A mixed-integer linear programming approach for the T-row and the multi-bay facility layout problem

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We introduce a new facility layout problem, the so-called T-Row Facility Layout Problem (TRFLP). The TRFLP consists of a set of one-dimensional departments with pairwise transport weights between them and two orthogonal rows which form a T such that departments in different rows cannot overlap. The aim is to find a non-overlapping assignment of the departments to the rows such that the sum of the weighted center-to-center distances measured rectilinear directions is minimized. The TRFLP is a generalization of the well-known Multi-Bay Facility Layout Problem with three rows (3-BFLP). Both problems, the TRFLP and the 3-BFLP, have wide applications, e.g., factory planning, semiconductor fabrication and arranging rooms in hospitals.

In this work we present a mixed-integer linear programming approach for the TRFLP and the 3-BFLP based on an extension of the well-known betweenness variables which now can be equal to one if the corresponding departments lie in different rows. One advantage of our formulation is the calculation of inter-row distances without big-M-type constraints. We provide cutting planes exploiting the crossroad structure in the layout, and hence T-row (3-Bay) instances with up to 18 (17) departments are solved to optimality in less than 7 hours. The best known approach for the 3-BFLP is clearly outperformed. Additionally, tight lower bounds for larger instances are calculated to evaluate our heuristically determined layouts.

Key words. Facilities planning and design; Mixed-Integer Linear Programming; Row Layout Problem

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1 Introduction

We introduce a new facility layout problem, the so-called T-Row Facility Layout Problem (`TRFLP`). We are given a set of departments $[n] := \{1, \dots, n\}$, $n \in \mathbb{N}$, with lengths $\ell_i, i \in [n]$, and symmetric pairwise weights $w_{ij} = w_{ji}, i, j \in [n], i < j$, and two orthogonal rows $\mathcal{R} := \{1, 2\}$ which form a T such that departments in different rows cannot overlap. One looks for an assignment $r\colon [n] \to \mathcal{R}$ of the departments to the rows $\mathcal{R}$ minimizing the weighted sum of the center-to-center distances between the departments measured in rectilinear directions such that departments in the same row do not overlap. By measuring the distances between departments in distinct rows, one has to take the width of the path $w^T_{path} \in \mathbb{R}_{\geq 0}$ into account.

The horizontal row is denoted by row 1 and the vertical row by row 2. We use distance variables $d_{ij} = d_{ji}, i, j \in [n], i < j$, to measure the center-to-center distances measured in rectilinear directions between i and j. Let $p_M \in \mathbb{R}$ denote the position in row 1 measured from a fixed left border which is directly opposite row 2. If $w^T_{path} = 0$, then p_M is the point of intersection of row 1 and row 2. We look for a vector $r \in \mathcal{R}^n$ of the assignment of the departments to the rows and for a vector $p \in \mathbb{R}^n$ of the center positions of the departments measured from a fixed left (upper) border if $r_i = 1$ ($r_i = 2$), $i \in [n]$, such that the following optimization problem is solved to optimality

$$
\begin{aligned}
\min_{r \in \mathcal{R}^n, p \in \mathbb{R}^n} & \sum_{\substack{i,j \in [n] \\ i<j}} w_{ij} d_{ij} \\
\text{s.t. } & |p_i - p_j| \geq \frac{\ell_i + \ell_j}{2}, && i, j \in [n], i < j, r_i = r_j, \\
& d_{ij} = |p_i - p_j|, && i, j \in [n], i < j, r_i = r_j, \\
& d_{ij} = |p_i - p_M| + p_j + w^T_{path}, && i, j \in [n], r_i = 1, r_j = 2.
\end{aligned} \tag{1}
$$

In this model for the `TRFLP` one might obtain $p_i \leq 0$ for some $i \in [n]$, depending on the position p_M. Setting $\mathcal{R} := [m], m \in \mathbb{N}$, as the set of m non-overlapping parallel rows and adapting the distance calculation in equations (1) with $1 \leq w^B_{path} \in \mathbb{R}$, such that

$$
d_{ij} = p_i + p_j + |r_i - r_j| w^B_{path}, \qquad i, j \in [n], i < j, r_i \neq r_j,
$$

where $p_i, i \in [n]$, denotes the position of the center of i, measured from a fixed left border such that all departments are to the right of this border, then we obtain the Multi-Bay Facility Layout Problem (`MBFLP`) with m rows[1], see, e. g., [43, 63, 67, 79]. The distance calculation can be interpreted in the following way. There are inner-row and inter-row material handling-systems, whereby the inter-row material handling-system, e. g., an overhead bridge crane [23, 67], is fixed at the left border. The rows might be separated by equipment, some free space for maintenance or passageways [25]. The inter-row transport is more costly than the inner-row transport, because two separate material handling systems have to be coordinated, the transfer mechanism is costly (typically a larger capacity crane is used) and at this point the most delay and damages occur, see [25, 67]. Calculating the distances between departments in row 1 (row 3) and row 2, we take w^B_{path} into account. Measuring the distances between departments in row 1 and row 3, we cross the path twice, so we take $2 \cdot w^B_{path}$ into account. The departments are given as one-dimensional objects, so we assume implicitly that the height of the departments equals one. Therefore, we assume $w^B_{path} \geq 1$ such that the height of the departments is included in the width of the path. We refer to Figure 1b for an illustration. If $w^B_{path} > 1$, then the width of the path is taken into account, and otherwise the width of the path is neglected. Therefore, we set $w^T_{path} + 1 = w^B_{path}$ in the following.

[1] In order to ensure a consistent terminology we use the term row. In the literature [67] usually the term bay is used.

In Section 4 we study the relation between the TRFLP and the MBFLP with $m = 3$ rows denoted by (3-BFLP). It turns out that the TRFLP is a generalization of the 3-BFLP and the optimal value of the TRFLP is less than or equal to the optimal value of the 3-BFLP if $w^T_{path} \leq w^B_{path}$.

The MBFLP and the TRFLP are of special interest in practice because these layouts are commonly used layouts in industry such as heavy manufacturing, e. g., steel production and bridge crane manufacturing, and semiconductor fabrication [63, 67, 80]. Many real-world factory layouts implicitly use these layout structures, see, e. g., [23, 63], and factory layout problems can often be decomposed, see [36, 67]. Therefore, real factory layouts often reduce to a combination of bay layouts, T-row layouts and further layouts. In real factory layout problems often a complex path structure arises containing several crossroads. One motivation of this paper is to study a facility layout problem with a complex path structure and the path structure in the TRFLP is more complex than in the 3-BFLP, because of the more complex crossroad structure. Hence, we extend the 3-BFLP in order to include realistic aspects. A second advantage of the TRFLP in comparison to the MBFLP is that the inter-row material handling-system is not fixed at the left border, and hence the weighted transport distances can be significantly smaller. The layout of the departments highly influences the costs of the production, see, e. g., [20, 48, 77].

Additionally, the TRFLP can be applied for arranging shops in shopping malls where two levels are separated by a moving stairway or an elevator. Further, the MBFLP and the TRFLP are relevant for arranging rooms in hospitals where often only one side of a corridor has windows [72]. The task is to assign the rooms of the patients along the window side such that the sum of the traveled distances between the rooms of the patients and nurses is minimized. In the TRFLP the moving stairway or the elevator is not fixed at the border of the level (row), and hence the sum of the traveled distances can be reduced in comparison to the 3-BFLP.

The special case of the MBFLP with $m = 1$ is called Single-Row Facility Layout Problem (SRFLP) and it is known to be $\mathcal{NP}$-hard [3, 44, 72]. Thus, the MBFLP and the TRFLP are $\mathcal{NP}$-hard as well. Besides its application in factory planning the SRFLP arises in the arrangement of rooms in hospitals [21, 40, 47] and the arrangement of books on a shelf [6]. The SRFLP is widely studied [1, 2, 55, 58].

The MBFLP where the distance between the rows equals a constant $c \in \mathbb{R}_+$ is called Pier-Type Material Flow Pattern (PMFP) and has application in designing the layout of cross docking warehouses, e. g., there are some dock departments for receiving incoming materials and the other departments are used for direct shipping of shipping supplies [29].

We illustrate the distance calculation of the SRFLP, the 3-BFLP and the TRFLP in the following example. We set $w^T_{path} + 1 = w^B_{path}$ and thus the height of the departments in the 3-BFLP is taken into account.

Example 1. *We consider an instance with $n = 5$ departments with lengths $\ell_1 = \ell_3 = \ell_4 = 4, \ell_2 = 5, \ell_5 = 2$, and non-zero weights $w_{12} = w_{23} = w_{24} = w_{34} = w_{45} = 1$, $w_{14} = 3$ with $w^T_{path} = 0$ and $w^B_{path} = 1$. In Figure 1 optimal layouts of the SRFLP, the 3-BFLP and the TRFLP are illustrated:*

a) An optimal single-row layout is illustrated in Figure 1a with an objective value of

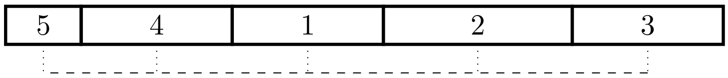
$$1 \cdot 4.5 + 3 \cdot 4 + 1 \cdot 4.5 + 1 \cdot 8.5 + 1 \cdot 13 + 1 \cdot 3 = 45.5;$$

b) An optimal 3-Bay layout is depicted in Figure 1b with an objective value of

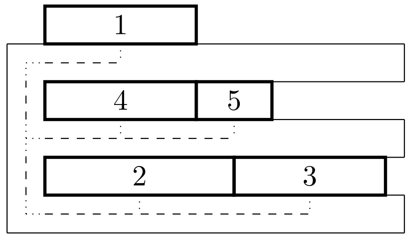
$$1 \cdot 6.5 + 3 \cdot 5 + 1 \cdot 4.5 + 1 \cdot 5.5 + 1 \cdot 10 + 1 \cdot 3 = 44.5;$$

c) An optimal T-row layout is shown in Figure 1c with an objective value of

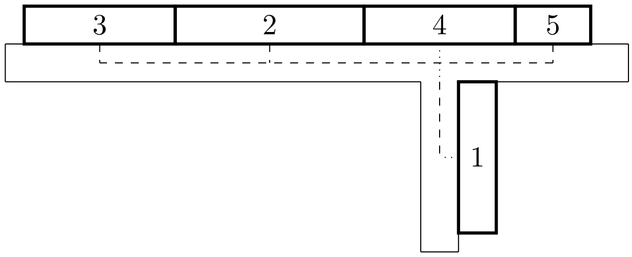
$$1 \cdot 6.5 + 3 \cdot 2 + 1 \cdot 4.5 + 1 \cdot 4.5 + 1 \cdot 9 + 1 \cdot 3 = 33.5.$$



(a) An optimal single-row layout with objective value 45.5.



(b) An optimal 3-Bay layout with objective value 44.5.



(c) An optimal T-row layout with objective value 33.5 where the center of 4 is on position p_M in row 1.

Figure 1: We are given an instance with $n = 5$ departments with lengths $\ell_1 = \ell_3 = \ell_4 = 4, \ell_2 = 5, \ell_5 = 2$, and non-zero weights $w_{12} = w_{23} = w_{24} = w_{34} = w_{45} = 1$, $w_{14} = 3$ with $w^T_{path} = 0, w^B_{path} = 1$. We illustrate optimal layouts for the `SRFLP`, the `3-BFLP` and the `TRFLP` where the center of 4 is on position p_M in row 1 in the illustrated T-row layout.

In the following example we consider three optimal T-row layouts with different departments on position p_M in row 1. Note that, if we consider only the departments in an optimal T-row layout, then such an optimal layout might have the shape of an L.

Example 2. *We are given a T-row instance with $n = 5$ departments with lengths $\ell_1 = \ell_2 = 5, \ell_3 = \ell_5 = 3, \ell_4 = 2$, and non-zero weights $w_{12} = w_{13} = 3, w_{23} = w_{34} = w_{35} = 2, w_{24} = w_{45} = 1, w^T_{path} = 1$, and $p_M = 15$. We obtain an optimal layout where the center of 1 (3) is on position p_M in row 1, see Figure 2a (2c). In the optimal T-row layout illustrated in Figure 2b the center positions of the departments are the following: $p_1 = 16.5, p_2 = 2.5, p_3 = 12.5, p_4 = 10, p_5 = 7.5$, i. e., $p_i \neq p_M, i \in [5]$.*

a) An optimal T-row layout is illustrated in Figure 2a with an objective value of

$$3 \cdot 3.5 + 3 \cdot 4 + 2 \cdot 7.5 + 1 \cdot 10 + 2 \cdot 2.5 + 2 \cdot 5 + 1 \cdot 2.5 = 65;$$

b) An optimal T-row layout is depicted in Figure 2b with an objective value of

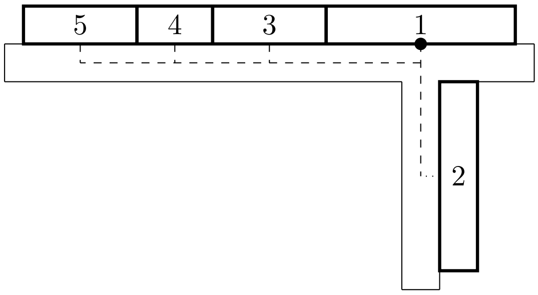
$$3 \cdot 5 + 3 \cdot 4 + 2 \cdot 6 + 1 \cdot 8.5 + 2 \cdot 2.5 + 2 \cdot 5 + 1 \cdot 2.5 = 65;$$

c) An optimal T-row layout is shown in Figure 2c with an objective value of

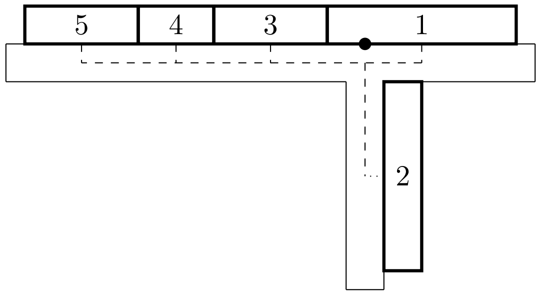
$$3 \cdot 7.5 + 3 \cdot 4 + 2 \cdot 3.5 + 1 \cdot 6 + 2 \cdot 2.5 + 2 \cdot 5 + 1 \cdot 2.5 = 65.$$

1.1 Literature Review

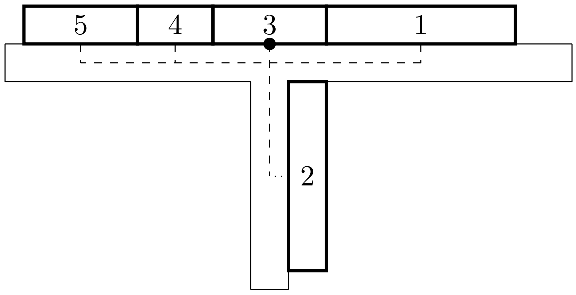
Almost all exact approaches for the `SRFLP` are either based on integer linear programming (`ILP`), see, e. g., [1, 2, 3, 11], or semidefinite programming (`SDP`), see [15, 16, 18, 55, 56]. The current



(a) An optimal T-row layout with $p_1 = p_M$ and 1 is in row 1.



(b) An optimal T-row layout with $p_i \neq p_M, i \in [n]$.



(c) An optimal T-row layout with $p_3 = p_M$ and 3 is in row 1.

Figure 2: We are given a T-row instance with $n = 5$ and $\ell_1 = \ell_2 = 5, \ell_3 = \ell_5 = 3, \ell_4 = 2$ and non-zero weights $w_{12} = w_{13} = 3, w_{23} = w_{34} = w_{35} = 2, w_{24} = w_{45} = 1$ with $w^T_{path} = 1$. Three optimal T-row layouts with objective value 65 are illustrated, and a black circle is displayed on position p_M in row 1.

best ILP approach of [3] makes use of betweenness variables and can solve instances with up to 35 departments in at most 6 hours while the current best SDP approach [55, 56] is able to solve instances with 36 departments in at most 20 minutes and one instance with 42 departments in less than 2 hours. Furthermore, lower and upper bounds are provided with gaps less than 2% for instances with up to 100 departments in around 200 hours. Note that the machine used in [55, 56] is faster than the machine used in [3]. Besides the exact approaches there have been various heuristic approaches presented in the last years [33, 37, 45, 60, 61, 70, 71, 74]. We refer to [58] for a recent survey.

The MBFLP and several extensions have been extensively studied in the literature [17, 38, 43, 62, 63, 67]. In [67], the MBFLP is considered where additionally the lengths of the rows are restricted. Then, a two-stage procedure is presented where in the first step the assignment of the departments is determined by solving a mixed-integer linear programming (MILP) approach neglecting inner-row distances and minimizing the weighted inter-row distances. In the second step a dummy department $n+1$ with length $\ell_{n+1} = 0$ and weights $w_{i(n+1)} = w_{(n+1)i}, i \in [n]$, which are equal to the sum of the weights of i to all departments which lie in a different row than i, is added. Then, the layout in each row is determined by setting the lengths of the departments to one and applying the dynamic programming algorithm of [72] to solve m single-row instances independently. This approach is extended by [24] to instances containing departments of the same type, i. e., departments with the same length and the same transport weights to the remaining departments. Two-stage procedures are often used for the MBFLP and its extensions, see, e. g., [24, 25, 67], but without knowledge about the quality of the obtained solution. In [43] an ILP model is presented for the MBFLP where the assignment of the departments to the rows is fixed, and instances with up to 25 departments and up to 5 rows are solved in less than one second. In order to compute an optimal solution for the MBFLP one has to enumerate over all distinguishable assignments of the departments to the rows. A survey for the MBFLP is given in [38].

The Flexible Multi-Bay Facility Layout Problem (FBFLP) is an extension of the MBFLP, see, e. g., [17, 25, 59, 63], where the departments are given as two-dimensional objects, i. e., the departments have a length and a height. The height of a row equals the height of a department with the highest height in that row. The vertical distance between two rows equals half of the heights of these two rows plus the sum of the heights of the rows between them. In contrast to the MBFLP, there is no upper bound for the number of rows given. The MILP approach of [59] can solve one instance with 12 departments in less than two hours but is not able to solve an instance with 11 departments within a given time limit of 24 hours. Besides the exact approach, the above described two-stage procedure is adapted to this case by [25] and several further heuristics are presented [27, 62, 63, 66, 79].

In [29] a MILP model for the PMFP is presented, and an instance with 12 departments and up to five rows is considered where a gap of around 95 % was obtained after a time limit of 2 hours. The authors conclude with the research question to develop a more efficient approach for the PMFP. We refer to [14, 31, 51, 53, 64, 80] for further extensions of the MBFLP.

Highly related to the MBFLP is the Multi-Row Facility Layout Problem (MRFLP) where distances are measured only in horizontal directions. For $m = 2$ this problem is called Double-Row Facility Layout Problem (DRFLP) and several MILP approaches are available for the DRFLP [6, 8, 9, 26, 30, 76] (see [82] for a correction of [30]). The enumerative approach of [42, 43] is the current best approach for the DRFLP and the MRFLP and is able to solve double-row instances with up to 16 departments in less than 12 hours and multi-row instances with up to 13 departments and 5 rows in at most 7 hours. Heuristics for the DRFLP and the MRFLP, including extensions, are given in [10, 19, 30, 46, 69, 73, 78, 83, 84]. The special case where all departments have the same length is considered in [4, 12, 13, 54], and the current best approach is given in [13] where instances with up to 25 departments and up to 5 rows are solved to optimality within a time limit of 3 hours. For further literature about facility layout planning we refer to the surveys [17, 39, 50].

1.2 Our Contribution

In this section, we describe our main contribution. We study the well-known betweenness variables and we show that it is sufficient to choose $(n-1)(n-2)$ of the betweenness variables as binary variables in the MILP model of [3]. Currently, all betweenness variables, i. e., $\frac{n(n-1)(n-2)}{3}$, are chosen to be binary, see, e. g. [3, 22, 43].

The TRFLP is a generalization of the SRFLP and the 3-BFLP with a more complex path structure, and hence we continue the line of research to extend facility layout problems in order to include realistic aspects. At first, we show that there always exists an optimal T-row layout where one department has its center position on position p_M in row 1. Then, we enumerate over each department with its center fixed on position p_M in row 1. We present a MILP model for this subproblem of the TRFLP based on an extension of the betweenness variables which can be in contrast to the literature equal to one if the corresponding departments lie in different rows. This extension has two advantages. At first, the distances between departments in different rows can be calculated without big-M-constraints. At second, we use transitivity constraints (instead of standard linearization) to ensure that we obtain the correct relation of the extended betweenness variables and the remaining variables. We provide cutting planes exploring the crossroad structure of the TRFLP to compute lower bounds for the extended betweenness variables. In order to evaluate the performance of our approach, we set up a MILP approach based on betweenness variables which can only be equal to one if the corresponding three departments are in the same row. Further, we use a variant which uses standard linearization instead of the transitivity inequalities and in another variant, we neglect some of the cutting planes. In our computational study we show that we clearly outperform these approaches.

Further, we adapt our approach and obtain a MILP model for the 3-BFLP. The current best known approach for the 3-BFLP can be adapted to the TRFLP, and for both problems our approach clearly outperforms the best known approach from the literature. In addition, we are able to compute tight lower bounds for even larger T-row and 3-Bay instances to evaluate the quality of our heuristically determined layouts.

2 The Single-Row Facility Layout Problem

In this paper we study betweenness variables and we present an extension in Section 3.2. In this section we prove that it is sufficient to choose $(n-1)(n-2)$ of the $\frac{n(n-1)(n-2)}{3}$ betweenness variables as binary variables. Betweenness variables are defined as follows in the literature, see, e. g., [3, 22, 28, 43]

$$x_{jki} = x_{ikj} = \begin{cases} 1, & \text{if } k \text{ lies between } i \text{ and } j \text{ in the same row,} \\ 0, & \text{otherwise,} \end{cases}$$

$i, j, k \in [n], |\{i, j, k\}| = 3, i < j$. Based on these betweenness variables the following ILP model is introduced in [3] for the SRFLP with the constant $C := \sum_{\substack{i,j \in [n] \\ i<j}} \frac{\ell_i + \ell_j}{2} w_{ij}$

$$C + \quad \min \sum_{\substack{i,j \in [n] \\ i<j}} w_{ij} \sum_{k \in [n] \setminus \{i,j\}} \ell_k x_{ikj} \tag{2}$$

$$\begin{aligned} \text{s.t.} \quad & x_{ikj} + x_{jik} + x_{ijk} = 1, && i, j, k \in [n], i < j < k, && (3) \\ & -x_{ihj} + x_{ihk} + x_{jhk} \geq 0, && i, j, k, h \in [n], |\{i, j, k, h\}| = 4, && (4) \\ & x_{ihj} + x_{ihk} + x_{jhk} \leq 2, && i, j, k, h \in [n], i < j < k, |\{i, j, k, h\}| = 4, && (5) \\ & x_{ikj} \in \{0, 1\}, && i, j, k \in [n], i < j, |\{i, j, k\}| = 3. && (6) \end{aligned}$$

Equations (3) ensure that, given three departments, exactly one of them lies in the middle. The departments satisfy certain transitivity conditions, see inequalities (4)–(5). There always exists an optimal space-free single-row layout, i. e., a single-row layout without spaces between two neighboring departments, so the distance between $i \in [n]$ and $j \in [n], i < j$, is calculated by summing up the lengths of all departments which lie between them, see (2). By inequalities (3)–(6) one obtains a feasible ordering of the departments, see [3]. A polyhedral study is given in [75].

In [3] a lower bounding strategy is used where the LP (2)–(5) with $x_{ikj} \in [0, 1], i, j, k \in [n], i < j, |\{i, j, k\}| = 3$, is solved and, while there exists violated cuts and the solution is not integral, the violated cuts are added and the resulting LP is solved again. All instances considered in [3] are solved to optimality by this method but in general, there is no guarantee to obtain an optimal solution. One reason for using this lower bounding approach instead of using a branch-and-cut algorithm might be that this model has $\mathcal{O}(n^3)$ binary variables. We provide the following result

Proposition 1. *Given inequalities* (3)–(5)*, it is sufficient to choose* $(n-1)(n-2)$ *betweenness variables as binary variables to obtain a correct ordering of the departments.*

Proof. Let $s \in [n]$ be fixed. We set $x_{sij}, x_{sji}, x_{isj} \in \{0, 1\}, i, j \in [n], i < j$, and $x_{ikj} \in [0, 1]$, $i, j, k \in [n] \setminus \{s\}, |\{i, j, k\}| = 3, i < j$. We will show that $x_{ikj}, i, j, k \in [n] \setminus \{s\}, |\{i, j, k\}| = 3, i < j$, is equal to one if $x_{ski} + x_{skj} = 1$, and equal to zero otherwise, i. e., $x_{ikj} \in \{0, 1\}$. Therefore, we distinguish between the following three cases

1) Let $x_{ski} + x_{skj} = 1$. Then, we obtain $x_{ikj} = 1$ by inequalities (4), as desired.

2) Let $x_{ski} = x_{skj} = 1$. By inequalities (5) we get $x_{ikj} = 0$.

3) Let $x_{ski} = x_{skj} = 0$. It follows by inequalities (4) that $x_{ikj} = 0$.

Therefore, we obtain $x_{ikj} \in \{0,1\}, i,j,k \in [n] \setminus \{s\}, |\{i,j,k\}| = 3, i < j$, and according to [3] we obtain a feasible ordering of the departments. Furthermore, one can use equations (3) to reduce the number of variables as done in [3], i. e., $x_{sji} = 1 - x_{sij} - x_{isj}, i,j \in [n] \setminus \{s\}, i < j$, and hence we obtain $(n-1)(n-2)$ binary variables. □

By fixing $s \in [n]$ and choosing only x_{sij} and $x_{isj}, i,j \in [n] \setminus \{s\}, i < j$, as binary variables, one influences the branching strategy of the `MILP` solver using a branch-and-cut algorithm, and hence the performance of the branch-and-cut algorithm can be improved. However, the number of branches is in general not reduced by this method, but it supports the idea to apply branch-and-cut algorithms on `MILP` models based on betweenness variables.

3 The T-Row Facility Layout Problem

Approaches for the `MBFLP` are often based on heuristics without knowledge about the quality of the solution, see, e. g., [24, 25, 67], and `MILP` models for the `MBFLP` and its extensions are not able to solve instances with 12 departments in reasonable time [29, 59]. Therefore, we present a new exact approach for the `TRFLP`. At first, in Section 3.1, we prove that there always exists an optimal T-row layout where the center of one department in row 1 is on position p_M. In a branch-and-cut approach we enumerate over each department with its center fixed on position p_M in row 1 and in Section 3.2 we present a `MILP` model for this problem based on ordering, assignment and betweenness variables. We extend the betweenness variables such that they can be equal to one if the corresponding three departments lie in distinct rows, and thus we are able to calculate the distance between departments in distinct rows without big-M-constraints. Transitivity constraints are used to ensure the correct relation of these betweenness variables and the remaining variables. In Section 3.3 we present various cutting planes especially designed for the `TRFLP`, where the center of $s_M \in [n]$ is fixed on position p_M in row 1, to strengthen our branch-and-cut algorithm and in Section 3.4 we derive diverse heuristic approaches. We describe our separation strategy in Section 3.5.

3.1 Our Algorithm

We start this section with the following proposition, which is essential for our algorithm

Proposition 2. *Given a T-row instance, then there exists an optimal T-row layout where the center of one department lies on position p_M in row 1.*

The proof is related to a proof of [34]. Given a `DRFLP` instance with objective function $\min \sum_{j \in [n] \setminus \{i\}} w_{ij} d_{ij}, i \in [n]$, there exists an double-row optimal layout where $k \in [n] \setminus \{i\}$ lies directly opposite i.

Proof. We assume, w. l. o. g., that an optimal T-row layout contains at least one department in row 1 because if all departments would be in row 2, one can simply shift these departments to row 1 without changing the order of the departments and without increasing the objective value of the layout. So let an optimal T-row layout with at least one department in row 1 be given and we assume for all $i \in [n]$ in row 1 that $p_i \neq p_M$. Let B_1 (B_2) denote the set of departments in row 1 with $p_i < p_M$ ($p_i > p_M$), $i \in [n]$, and let B_3 denote the set of departments in row 2. We assume, w. l. o. g., $\sum_{\substack{k \in B_3 \\ i \in B_1}} w_{ik} \geq \sum_{\substack{k \in B_3 \\ i \in B_2}} w_{ik}$, and we shift all departments in row 1 to the right until the center of the first department lies on position p_M. By this method we do not increase the objective value, and the desired result is proven. □

Therefore, we fix the center of $s_M \in [n]$ on position p_M in row 1 and we denote the resulting problem by (s_M-`TRFLP`). Our algorithm for the `TRFLP` works as follows. For each $s_M \in [n]$ we heuristically determine a s_M-T-row layout, and we sort the departments by increasing order of the objective values of the s_M-T-row layouts. Let $s_M \in [n]$ be the first not yet considered department in this sorting. Then, we solve the s_M-TRFLP with our branch-and-cut algorithm where the current best objective value is set as an upper bound, i. e., the branch-and-cut algorithm is interrupted if the best lower bound exceeds the upper bound. We repeat this until the s_M-TRFLP with the described upper bound is solved for each $s_M \in [n]$. Our algorithm is summarized in Algorithm 1.

Algorithm 1: Exact solution approach for the TRFLP

Input : T-row instance with departments $[n]$ with lengths $\ell_i, i \in [n]$, and pairwise weights $w_{ij} = w_{ji}, i, j \in [n], i < j, w_{path}^T \in \mathbb{R}_{\geq 0}$.

Output : Optimal value v^* of the TRFLP.

for $k = 1, \ldots, n$ **do**
  $s_M \leftarrow k$.
  Determine an upper bound u_k for the s_M-TRFLP heuristically.
Sort the departments in ascending order according to their upper bounds $\hookrightarrow (s_1, \ldots, s_n)$.
Set $v^* \leftarrow u_{s_1}$.
for $k = 1, \ldots, n$ **do**
  $s_M \leftarrow s_k$.
  Compute optimal value v of the s_M-TRFLP with the additional constraint that the optimal value is smaller than or equal to the upper bound v^* ($v \leftarrow \infty$ otherwise).
  if $v < v^*$ **then**
    $v^* \leftarrow v$.
return v^*.

The question arises, whether the number of departments where the centers are fixed on position p_M in row 1 can be reduced, e. g., one could try to arrange only the centers of the three departments on position p_M in row 1, which are at the left border of an optimal 3-Bay layout

Example 3. *We consider an instance with $n = 4$ departments with lengths $\ell_1 = \ell_3 = 5, \ell_2 = \ell_4 = 7$, and non-zero weights $w_{12} = w_{13} = w_{23} = 1, w_{14} = 3$, with $w_{path}^T = 0$ and $w_{path}^B = 1$. In an optimal 3-Bay layout, 1 and 4 are neighboring without free-space in the same row and 1, 2 and 3 are on the left border of a row, see Figure 3a for an illustration. Therefore, the optimal value of the 3-BFLP is 39. In an optimal T-row layout the center of 4 lies on position p_M in row 1, illustrated in Figure 3b, with objective value 29.5.*

However, if there are departments of the same type as recently considered in [20, 24, 46] for related layout problems, one just has to fix the center of one department of each type on position p_M in row 1.

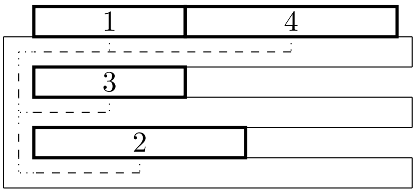
3.2 A MILP model for the s_M-TRFLP

In this section we present a MILP model for the s_M-TRFLP, $s_M \in [n]$. So, in the following, let $s_M \in [n]$ be fixed. Our MILP model makes use of the well-known ordering and assignment variables. If $i \in [n]$ and $j \in [n] \setminus \{i\}$ lie in row 2 and i is above (below) j, we say that i is left (right) to j. Then, we define

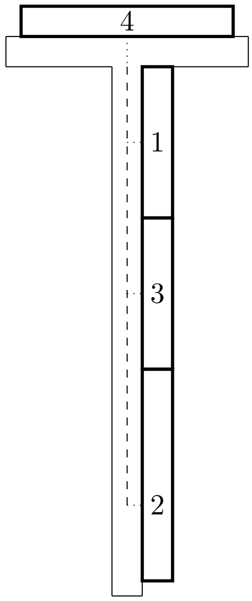
$$z_{ij} = \begin{cases} 1, & i \text{ is left to } j \text{ and } i \text{ and } j \text{ are in the same row} \\ 0, & \text{otherwise,} \end{cases}$$

$i, j \in [n], i \neq j$. Furthermore, we use assignment variables

$$y_i = \begin{cases} 1, & \text{if } i \text{ lies in row 1} \\ 0, & \text{otherwise,} \end{cases}$$



(a) optimal 3-Bay layout with 1,2 and 3 at the left border.



(b) optimal T-row layout where the center of 4 is on position p_M in row 1.

Figure 3: We are given an instance with $n = 4$ departments with $\ell_1 = \ell_3 = 5, \ell_2 = \ell_4 = 7$, and non-zero weights $w_{12} = w_{13} = w_{23} = 1, w_{14} = 3$, with $w^T_{path} = 0, w^B_{path} = 1$. An optimal 3-Bay layout is illustrated on the left-hand side with objective value 39 and an optimal T-row layout is illustrated on the right-hand side with objective value 29.5.

$i \in [n]$. At first, we fix s_M to row 1, see equation (7), and we ensure that $i \in [n] \setminus \{s_M\}$ is left or right to s_M if and only if i is in row 1, see equations (8)

$$y_{s_M} = 1, \tag{7}$$

$$z_{is_M} + z_{s_M i} - y_i = 0, \qquad i \in [n] \setminus \{s_M\}. \tag{8}$$

We add two dummy departments $n + 1$ and $n + 2$ with lengths $\ell_{n+1} = \ell_{n+2} = 0$ and weights $w_{i(n+1)} = w_{(n+1)i} = w_{(n+2)i} = w_{i(n+2)} = 0, i \in [n] \setminus \{s_M\}$, to our model, and we fix $n + 1$ at the left border of row 1 and $n + 2$ at the left (upper) border of row 2. Then, we define betweenness variables in the following way where at least one of the three departments is a dummy department or s_M

$$x_{ikj} = x_{jki} = \begin{cases} 1, & \text{if } k \text{ lies between } i \text{ and } j \text{ in the same row} \\ 0, & \text{otherwise,} \end{cases}$$

$i, j, k \in [n + 2], |\{i, j, k\} \cap \{s_M, n + 1, n + 2\}| \geq 1, i < j, |\{i, j, k\}| = 3$. Remark that, given three departments and one of them is equal to s_M, $n + 1$ or $n + 2$, then the corresponding betweenness variables are only equal to one if all three departments lie in the same row. In this way betweenness variables were used successfully in the literature, see, e. g., [3, 22, 43, 76]. The correct relation of the betweenness and the assignment variables are ensured by the following inequalities

$$x_{isj} + x_{sij} + x_{sji} - y_i \leq 0, \qquad i, j \in [n] \setminus \{s_M\}, i \neq j, s \in \{s_M, n + 1\}, \tag{9}$$

$$x_{(n+2)ij} + x_{(n+2)ji} + y_i \leq 1, \qquad i, j \in [n] \setminus \{s_M\}, i \neq j, \tag{10}$$

$$x_{isj} + x_{sij} + x_{sji} - y_i - y_j \geq -1, \qquad i, j \in [n] \setminus \{s_M\}, i < j, s \in \{s_M, n + 1\}, \tag{11}$$

$$x_{(n+2)ij} + x_{(n+2)ji} + y_i + y_j \geq 1, \qquad i, j \in [n] \setminus \{s_M\}, i < j. \tag{12}$$

Inequalities (9) ensure that betweenness variables containing i, j and s_M or $n+1$, $i, j \in [n] \setminus \{s_M\}, i \neq j$, are equal to zero if i or j (or both) are assigned to row 2. The sum of the corresponding three betweenness variables equals one if i and j are assigned to row 1, see inequalities (11). Similar inequalities are used for the dummy department $n+2$, see inequalities (10) and inequalities (12).

Betweenness variables containing s_M are equal to the sum of two products of ordering variables, e. g., $x_{s_M ij} = z_{s_M i} \cdot z_{ij} + z_{ji} \cdot z_{i s_M}, i, j \in [n] \setminus \{s_M\}, i \neq j$. Therefore, we use lower bounds related to the standard linearization, see inequalities (13)–(15). Note that the corresponding upper bounds are implied by inequalities (7)–(10)

$$
\begin{aligned}
& x_{i s_M j} \geq z_{i s_M} + z_{s_M j} - 1, && i, j \in [n] \setminus \{s_M\}, i \neq j, && (13)\\
& x_{s_M ij} \geq z_{s_M i} + z_{ij} - 1, && i, j \in [n] \setminus \{s_M\}, i \neq j, && (14)\\
& x_{s_M ij} \geq z_{ji} + z_{i s_M} - 1, && i, j \in [n] \setminus \{s_M\}, i \neq j, && (15)\\
& x_{(n+1)ij} + x_{(n+2)ij} - z_{ij} = 0, && i, j \in [n] \setminus \{s_M\}, i \neq j, && (16)\\
& x_{i s_M j} + x_{s_M ij} + x_{s_M ji} - x_{(n+1)ij} - x_{(n+1)ji} = 0, && i, j \in [n] \setminus \{s_M\}, i < j, && (17)\\
& x_{i(n+1)j} + x_{i(n+2)j} = 0, && i, j \in [n], i < j. && (18)
\end{aligned}
$$

Let $i, j \in [n] \setminus \{s_M\}, i \neq j$. Then, i and j are in the same row and i is left to j if and only if i is left to j in row 1 or row 2, see equations (16). Further, i is left or right to j in row 1 if and only if $x_{i s_M j} + x_{s_M ij} + x_{s_M ji}$ equals one, see equations (17). Equations (18) ensure that the dummy departments are fixed at the border. While the dummy department $n+2$ is necessary for our formulation for the s_M-TRFLP, we mainly use the dummy department $n+1$ to relate $x_{(n+2)ij}$ and $x_{(n+2)ji}, i, j \in [n] \setminus \{s_M\}, i < j$, with the ordering variables, see equations (16). Hence, we avoid the usage of standard linearization in this case.

In [8] a MILP model for the DRFLP is presented which consists of ordering variables σ_{ij} which are equal to one if $i \in [n]$ is to the left of $j \in [n], i \neq j$, and i and j are in the same row or i is in row 2 and j is in row 1. The distances between the centers of the departments are calculated via big-M-type constraints. Let R_1 (R_2) denote the set of departments assigned to row 1 (row 2) in a double-row layout. Then, we assign the departments in R_1 to row 2 such that the departments in R_1 are to the right of the departments in R_2 without changing the order of the departments in R_1 and in R_2. By this method, the value of the ordering variables is not influenced, so the ordering variables σ are equal to ordering variables in a single-row layout. Therefore, this method differs significantly from our method described below.

The distance between $i \in [n]$ and $j \in [n] \setminus \{i\}$ in the s_M-TRFLP can be expressed by $|p_i - p_M| + p_j + w^T_{path}$ if i lies in row 1 and j in row 2. Using this idea in a branch-and-cut algorithm coupled with big-M-constraints to couple the row assignment of i and j, leads to a rather bad root relaxation and to a poorly performing algorithm as we will see in Section 5. We aim to calculate the distances between the departments by betweenness, ordering and assignment variables without big-M-constraints. Therefore, we extend the definition of the betweenness variables. Given a s_M-T-row layout, we define B_1 (B_2) as the set of departments left (right) to s_M and B_3 as the set of departments in row 2. Let i' (j') be the department in B_a (B_b) closest to $p_M, a, b \in \{1, 2, 3\}, a \neq b$. We arrange the departments $B_a \cup B_b$ on a straight line such that i' and j' are neighboring as well as neighboring departments in the s_M-T-row layout are neighboring. Let $i, j, k \in [n] \setminus \{s_M\}, |\{i, j, k\}| = 3, i < j$. We say that $k \in [n] \setminus \{s_M\}$ lies between i and j if there exists $a, b \in \{1, 2, 3\}$ such that k lies between i and j on the with $B_a \cup B_b$ associated straight line. If $a = b$, then the departments in the set B_a are already arranged on a straight line. In the s_M-TRFLP an automated guided vehicle has to traverse every k which lies between i and j. This motivates the following extension of the betweenness variables

$$
x_{ikj} = x_{jki} = \begin{cases} 1, & \text{if } k \text{ lies between } i \text{ and } j \\ 0, & \text{otherwise,} \end{cases}
$$

$i,j,k \in [n] \setminus \{s_M\}, |\{i,j,k\}| = 3, i < j$. So our new betweenness variables x_{ikj} might also be equal to one if $i \in [n]$ and $j \in [n], i < j$, lie in distinct rows, $k \in [n] \setminus \{i,j\}$. Since this version of betweenness variables seems to be a novelty in the layout planning literature, we illustrate their usage by an example

Example 4. *We are given a T-row instance with $n = 6$ departments with lengths $\ell_1 = \ell_5 = 3, \ell_2 = 4, \ell_3 = 1, \ell_4 = \ell_6 = 2$. A s_M-T-row layout with $s_M = 2$ is illustrated in Figure 4 including the dummy departments 7 and 8. We summarize all betweenness variables which are equal to one. We start with $x_{ikj}, i,j,k \in [n], i < j, |\{i,k,j\}| = 3$, where i and j lie in distinct rows: $x_{156} = x_{356} = x_{435} = x_{436} = x_{456} = 1$. For betweenness variables containing s_M or at least one of the dummy departments, we get: $x_{123} = x_{124} = x_{234} = x_{217} = x_{317} = x_{327} = x_{417} = x_{427} = x_{437} = x_{856} = 1$. Besides that we get $x_{134} = 1$ and the remaining betweenness variables are equal to zero.*

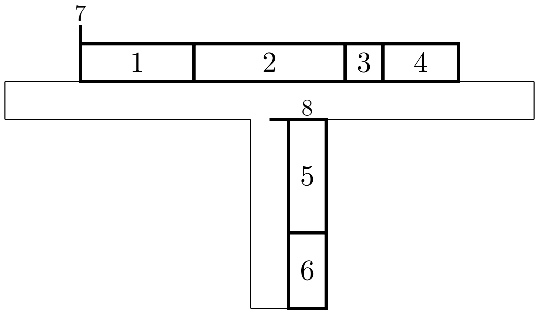


Figure 4: Illustration of a s_M-TRFLP layout with $s_M = 2$ where the dummy departments 7 (8) is at the left (upper) border of row 1 (row 2). We summarize all extended betweenness variables which are equal to one: $x_{156} = x_{356} = x_{435} = x_{436} = x_{456} = 1$.

We present the following inequalities to obtain the correct relation between our new extended betweenness variables and betweenness variables containing s_M or a dummy department

$$x_{ikj} - x_{s_M ki} + x_{s_M kj} - x_{(n+2)ki} + x_{(n+2)kj} \geq 0, \quad i,j,k \in [n] \setminus \{s_M\}, |\{i,j,k\}| = 3, \tag{19}$$

$$x_{ikj} - x_{s_M ki} - x_{s_M kj} - x_{(n+2)ki} - x_{(n+2)kj} \leq 0, \quad i,j,k \in [n] \setminus \{s_M\}, |\{i,j,k\}| = 3, i < j, \tag{20}$$

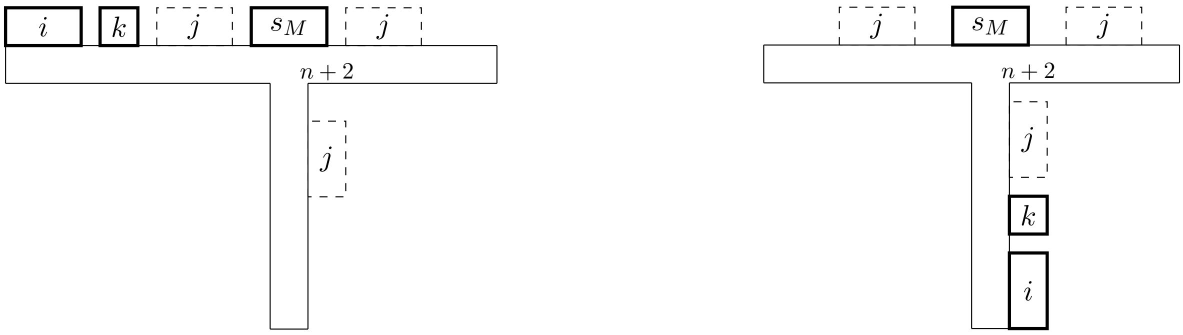
$$x_{ikj} + x_{jik} + x_{ijk} \leq 1, \quad i,j,k \in [n] \setminus \{s_M\}, i < j < k. \tag{21}$$

Considering three departments, at most one of them lies in the middle, see inequalities (21). Let $i,j,k \in [n] \setminus \{s_M\}, |\{i,j,k\}| = 3$, be fixed, and we consider inequalities (19). Recall that $x_{s_M ki} + x_{(n+2)ki} \leq 1$ by inequalities (9)–(10). Let k lie between i and s_M $(n+2)$ and let k not lie between s_M $(n+2)$ and j. Then, k lies between i and j, we refer to Figure 5a (5b) for an illustration. Otherwise, inequalities (19) are redundant. Now, let $i,j,k \in [n] \setminus \{s_M\}, |\{i,j,k\}| = 3, i < j$. Recall that B_1 (B_2) denotes the set of departments left (right) to s_M and B_3 denotes the set of departments in row 2. If $k \in B_a, a \in \{1,2,3\}$, and $i,j \notin B_a$, then k does not lie between i and j, see inequalities (20). Now, let $i,k \in B_a, a \in \{1,2,3\}$, and we set $s = s_M$ if $a \in \{1,2\}$ and $s = n+2$ if $a = 3$. Let $j \in B_b, b \in \{1,2,3\}, b \neq a$, and let k lie between i and j, then k lies between i and s, see inequalities (20). Let $j \in B_a$ and let k lie between i and j, then k lies between i and s or j and s and inequalities (9) are satisfied. This leads to the following remark

Remark 1. *Let i,j and k lie in the same row, $i,j,k \in [n] \setminus \{s_M\}, |\{i,j,k\}| = 3, i < j$. We set $s = s_M$ $(s = n+2)$ if i lies in row 1 (row 2) and then we obtain by inequalities* (19)–(20)

$$x_{ikj} - x_{ski} + x_{skj} \geq 0,$$
$$x_{ikj} - x_{skj} + x_{ski} \geq 0,$$
$$x_{ikj} - x_{ski} - x_{skj} \leq 0.$$

Hence, inequalities (19)–(20) *are an extension of inequalities* (4) *with departments* $[n] \setminus \{s_M\}$.



(a) Let $x_{iks_M} = 1$. Then, k lies between i and j if and only if k does not lie between s_M and j.

(b) Let $x_{(n+2)ki} = 1$. Then, k lies between i and j if and only if k is not left to j in row 2, i. e., $x_{(n+2)kj} = 0$.

Figure 5: Let $i, j, k \in [n] \setminus \{s_M\}, |\{i, j, k\}| = 3$. Visualization of inequalities (19) with $x_{s_M ki} = 1$ in Figure 5a and $x_{(n+2)ki} = 1$ in Figure 5b. We illustrate possible positions for j in dashed rectangles such that $x_{ikj} = 1$. The dummy department $n + 1$ is not illustrated here.

It remains to calculate the rectilinear center-to-center distances between the departments. The minimal distance between i and j, $i, j \in [n] \setminus \{s_M\}, i < j$, in the s_M-`TRFLP` equals $\frac{\ell_i+\ell_j}{2}$ and, assuming that i and j lie in different rows, one can add $\frac{\ell_{s_M}}{2} + w^T_{path}$. Further, the minimal distance between $i \in [n] \setminus \{s_M\}$ and s_M is $\frac{\ell_i}{2}$ and assuming that i lies in row 2 we add w^T_{path}, so altogether we exclude the following constant value from our model

$$W_{s_M} := \sum_{i\in[n]\setminus\{s_M\}} w_{is_M} \left(\frac{\ell_i}{2} + w^T_{path}\right) + \sum_{\substack{i,j\in[n]\setminus\{s_M\}\\ i<j}} w_{ij} \left(\frac{\ell_i}{2} + \frac{\ell_j}{2} + \frac{\ell_{s_M}}{2} + w^T_{path}\right). \qquad (22)$$

For the s_M-`TRFLP` there always exists an optimal space-free layout, i. e., an optimal layout without free-spaces between neighboring departments in the same row. So we neglect distance variables and we compute the distances between the departments in the following way

$$\left(-w^T_{path} - \frac{\ell_{s_M}}{2}\right)(z_{ij} + z_{ji}) + \sum_{k\in[n]\setminus\{i,j\}} \ell_k x_{ikj}, \qquad i, j \in [n] \setminus \{s_M\}, i < j, \qquad (23)$$

$$\left(\frac{\ell_{s_M}}{2} - w^T_{path}\right) y_i + \sum_{k\in[n]\setminus\{s_M,i\}} \ell_k(x_{iks_M} + x_{(n+2)ki}), \qquad i \in [n] \setminus \{s_M\}. \qquad (24)$$

By (23) and (22) we measure the rectilinear center-to-center distance between $i \in [n] \setminus \{s_M\}$ and $j \in [n] \setminus \{s_M\}, i < j$. We subtract $\frac{\ell_{s_M}}{2} + w^T_{path}$ if i and j lie in the same row, see (22), and we add the sum of the lengths of the departments between i and j. If i and j lie in row 1 and s_M lies between them, then $x_{is_Mj} = 1$ and we take ℓ_{s_M} into account. By (24) and (22) we calculate the rectilinear center-to-center distance between s_M and $i \in [n] \setminus \{s_M\}$ by summing up the lengths of all departments which are between i and s_M and i and $n + 2$. Additionally, we add $\frac{\ell_{s_M}}{2} - w^T_{path}$ if i lies in row 1. This leads to the following result

Theorem 3. *Let $s_M \in [n]$ and W_{s_M} be calculated as described in equation* (22)*. Then*

$$W_{s_M} + \quad \min \sum_{\substack{i,j\in[n]\setminus\{s_M\}\\ i<j}} w_{ij} \left(-w^T_{path} - \frac{\ell_{s_M}}{2}\right)(z_{ij} + z_{ji})$$

$$+ \sum_{\substack{i,j\in[n]\setminus\{s_M\}\\ i<j}} w_{ij} \sum_{k\in[n]\setminus\{i,j\}} \ell_k x_{ikj}$$

$$
\begin{aligned}
&+ \sum_{i\in[n]\setminus\{s_M\}} w_{is_M} \left(\frac{\ell_{s_M}}{2} - w_{path}^T\right) y_i \\
&+ \sum_{i\in[n]\setminus\{s_M\}} w_{is_M} \sum_{k\in[n]\setminus\{s_M,i\}} \ell_k (x_{iks_M} + x_{(n+2)ki})
\end{aligned}
$$

$$
\begin{aligned}
s.\,t.\quad & y_i \in \{0,1\}, && i \in [n], \\
& z_{ij} \in \{0,1\}, && i,j \in [n]\setminus\{s_M\}, i<j, \\
& z_{ij} \in [0,1], && i,j \in [n]\setminus\{s_M\}, i>j, \\
& z_{is_M} \in \{0,1\}, && i \in [n]\setminus\{s_M\}, \\
& z_{s_Mi} \in [0,1], && i \in [n]\setminus\{s_M\}, \\
& 0 \le x_{ikj} \le 1, && i,j,k \in [n+2], i<j, \\
& && |\{i,j,k\}| = 3,
\end{aligned}
$$

subject to inequalities (7)–(21) *is a* `MILP` *model for the* s_M*-*`TRFLP`*.*

Proof. Let $s_M \in [n]$ be fixed. This proof is structured as follows. At first, we show that the z variables are set to binary values and we prove that the z and y variables satisfy inequalities of a model from the scheduling literature [32]. Hence, we obtain the correct relation of the z and y variables and a correct ordering of the departments in the same row. Then we show that the x variables are set to binary values as well. We show the correct relation of the x and y and z variables where we distinguish between our new extended x variables and x variables containing s_M or dummy departments.

By equations (8) we obtain binary values for $z_{s_Mi}, i \in [n]\setminus\{s_M\}$. Let $i,j \in [n]\setminus\{s_M\}, i<j$. By equations (16) and inequalities (9)–(10) together with $z_{ij} \in \{0,1\}$, we obtain $x_{(n+1)ij}, x_{(n+2)ij} \in \{0,1\}$. If $y_i \neq y_j$ we obtain by inequalities (9)–(10) that $0 = x_{(n+1)ji} + x_{(n+2)ji} = z_{ji}$, see equations (16). Otherwise let, w. l. o. g., $y_i = y_j = 1$. Then, by inequalities (9) and inequalities (11) we get $x_{(n+1)ij} + x_{(n+1)ji} = 1$. Since $x_{(n+1)ij} \in \{0,1\}$, we obtain $x_{(n+1)ji} \in \{0,1\}$ as well. This implies $z_{ji} \in \{0,1\}$, see equations (16). In conclusion, the z variables and the x variables containing dummy departments are set to binary values and these x variables are coupled correctly to the z variables.

Next we show that the following inequalities are satisfied

$$
\begin{aligned}
& z_{ij} + z_{ji} + y_i - y_j \le 1, && i,j \in [n], i \neq j, && (25)\\
& y_i + y_j - z_{ij} - z_{ji} \le 1, && i,j \in [n], i<j, && (26)\\
& y_i + y_j + z_{ij} + z_{ji} \ge 1, && i,j \in [n], i<j, && (27)\\
& z_{ki} + z_{ij} + z_{jk} - z_{ik} - z_{ji} - z_{kj} \le 1, && i,j,k \in [n], |\{i,j,k\}| = 3. && (28)
\end{aligned}
$$

Let $i,j \in [n]\setminus\{s_M\}, i \neq j$, be given. By equations (16) and inequalities (9)–(10) we obtain

$$
z_{ij} + z_{ji} + y_i - y_j = x_{(n+1)ij} + x_{(n+1)ji} + x_{(n+2)ij} + x_{(n+2)ji} + y_i - y_j \le 1,
$$

and thus inequalities (25) are satisfied. By similar arguments, inequalities (16) together with inequalities (11)–(12) imply inequalities (26)–(27). If $i,j \in [n]$ and $|\{i,j\} \cap \{s_M\}| = 1$, then inequalities (25)–(27) are implied by inequalities (7)–(8).

It remains to prove that inequalities (28) are satisfied if $y_i \in \{0,1\}, i \in [n]$, and $z_{ij} \in \{0,1\}, i,j \in [n], i \neq j$. Assume, on the contrary, inequalities (28) are violated by some $i,j,k \in [n], |\{i,j,k\}| = 3$. Let $z_{ij} = z_{jk} = 1$ and by inequalities (25) we obtain $y_i = y_j = y_k$. If $z_{ki} = 0$, we obtain by inequalities (26)–(27) that $z_{ik} = 1$, and thus inequalities (28) are satisfied. It remains to consider the case $z_{ki} = 1$.

a) Let $y_i = y_j = y_k = 0$. It follows by equation (7) that $|\{i,j,k\} \cap \{s_M\}| = 0$. Then, we obtain by inequalities (19), inequalities (9) and inequalities (16) that

$$x_{ikj} \geq x_{(n+2)ki} - x_{(n+2)kj} = z_{ki} - z_{kj} = 1$$
$$x_{jik} \geq x_{(n+2)ij} - x_{(n+2)ik} = z_{ij} - z_{ik} = 1,$$

a contradiction to inequalities (21).

b) Let $y_i = y_j = y_k = 1$, $|\{i,j,k\} \cap \{s_M\}| = 1$, and we assume, w. l. o. g., $k = s_M$. Then, by inequalities (13)–(14) we obtain

$$x_{js_Mi} \geq z_{js_M} + z_{s_Mi} - 1 = 1,$$
$$x_{s_Mij} \geq z_{s_Mi} + z_{ij} - 1 = 1,$$

a contradiction to inequalities (9).

c) Let $y_i = y_j = y_k = 1$ and $|\{i,j,k\} \cap \{s_M\}| = 0$. By symmetry, it is sufficient to consider the case that two or three departments of the set $\{i,j,k\}$ are to the right of s_M.

1) Let $z_{s_Mk} = z_{s_Mi} = z_{s_Mj} = 1$. Then, $x_{s_Mij} = x_{s_Mki} = x_{s_Mjk} = 1$, see inequalities (14), and thus we get by inequalities (9)–(10) and inequalities (19) that

$$x_{ikj} \geq x_{s_Mki} - x_{s_Mkj} = 1,$$
$$x_{kij} \geq x_{s_Mij} - x_{s_Mik} = 1,$$

a contradiction to inequalities (21).

2) Let $z_{ks_M} = 1$ and $z_{s_Mi} = z_{s_Mj} = 1$. Then, we obtain by inequalities (13)–(14) that $x_{ks_Mj} = 1$ and $x_{s_Mkj} \geq z_{ks_M} + z_{jk} - 1 = 1$, a contradiction to inequalities (9).

According to [32], we obtain by inequalities (25)–(28) together with $y_i \in \{0,1\}, i \in [n], z_{ij} \in \{0,1\}, i,j \in [n], i \neq j$, the correct relation between the z and y variables and a correct ordering of the departments in the same row.

The x variables containing s_M are coupled by inequalities (13)–(15), which are highly related to the standard linearization, see, e. g., [14, 52, 76], to the z variables. Inequalities (7)–(10) imply the upper bounds from the standard linearization. Hence, these variables obtain binary values and are coupled correctly to the z and y variables.

It remains to consider our new extended x variables. Let $i,j,k \in [n] \setminus \{s_M\}, |\{i,j,k\}| = 3$, be fixed. We use the following notation: $s \in \{s_M, n+2\}$ with $s = s_M$ if $y_i = 1$ and $s = n+2$ if $y_i = 0$. Note that if $y_i = y_k$, we obtain $x_{tki} = x_{tkj} = 0, t \in \{s_M, n+2\} \setminus \{s\}$, by inequalities (9)–(10). Similar to the proof of Proposition 1, it is sufficient to show that x_{ikj} equals one if $x_{ski} + x_{skj} = 1$ and equals zero otherwise. We distinguish between the following three cases depending on the assignment of i, j and k to the rows

a) Let $y_i = y_j = y_k$. The following three cases can be distinguished

1) Let $x_{ski} + x_{skj} = 1$. By inequalities (19) we obtain $x_{ikj} = 1$.

2) Let $x_{ski} = x_{skj} = 1$. The x variables containing s have the correct relation to the z variables, so they satisfy transitivity properties, i. e., $x_{isj} = 0$, and thus $x_{sij} + x_{sji} = 1$ by inequalities (11)–(12). Further, by inequalities (21) we get $x_{sik} + x_{sjk} = 0$. As a result, we obtain by inequalities (19)

$$x_{jik} + x_{ijk} \geq x_{sij} + x_{sji} = 1,$$

and thus $x_{ikj} = 0$, see inequalities (21).

3) Let $x_{ski} = x_{skj} = 0$. By inequalities (20) it follows immediately that $x_{ikj} = 0$.

b) Let, w. l. o. g., $y_k = y_i$ and $y_k \neq y_j$. By inequalities (9)–(10) we get $x_{skj} = 0$. So by inequalities (19)–(20) we obtain $x_{ikj} = x_{ski}$.

c) Let $y_i = y_j$ and $y_i \neq y_k$. Then, we obtain $x_{ikj} = 0$ by inequalities (20).

There always exists an optimal space-free T-row layout, the x, z and y variables are set to binary values and are coupled correctly, so the distance calculation, see (23)–(24), is correct. This concludes the proof. □

Let $S = (\pi_1, \dots, \pi_{n-1})$ denote an arbitrary sorting of the departments $[n] \setminus \{s_M\}$. In order to eliminate symmetrical layouts, we fix one department in row 1 to the left of s_M, see equation (29). However, if this department is assigned to row 2, we fix another department in row 1 to the left of s_M. We continue in this manner, see inequalities (30). These inequalities are related to symmetry breaking constraints for the graph coloring problem, see, [65, 68]. We set

$$z_{\pi_1 s_M} - y_{\pi_1} = 0, \tag{29}$$

$$z_{\pi_j s_M} - y_{\pi_j} + \sum_{k=1}^{j-1} y_{\pi_k} \geq 0, \qquad j = 2, \dots, n-1. \tag{30}$$

If $\frac{\ell_{s_M}}{2} = w^T_{path}$, one can use stronger symmetry breaking constraints, let $S = (\pi_1, \dots, \pi_{n-1})$ denote an arbitrary sorting of the departments $[n] \setminus \{s_M\}$. Then,

$$y_{\pi_1} = 1, \tag{31}$$

$$z_{\pi_1 s_M} = 1, \tag{32}$$

$$y_{\pi_2} = 1, \tag{33}$$

$$y_{\pi_j} - \sum_{k=2}^{j-1} z_{\pi_k s_M} \geq 3 - j, \qquad j = 3, \dots, n-1. \tag{34}$$

In the case $\frac{\ell_{s_M}}{2} = w^T_{path}$, we may fix $\pi_1 \in [n] \setminus \{s_M\}$ in row 1 to the left of s_M, see equations (31)–(32), and in addition, we fix $\pi_2 \in [n] \setminus \{s_M\}$ to row 1, see equation (33). If π_2 is to the left of s_M, we fix another department to row 1 until one department is to the right of s_M, see inequalities (34). Note that inequalities (34) are redundant for $j \in [n-1], j \geq 3$, if $z_{\pi_k s_M} = 0$ for some $k \in [n] \setminus \{s_M\}, 2 \leq k < j$.

Further, in an optimal s_M-T-row layout at least one department is contained in row 2 if $w^T_{path} = 0$, so in this case we set

$$\sum_{i \in [n] \setminus \{s_M\}} y_i \leq n - 2. \tag{35}$$

3.3 Cutting Planes

In the following, let $s_M \in [n]$ be fixed. In this section we describe further inequalities to strengthen our formulation for the s_M-TRFLP. Let $i, j, k \in [n] \setminus \{s_M\}, i < j < k$, be given. Remark again that in the SRFLP the sum of the corresponding three betweenness variables, i. e., $x_{ikj} + x_{jik} + x_{ijk}$, equals one, see equations (3), but might be equal to zero in the s_M-TRFLP. This is a significant difference because in the root relaxation of our branch-and-cut algorithm the value of the betweenness variables might be close to zero. Therefore, in this section, we present various lower bounds for the sum of the betweenness variables

Proposition 4. *Let $s_M \in [n]$. The following inequalities are valid for the s_M-TRFLP*

$$x_{ikj} + x_{jik} + x_{ijk} + x_{is_Mj} + x_{is_Mk} + x_{js_Mk} \geq 1, \quad i, j, k \in [n] \setminus \{s_M\}, i < j < k, \tag{36}$$

$$x_{ikj} + x_{jik} + x_{ijk} + x_{is_Mk} + x_{js_Mk} - y_k \geq 0, \quad i, j, k \in [n] \setminus \{s_M\}, |\{i, j, k\}| = 3, i < j, \tag{37}$$

$$x_{ikj} + x_{jik} + x_{ijk} - y_i - y_j - y_k \geq -2, \quad i, j, k \in [n] \setminus \{s_M\}, i < j < k. \tag{38}$$

Proof. Let $S := \{i, j, k\} \subseteq [n] \setminus \{s_M\}, |\{i, j, k\}| = 3, i < j$, be given. Note first that inequalities (36)–(38) are satisfied if $x_{ikj} + x_{ijk} + x_{jik}$ equals one. It turns out that $x_{ikj} + x_{ijk} + x_{jik}$ equals zero if and only if s_M lies between two departments of the set S and the remaining department in S lies in row 2. Otherwise, two or three departments are left or right to s_M $(n + 2)$ and then by inequalities (19) one of the betweenness variables equals one. Thus, inequalities (36) are valid. Consider inequalities (37) with $y_k = 1$ and let the sum of the corresponding betweenness variables be equal to zero, then s_M lies between i and k or between j and k. Inequalities (37) are satisfied if $y_k = 0$. Given three departments which are assigned to row 1, then the sum of the corresponding betweenness variables equals one, see inequalities (38). □

Note that inequalities (36)–(37) are not valid for betweenness variables known in literature and in Section 5 we demonstrate that inequalities (36)–(37) significantly improve the performance of our branch-and-cut algorithm.

In the `SRFLP` we obtain $\sum_{\substack{i,j,k\in[n] \\ i<j \\ i\neq k\neq j}} x_{ikj} = \binom{n}{3}$ by equations (3). However, this equation is not valid for the s_M-`TRFLP`, and thus we present lower bounds for the sum of the betweenness variables in the s_M-`TRFLP` distinguishing whether s_M or $n + 2$ are contained or not.

Proposition 5. *Let $s_M \in [n]$ and $S \subseteq [n] \setminus \{s_M\}, |S| \geq 4$. Then, the following inequality is valid for the s_M-TRFLP*

$$\sum_{\substack{i,j\in S \\ i<j}} \left(x_{is_Mj} + x_{s_Mij} + x_{s_Mji} + x_{(n+2)ij} + x_{(n+2)ji} \right) \geq \binom{\left\lceil \frac{|S|}{2} \right\rceil}{2} + \binom{\left\lfloor \frac{|S|}{2} \right\rfloor}{2}, \tag{39}$$

and for $S \subseteq [n] \setminus \{s_M\}, n_1 = \left\lceil \frac{|S|}{3} \right\rceil, n_2 = \left\lfloor \frac{|S|}{3} \right\rfloor, n_3 = |S| - n_1 - n_2$, the following inequality

$$\sum_{\substack{i,j,k\in S \\ i<j \\ i\neq k\neq j}} x_{ikj} \geq \binom{|S|}{3} - n_1 \cdot n_2 \cdot n_3 \tag{40}$$

is valid for the s_M-TRFLP.

Proof. Let $S \subseteq [n] \setminus \{s_M\}, |S| \geq 4$. Recall that, given a s_M-T-row layout, B_1 (B_2) denotes the set of departments left (right) to s_M and B_3 denotes the set of departments in row 2. Then, we obtain

$$\sum_{\substack{i,j\in S \\ i<j}} \left(x_{is_Mj} + x_{s_Mij} + x_{s_Mji} + x_{(n+2)ij} + x_{(n+2)ji} \right) = \binom{|B_1| + |B_2|}{2} + \binom{|B_3|}{2},$$

if $|B_1| + |B_2| \geq 2, |B_3| \geq 2$ (otherwise we can neglect the corresponding binomial coefficient on the right-hand side). Interpreting $B' = B_1 \cup B_2$ and computing the minimum value of the right-hand side subject to $B' \dot\cup B_3 = S,\ B', B_3 \subseteq S, B' \cap B_3 = \emptyset$, we obtain the desired result, see inequalities (39).

Now we consider inequalities (40), so let $S \subseteq [n] \setminus \{s_M\}$ be given. We are given a s_M-T-row layout, then $x_{ikj} + x_{ijk} + x_{jik}, i, j, k \in [n], i < j < k$, equals zero if and only if i, j and k lie pairwise in distinct sets B_1, B_2 and B_3. So we obtain

$$\sum_{\substack{i,j,k\in S \\ i<j \\ i\neq k\neq j}} x_{ikj} = \binom{|S|}{3} - |B_1| \cdot |B_2| \cdot |B_3|.$$

Computing the minimum value of the right-hand side is equivalent to maximize $|B_1| \cdot |B_2| \cdot |B_3|$ subject to $B_1 \dot\cup B_2 \dot\cup B_3 = S,\ B_1, B_2, B_3 \subseteq S, B_1 \cap B_2 \cap B_3 = \emptyset$, and thus we obtain the desired result. □

In addition, we are able to adapt inequalities (39) to the case with $|S| = 3$ in the following way

$$\sum_{\substack{i,j\in S\\ i<j}} \left(x_{(n+2)ij} + x_{(n+2)ji}\right) + \sum_{i\in S} y_i \geq 2, \qquad S \in [n] \setminus \{s_M\}, |S| = 3. \tag{41}$$

Let $S \subseteq [n] \setminus \{s_M\}, |S| = 3$, and a s_M-T-row layout be given. Inequalities (41) are satisfied if at least two departments of the set S are in row 1. So let two (three) departments be in row 2. Then, $\sum_{\substack{i,j\in S\\ i<j}} x_{(n+2)ij} + x_{(n+2)ji} = 1 \ (= 3)$ and inequalities (41) are satisfied.

Next we consider the special case of inequalities (40) with $|S| = 4, S \subseteq [n] \setminus \{s_M\}$, and we obtain $\sum_{\substack{i,j,k\in S\\ i\neq k\neq j\\ i<j}} x_{ikj} \geq 2$. This lower bound is tight if and only if each of the sets B_1, B_2 and B_3 contains at least one department of the set S, because otherwise $S \subseteq B_a \cup B_b, a, b \in \{1, 2, 3\}$, and hence the sum of the betweenness variables of the departments in S equals 4. So we present further lower bounds for the case with $|S| = 4$.

Proposition 6. *Let $s_M \in [n]$ and $S \subseteq [n] \setminus \{s_M\}, |S| = 4$. Then, the following inequalities are valid for the s_M-TRFLP*

$$\sum_{\substack{i,j,k\in S\\ i<j\\ i\neq k\neq j}} x_{ikj} + \sum_{i\in S} y_i \geq 4, \tag{42}$$

$$\sum_{\substack{i,j,k\in S\\ i<j\\ i\neq k\neq j}} x_{ikj} - \sum_{i\in S} y_i \geq -1, \tag{43}$$

$$\sum_{\substack{i,j,k\in S\\ i<j\\ i\neq k\neq j}} x_{ikj} - \sum_{s\in\{s_M,n+2\}} 2\left(x_{spq} + x_{sqp} + x_{spt} + x_{stp}\right) \geq 0, \quad p, q, t \in S, |\{p, q, t\}| = 3, q < t, \tag{44}$$

$$\sum_{\substack{i,j,k\in S\\ i<j\\ i\neq k\neq j}} x_{ikj} + \sum_{\substack{i,j\in S\\ i<j}} x_{is_Mj} - \sum_{i\in S} y_i + x_{s_Mqp} + x_{s_Mpq} \geq 1, \qquad p, q \in S, p < q. \tag{45}$$

Proof. Let $S = \{i, j, k, h\} \subseteq [n] \setminus \{s_M\}, |S| = 4$, and let a s_M-T-row layout be given. Recall that B_1 (B_2) denotes the set of departments left (right) to s_M and B_3 denotes the set of departments in row 2. If $\sum_{p\in S} y_p \geq 2$, then inequalities (42) are satisfied by inequalities (40) with $|S| = 4$. Otherwise, at most one department is assigned to row 1. Then, $S \subseteq B_a \cup B_3, a \in \{1, 2\}$, and thus the sum of the corresponding betweenness variables equals 4. Inequalities (43) are satisfied by inequalities (40) if at most three departments are assigned to row 1. Otherwise, all departments are in row 1 and inequalities (43) are satisfied. Inequalities (44) ensure that if three departments lie in the same set $B_a, a \in \{1, 2, 3\}$, then the sum of the corresponding betweenness variables is equal to 4. Otherwise, inequalities (44) are satisfied by inequalities (40).

It remains to consider inequalities (45), let $p, q \in S, p < q$. If at most one department is assigned to row 1, then inequalities (45) are satisfied by inequalities (40). So let two (three) departments be in row 1, and we assume at first, that the departments are, w. l. o. g., left to s_M. Then, $S \subseteq B_1 \cup B_3$ and the sum of the corresponding betweenness variables equals 4. Otherwise we obtain $\sum_{\substack{i,j\in S\\ i<j}} x_{is_Mj} = 1$ ($\sum_{\substack{i,j\in S\\ i<j}} x_{is_Mj} = 2$), and thus inequalities (45) are satisfied. It remains to consider the case where all departments are assigned to row 1. Then, the sum of the corresponding betweenness variables equals 4 and in addition, we get by inequalities (11) $x_{ps_Mq} + x_{s_Mpq} + x_{s_Mqp} = 1$, which proves the desired result. □

In the next proposition, we show that inequalities from the SRFLP can be used for the s_M-TRFLP, although an extension of the betweenness variables is used. The slightly adapted inequalities (46)–(47) and inequalities (49) were introduced in [3] for the SRFLP.

Proposition 7. *Let $s_M \in [n]$. The following inequalities are valid for the s_M-TRFLP*

$$-x_{ihj} + x_{ihk} + x_{jhk} \geq 0, \qquad i,j,k,h \in [n] \setminus \{s_M\}, |\{i,j,k,h\}| = 4, \tag{46}$$

$$x_{ihj} + x_{ihk} + x_{jhk} \leq 2, \qquad i,j,k,h \in [n], |\{i,j,k,h\}| = 4, i < j < k, h \neq \{s_M\}, \tag{47}$$

$$x_{is_Mj} + x_{is_Mk} - x_{s_Mkj} - x_{s_Mjk} - y_i \leq 0, \quad i,j,k \in [n] \setminus \{s_M\}, i \neq j, j < k. \tag{48}$$

Let $\beta \geq 6$ be an even integer, $k \in S \subseteq [n] \setminus \{s_M\}$ and let $S_1, S_2 \subseteq S \setminus \{k\}$ with $|S_1| = \frac{\beta}{2}$ such that $S_1 \dot\cup S_2 = S \setminus \{k\}, S_1 \cap S_2 = \emptyset$. Then, the following inequalities

$$\sum_{\substack{i,j \in S_1 \\ i<j}} x_{ikj} + \sum_{\substack{i,j \in S_2 \\ i<j}} x_{ikj} - \sum_{\substack{i \in S_1 \\ j \in S_2}} x_{ikj} \leq 0, \tag{49}$$

are valid for the s_M-TRFLP.

The proof of the correctness of inequalities (49) for the s_M-TRFLP is an extension of the proof of [41] for the correctness of inequalities (49) for the SRFLP.

Proof. Let $i,j,k,h \in [n] \setminus \{s_M\}, |\{i,j,k,h\}| = 4$, and a s_M-T-row layout be given. Recall that B_1 (B_2) denotes the set of departments in row 1 left (right) to s_M and B_3 the set of departments in row 2. If $i,j,k,h \in B_a \cup B_b, a,b \in \{1,2,3\}$, inequalities (46) are satisfied since these inequalities are valid for the SRFLP. If $x_{ihj} = 0$, then inequalities (46) are redundant, so it remains to consider the case $x_{ihj} = 1$. We assume, w. l. o. g., $i,h \in B_a, j \in B_b, a,b \in \{1,2,3\}, a \neq b$, and $k \in B_c, c \in \{1,2,3\} \setminus \{a,b\}$. It follows immediately that $x_{ihk} = 1$, see inequalities (19), and thus inequalities (46) are satisfied.

Next we consider inequalities (47). Let a s_M-T-row layout and $i,j,k,h \in [n]$, $|\{i,j,k,h\}| = 4, i < j < k, h \neq s_M$, be given. We shift the departments in the set $B_2 \cap \{i,j,k,h\}$ to the left of s_M without changing the order of the departments $B_1 \cup (B_2 \cap \{i,j,k,h\})$. By this method we only increase the sum of the corresponding betweenness variables. Then, $i,j,k,h \in B_1 \cup B_3$ and inequalities (47) are satisfied because these inequalities are valid for the SRFLP. Now we investigate inequalities (48). Let i lie in row 1, let s_M lie between i and j and let s_M lie between i and k. Then, k lies between s_M and j or j lies between k and s_M, see inequalities (48). Otherwise, inequalities (48) are redundant.

Next we consider inequalities (49). We are given a s_M-T-row layout and let $k \in S \subseteq [n] \setminus \{s_M\}$ lie in row 1, w. l. o. g., left to s_M. Let $\beta \geq 6$ be an even integer and let $S_1, S_2 \subseteq S \setminus \{k\}, |S_1| = \frac{\beta}{2}, S_1 \dot\cup S_2 = S \setminus \{k\}, S_1 \cap S_2 = \emptyset$.

Then, we denote by ℓ^1 (ℓ^2) the number of departments in S_1 (S_2) left to k. Let r^1 (r^2) denote the number of departments right to k in row 1 and let u^1 (u^2) denote the number of departments in S_1 (S_2) in row 2. Then

$$\begin{aligned} &\sum_{\substack{i,j \in S_1 \\ i<j}} x_{ikj} + \sum_{\substack{i,j \in S_2 \\ i<j}} x_{ikj} - \sum_{\substack{i \in S_1 \\ j \in S_2}} x_{ikj} \\ &= \ell^1(r^1 + u^1) + \ell^2(r^2 + u^2) - \ell^1(r^2 + u^2) - \ell^2(r^1 + u^1) \\ &= (\ell^1 - \ell^2)(r^1 + u^1 - r^2 - u^2) \leq 0, \end{aligned}$$

where the last inequality is satisfied since $\ell^1 + r^1 + u^1 = 1 + \ell^2 + r^2 + u^2$. The proof is similar if k lies in row 2. □

3.4 Heuristic approaches

Let $s_M \in [n]$. In this section we describe heuristic approaches for the s_M-TRFLP which are applied in Algorithm 1. We use five heuristics to compute start layouts for the s_M-TRFLP. Afterwards we apply exchange approaches to further improve these start layouts.

We fix the center of $s_M \in [n]$ on position p_M in row 1, and then iteratively add the remaining departments to the layout, see [30] for a related heuristic approach for the DRFLP. Let B_1 (B_2) denote the set of departments left (right) to s_M and B_3 the set of departments in row 2 in the current constructed layout. We set $\bar{\ell}_i := \frac{\ell_{s_M}}{2} + \sum_{j \in B_i} \ell_j, i = 1, 2,\ \bar{\ell}_3 := w^T_{path} + \sum_{j \in B_3} \ell_j$ and $a := \arg\min\{\bar{\ell}_1, \bar{\ell}_2, \bar{\ell}_3\}$. In the first heuristic we sort the departments $[n] \setminus \{s_M\}$ in ascending order according to their lengths. The sorted departments are sequentially added space-free to the layout, if $a = 1$ at the leftmost position in row 1 and otherwise at the rightmost position in row 1 or row 2, respectively, see Algorithm 2.

Algorithm 2: Heuristic approach for the s_M-TRFLP

Input : $s_M \in [n]$, $S = (s_1, \dots, s_{n-1})$ is a sorting of the departments $[n] \setminus \{s_M\}$ with lengths $\ell_i \in \mathbb{R}_+, i \in [n]$, $w^T_{path} \in \mathbb{R}_{\geq 0}$.

Output : positions p_{s_k} and row assignment $r_{s_k}, s_k \in S$, of the departments $[n] \setminus \{s_M\}$.

Initialize $(\bar{\ell}_1, \bar{\ell}_2, \bar{\ell}_3) \leftarrow (\frac{\ell_{s_M}}{2}, \frac{\ell_{s_M}}{2}, w^T_{path})$.
for $k = 1, \dots, n-1$ **do**
  Choose $a \in \arg\min\{\bar{\ell}_o \colon o \in \{1, 2, 3\}\}$.
  if $a = 1$ **then**
    $p_{s_k} \leftarrow p_M - \bar{\ell}_a - \frac{\ell_{s_k}}{2}$.
    $\bar{\ell}_a \leftarrow \bar{\ell}_a + \ell_{s_k}$.
    $r_{s_k} \leftarrow 1$.
  if $a = 2$ **then**
    $p_{s_k} \leftarrow p_M + \bar{\ell}_a + \frac{\ell_{s_k}}{2}$.
    $\bar{\ell}_a \leftarrow \bar{\ell}_a + \ell_{s_k}$.
    $r_{s_k} \leftarrow 1$.
  if $a = 3$ **then**
    $p_{s_k} \leftarrow \bar{\ell}_a + \frac{\ell_{s_k}}{2} - w^T_{path}$.
    $\bar{\ell}_a \leftarrow \bar{\ell}_a + \ell_{s_k}$.
    $r_{s_k} \leftarrow 2$.
return p_{s_k}, r_{s_k}, $s_k \in S$.

In the following heuristic approaches we assign the departments in the following way. Let, w. l. o. g., the departments $1, \dots, j \in [n] \setminus \{s_M\}$ be assigned to the s_M-T-row layout and let, $R_1 = (1, 2, \dots, s_M, \dots, i)$, $i \in [n] \setminus \{s_M\}, i \leq j$, ($R_2 = (i+1, \dots, j)$) denote the set of departments assigned to row 1 (row 2), as well as the sorting of the departments in row 1 (row 2). We assign $h \in [n], h > j$, on all possible positions in R_1 and R_2 and then we choose a layout with minimal objective value, i. e., h is assigned at its current best position (note that the center of h is not assigned on position p_M in row 1).

In our second heuristic approach we sort the departments in ascending order according to their lengths and, in contrast to the first heuristic, in each step a department is placed at its current best position. Next we divide the departments $[n] \setminus \{s_M\}$ into two sets depending whether $w_{is_M} = 0$ or $w_{is_M} > 0$, $i \in [n] \setminus \{s_M\}$, i. e., $S_1 := \{i \in [n] \setminus \{s_M\} \colon w_{is_M} = 0\}$ and $S_2 := \{i \in [n] \setminus \{s_M\} \colon w_{is_M} > 0\}$. Then, we sort the departments in each set in decreasing order according to their relative weights, i. e., $\sum_{\substack{j \in [n] \\ i \neq j}} \frac{w_{ij}}{\ell_i}, i \in [n] \setminus \{s_M\}$. In the third (fourth) heuristic we assign the departments in the set S_2 (S_1), and afterwards the departments in the set S_1 (S_2) in the determined order to the layout. In each step the considered department is assigned at its current best position.

In [35] a good or optimal single-row layout is given and according to different rules a double-row layout is constructed. Given an optimal or near-optimal single-row layout, in the fifth heuristic we assign departments in that order space-free to row 1, where the center of s_M is fixed on position p_M in row 1. In the first step we arrange one department to row 2 which reduces the objective value of the current layout at most. Then, in further steps, we choose one department from row 1 which reduces the objective value at most when we assign it on its best position in row 2, and we arrange it at that position. We apply these steps until we do not improve the objective value by rearranging one department from row 1 to row 2.

After determining start layouts, we use a 1-opt and a 2-opt improvement heuristic. At first we apply a 1-opt heuristic where we place in each step one department on its current best position. Afterwards, in our 2-opt heuristic, we simply exchange the position of two departments. Whenever we obtain a better solution during the opt heuristics, we swap the departments and we apply the k-opt heuristic until the layout is k-optimal, $k \in \{1, 2\}$.

3.5 Separation

In this section we describe three variants for branch-and-cut algorithms based on inequalities (7)–(21) and inequalities (29)–(49). In the first variant we include inequalities (7),(9)–(16), (18)–(19), (35) and inequalities (39) with $S = [n] \setminus \{s_M\}$ from the beginning. Due to equations (8) we can eliminate $n-1$ ordering variables by setting

$$z_{s_M i} = y_i - z_{i s_M}, \qquad i \in [n] \setminus \{s_M\}.$$

The sorting of the departments in our symmetry breaking constraints (29)–(30) or, if $\frac{\ell_{s_M}}{2} = w_{path}^T$, inequalities (31)–(34), is determined in the following way. We choose a heuristically determined s_M-T-row layout with minimal objective value, see Section 3.4, and denote B_1 (B_2) as the set of departments left (right) to s_M and let B_3 denote the set of departments in row 2. Then, we compute $v(i) := \sum_{\substack{j \in B_i \\ k \in [n] \\ k \neq j}} \frac{w_{jk}}{\ell_k}, i \in \{1, 2\}$, and we set $a := \arg\min\{v(1), v(2)\}$. In inequalities (29)–(30) we assign the departments in the set B_a first, then $B_b, b \in \{1, 2\} \setminus \{a\}$, and at last B_3 where the departments in each set $B_m, m \in \{1, 2, 3\}$, are sorted in ascending order according of their positions, i. e., we start with the leftmost department of each set.

In the next part we describe the separation of the remaining inequalities. We only add inequalities for a set $S \subseteq [n]$ which are violated (at least) by the constant 0.4. We separate inequalities (20)–(21) and inequalities (41) by brute-force enumeration.

Given a relaxation $\bar{x}, \bar{z}$ and $\bar{y}$. Preliminary tests indicate to separate equations (17) if $i, j \in [n] \setminus \{s_M\}, i < j$, lie in the same row, hence, due to equations (16), we add equations (17) if $\bar{x}_{i s_M j} + \bar{x}_{s_M i j} + \bar{x}_{s_M j i} - \bar{x}_{(n+2) i j} - \bar{x}_{(n+2) j i} < -0.4$.

Let $i, j, k \in [n] \setminus \{s_M\}, i < j < k$, be given. Inequalities (36)–(38) are mainly used for computing lower bounds for $x_{ikj} + x_{ijk} + x_{jik}$, so, given a relaxation $\bar{x}, \bar{y}$, we compute

$$\max \begin{cases} 1 - \bar{x}_{i s_M j} - \bar{x}_{i s_M k} - \bar{x}_{j s_M k}, \\ -2 + \bar{y}_i + \bar{y}_j + \bar{y}_k, \\ \bar{y}_k - \bar{x}_{i s_M k} - \bar{x}_{j s_M k}, \\ \bar{y}_j - \bar{x}_{i s_M j} - \bar{x}_{k s_M j}, \\ \bar{y}_i - \bar{x}_{k s_M i} - \bar{x}_{j s_M i}. \end{cases}$$

Then, we add one of the five inequalities (36)–(38) with $S = \{i, j, k\}$ where the maximum value above is attained. Recall that we only add the inequality if it is violated by at least 0.4.

In variant 2 of our separation strategy we use the same separation strategy as in variant 1 but we neglect inequalities (36)–(37) in order to verify that these inequalities improve the performance of our branch-and-cut algorithm.

In variant 3 we use the same strategy as in variant 1, and additionally we add inequalities (40) with $S = [n] \setminus \{s_M\}$ from the beginning and we separate inequalities (40) by brute-force enumeration for $S \subseteq [n] \setminus \{s_M\}$ with $|S| = 4, 5, 6, 7$, where we only add inequalities which are violated by at least $1, 2, 4$ or 8, respectively. We separate inequalities (42)–(48) by brute-force enumeration where we only add inequalities (42)–(45) and inequalities (46)–(48) violated by at least 1 or 0.5, respectively. We separate inequalities (49) with $\beta = 6$ by complete brute-force enumeration and we only add inequalities violated by at least 2.

4 The Multi-Bay Facility Layout Problem with three rows

The current known MILP models for the MBFLP and its extensions, are not able to solve instances with 12 departments in reasonable time [29, 59], so we describe the adaption of our MILP model for the s_M-TRFLP, $s_M \in [n]$, to the 3-BFLP.

Given a T-row instance, we add an additional dummy department $n+3$ with length $\ell_{n+3} = 2w^B_{path}$ and weights $w_{i(n+3)} = w_{(n+3)i} = 0, i \in [n+2]$, to our model and we fix the center of the dummy department $n+3$ on position p_M in row 1, the obtained problem is denoted by ($(n+3)$-TRFLP). We use the following connecting between the 3-BFLP and the $(n+3)$-TRFLP. Let an optimal $(n+3)$-T-row layout be given with $w^T_{path} = 0$. We assign the departments in the T-row layout to the left (right) of $n+3$ to row 1 (row 3) in the 3-Bay layout and the departments in row 2 in the T-row layout to row 2 in the 3-Bay layout without changing the order of the departments in the same row. Hence, we obtain an optimal 3-Bay layout and vice versa. An immediate consequence is the following

Proposition 8. *Given a 3-BFLP instance with $w^B_{path} \in \mathbb{R}_{\geq 0}$ and a dummy department with length $\ell_{n+3} = 2w^B_{path}$ and weights $w_{i(n+3)} = w_{(n+3)i} = 0, i \in [n+2]$. Then, the 3-BFLP is equivalent to the $(n+3)$-TRFLP with $w^T_{path} = 0$.*

Now we compare the optimal values of the SRFLP, the TRFLP and the 3-BFLP. Note first that if $w^T_{path} \leq w^B_{path}$, then the optimal value of the TRFLP is less than or equal to the optimal value of the $(n+3)$-TRFLP with $w^T_{path} = 0$, and thus smaller than or equal to the optimal value of the 3-BFLP by Proposition 8. In the following, we consider an instance which cannot be divided into two or more smaller independent instances. If $w^T_{path} \leq w^B_{path}$ and $w^B_{path} \geq 1$, then the optimal value of the TRFLP is equal to the optimal value of the 3-BFLP if and only the optimal value of the TRFLP is equal to the optimal value of the SRFLP. If $w^T_{path} < \frac{\max_{i \in [n]} \ell_i}{2}$, then the optimal value of the TRFLP is smaller than the optimal value of the SRFLP. So for $w^T_{path} = 0$ and $w^B_{path} = 1$, the optimal value of the TRFLP is smaller than the optimal value of the 3-BFLP. However, one can construct instances such that the optimal value of the TRFLP is equal to the optimal value of the SRFLP if $w^T_{path} \geq \frac{\max_{i \in [n]} \ell_i}{2}$.

One main difference between solving the TRFLP and the 3-BFLP with our approach is that for the 3-BFLP only one MILP model has to be solved. We adapt our MILP model for the s_M-TRFLP, $s_M \in [n]$, to the $(n+3)$-TRFLP with $w^T_{path} = 0$ and we only describe the differences in the following. We use the z, y and x variables in the same manner as above including the dummy department $n+3$, i. e., we define z_{ij} for $i, j \in [n] \cup \{n+3\}, i \neq j$, and x_{ikj} with $i, j, k \in [n+3], i < j, |\{i, j, k\}| = 3$. We neglect inequality (35). The distances between the $i \in [n]$ and $j \in [n], i < j$, are calculated as described in (23), see (22) for the calculation of the excluded constant. We are able to use symmetry breaking constraints (29)–(30) from the s_M-TRFLP and the departments are sorted as described in Section 3.5.

To further strengthen our branch-and-cut algorithm, we use the following inequalities for the $(n+3)$-TRFLP where inequalities (50) are used in [6, 76] for the DRFLP

$$z_{ij} + z_{ji} + z_{ik} + z_{ki} + z_{jk} + z_{kj} \geq 1, \qquad i, j, k \in [n], i < j < k, \tag{50}$$

$$x_{ikj} + x_{ijk} + x_{jik} + x_{i(n+3)j} - y_k \leq 1, \qquad i, j, k \in [n], i < j < k. \tag{51}$$

Given three departments, at least two of them lie in the same row, see inequalities (50). Let $i, j, k \in [n], i < j < k$, and let $y_k = 0$, then either the sum of the corresponding three betweenness variables equals zero or $n + 3$ does not lie between i and j, see inequalities (51). Inequalities (51) are redundant if $y_k = 1$, see inequalities (21).

In order to get upper bounds on the optimal value of the $(n + 3)$-`TRFLP`, we use the first three heuristic approaches for the s_M-`TRFLP`, $s_M \in [n]$, presented in Section 3.4. Since $w_{i(n+3)} = w_{(n+3)i} = 0, i \in [n]$, the fourth heuristic equals the third heuristic so we exclude it. In the fifth heuristic, we arrange the dummy department $n + 3$ to the right of the $\lfloor \frac{n}{2} \rfloor$ leftmost department in a given single-row layout. Then, we proceed as described in the fifth heuristic.

We mainly use the same separation strategy for the $(n+3)$-`TRFLP` as for the s_M-`TRFLP`, $s_M \in [n]$, so we only describe the differences here. We include inequalities (20) from the beginning if $w^B_{path} > 1$, otherwise we separate inequalities (20) as described in Section 3.5. We separate inequalities (50) by brute-force enumeration. Given a relaxation $\bar{x}, \bar{y}$ and $i, j, k \in [n], i < j < k$, we separate inequalities (21) and inequalities (51) in the following way. We calculate

$$\min \begin{cases} 1 \\ \bar{y}_k - \bar{x}_{i(n+3)j}, \\ \bar{y}_j - \bar{x}_{i(n+3)k}, \\ \bar{y}_i - \bar{x}_{j(n+3)k}, \end{cases}$$

and we add one of the four corresponding inequalities where the minimum value is attained.

5 Computational Experiments

In this section we present our computational results. The computational experiments are implemented in C++, and we use Cplex 12.9 as an `MILP` Solver [57]. All results were conducted on a 2.10GHz quad-core using Virtual Box 6 running on Debian GNU/Linux 8 in single processor mode. In all tests Cplex generated cuts are not added and we set $w^T_{path} + 1 = w^B_{path}$, as discussed in the introduction.

5.1 Computational Results for the `TRFLP` and the `3-BFLP`

In this paper we focus on the `TRFLP` and the `3-BFLP`, so applying Proposition 1 is left for future work. At first, we describe the usage of our heuristic approaches in Algorithm 1. For $s_M \in [n]$, we determine five start layouts with our heuristics for the s_M-`TRFLP` and we apply a 1-opt and 2-opt improvement heuristic on each start layout. Our heuristic approach for the `TRFLP` simply chooses a s_M-T-row layout with minimal objective value, $s_M \in [n]$, and the minimal objective value is used as an upper bound in Algorithm 1. The single-row instances in the fifth heuristic were solved to optimality using the model summarized in Section 2. We generate new instances which are larger than literature instances in the same way as described in [7]. All instances are available from the author.

In the first two columns in Table 1 the instance, whereby the number denotes the number of departments in that instance, and its source is given. In the next three columns we compare the optimal solution values of the `SRFLP`, the `3-BFLP` and the `TRFLP`, and we write "TL" if the time limit of 8 hours is exceeded. If the `3-BFLP` is not solved to optimality, the obtained lower bound is displayed and marked with " ' ". Our heuristically determined upper bounds for the `TRFLP` (`3-BFLP`) are given in the sixth (seventh) column denoted by "$\text{H}_{\texttt{TRFLP}}$" ("$\text{H}_{\texttt{3-BFLP}}$"). Due to computational accuracy, some $z_{ij} \in [0, 1], i, j \in [n] \setminus \{s_M\}, j > i$, achieve values with very small gaps to zero or one. Hence, the values in Table 1, Table 3 and Table 4 are optimal neglecting computational accuracy. The corresponding values are rounded up such that $2 \cdot OPT(\mathcal{I}) \in \mathbb{Z}$ for T-row and 3-Bay instances $\mathcal{I}$ since $w_{ij} \in \mathbb{Z}, i, j \in [n], i < j$.

Instances	Source	SRFLP	3-BFLP	TRFLP	$\text{H}_{\text{3-BFLP}}$	H_{TRFLP}
$Am11a$	[9]	10630.5	8795.5	8407.0	8814.5	8411.5
$Am11b$	[9]	7375.5	6021.5	5665.0	6021.5	5667.5
$Am12a$	[5, 6]	2901.0	2508.0	2354.5	2515.0	2354.5
$Am12b$	[5, 6]	3280.5	2691.5	2539.5	2697.5	2548.5
$Am13a$	[5]	4902.5	4021.5	3836.0	4204.5	3836.0
$Am13b$	[5]	5698.0	4529.0	4362.5	4529.0	4362.5
$Am14\_1$	[43]	5481.5	4560.5	4350.5	4565.5	4358.0
$Am14a$	[76]	5673.0	4687.0	4446.5	4734.0	4448.0
$Am14b$	[76]	5595.0	4665.0	4430.5	4683.0	4433.0
$Am15$	[1]	6305.0	5291.0	5071.0	5294.0	5079.0
$HK15$	[49]	33220.0	26494.0	26124.0	26495.0	26125.0
$P16a$	[7]	14829.0	12287.5	11943.0	12326.0	11943.0
$P16b$	[7]	11878.5	9781.0	9469.5	9784.5	9469.5
$P17a$	new	14436.5	11852.0	11524.5	11888.5	11528.0
$P17b$	new	15682.0	12691.5	12317.0	12874.0	12389.0
$Am17$	[2]	9254.0	7647.0	7315.0	7690.0	7318.0
$P18a$	new	16118.5	12022.0’	TL	12863.5	12516.0
$P18b$	new	17716.5	12972.5’	TL	14616.5	14072.0
$Am18$	[2]	10650.5	7990.5’	8413.5	8835.5	8413.5

Table 1: Optimal values of the SRFLP, the 3-BFLP and the TRFLP as well as heuristically determined upper bounds are displayed for instances from the literature with $w^T_{path} = 0, w^B_{path} = 1$. We write “TL” if the time limit of 8 hours is exceeded. Lower bounds for the 3-BFLP are marked with “ ’ ” if the time limit is exceeded or if we run out of memory storage. The optimal value of the TRFLP is up to 6.1 % smaller than the optimal value of the 3-BFLP and up to 23.4 % than the optimal value of the SRFLP.

In Table 1 we set $w^T_{path} = 0$ and $w^B_{path} = 1$. The optimal value of the TRFLP (3-BFLP) is between 18.8 % and 23.4 % (13.5 % and 20.5 %) smaller than the optimal value of the SRFLP, so the optimal values of the TRFLP and the 3-BFLP are significantly smaller than the optimal value of the SRFLP. Further, the optimal value of the TRFLP is between 1.4 % and 6.1 % smaller than the optimal value of the 3-BFLP, see Table 1. So for these instances the TRFLP is preferable to the 3-BFLP and the SRFLP. These reductions of the optimal values are remarkable since factories are built for a long period, and the rearrangement of the departments is expensive. For the TRFLP, the heuristic derives gaps with less than 1 % to the optimal solution values and five optimal layouts. In addition, in eleven instances the department with its center arranged first on position p_M in Algorithm 1 for the TRFLP has its center position on p_M in the calculated optimal T-row layout. The heuristics were computed in less than one second. For the 3-BFLP the heuristic derives two optimal layouts and for the instance $Am13a$ the gap is around 4.4 %. For the remaining instances, the gaps are less than 1.5 %. The 3-Bay instances with 18 departments are not solved to optimality, but we obtain lower bounds with tight gaps to the heuristically determined upper bounds, i. e., the gaps are between 4.0 % and 7.9 %.

In Table 2 we compare the running times of several approaches for the 3-BFLP and the TRFLP, given in sec, min:sec and h:min:sec. In column 2 we display the running time of the current best exact approach for the 3-BFLP by [43] where the 3-BFLP is solved with fixed row assignment and we enumerate over all distinguishable row assignments as suggested in [43]. Adapting this approach to the s_M-TRFLP and using Algorithm 1, we obtain an optimal solution for the TRFLP and the running time is displayed in column 4. In both approaches, a heuristically determined upper bound is used. In the fifth, sixth and seventh column we display the running time of our separation variant $i \in [3]$, see Section 3.5, denoted by “MILP_i”. In the eights column we mainly use the same separation strategy as in variant 1, but we use standard linearization, see inequalities (59)–(61) in the appendix, to couple our new extended betweenness variables to the ordering, assignment and betweenness variables containing dummy departments or s_M. This variant is denoted by variant 4 and we do not use the transitivity inequalities (19)–(20). Additionally, we set up a MILP model for the s_M-TRFLP which consists of betweenness variables which are only equal to one if all departments are in the same row (in this version, betweenness variables are known in the literature). The distances between $i \in [n] \setminus \{s_M\}$ and $j \in [n] \setminus \{s_M\}, i < j$, are calculated via big-M-constraints and the betweenness variables are coupled via standard linearization to the ordering variables. Note that the transitivity inequalities (19)–(20) are not valid for this MILP model. We refer to the appendix for a description and the running time of Algorithm 1 using this MILP model is summarized in the ninth column. We use a time limit of 8 hours and we write "TL" if the time limit is exceeded.

Considering the TRFLP, variant 1 clearly outperforms the other approaches because all considered T-row instances with 17 departments and even one instance with 18 departments were solved to optimality within the time limit of 8 hours. The second best approach is variant 2 where one T-row instance with 17 departments is solved to optimality. Note that inequalities (36)–(37), which are neglected in variant 2, are not valid for betweenness variables used in the literature, i. e., betweenness variables which can only be equal to one if the three departments are in the same row. The running time of our approach using variant 2 is for almost all instances with at least 14 departments more than twice as high as the running time of our approach using variant 1. Thus, the inequalities (36)–(37) significantly improve the performance of our algorithm.

The s_M-TRFLP model summarized in the appendix is significantly weaker than variant 1 because standard linearization is used to couple the betweenness variables and the ordering variables and big-M-constraints are used to calculate distances between departments in distinct rows. Using this approach, only instances with up to 15 departments were solved to optimality within the time limit of 8 hours while in the approach of variant 1 instances with 15 departments were solved to optimality in less than 20 minutes. Considering variant 4, the performance is even worse than the performance of the s_M-TRFLP model summarized in the appendix. This shows that the transitivity inequalities (19)–(20) significantly improve the performance of our approach in comparison to the standard linearization. In comparison to variant 1, in variant 3 significantly more inequalities are added, and thus the gap at the root node is smaller. However, this increases the running time in the further branching steps, and hence variant 3 is slower than variant 1. In conclusion, our experiments show that our extension of the betweenness variables in combination with the transitivity inequalities (19)–(20) and inequalities (36)–(37) significantly improve the performance of our branch-and-cut algorithm. For all instances which were solved to optimality, our best approach for the TRFLP is faster than our approach for the 3-BFLP.

The enumerative approach of [43] is only able to solve 3-Bay instances with up to 14 departments and T-row instances with up to 13 departments within 8 hours. For larger instances, the running time is exceeded. In the corresponding TRFLP approach one enumerates over each department with its center position fixed on position p_M in row 1, and therefore one has to consider more MILP models than in the 3-BFLP. Thus, the running time is higher. In contrast, our approach (variant 1) is able to solve T-row instances with 13 departments in a few minutes and 3-Bay instances with 14 departments in less than 15 minutes. So for both problems we clearly outperform the approach of [43].

The inter-row transport is more costly than the inner-row transport, see, e. g.,[25, 67]. Therefore, we investigate the effect of enlarging $w^T_{path} + 1 = w^B_{path}$ on the optimal value of the TRFLP and the 3-BFLP. In the previous results we observed that our approaches clearly outperformed the enumerative approaches, so in Table 3 and Table 4 we apply only our best fitting approaches, i. e., variant 1 for the TRFLP. The notation in Table 3 and Table 4 is similar to the notation above and we denote by "$\text{Time}_{\texttt{3-BFLP}}$" and "$\text{Time}_{\texttt{TRFLP}}$" the running time of our approach for the 3-BFLP and the TRFLP, respectively, and by "TRFLP" and "3-BFLP" the objective value of the TRFLP and 3-BFLP.

We consider literature instances in Table 3 with $w^T_{path} + 1 = w^B_{path} = 4$ and $w^T_{path} + 1 = w^B_{path} =$

Instances	$\text{Enu}_{\texttt{3-BFLP}}$	$\texttt{MILP}_{\texttt{3-BFLP}}$	$\text{Enu}_{\texttt{TRFLP}}$	$\texttt{MILP}_1$	$\texttt{MILP}_2$	$\texttt{MILP}_3$	$\texttt{MILP}_4$	$\texttt{MILP}_{\text{same-row}}$
$Am11a$	3:26	48	9:31	33	54	34	1:04	1:37
$Am11b$	3:35	1:19	10:11	20	41	16	41	1:27
$Am12a$	13:26	1:53	38:29	51	1:29	1:20	2:44	6:18
$Am12b$	13:30	1:01	40:19	42	1:15	50	2:35	6:40
$Am13a$	53:46	4:16	2:58:52	2:17	3:05	3:56	14:03	24:18
$Am13b$	53:34	9:31	3:01:51	2:17	2:58	5:02	10:16	18:33
$Am14\_1$	3:45:33	1:28:24	TL	9:11	16:27	24:47	1:16:10	1:17:49
$Am14a$	3:28:30	13:23	TL	6:00	15:32	12:13	52:54	58:26
$Am14b$	3:28:47	13:37	TL	6:51	15:59	17:34	1:22:07	1:17:58
$Am15$	TL	25:24	TL	15:31	33:45	1:49:33	3:20:07	3:35:25
$HK15$	TL	17:51	TL	13:48	20:05	58:54	3:54:52	2:56:30
$P16a$	TL	44:00	TL	1:12:31	3:27:04	TL	TL	TL
$P16b$	TL	1:21:35	TL	1:09:58	2:42:40	6:04:38	TL	TL
$P17a$	TL	3:51:48	TL	3:32:27	TL	TL	TL	TL
$P17b$	TL	6:31:46	TL	3:37:59	TL	TL	TL	TL
$Am17$	TL	4:58:37	TL	2:19:20	5:53:06	TL	TL	TL
$P18a$	TL	TL	TL	TL	TL	TL	TL	TL
$P18b$	TL	5:23:22†	TL	TL	TL	TL	TL	TL
$Am18$	TL	TL	TL	6:01:58	TL	TL	TL	TL

Table 2: Running times are given in sec, min:sec or in h:min:sec for instances from the literature with $w^T_{path} = 0, w^B_{path} = 1$. We write "TL" if the time limit of 8 hours is exceeded and the running time is marked with "†" if we run out of memory storage. For the TRFLP, variant 1 delivers the fastest approach. We clearly outperform the enumerative approach of [43] for the TRFLP and the 3-BFLP.

11. By enlarging w^B_{path}, the running time of the 3-BFLP approach is for some instances significantly increased and for some instances significantly decreased. In contrast to the case of $w^B_{path} = 1$, we are able to solve one 3-Bay instance with 18 departments to optimality. Note that for larger values of w^B_{path} we run more often out of memory storage. For the TRFLP the running time is only slightly influenced (neglecting instance $P17b$) by enlarging w^T_{path}, so our approach works well with large values of w^T_{path} and w^B_{path}. For $w^T_{path} = 3$ and $w^B_{path} = 4$ ($w^T_{path} = 10$ and $w^B_{path} = 11$), the optimal value of the TRFLP is between 2.4 % and 11.4 % (3.7 % and 12.2 %) smaller than the optimal value of the 3-BFLP. So by enlarging w^T_{path} and w^B_{path}, the gap between the optimal value of the TRFLP and the optimal value of the 3-BFLP is increased in our computational results. The optimal value of the TRFLP with $w^T_{path} = 3$ ($w^T_{path} = 10$) is between 1.2 % and 7.7 % (3.9 % and 18.3 %) greater than the optimal value of the TRFLP with $w^T_{path} = 0$. The optimal value of the 3-BFLP with $w^B_{path} = 4$ is between 2.3 % and 13.0 % greater than the optimal value of the 3-BFLP with $w^B_{path} = 0$ and the 3-BFLP with $w^B_{path} = 11$ has 13 times the same optimal value as the SRFLP. So with $w^B_{path} \geq 11$ the departments are often arranged in one row. In contrast, the optimal value of the TRFLP with $w^T_{path} = 10$ is up to 12.3 % smaller than the optimal value of the SRFLP.

In addition, we generate star instances with $\ell_1 = 20$ and we choose the remaining integer department lengths randomly between 1 and 15. We set integer transport weights $w_{1i}, i \in [n], i \geq 2$, randomly between 10 and 20 and for the remaining departments we set the transport density of 20 % and we choose integer transport weights randomly between 1 and 10. We generate 5 instances for each n and we use a time limit of 4 hours for our branch-and-cut algorithm. If the 3-BFLP is not solved to optimality, the obtained lower bound is displayed and marked it with " ' ".

The TRFLP was solved to optimality for all instances with up to 19 departments and we were able to solve four of the five instances with 20 departments in less than four hours, see Table

	$w^T_{path} = 3, w^B_{path} = 4$				$w^T_{path} = 10, w^B_{path} = 11$			
Instances	3-BFLP	TRFLP	Time$_{\texttt{3-BFLP}}$	Time$_{\texttt{TRFLP}}$	3-BFLP	TRFLP	Time$_{\texttt{3-BFLP}}$	Time$_{\texttt{TRFLP}}$
*Am*11*a*	9774.5	8902.0	25	28	10630.5	9852.5	13	26
*Am*11*b*	6890.5	6118.5	10	16	7375.5	6930.5	10	12
*Am*12*a*	2862.0	2552.0	1:29	51	2901.0	2793.5	20	37
*Am*12*b*	3093.5	2740.5	1:38	40	3280.5	3081.5	40	38
*Am*13*a*	4489.5	4077.0	5:02	2:01	4902.5	4517.5	1:26	2:04
*Am*13*b*	4956.0	4581.5	1:41	1:54	5698.0	4999.0	8:07	1:54
*Am*14_1	5114.5	4642.0	31:33	8:53	5481.5	5169.5	3:17	10:17
*Am*14*a*	5296.0	4751.0	17:11	5:35	5673.0	5327.5	4:43	5:10
*Am*14*b*	5248.0	4739.5	8:36	6:47	5595.0	5323.0	3:28	7:45
*Am*15	5869.0	5378.0	22:30	15:53	6305.0	5946.5	12:20	17:26
*HK*15	27107.0	26446.0	52:28	13:04	28486.0	27180.0	15:57	12:04
*P*16*a*	13142.5	12381.0	2:22:28	1:18:48	14828.5	13233.0	2:38:45	1:13:21
*P*16*b*	10583.5	9882.5	2:00:08	1:12:04	11878.5	10627.5	2:57:12	1:05:12
*P*17*a*	10770.0’	11956.5	4:04:58†	3:57:30	13429.5’	12871.0	5:46:33†	3:48:33
*P*17*b*	13596.5	12779.0	6:51:30	3:06:40	14332.5’	13761.0	4:46:06†	5:09:02
*Am*17	8516.0	7767.5	3:32:13	2:20:52	9254.0	8590.0	2:19:17	1:51:48
*P*18*a*	11750.0’	TL	TL	TL	13496.5’	TL	6:22:30†	TL
*P*18*b*	13732.0’	5289.5	6:27:55†	TL	14250.5’	TL	4:19:38†	TL
*Am*18	9744.5	8911.5	5:51:56	5:41:57	10650.5	9807.5	4:00:48	5:49:05

Table 3: Optimal values of the 3-BFLP and the TRFLP for instances from the literature with $w^T_{path} + 1 = w^B_{path} = 4$ as well as $w^T_{path} + 1 = w^B_{path} = 11$. We write “TL” if the time limit of 8 hours is exceeded and the running time is marked with “†” if we run out of memory storage. The running times are given in sec, min:sec or h:min:sec. If the 3-BFLP is not solved to optimality, the obtained lower bounds are marked with “ ’ ”.

4. For the 3-BFLP, all instances with 18 departments and four (two) instances with 19 (20) departments were solved to optimality. For most star instances, the TRFLP is solved faster than the 3-BFLP. The optimal value of the TRFLP is between 26.2 % and 40.8 % smaller than the optimal value of the SRFLP and by 8.5 % to 33.1 % smaller than the optimal value of the 3-BFLP. So for the considered instances, the TRFLP and the 3-BFLP are preferable to the SRFLP and the TRFLP is preferable to the 3-BFLP. Again, our heuristic approaches obtain small gaps, i. e., less than 6.3 % for the 3-Bay instances and for almost all T-row instances less than 2.0 %.

Furthermore, we use variant 3 to compute lower bounds for larger T-row instances and all additionally added inequalities in variant 3 in comparison to variant 1 are also added to the 3-BFLP using the same separation strategy. The calculation of heuristics by Cplex is disabled. We interrupt the branch-and-cut algorithm when we reach the root node, i. e., at first the LP consisting of the inequalities included in the beginning is solved with $x_{ikj} \in [0,1], |\{i,j,k\}| = 3, i < j, z_{ij} \in [0,1], i,j \in [n] \setminus \{s_M\}, i \neq j, z_{is_M} \in [0,1], i \in [n] \setminus \{s_M\}, y_i \in [0,1], i \in [n]$, and then violated cutting planes are added according to our separation strategy. This LP is solved again until we obtain a binary solution, i. e., the y, z and x variables are binary, or until no violated cutting plane can be found. Besides this, we use Algorithm 1 as described above for the TRFLP. We set a time limit of 4 hours and if we exceed this time limit we display the current best lower bound for the 3-BFLP. The instances $P19a$, $P19b$ and $P19c$ are generated as described in [7], the instances with $n \in \{20, 21, 22, 23\}$ are taken from [7] and the instances with $n = 24$ are taken from [81]. In Table 5 we denote by “ $\mathrm{L}_{\texttt{3-BFLP}}$” (“$\mathrm{L}_{\texttt{TRFLP}}$”) the obtained lower bound for the 3-BFLP (TRFLP) in the second and third column and we rounded the lower bounds such that $2 \cdot \mathrm{L}_{\texttt{3-BFLP}} \in \mathbb{Z}$ ($2 \cdot \mathrm{L}_{\texttt{TRFLP}} \in \mathbb{Z}$). The objective values of our heuristically determined upper bounds are denoted by $\mathrm{H}_{\texttt{3-BFLP}}$ and $\mathrm{H}_{\texttt{TRFLP}}$ for the 3-BFLP and the TRFLP, respectively. The gap

Instance	SRFLP	3-BFLP	TRFLP	$H_{3\text{-BFLP}}$	H_{TRFLP}	$\text{Time}_{3\text{-BFLP}}$	$\text{Time}_{\text{TRFLP}}$
11a	2875.0	2543.0	1702.0	2550.0	1702.0	5	0
11b	4346.5	3482.5	2847.5	3513.5	2847.5	12	2
11c	3417.0	3019.0	2301.0	3112.0	2301.0	10	1
11d	4180.0	3412.0	2878.0	3412.0	2878.0	5	2
11e	4334.5	3679.5	3098.5	3720.5	3122.5	8	4
12a	7903.0	6333.0	5540.0	6403.0	5540.0	2:20	13
12b	5587.0	4598.0	3911.0	4642.0	3925.0	10	6
12c	3914.0	3286.0	2529.0	3338.0	2529.0	15	2
12d	5876.5	4803.5	4027.5	4892.5	4040.5	21	8
12e	7809.0	6199.0	5583.0	6299.0	5583.0	1:08	15
13a	5584.5	4700.5	3823.5	4779.5	3829.5	40	11
13b	5036.5	4247.5	3290.5	4247.5	3290.5	18	4
13c	6023.5	4877.5	4040.5	4877.5	4040.5	50	20
13d	5952.0	4838.0	4036.0	4956.0	4036.0	55	27
13e	4944.0	4170.0	3266.0	4220.0	3281.0	51	8
14a	7153.5	5900.5	5276.5	6023.5	5276.5	1:45	1:01
14b	7933.0	6524.0	5640.0	6703.0	5640.0	1:36	48
14c	6251.0	5195.0	4150.0	5259.0	4185.0	1:32	23
14d	6782.0	5784.0	4884.0	5830.0	4884.0	1:31	26
14e	6913.5	5800.5	4935.5	5874.5	5034.5	55	49
15a	7668.0	6120.0	5312.0	6208.0	5312.0	2:02	1:31
15b	7730.5	6317.5	5298.5	6317.5	5355.5	5:38	1:36
15c	6387.0	5199.0	4225.0	5199.0	4225.0	7:02	1:27
15d	6559.5	5608.5	4609.5	5662.5	4609.5	3:06	48
15e	6929.0	5567.0	4643.0	5601.0	4662.0	8:09	3:37
16a	9461.0	7663.0	6564.0	8142.0	6779.0	7:48	3:54
16b	11912.0	9376.0	8356.0	9376.0	8356.0	10:00	10:15
16c	11351.5	9122.5	8082.5	9255.5	8113.5	8:58	6:27
16d	8073.5	6548.5	5521.5	6609.5	5535.5	8:28	3:23
16e	8184.0	6541.0	5561.0	6541.0	5577.0	3:49	4:49
17a	11063.0	8985.0	7853.0	9036.0	7853.0	1:50:14	13:47
17b	13692.5	10913.5	9876.5	10970.5	10009.5	36:31	34:34
17c	11101.0	8812.0	7640.0	8865.0	7640.0	27:24	12:06
17d	9753.0	7981.0	6823.0	8213.0	6929.0	23:10	9:59
17e	9879.4	7957.5	6736.5	8027.5	6810.5	26:19	9:37
18a	15157.5	12288.5	11108.5	12455.5	11214.5	1:16:55	36:08
18b	10254.5	8142.5	7037.5	8315.5	7088.5	1:01:53	25:32
18c	12847.5	10370.5	9264.5	10406.5	9285.5	1:14:52	38:36
18d	9437.0	7638.0	6464.0	7720.0	6483.0	1:58:08	21:11
18e	11769.0	9661.0	8538.0	9661.0	8538.0	1:00:53	32:16
19a	14371.5	11353.0	10046.5	11433.5	10148.5	1:48:11	1:40:54
19b	14110.5	11414.5	10203.5	11693.5	10203.5	3:53:33	1:19:15
19c	10119.0	8100.0	6637.0	8337.0	6731.0	1:11:44	25:56
19d	12821.0	8805.5’	8848.0	10200.0	8848.0	TL	1:58:16
19e	13209.0	10380.0	9495.0	10484.0	9575.0	1:56:34	2:37:10
20a	11826.0	9529.0	8257.0	9828.0	8288.0	2:08:43	1:27:44
20b	15956.5	12397.0	TL	12446.5	12127.5	3:12:02	TL
20c	14191.0	8471.0’	9826.0	11192.0	9827.0	TL	3:28:52
20d	16047.5	11876.0’	11540.5	12826.5	11540.5	TL	3:51:36
20e	11882.5	8384.0’	8097.5	9462.5	8105.5	TL	2:39:25

Table 4: Optimal values for the SRFLP, the 3-BFLP and the TRFLP for randomly generated star instances are displayed. The running times are given in sec, min:sec or in h:min:sec. Note that the TRFLP was solved to optimality for four instances with $n = 20$ departments. The optimal value of the TRFLP is up to 33.1 % smaller than the optimal value of the 3-BFLP and up to 40.8 % smaller than the optimal value of the SRFLP.

is calculated by

$$\text{Gap}_a = \frac{H_a - L_a}{H_a} \cdot 100, \qquad a \in \{\texttt{3-BFLP}, \texttt{TRFLP}\}.$$

and the gap for the 3-BFLP (TRFLP) is given in the sixth (seventh) column denoted by "$\text{Gap}_{\texttt{3-BFLP}}$" ("$\text{Gap}_{\texttt{TRFLP}}$") . Considering our results in Table 5, we obtain lower bounds for T-row instances with up to 22 departments with gaps less than 9.68 % to heuristically determined upper bounds. For four of the five instances with 22 departments and for all larger instances the time limit was exceeded, and hence we are not able to derive non-trivial lower bounds for the TRFLP. If the root node is reached in the time limit of 4 hours, we obtain gaps between 2.23% and 7.90% to heuristically determined upper bounds for 3-Bay instances with up to 24 departments. Considering the instances $P24a$ and $P24c$, the root node was not reached and therefore the gaps are higher, i. e., up to 10.12 %. In conclusion, our approach is able to calculate tight lower bounds for the TRFLP and the 3-BFLP whereby the lower bounds for the 3-BFLP are generally better.

6 Conclusion and Future Work

In this paper we introduced a new facility layout problem, the so-called T-Row Facility Layout Problem (TRFLP), which is a generalization of the Multi-Bay Facility Layout Problem (MBFLP) with $m = 3$ rows denoted by (3-BFLP). The TRFLP and the MBFLP have several applications, e. g., in heavy manufacturing and semiconductor fabrication. The TRFLP has a more complex path structure than the 3-BFLP and we proved there always exists an optimal T-row layout where one department has its center position on p_M in row 1. In a branch-and-cut approach we enumerated over each department with its center fixed on position p_M in row 1 and we presented a mixed-integer linear programming (MILP) model for this problem based on an extension of the betweenness variables which might be in contrast to the literature equal to one if the corresponding departments lie in different rows. Transitivity inequalities are used to ensure the correct relation of these extended betweenness variables to the remaining variables. To strengthen our formulation we provided cutting planes exploring the crossroad structure of the TRFLP. In addition, we adapted our MILP model to the 3-BFLP.

We were able to solve T-row and 3-bay instances from the literature with up to 18 departments within a given time limit of 8 hours and we clearly outperform the current best approach from the literature. Further, we outperform a MILP approach based on betweenness variables as known in the literature and we observed in our computational results that the transitivity inequalities (in comparison to the standard linearization) and the cutting planes significantly improve the performance of our approach. Additionally, we considered star instances and we were able to solve even larger instances of this type for the TRFLP and the 3-BFLP. According to our computational results, the TRFLP is preferable to the 3-BFLP and the SRFLP if a factory is built from the ground up, because the optimal value is significantly smaller and in factory planning the departments are arranged for a long period since the rearrangement of the departments is expensive. So even a small reduction of the yearly production costs can reduce the total production costs over a long period significantly. In addition, our approach is able to calculate tight lower bounds for even larger T-row and 3-Bay instances to evaluate the quality of heuristically determined layouts.

It remains for future work to consider facility layouts in the shape of an X or an U and the MBFLP with $m = 4$ and $m = 5$. Therefore, one can use our extension of the betweenness variables and one can adapt our MILP approach as well as our cutting planes. From a practical point of view, it is interesting to extend the betweenness model for the SRFLP such that it is capable of more realistic extensions such as individual in- and output positions of the departments and to use the result of Proposition 1 to improve the performance of the branch-and-cut algorithm for the SRFLP.

Instance	$L_{3\text{-BFLP}}$	L_{TRFLP}	$H_{3\text{-BFLP}}$	H_{TRFLP}	$Gap_{3\text{-BFLP}}$	Gap_{TRFLP}	$Time_{3\text{-BFLP}}$	$Time_{TRFLP}$
$Am11a$	8450.0	7823.5	8814.5	8411.5	4.14 %	6.99 %	5	8
$Am11b$	5882.0	5431.0	6021.5	5667.5	2.32 %	4.17 %	5	4
$Am12a$	2382.5	2207.5	2515.0	2354.5	5.27 %	6.24 %	18	13
$Am12b$	2621.5	2482.0	2697.5	2548.5	2.82 %	2.61 %	15	25
$Am13a$	3872.5	3688.0	4204.5	3836.0	7.90 %	3.86 %	37	43
$Am13b$	4357.0	4215.5	4529.0	4362.5	3.80 %	3.37 %	38	1:12
$Am14\_1$	4359.0	4079.5	4565.5	4358.0	4.52 %	6.39 %	1:21	2:03
$Am14a$	4560.0	4273.5	4734.0	4448.0	3.68 %	3.92 %	59	1:43
$Am14b$	4482.5	4200.5	4683.0	4433.0	4.28 %	5.24 %	1:03	2:03
$Am15$	4953.0	4744.5	5294.0	5079.0	6.44 %	6.59 %	1:06	4:33
$HK15$	25521.5	24528.5	26495.0	26125.0	3.67 %	6.11 %	2:04	4:30
$P16a$	11536.0	10911.0	12326.0	11943.0	6.41 %	8.64 %	1:58	12:02
$P16b$	9305.5	8677.5	9784.5	9469.5	4.90 %	8.36 %	2:25	14:26
$P17a$	11344.5	10428.5	11888.5	11528.0	4.58 %	9.54 %	3:23	21:26
$P17b$	12207.5	11388.0	12874.0	12389.0	5.18 %	8.08 %	5:06	33:43
$Am17$	7267.5	6788.5	7690.0	7318.0	5.49 %	7.24 %	5:02	18:38
$P18a$	12333.0	11611.0	12863.5	12516.0	4.12 %	7.23 %	8:52	56:12
$P18b$	13746.0	12578.0	14616.5	14072.0	6.00 %	0.62 %	6:50	25:47
$Am18$	8318.0	7758.0	8835.5	8413.5	5.86 %	7.79 %	6:56	36:06
$P19a$	14049.5	13140.5	14715.5	14289.5	4.53 %	8.04 %	6:26	53:53
$P19b$	18074.5	16975.5	19166.5	18660.0	5.70 %	9.03 %	10:24	1:14:46
$P19c$	9865.0	9230.0	10455.0	9989.5	5.64 %	7.60 %	6:53	59:02
$P20a$	18485.5	17297.0	19204.5	18716.5	3.74 %	7.58 %	20:46	2:46:18
$P20b$	19277.5	17897.0	20427.0	19816.0	5.63 %	9.68 %	18:18	1:52:04
$H20$	11882.5	11138.0	12576.0	12087.0	5.51 %	7.85 %	38:06	2:52:48
$P21a$	10382.5	9645.5	10862.5	10439.0	4.42 %	7.60 %	23:50	3:10:40
$P21b$	17646.0	16354.5	18233.0	17806.0	3.22 %	8.15 %	17:07	1:26:56
$P21c$	16811.0	15799.0	17806.0	17100.0	5.59 %	7.61 %	21:22	2:55:49
$P22a$	12912.5	-	13757.0	13238.0	6.14 %	-	32:21	TL
$P22b$	22686.0	-	24096.0	23359.5	5.85 %	-	36:59	TL
$P22c$	21413.0	19922.5	22940.0	22036.0	6.66 %	9.59 %	35:55	3:28:12
$P23a$	14790.5	-	15975.0	15248.5	7.41 %	-	3:39:59	TL
$P23b$	22532.5	-	23484.5	22968.0	4.05 %	-	1:34:25	TL
$P23c$	22206.0	-	23617.0	22956.5	5.97 %	-	1:34:36	TL
$P24a$	16529.0	-	18390.0	17728.0	10.12 %	-	TL	TL
$P24b$	23478.5	-	25219.5	24578.5	6.90 %	-	2:13:54	TL
$P24c$	25114.0	-	27020.0	26058.5	7.05 %	-	TL	TL
$P24d$	23519.0	-	25113.0	24366.0	6.35 %	-	3:37:30	TL
$P24e$	27066.0	-	28809.0	28200.0	6.05 %	-	2:00:17	TL

Table 5: Calculation of lower bounds for the 3-BFLP and the TRFLP with a given time limit of 4 hours and the computation of the branch-and-cut algorithm is interrupted at the root node. We use variant 3 of our separation strategy. The gaps of the lower bounds for the 3-BFLP are less than 10.12 % to heuristically determined upper bounds for the 3-BFLP and even better if the root node is reached during the time limit. The computation time in the last two columns are given in sec, min:sec, h:min:sec and we write " TL " if the time limit of 4 hours is exceeded.

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Appendix

We present a MILP model for the s_M-TRFLP, based on betweenness variables which can only be equal to one, if the corresponding three departments are in the same row, see, e. g., [3, 22, 28, 43]. The dummy departments $n+1$ and $n+2$ are added to the model as described in Section 3.2. Then

$$x_{jki} = x_{ikj} = \begin{cases} 1, & \text{if } k \text{ lies between } i \text{ and } j \text{ in the same row,} \\ 0, & \text{otherwise,} \end{cases}$$

$i, j, k \in [n+2], |\{i, j, k\}| = 3, i < j$. The ordering and assignment variables are used as described in Section 3.2

$$z_{ij} = \begin{cases} 1, & i \text{ is left to } j \text{ and } i \text{ and } j \text{ are in the same row} \\ 0, & \text{otherwise,} \end{cases}$$

$i, j \in [n], i \neq j$, and

$$y_i = \begin{cases} 1, & \text{if } i \text{ lies in row 1} \\ 0, & \text{otherwise,} \end{cases}$$

$i \in [n]$. Further, let $d_{ij} = d_{ji}$ denote the distance between $i \in [n]$ and $j \in [n], i < j$, measured in rectilinear directions. Let $s_M \in [n]$ be fixed. We exclude the constant $W^T := \sum_{\substack{i,j \in [n] \setminus \{s_M\} \\ i<j}} \frac{\ell_i + \ell_j}{2} w_{ij} + \sum_{i \in [n] \setminus \{s_M\}} w_{is_M} \frac{\ell_i}{2}$. Then our MILP model for the s_M-TRFLP with $M := 2 \cdot \sum_{k \in [n]} \ell_k$ reads as follows

$$\sum_{\substack{i,j \in [n] \\ i<j}} w_{ij} d_{ij}$$

$$\text{s. t.} \quad (7) - (18), (21),$$

$$x_{ikj} - z_{ik} - z_{kj} \geq -1, \qquad i,j,k \in [n] \setminus \{s_M\}, i < j, |\{i,j,k\}| = 3, \tag{52}$$

$$x_{ikj} - z_{jk} - z_{ki} \geq -1, \qquad i,j,k \in [n] \setminus \{s_M\}, i < j, |\{i,j,k\}| = 3, \tag{53}$$

$$d_{ij} - \sum_{k \in [n] \setminus \{i,j\}} \ell_k x_{ikj} \geq 0, \qquad i,j \in [n], i < j, \tag{54}$$

$$d_{is_M} + y_i \left(w^T_{path} - \frac{\ell_{s_M}}{2} \right) - \sum_{k \in [n] \setminus \{i,s_M\}} \ell_k \left(x_{iks_M} + x_{(n+2)ki} \right) = w^T_{path}, \qquad i \in [n] \setminus \{s_M\}, \tag{55}$$

$$d_{ij} \geq d_{is_M} + d_{js_M} - M(1 - y_i + y_j), \qquad i,j \in [n] \setminus \{s_M\}, i < j, \tag{56}$$

$$d_{ij} \geq d_{is_M} + d_{js_M} - M(1 + y_i - y_j), \qquad i,j \in [n] \setminus \{s_M\}, i < j, \tag{57}$$

$$d_{ij} \geq 0, \qquad i,j \in [n], i \neq j. \tag{58}$$

Inequalities (52)–(53) are related to the standard linearization to calculate lower bounds for the betweenness variables. Note that upper bounds are obtained by inequalities (21). Previous tests, which we do not mention here, indicate that upper bounds obtained by the standard linearization do not improve this approach. The distance between $i \in [n] \setminus \{s_M\}$ and $j \in [n] \setminus \{s_M\}, i < j$, is calculated via inequalities (54) if i and j are in the same row, otherwise inequalities (54) are redundant. The distance between $i \in [n] \setminus \{s_M\}$ and s_M is calculated by inequalities (55). If $i \in [n] \setminus \{s_M\}$ lies in row 1 (row 2) and $j \in [n] \setminus \{s_M\}, i < j$, in row 2 (row 1), their distance is calculated by inequalities (56) ((57)). The y, z and x variables are chosen as described in Theorem 3. The z variables are set to binary values as shown in the proof of Theorem 3. Further, $x_{ikj}, i,j,k \in [n], i < j, |\{i,j,k\}| = 3$, is set to a binary value if $w_{ij} > 0$ by inequalities (52)–(53), inequalities (21) and since the objective function is minimized. If $w_{ij} = 0$, the objective function is not influenced by the value of x_{ikj}.

Inequalities (7), inequalities (9)–(18) and inequalities (52)-(57) are included from the beginning and inequalities (21) are separated as described in Section 3.3. For the usage of equations (8) we refer to Section 3.3. Additionally, we use inequalities (38), (39), (41) as described in Section 3.3 and we use the symmetry breaking constraints (29)–(30). Here, betweenness variables can only be equal to one if the corresponding departments lie in the same row, hence inequalities (19)-(20) and inequalities (36)-(37) are not valid for this `MILP` model.

In the following, we describe the standard linearization used in the approach denoted by variant 4 for the `TRFLP` in Table 2. We use standard linearization (instead of inequalities (19)–(20)) to couple the extended betweenness variables, i. e.,

$$x_{ikj} - z_{ik} - z_{kj} \geq -1, \qquad i,j,k \in [n] \setminus \{s_M\}, |\{i,j,k\}| = 3, \tag{59}$$

$$x_{ikj} - x_{s_M ki} + y_j \geq 0, \qquad i,j,k \in [n] \setminus \{s_M\}, |\{i,j,k\}| = 3, \tag{60}$$

$$x_{ikj} - x_{(n+2)ki} - y_j \geq -1, \qquad i,j,k \in [n] \setminus \{s_M\}, |\{i,j,k\}| = 3. \tag{61}$$

The inequalities (59)–(61) are included from the beginning and the inequalities (19)–(20) are neglected. Besides this, we use variant 1 as described in Section 3.5. The corresponding upper bounds are obtained by inequalities (21) and previous tests, which we do not mention here, indicate that upper bounds related to the standard linearization do not improve this approach. The y, z and x variables are chosen as described in Theorem 3. The z variables are set to binary values as shown in the proof of Theorem 3. Further, $x_{ikj}, i,j,k \in [n], i < j, |\{i,j,k\}| = 3$, is set to a binary value if $w_{ij} > 0$ by inequalities (59)–(61), inequalities (21) and since the objective function is minimized. If $w_{ij} = 0$, the objective function is not influenced by the value of x_{ikj}.