Moreover, we observe that Lagrangian relaxation with the test problems aint11, aint12 and aint13 provides the optimal solution (in addition to duality gap equal to 0), although for test problems s500_1, s800_1 and s1000_1, it provides solutions that differ in percentage from the optimal solution of at most about 5%.

Clearly, GRASP is by far the best method of the three, as here it always provides the optimal solution in a very short time.

Table 1: Computational results

	XPRESS			Lagrangian relaxation			GRASP		
	z	t	d^*_{max}	z^{LR}	t^{LR}	d^{LR}	z^G	t^G	d^G
aint11									
(p,s)									
$(4,300)$	57.62	0.08	292	57.62	0.125	0	57.62	0.0066	0
$(4,350)$	50.32	0.09	305	50.32	0.125	0	50.32	0.0066	0
$(5,300)$	46.68	0.08	292	46.68	0.078	0	46.68	0.0067	0
$(5,350)$	40.84	0.08	305	40.84	0.125	0	40.84	0.0070	0
$(6,300)$	37.20	0.08	235	37.20	0.125	0	46.68	0.0073	0
$(6,350)$	34.54	0.09	305	34.54	0.124	0	34.54	0.0072	0
$(7,220)$	40.32	0.08	209	40.32	0.125	0	40.32	0.0077	0
$(7,230)$	36.32	0.06	222	36.32	0.125	0	36.32	0.0059	0
aint12									
(p,s)									
$(10,250)$	41.12	0.27	235	41.12	0.359	0	41.12	0.01443	0
$(10,300)$	37.91	0.30	255	37.91	0.374	0	37.91	0.01384	0
$(11,250)$	38.30	0.27	235	38.30	0.92	0	38.3	0.01477	0
$(11,300)$	35.35	0.30	255	35.35	0.375	0	41.12	0.01452	0
$(12,250)$	35.75	0.27	235	35.75	2.512	0	35.75	0.01598	0
$(12,300)$	33.06	0.38	255	33.06	2.652	0	33.06	0.01599	0
aint13									
(p,s)									
$(7,380)$	80.94	0.33	369	80.94	0.265	0	80.94	0.01244	0
$(7,400)$	75.49	0.37	383	75.49	0.578	0	75.49	0.01253	0
$(10,300)$	63.68	0.31	295	63.68	0.53	0	63.68	0.01656	0
$(10,350)$	61.77	0.30	320	61.77	0.546	0	61.77	0.01408	0
$(12,300)$	54.45	0.28	295	54.45	0.39	0	54.45	0.01467	0
$(12,400)$	51.95	0.28	383	51.95	0.39	0	51.95	0.0154	0
s500_1									
(p,s)									
$(15,21)$	9.20	15.44	20	9.20	28.61	0	9.20	5.975	0
$(15,50)$	9.20	24.06	22	9.20	21.70	0	9.20	4.99	0
$(20,21)$	7.73	11.25	17	8.05	26.29	4.13	7.73	4.314	0
$(20,50)$	7.73	20.93	17	7.99	32.84	3.25	7.73	5.124	0
s800_1									
(p,s)									
$(15,21)$	9.44	140.17	21	9.46	105.27	0.21	9.44	11.71	0
$(15,50)$	9.44	293.76	21	9.47	122.98	0.32	9.44	15.83	0
$(20,21)$	8.11	147.19	18	8.17	105.12	0.73	8.11	16.08	0
$(20,50)$	8.11	185.43	18	8.18	113.90	0.86	8.11	17.13	0
s1000_1									
(p,s)									
$(15,21)$	9.55	389.85	20	10.11	213.97	5.53	9.55	23.34	0
$(15,50)$	9.55	544.34	22	9.67	234.98	1.24	9.55	27.08	0
$(20,21)$	8.12	841.61	19	8.53	194.53	4.81	8.12	31.65	0
$(20,50)$	8.12	1540.75	19	8.56	219.50	5.14	8.12	35.74	0

6 Application to a bi-objective problem. The p-median-center tradeoff problem.

The p-median-center tradeoff problem is an important problem in Location Theory. As far as we know, there have been no other computational studies regarding the exact calculation of the Pareto front for this kind of bi-objective localization problem. We provide a very fast procedure for the exact calculation of the Pareto front for the p-median-center tradeoff problem. We recall some basic concepts about Bi-objective programming:

6.1 Basic concepts about Bi-objective programming

This section aims to present a series of concepts and basic results of Multi-Objective Programming necessary for this part of the article. For this, we will follow Sáez-Aguado and Trandafir (2017). For a general reference dedicated to Multi-objective Programming, see Ehrgott (2005).

The Biobjective Integer Linear Programming (**BOIP**) is as follows

$$(\mathbf{BOIP}) \quad \begin{array}{ll} \text{Minimize} & f(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x})) \\ s.t. & \mathbf{x} \in X \end{array}$$

where every feasible solution $\mathbf{x} = (x_1, \ldots, x_n) \in X$ is such that $x_j \geqslant 0$ is an integer, for all $j = 1, \ldots, n$. Moreover, we suppose that the objective functions are linear and of the form

$$f_1(\mathbf{x}) = \sum_{j=1}^{n} c_{1j} x_j$$
$$f_2(\mathbf{x}) = \sum_{j=1}^{n} c_{2j} x_j$$

where $c_{ij} \in \mathbb{Z}$, for all $i = 1, 2$ and $j = 1, 2$.

Let $\mathbf{x}^* \in X$ be a feasible solution of (**BOIP**). Therefore, we say that $\mathbf{x}^*$ is *efficient* or *Pareto optimal* if there is no $\mathbf{x} \in X$ such that $f_i(\mathbf{x}) \leqslant f_i(\mathbf{x}^*)$, $i = 1, 2$, where strict inequality holds for some $i \in \{1, 2\}$. If there is no $\mathbf{x} \in X$ such that $f_i(\mathbf{x}) < f_i(\mathbf{x}^*)$, $i = 1, 2$, then $\mathbf{x}^*$ is said to be *weakly efficient*.

Given $\mathbf{y} = f(\mathbf{x}) \in Y = f(X) \subset \mathbb{R}^2$, we say that $\mathbf{y}$ is *non-dominated* if $\mathbf{x}$ is efficient. Moreover, let $X_E \subset X$ be the set of efficient solutions to the biobjective problem and $Y_E = f(X_E) \subset Y$ the set of all non-dominated points in the objective space, also known as the *Pareto front*. Suppose that there are N non-dominated points and $Y_E = \left\{\mathbf{y}^1, \ldots, \mathbf{y}^N\right\}$ is indexed so that

$$y_1^1 < y_1^2 < \cdots < y_1^N$$

and therefore

$$y_2^1 > y_2^2 > \cdots > y_2^N$$

In this case, the two points $\mathbf{y}^1$ and $\mathbf{y}^N$ are known as the *extremal non-dominated points*. From these points, the ideal and nadir points, ip and np, whose components provides the lower and upper bounds, respectively, for the objective functions. More explicitly, $ip = (y_1^1, y_2^N)$, $np = (y_1^N, y_2^1)$. These four points determine a rectangle which includes the rest of the non-dominated points The objective is find the set Y_E, and for all $\mathbf{y} \in Y_E$ have $\mathbf{x} \in X_E$ such that $\mathbf{y} = f(\mathbf{x})$. In the existing literature, it is common to find different procedures for finding non-dominated points. The choice of procedure depends on the characteristics of the particular multiobjective problem being dealt with.

6.2 Bi-objective p-median-center problem. The ε-constraint method

We consider the bi-objective problem based on the (classical) p-median problem and the p-center problem (**BO-MCP**):

$$\begin{aligned}(\mathbf{BO\text{-}MCP}) \quad \text{Minimize} \ & \sum_{i=1}^{m}\sum_{j=1}^{n} h_i d_{ij} y_{ij} \\ \text{Minimize} \ & w\end{aligned} \tag{21}$$

$$\text{s.t.} \quad \sum_{j=1}^{n} y_{ij} = 1, \ i = 1, \ldots, m \tag{22}$$

$$\sum_{j=1}^{n} d_{ij} y_{ij} \leqslant w, \ i = 1, \ldots, m \tag{23}$$

$$y_{ij} \leqslant x_j, \ i = 1, \ldots, m, \ j = 1, \ldots, n \tag{24}$$

$$\sum_{j=1}^{n} x_j = p, \tag{25}$$

$$x_j \in \{0, 1\}, \ j = 1, \ldots, n \tag{26}$$

$$y_{ij} \in \{0, 1\}, \ i = 1, \ldots, m, \ j = 1, \ldots, n \tag{27}$$

This bi-objective problem can be solved by the ε-constraint method (see Erhgott (2005) for general theoretical questions of this method). The basic idea is to minimize one objective function and add constraints to the other to get the efficient points. Consequently, the following subproblem (**BO-MCP**(ε)) is obtained:

$$\begin{aligned}(\mathbf{BO\text{-}MCP}(\varepsilon)) \quad \text{Minimize} \ & \sum_{i=1}^{m}\sum_{j=1}^{n} h_i d_{ij} y_{ij} \\ s.t. & \sum_{j=1}^{n} d_{ij} y_{ij} \leqslant \varepsilon, \ i = 1, \ldots, m \\ & \mathbf{x} \in X\end{aligned} \tag{28}$$

where ε is an upper bound of maximum distance and X is the set of constraints based on (22),(24), (25),(26) and (27).

Fortunately, with the ε-constraint method, we have fallen into a p-median problem with maximum distance constraints, which we have discussed before and solved using several methods, with GRASP finally being the appropriate method. Later, we will observe the importance of the fact that subproblems are a p-median problem with maximum distance constraints, since there will be a relation between the efficient points and the subintervals of invariance corresponding to the subproblems. In the literature (see Ehrgott (2005)), there are other methods to solve bi-objective problems, such as those based on Chebychev norms, or those based on weights or lexicographic methods , though they are not suitable for our problem because they are difficult to solve or do not allow all the efficient points to be obtained. More specifically, a very common method found in the literature to solve the bi-objective problem that we are dealing with here is the weighted sum method, which is based on

$$\textbf{(WM-BOIP)} \quad \begin{array}{ll} \text{Minimize} & f(\mathbf{x}) = (\lambda_1 f_1(\mathbf{x}) + \lambda_2 f_2(\mathbf{x})) \\ s.t. & \mathbf{x} \in X \end{array}$$

where $\lambda_1, \lambda_2 \geqslant 0$. If $\lambda > 0,\ i = 1, 2$ then all optimal solution of (**WM-BOIP**) is efficient.

Efficient solutions that can be found by the method of weights with positive coefficients are known as *supported solutions* and the corresponding points are called *supported non-dominated points*. Graphically, the supported points are between the extreme points of the lower envelope of the set Y_E, as for example the points $\mathbf{y}^1, \mathbf{y}^3$ and $\mathbf{y}^5$ in Figure 8. The major drawback of the weights method is that, for example for non-convex problems such as (**BOIP**), nonsupported points, such as $\mathbf{y}^2$ and $\mathbf{y}^4$ in Figure 8, cannot be obtained by this method.

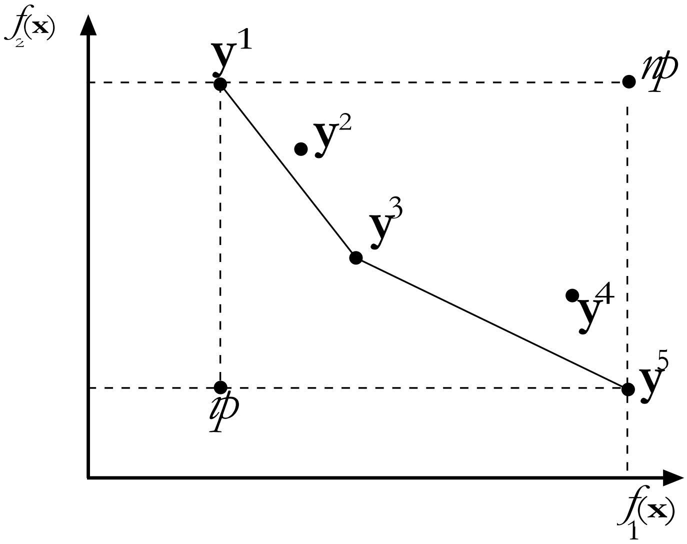

Figure 8: (From Sáez-Aguado and Trandafir (2017)).

In this problem, we can solve the difficulties that other authors find with the supported solutions that appear when using weighted methods. Daskin (2010, 2013) points out that, with the weights method not all the efficient points can be obtained, and that is why it uses the ε-constraint method and solves the problem with the software that accompanies the texts of Daskin, known as SITATION. In addition, SITATION is characterized by solving a limited type of problems and examples of existing ones in Daskin (2010,2013). These also have the inconvenience that it is only able to solve up to a limited number of nodes. Next, we should point out how Daskin (2010,2013) tackles the problem we are dealing with:

- In Daskin (2010), the following approach appears based on introducing

$$\beta_{ij} = \begin{cases} 1, & \text{if } d_{ij} > s \\ 0, & \text{if } d_{ij} \leqslant s \end{cases}$$

where we assume that the maximum distance limit is equal to the value s for all $i = 1, \ldots, m$ and we solve the problem

$$\text{Minimize} \sum_{i=1}^{m} \sum_{j=1}^{n} h_i \left(d_{ij} + W\beta_{ij} \right) y_{ij} \tag{29}$$

subject to the constraints of the (classical) p-median problem, where W is a penalty parameter. Morever, the quality of the solution obtained by this approach depends of the choice of penalty parameter W. This approach implies a long computation time.

- In Daskin (2013), the problem we solve here is solved with the ε-constraint method, also pointing to the indirect approach (based on modifying the distances matrix), resulting in a long computation time because is neccesary to solve a high number of problems of p-median with maximum distance contraints.

Daskin (2010,2013), having no efficient method for solving the p -median problem with maximum distance constraints, resorts to the use of SITATION software, which is therefore not a new approach of heuristic methods of resolution. In Daskin and Maass (2015), appears "*For small instances it is often possible to solve bi-objective problems using extensions of the Lagrangian algorithm outlined above. For larger instances, using a genetic algorithm is often advisable since the population of solutions in a genetic algorithm automatically gives an initial approximation of the non-dominated set of solutions*". Therefore, Daskin and Maass (2015) solve the bi-objective problem using the SITATION software with a Lagrangian relaxation + Branch & Bound algorithm for small size problems, and for larger problems, a genetic algorithm, which provides an *initial approximation* to the set of non-dominated points. Note that Daskin and Maass (2015, p. 41) use the indirect approach, based on modifying the distances matrix, without taking into account the dangers involved in heuristic methods. In fact, for the problem with 250 nodes set out in Daskin and Maass (2015, p.42), they state "*Obtaining the 22 solutions shown in the Figure took nearly 16 h of computing time*". Later, we will see that our heuristic procedures greatly improve those existing in the literature, calculating all non-dominated points in a few seconds, without obtaining approximations, unlike Daskin and Maass (2015). Note that in Daskin (2010,2013) and Daskin and Maass (2015), no effective heuristic method of resolution is provided (in addition to using SITATION software), nor does it indicate the problems involved in the indirect approach in heuristic methods. Here are some properties related to the ε-constraint method:

- All efficient solutions can be obtained by solving (**BO-MCP**(ε)) for some value of ε.
- All solutions $\mathbf{x}^*$ of (**BO-MCP**(ε)) are weakly efficient, and morever, there is some optimal solution of (**BO-MCP**(ε)) which is efficient. In particular, if $\mathbf{x}^*$ is the unique optimal solution of (**BO-MCP**(ε)), then $\mathbf{x}^*$ is efficient.
- The greatest advantage of the ε-constraint method is that all efficient solutions are obtained, although there is a drawback, since dominated points can be obtained. In Figure 9, the points A, B and C simultaneously minimize the first objective, but only C is not dominated. That

is why, when implementing the same, there must be a procedure or routine to eliminate such dominated points. For a detailed review of the different implementations of the ε-constraint method, see Sáez-Aguado and Trandafir (2017).

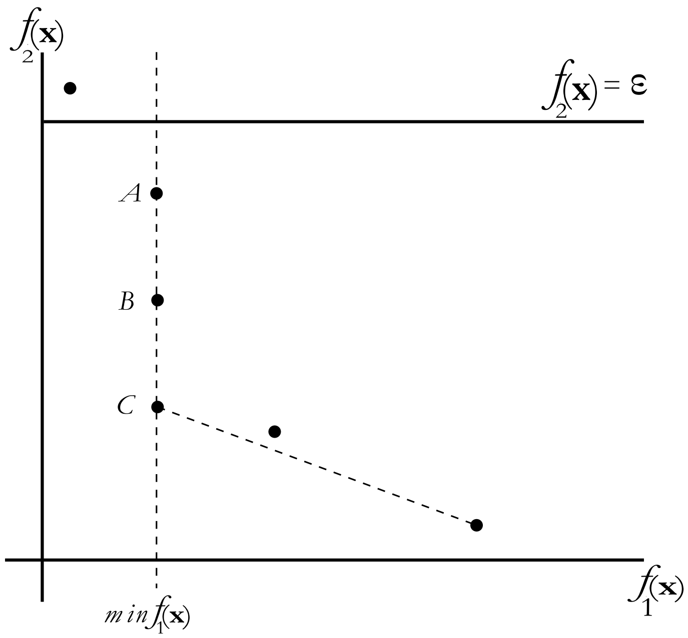

Figure 9: (From Sáez-Aguado and Trandafir (2017)).

Next, we describe the steps of the ε-constraint algorithm with the elimination of dominated points applied to our problem:

ε-constraint ALGORITHM WITH THE ELIMINATION OF DOMINATED POINTS

- **Step 1** (Initialization): Set $Y_E = \emptyset, X^* = \emptyset, \varepsilon = \infty$ and $k = 0$.
- **Step 2** (Resolution of subproblems) : Solve (**BO-MCP**(ε)). If the problem is not feasible, let $N = k$, and go to step 4. Otherwise, set $\overline{\mathbf{x}}$ as the final solution obtained

  and go to step 3.
- **Step 3**: Set $k := k + 1$, $\mathbf{x}^k = \overline{\mathbf{x}}$, $\mathbf{y}^k = (f_1(\overline{\mathbf{x}}), f_2(\overline{\mathbf{x}}))$, $Y_E = Y_E \cup \{\mathbf{y}^k\}$, $X^* = X^* \cup \{\mathbf{x}^k\}$, $\varepsilon = f_2(\overline{\mathbf{x}}) - 1$ and go to step 2.
- **Step 4**: Remove possible dominated points of Y_E, and update N, Y_E and X^* and STOP.

6.3 Computational results. Bi-objective analysis

We will perform a bi-objective analysis corresponding to the aint12 test problems with $p = 7$ and aint13, with $p = 10$ to illustrate much of the above.

- Problem aint12, with parameters $m = 266, n = 39, p = 7$, and maximum minimum distance equal to 185 and computation time (in seconds) equal to 0.202 sec.

ε	**OBJ. 1 (Average distance)**	**OBJ. 2 (Maximum distance)**
9999999	45.72	337
336	249.088811	309
308	250.551789	305
304	251.428772	292
291	253.618036	261
260	257.046515	255
254	264.324902	235

Next, the maximum distance versus average distance for the p-median problem with maximum distance constraints associated with the same data is shown in a graph. The subintervals of invariance are: [235,254], [255,260], [261,291], [292, 304], [305,308], [309,336] and 337. The relation

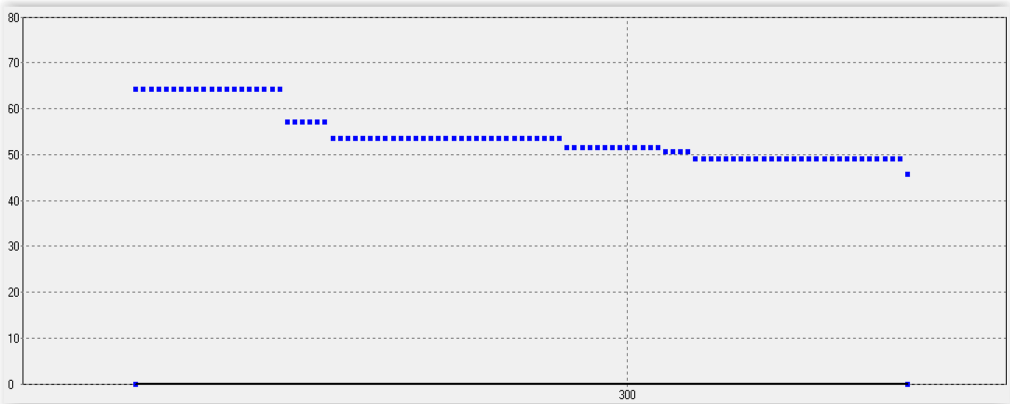

Figure 10:

between the subintervals of invariance and the bi-objective problem is the following:

- The number of points of the Pareto front coincides with the number of subintervals of invariance.
- The lower extremes of the invariance subintervals coincide (we adopt the convention that if a subinterval of invariance degenerates at one point, then the lower extreme of it is the same point) with the found values of objective 2.
- The values of ε corresponding to an efficient point of Pareto front match except for the first value of that with the upper extremes of the invariance subintervals (recall that ε is taken as a sufficiently *large* value of objective 1, we have taken 9999999).
- The first point obtained corresponds to the solution of the (classical) p-median problem, and the last point corresponds to the solution of the p-center problem. The rest of the intermediate points (if any) correspond to intermediate solutions of possible interest to the decision maker.
- Note that a priori (except when ε is large enough), there is no relation between the average distance associated with a subinterval of invariance and the average distance found in objective 1. This is the difference that the bi-objective analysis contributes to the analysis of the feasibility intervals. With the bi-objective analysis, more information is obtained for the decision maker, and this is

mainly due to the influence of objective 2. At the end, the decision maker has to select an efficient solution. We will call this the *best compromise solution*.

Next, we will show in a graph the efficient points found for this test problem:

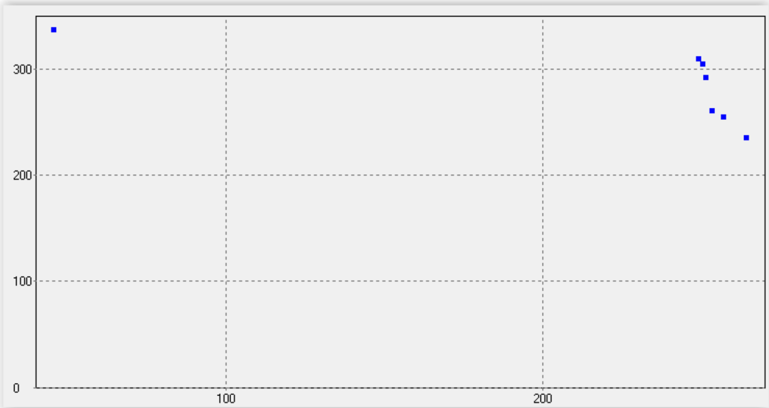

Figure 11:

It can be seen that the second point obtained greatly increases the first objective (average distance) by keeping the second objective (maximum distance) almost invariant. This seems to indicate that the second point obtained is not an advisable solution option, since having the first point, with a little greater maximum distance, one has quite a smaller average distance and this indicates that it is a good candidate for the best compromise solution.

- Problem aint13, with parameters $m = 430, n = 22, p = 10$, and maximum minimum distance equal to 295 and computation time (in seconds) equal to 0.083 sec.

ε	**OBJ. 1 (Average distance)**	**OBJ. 2 (Maximum distance)**
9999999	58.44	383
382	261.77	320
319	263.68	295

Next, the maximum distance versus average distance for the p-median problem, with maximum distance constraints associated with the same data, is shown in a graph. The subintervals of invariance are: [295,319], [320,382] and 383.

Next, we will show in a graph the efficient points found for this test problem:

It can be seen that the second point obtained greatly increases the first objective (average distance) by keeping the second objective (maximum distance) almost invariant. This seems to indicate that the second point obtained is not an advisable solution option, since having the first point with a little more distance maximum, one has a smaller average distance and this indicates that it is a good candidate for the best compromise

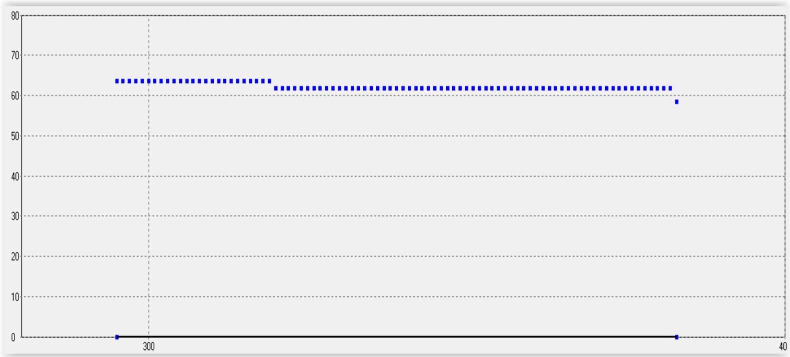

Figure 12:

Figure 13:

solution. It can be seen that there are fewer efficient points than in the previous test problem, and that basically the decision maker has to select a solution to his/her problem either by taking into account the efficiency or the equity, since the intermediate solution found is quite close to the solution of the p-center problem.

7 Summary and conclusions

In this paper, an extension of the classical p-median problem has been considered, the p-median problem with maximum distance constraints. The literature on this question is scarce, and to date, it has been found that for this problem (unless further knowledge arises) the heuristic procedure of Teitz and Bart provided the best method of resolution. For this reason, since GRASP metaheuristics is perhaps one of the best procedures to solve the problem of the classical p-median problem, we have tried to apply it to this problem. In addition, a detailed study of the feasibility of the problem and its relation to the parameters of the problem has been carried out, and two new methods of resolution have been proposed: a Lagrangian relaxation procedure, which differs from that existing in the literature for this problem, and a procedure based on the methodology of GRASP metaheuristics. Finally, we have verified that the latter has been very successful in the numerical experiments performed, Lagrangian relaxation and XPRESS in computing times. Moreover, we study and solve efficiently the bi-objective problem corresponding to the p-median and the p-center problems. Applying the ε-constraint method, we fall into subproblems of p-median with maximum distance constraints, which have been solved very fast with the GRASP metaheuristic to obtain all the efficient points of the Pareto front. In addition to analyzing them and their relationship with the invariance subintervals, we have obtained intermediate solutions that are of great interest to the decision maker, besides the extremes, which would be obtained by solving the two objectives separately.

Future research that we are considering is to include theoretical results on the feasibility of the p-median with maximum distance constraints, where we have a classification of points of demand according to priority of service and different values of maximum distance limit associated with them. This extension is of interest and has applications for the location of public (healthcare location services) and private (location of logistic centers) services.

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