

The Magic of Nash Social Welfare in Optimization: Do Not Sum, Just Multiply!

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Abstract

In this paper, we explain some key challenges when dealing with a single/multi-objective optimization problem in practice. To overcome these challenges, we present a mathematical program that optimizes the Nash Social Welfare function. We refer to this mathematical program as the Nash Social Welfare Program (NSWP). An interesting property of the NSWP is that it can be constructed for any single/multi-objective optimization problem. We show that solving the NSWP could result in more desirable solutions in practice than its single/multi-objective counterpart. We also discuss several promising approaches that could be employed to solve the NSWP in practice.

Keywords: Nash Social Welfare, Equity, Single-objective Optimization, Multi-objective Optimization, Multiplicative Program

1. Introduction

Without loss of generality, a single-objective optimization problem can be stated as,

$$\max \left\{ \sum_{i=1}^{p'} w_i f_i(\mathbf{x}) - \sum_{i=p'+1}^p w_i f_i(\mathbf{x}) : \mathbf{x} \in \mathcal{X} \right\}, \quad (1)$$

where $p \geq p' \geq 0$ and $\mathcal{X} \subseteq \mathbb{R}^n$ represents the set of feasible solutions and is assumed to be bounded. Moreover, $\mathbf{f}(\mathbf{x}) := (f_1(\mathbf{x}), \dots, f_p(\mathbf{x}))$ is a vector of arbitrary functions and $\mathbf{w} := (w_1, \dots, w_p)$ is a vector of weights with $w_i > 0$ for all $i = 1, \dots, p$. In the remaining of this paper, we refer to the objective function of Problem (1) as the (weighted) social welfare function. Also, we refer to Problem (1) as the Social Welfare Program (SWP).

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Similarly, a multi-objective optimization problem can be stated as,

$$\begin{aligned}
& \max \quad \{f_i(\mathbf{x}) : i \in S^+\} \\
& \min \quad \{f_i(\mathbf{x}) : i \in S^-\} \\
& \text{s.t.} \quad \mathbf{x} \in \mathcal{X},
\end{aligned} \tag{2}$$

where S^+ and S^- are the (index) sets of objective functions that need to be maximized and minimized, respectively. For simplicity, we assume that $S^+ := \{1, \dots, p'\}$ and $S^- := \{p' + 1, \dots, p\}$ which implies that $S^+ \cup S^- := \{1, \dots, p\}$. In the remaining of this paper, we refer to Problem (2) as the Multi-Objective Optimization Program (MOOP).

The main contribution of this article is to present a different problem, the Nash Social Welfare Program (NSWP), that can result in solutions with desirable properties for both Problems (1) and (2). In other words, instead of solving the SWP or MOOP, modelers and/or practitioners can solve the NSWP to obtain more practical solutions. We show that for any single/multi-objective optimization problem, the NSWP can be easily constructed by using its corresponding SWP/MOOP. We also show that the NSWP can be solved in polynomial time if its corresponding SWP/MOOP is a (continuous) convex optimization problem (see Remark 7).

We would like to highlight that there are some studies in the literature about some variations of the NSWP. To the best of our knowledge, the first study regarding the NSWP is conducted by Rao (1987) in the context of multi-objective optimization. However, there is no article in the literature that provides some insights as to why the NSWP should be employed, how it can be generated for any single/multi-objective optimization problem, and more importantly, how it can be solved. The present work is an attempt to fill this gap in the literature. The rest of this paper is organized as follows. In Section 2, we discuss the motivation for writing this paper by providing some reasons as to why solving the SWP and NSWP can be problematic in practice. In Section 3, the NSWP is explained in detail. In Section 4, our proposed solution approaches for solving the NSWP are introduced. Finally, in Section 5, some concluding remarks are given.

2. Motivation

In this section, we will explain the main weaknesses of the SWP and MOOP. The main motivation for developing the NSWP is to overcome these weaknesses.

2.1. The Multi-Objective Optimization Program

In general, when solving the MOOP, decision makers are only interested in the so-called *Pareto-optimal* solutions, see Definition 1.

Definition 1. A feasible solution $\mathbf{x} \in \mathcal{X}$ is called *Pareto-optimal*, if there is no other $\mathbf{x}' \in \mathcal{X}$ such that

$$\begin{aligned} f_i(\mathbf{x}) &\leq f_i(\mathbf{x}') && \forall i \in S^+, \\ f_i(\mathbf{x}) &\geq f_i(\mathbf{x}') && \forall i \in S^-, \\ f_i(\mathbf{x}) &\neq f_i(\mathbf{x}') && \exists i \in S^+ \cup S^-. \end{aligned}$$

The image of each Pareto-optimal solution in the criterion space is called a *Pareto-optimal point*. The set of all Pareto-optimal points, also known as the *Pareto-optimal frontier*, show the trade-off between objectives. An illustration of the trade-off of the MOOP involving only two objectives (in the form of maximization) can be found in Figure 1.

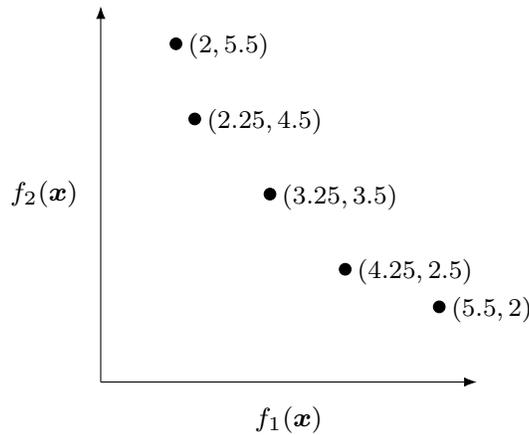


Figure 1: An illustration of a Pareto-optimal frontier with two (maximization) objective functions in the criterion space

In practice a typical challenge when dealing with the MOOP is to answer how a Pareto-optimal solution should be selected for implementation in practice? A natural answer to this question is that modelers have no responsibility in identifying a desirable solution and the decision makers (or managers) are expected to do that in practice. In other words, modelers need to compute the trade-off (between Pareto-optimal solutions) and ask the decision makers to select the one which is suitable. Such a process is typically implemented using either a two-phase approach or a step-by-step approach. In the two-phase approach, the complete trade-off will be computed in the first phase and then the entire information will be sent to the decision

makers. However, the step-by-step approach is interactive meaning that in each iteration one (or some) Pareto-optimal solutions will be computed by modelers and then the opinion of decision makers will be obtained. If decision makers are happy with the solution then the search terminates, otherwise the search will be modified for finding some other Pareto-optimal solutions (if there exist anymore). This process repeats until a desirable solution is found.

Although the approaches described above can be quite effective in many practical situations for selecting a desirable Pareto-optimal solution, they do not work in many situations because they suffer from two main weaknesses described below.

Weakness 1. *In some practical situations, computing even a single Pareto-optimal solution is computationally infeasible/expensive. So, selecting a desirable Pareto-optimal solution using the existing approaches can be infeasible.*

Weakness 2. *In some practical situations, either there is no decision maker or the decision makers do not know how to select a desirable solution (Jorge, 2009). So, selecting a desirable Pareto-optimal solution using the existing approaches can be infeasible.*

For Weakness 1, a possible remedy is heuristic/evolutionary solution methods. However, inexact methods do not guarantee optimality and may perform poorly. More importantly, they still do not address Weakness 2. Our proposed technique in this study, i.e., the NSWP, addresses both weaknesses at the same time because it is designed to directly generate a Pareto-optimal solution (see Proposition 1) that can create a desirable balance between objectives.

2.2. The Social Welfare Program

To explain the main weaknesses of the SWP, we first show that the SWP itself is trying to solve the MOOP. Observe that the MOOP can be written in the following form,

$$\begin{aligned}
 & \max \quad \{f_i(\mathbf{x}) : i \in S^+\} \\
 & \max \quad \{-f_i(\mathbf{x}) : i \in S^-\} \\
 & \text{s.t.} \quad \mathbf{x} \in \mathcal{X},
 \end{aligned} \tag{3}$$

Now, a popular solution technique to deal with the MOOP is to aggregate its objective functions with some positive weights and then solving the following single-objective optimization problem (instead),

$$\max \left\{ \sum_{i \in S^+} w_i f_i(\mathbf{x}) - \sum_{i \in S^-} w_i f_i(\mathbf{x}) : \mathbf{x} \in \mathcal{X} \right\}. \quad (4)$$

Problem (4) is sometimes called as the weighted sum optimization in the literature of multi-objective optimization (Ehrgott, 2005). We can observe that the weighted sum optimization is precisely Problem (1) because $S^+ = \{1, \dots, p'\}$ and $S^- = \{p' + 1, \dots, p\}$. So, when modelers are trying to solve the SWP, they are basically trying to find a desirable Pareto-optimal solution for Problem (2) by setting the weight of $f_i(\mathbf{x})$ to w_i for all $i = 1, \dots, p$. In other words, they implicitly have p number of criteria in their mind, i.e., $f_1(\mathbf{x}), \dots, f_p(\mathbf{x})$, in practice with w_1, \dots, w_p as their corresponding degrees of importance.

Given that the SWP can be viewed as a solution approach for computing a desirable Pareto-optimal solution for the MOOP, it is natural to ask whether it actually results in finding a desirable solution in practice? Note that one may argue that the SWP can remove Weaknesses 1 and 2 that the MOOP causes. While for many practical situations this can be indeed true, we will argue in the rest of this section that the SWP should be generally avoided because it may not generate a desirable Pareto-optimal solution, i.e., it may not provide a good balance between the values of different objectives in practice.

Weakness 3. *For non-convex multi-objective optimization problems, there can exist many (and possibly infinite) Pareto-optimal points that cannot be obtained by optimizing a (positive) weighted summation of all objective functions over the feasible set (even by testing all possible positive weights). Such points are called unsupported Pareto-optimal points (Ehrgott, 2005). So, a weakness of the SWP is that it completely ignores the existence of unsupported Pareto-optimal points that can possibly better balance different objectives.*

To better understand Weakness 3 consider Figure 1 again. In the figure, we have $p = 2$ and $S^- = \emptyset$, i.e., there are two objectives in the form of maximization. It is not hard to see that if we use the SWP then we will find either the point (2,5.5) or point (5.5,2) for any possible values of w_1 and w_2 (note that $w_1, w_2 > 0$). So, none of the three middle points can be obtained using the SWP because they are unsupported Pareto-optimal points. Observe that this can be problematic (in some practical cases) because from the view point of multi-objective optimization those middle points balance the conflicting objectives better than the endpoints.

Weakness 4. *In many practical situations, $f_1(\mathbf{x}), \dots, f_p(\mathbf{x})$ represent the objective functions of p independent players/components of an environment/system. So, by developing the SWP for such cases, a modeler*

is basically trying to coordinate these players with respect to their powers, denoted by w_1, \dots, w_p , and make sure that the benefit of the system is distributed (almost) fairly among players with respect to their powers. However, a weakness of the SWP is that it does not necessarily ensure the fairness.

To explain Weakness 4 better consider Figure 1 again where $p = 2$ and $S^- = \emptyset$. We know that in this example, we will find either the point (2,5.5) or point (5.5,2) for any possible values of w_1 and w_2 . So, suppose that $w_1 = w_2 = 1$, in this case, both (2,5.5) and (5.5,2) are optimal and so the SWP returns one of them randomly. However, the power of players are exactly the same. So no matter which point is returned by the SWP it will not be fair and make (one of) the players unhappy. For this particular example, we can easily see that the point (3.25, 3) is indeed the only fair point that can make both players happy at the same time. Overall, our proposed technique in this study, i.e., the NSWP, addresses both Weaknesses 3-4 at the same time because it is designed to directly generate a Pareto-optimal solution that can create a desirable balance between objectives.

2.3. A Practical Example

To better illustrate the weaknesses mentioned earlier, we now provide a (simplified) real-world problem/scenario. In the field of conservation planning, one of the most important problems is the so-called reserve selection problem (Sierra-Altamiranda et al., 2020). Suppose that there are n parcels of land and parcel $j \in \{1, \dots, n\}$ has an associated cost denoted by c_j . As a modeler you have a budget, denoted by b , to spend on buying some of these parcels and you want to develop an optimization model that can help natural resource managers in identifying which parcels should be selected to be reserved. Suppose that there are (in total) p species where S^+ represents the index set of engendered/important species and S^- represents the index set of invasive species. Accruing parcel $j \in \{1, \dots, n\}$ has an impact on species $i \in \{1, \dots, p\}$, e.g., increases its population size, which is denoted by parameter $u_{ij} \geq 0$. Observe that for this problem, we have that,

$$\mathcal{X} = \{\mathbf{x} \in \{0, 1\}^n : \sum_{j=1}^n c_j x_j \leq b\},$$

where x_j is a binary variable that shows whether parcel j should be bought or not. Also, for each species $i \in \{1, \dots, p\}$, we have that,

$$f_i(\mathbf{x}) = \sum_{j=1}^n u_{ij} x_j.$$

Modelers in the field of conservation planning address the problem mentioned above by either using the SWP or the MOOP (Sierra-Altamiranda et al., 2020). However, it is easy to see that if they use the MOOP, Weaknesses 1 and 2 can arise when n and p are large, respectively. Also, if they use the SWP, Weakness 3 can arise because this problem is non-convex. Moreover, Weakness 4 can arise because the modeler is indeed working on a system where each species is a player on its own.

3. The Nash Social Welfare Program

The underlying idea of the NSWSP comes from the field of cooperative game theory. A well-known problem in the field of cooperative game theory is the so-called bargaining problem. The bargaining problem is a game where all (competing) players agree to create a grand coalition, instead of competing with each other, to get a higher payoff (Serrano, 2005). To be able to create a grand coalition, the agreement of all players is necessary. Therefore, the main concern when dealing with a bargaining problem is what the payoff of each player should be in a grand coalition (and how it should be computed). One of the well-known solution techniques to this problem is introduced by Nash (1950, 1953) and extended further by Kalai (1977) that we use in this paper. Specifically, we assume that there are p number of (imaginary) players and each player $i \in S^+ \cup S^-$ has an objective function $f_i(\mathbf{x})$. Note that $|S^+ \cup S^-| = p$. Hence, we try to create a coalition between these players by solving the following optimization problem,

$$\begin{aligned}
 \max \quad & \prod_{i \in S^+} (f_i(\mathbf{x}) - d_i)^{w_i} \times \prod_{i \in S^-} (d_i - f_i(\mathbf{x}))^{w_i} \\
 \text{s.t.} \quad & \mathbf{x} \in \mathcal{X}, \\
 & f_i(\mathbf{x}) \geq d_i \quad \forall i \in S^+, \\
 & f_i(\mathbf{x}) \leq d_i \quad \forall i \in S^-,
 \end{aligned} \tag{5}$$

where $\mathbf{d} \in \mathbb{R}^p$ is the reference point, an input parameter for the NSWSP. The reference point in the literature of the bargaining problem is sometimes referred to as the disagreement point and it basically indicates the payoff of each player under no coalition. We will later in Section 3.1 explain some natural choices for the reference point (if it is unknown to modelers). The following remark is helpful.

Remark 1. *For changing the MOOP to the NSWSP, one can assume $w_i = 1$ for all $i \in S^+ \cup S^-$ unless it is known that the objective functions do not have the same preference.*

The objective function of Problem (5) is the (weighted) Nash social welfare function which is why we refer to Problem (5) as the NSWP. Note that since player $i \in S^+$ is interested in maximizing its objective function, $(f_i(\mathbf{x}) - d_i)$ captures the benefit that it will obtain as the result of creating a coalition (with respect to the reference point). Similarly, since player $i \in S^-$ is interested in minimizing its objective function, $(d_i - f_i(\mathbf{x}))$ captures the benefit that it obtains as the result of creating a coalition (with respect to the reference point). So, the (weighted) Nash Social Welfare function is the product of benefits of players (considering their negotiating powers/weights) and the goal of the NSWP is to maximize the value of this function.

Assumption 1. *In the remaining of this paper, we assume that the optimal objective value of the NSWP is non-zero, i.e., strictly positive. This implies that $\mathcal{X} \neq \emptyset$ and the disagreement point is selected such that there exists a feasible solution $\mathbf{x} \in \mathcal{X}$ with $f_i(\mathbf{x}) > d_i$ for all $i \in S^+$ and $f_i(\mathbf{x}) < d_i$ for all $i \in S^-$.*

Proposition 1. *The NSWP returns a Pareto-optimal solution.*

PROOF. Suppose that \mathbf{x}^* is an optimal solution of the NSWP but it is not a Pareto-optimal solution. By definition of Pareto-optimality (Definition 1), this implies that there must exist a feasible solution denoted by $\mathbf{x} \in \mathcal{X}$ that *dominates* \mathbf{x}^* , i.e.,

$$\begin{aligned} f_i(\mathbf{x}^*) &\leq f_i(\mathbf{x}) && \forall i \in S^+, \\ f_i(\mathbf{x}^*) &\geq f_i(\mathbf{x}) && \forall i \in S^-, \\ f_i(\mathbf{x}^*) &\neq f_i(\mathbf{x}) && \exists i \in S^+ \cup S^-. \end{aligned}$$

In addition, by assumptions, we have $w_i > 0$ for all $i = 1, \dots, p$. This combined with Assumption 1 implies that,

$$0 < \prod_{i \in S^+} (f_i(\mathbf{x}^*) - d_i)^{w_i} \times \prod_{i \in S^-} (d_i - f_i(\mathbf{x}^*))^{w_i} < \prod_{i \in S^+} (f_i(\mathbf{x}) - d_i)^{w_i} \times \prod_{i \in S^-} (d_i - f_i(\mathbf{x}))^{w_i}.$$

Consequently, \mathbf{x}^* cannot be an optimal solution (a contradiction). □

To illustrate what the NSWP actually does, consider the same example described in Figure 1 where $S_i^- = \emptyset$ and $S_i^+ = \{1, 2\}$. Suppose that $w_1 = w_2 = 1$ and $\mathbf{d} = (2, 2)$, the Pareto-optimal point that NSWP selects for this example is (3.25, 3.5) as shown in Figure 2, which is not only an unsupported Pareto-optimal

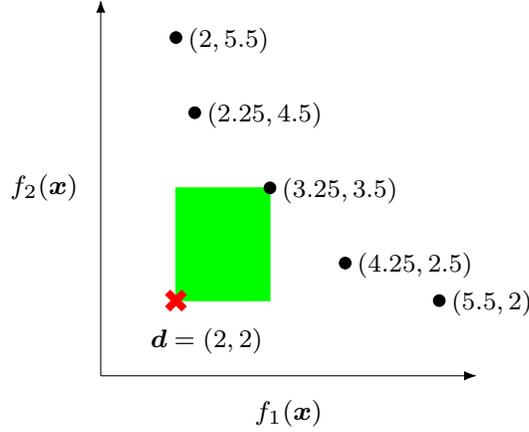


Figure 2: Selection of the Pareto-optimal point by the NSWP

point, but also the only fair Pareto-optimal point from the view point of both players. Note that since $w_1 = w_2 = 1$, we observe that, geometrically, the NSWP attempts to find a Pareto-optimal point such that the area of the box between this point and \mathbf{d} is maximized. In other words, the objective function of the NSWP captures the area of the box. The green box in Figure 2 shows the box with the maximum area. The following remarks and proposition are helpful.

Remark 2. *The NSWP is global-power-scale-free, i.e., the NSWP with negotiation powers/weights w_1, \dots, w_p is equivalent to the NSWP with powers $\alpha w_1, \dots, \alpha w_p$ where α is an arbitrary positive constant. This remark can be directly obtained from Proposition 3 (see Section 4).*

Note that from Remark 2, one can also infer that the NSWP with equal negotiating powers, i.e., $w_i = w_j$ for all $i, j \in S^+ \cup S^-$, is equivalent to the NSWP with unit negotiating powers, i.e., $w_i = 1$ for all $i \in S^+ \cup S^-$.

Proposition 2. *The NSWP is local-benefit-scale-free, i.e., by replacing the objective function of the NSWP with the following function,*

$$\max \prod_{i \in S^+} (\alpha_i f_i(\mathbf{x}) - \alpha_i d_i)^{w_i} \times \prod_{i \in S^-} (\alpha_i d_i - \alpha_i f_i(\mathbf{x}))^{w_i}, \quad (6)$$

an equivalent problem will be constructed where $\alpha_1, \dots, \alpha_p$ are arbitrary positive constants.

PROOF. Observe that Equation (6) is equivalent to,

$$\max \prod_{i=1}^p \alpha_i^{w_i} \times \prod_{i \in S^+} (f_i(\mathbf{x}) - d_i)^{w_i} \times \prod_{i \in S^-} (d_i - f_i(\mathbf{x}))^{w_i}.$$

Since $\prod_{i=1}^p \alpha_i^{w_i}$ is a positive constant, it has no impact on the optimization and can be dropped. \square

Again note that in the NSWP, the term $(f_i(\mathbf{x}) - d_i)$ captures the benefit of player $i \in S^+$ and the term $(d_i - f_i(\mathbf{x}))$ captures the benefit of player $i \in S^-$ with respect to the reference point. Hence, if one uses Proposition 2 by setting

$$\alpha_i = \frac{1}{\max_{\mathbf{x} \in \mathcal{X}'} f_i(\mathbf{x}) - d_i} \quad \forall i \in S^+,$$

$$\alpha_i = \frac{1}{\max_{\mathbf{x} \in \mathcal{X}'} d_i - f_i(\mathbf{x})} \quad \forall i \in S^-,$$

where $\mathcal{X}' := \{\mathbf{x} \in \mathcal{X} : f_i(\mathbf{x}) \geq d_i \forall i \in S^+ \text{ and } f_i(\mathbf{x}) \leq d_i \forall i \in S^-\}$, then the maximum benefit that each player can obtain will be precisely one in the equivalent problem. This implies that from any NSWP, an equivalent NSWP with unit-maximum-benefit can be constructed because for each $i \in \{1, \dots, p\}$, $\alpha_i f_i(\mathbf{x})$ and $\alpha_i d_i$ can be viewed as the new $f_i(\mathbf{x})$ and the new d_i , respectively.

Remark 3. *From Remark 2 and Proposition 2, we know that the NSWP with equal negotiation powers/weights can be rewritten as the NSWP with unit-weights and unit-maximum-benefits. When solving the latter, the NSWP attempts to find a Pareto-optimal point where the area of the box between the reference point and the Pareto-optimal point is maximized. The area of the box cannot be more than one because the length of each side of the box indicates the benefit obtained by a player and the maximum benefit of each player is one. So, the NSWP with equal negotiating powers basically attempts to not only make the benefit of each player equal to one but also make them relatively the same. So, the NSWP with equal negotiating powers is naturally designed to create equity in benefits, i.e., fairness (Cole and Gkatzelis, 2018; Moulin, 2003; Caragiannis et al., 2016; Mariotti, 1999).*

In summary, our proposed mathematical program, i.e., the NSWP, is a single-objective optimization problem. It resolves Weakness 1 because it returns a single Pareto-optimal solution (just like the SWP) if being solved to optimality (see Proposition 1). Evidently, generating only a single Pareto-optimal solution can decrease the computational time significantly. Even for the cases that the NSWP is computationally expensive, one can develop a heuristic method for finding high-quality feasible solutions for it. The NSWP resolves Weakness 2 because it generally tries to avoid the endpoints of the Pareto-optimal frontier and generates a good balance between objectives (which is generally what decision makers look for). The NSWP resolves Weakness 3 because it does not ignore unsupported Pareto-optimal points and its output can be even an unsupported Pareto-optimal point. Finally, the NSWP resolves Weakness 4 because it attempts to ensure that the benefits are distributed fairly (based on Remarks 3) with respect to the power of players.

Specifically, if the negotiating powers are equal then the NSWP attempts to create equity in benefits. Otherwise, it attempts to deviate from the equity in benefits based on the power of players.

3.1. The reference point

As mentioned earlier, in the context of game theory, the reference point should denote the payoff of each player under no coalition, i.e., the status quo of the game. While in many applications the reference point can be identified easily, in some cases it is not clear how the reference point should be determined. For such cases, one natural choice for the reference point can be the so-called *nadir* point, i.e., a (possibly imaginary) point in the criterion space that denotes the worst value for each objective function in the Pareto-optimal frontier. For example, in Figure 2, both objectives are in the form of maximization. So, the worst value for both $f_1(x)$ and $f_2(x)$ in the Pareto-optimal frontier is 2. Hence, the nadir point is (2,2).

However, computing the nadir point is computationally expensive, especially for cases with $p > 2$. For cases with $p = 2$, the nadir point can be generated by computing the two endpoints of the Pareto-optimal frontier (see Figure 2). However, for cases with $p > 2$, this is not necessarily true in general (Ehrgott, 2005). Hence, alternatively, one can set the reference point to an approximation of the nadir point. For example, for each $i \in S^+$, one can set

$$d_i = \min \left\{ f_i(\mathbf{x}) : \mathbf{x} \in \mathcal{X} \right\},$$

or

$$d_i = \min \left\{ f_i(\mathbf{x}) : \mathbf{x} \in \mathcal{X}^R \right\},$$

where \mathcal{X}^R is a relaxation of \mathcal{X} , e.g., linear programming relaxation. Similarly, for each $i \in S^-$, one can set

$$d_i = \max \left\{ f_i(\mathbf{x}) : \mathbf{x} \in \mathcal{X} \right\},$$

or

$$d_i = \max \left\{ f_i(\mathbf{x}) : \mathbf{x} \in \mathcal{X}^R \right\}.$$

4. Solution Methods for the Nash Social Welfare Program

Observe that, in the NSWP, we have $f_i(\mathbf{x}) - d_i \geq 0$ for all $i \in S^+$ because $f_i(\mathbf{x}) \geq d_i$. Similarly, we have $d_i - f_i(\mathbf{x}) \geq 0$ for all $i \in S^-$ because $f_i(\mathbf{x}) \leq d_i$. Hence, for notational convenience, let

$$\begin{aligned} \mathcal{Y} &:= \{\mathbf{y} \in \mathbb{R}^p : \mathbf{x} \in \mathcal{X}, \\ &\quad y_i = f_i(\mathbf{x}) - d_i \text{ and } y_i \geq 0 \quad \forall i \in S^+, \\ &\quad y_i = d_i - f_i(\mathbf{x}) \text{ and } y_i \geq 0 \quad \forall i \in S^-\}, \end{aligned}$$

be the feasible set of the NSWP in the criterion space. Using this notation the NSWP can be rewritten as follows,

$$\max_{\mathbf{y} \in \mathcal{Y}} \prod_{i \in S^+ \cup S^-} y_i^{w_i}. \quad (7)$$

Assumption 2. *In the rest of this paper, we assume that \mathcal{Y} can be represented by only convex (and ideally linear) constraints involving continuous and/or integer decision variables.*

We now describe a few solution approaches for solving the NSWP using Assumption 2. The first approach is straightforward and it basically applies the log-transformation to the objective function of the NSWP. So, the NSWP can be rewritten as follows,

$$\max_{\mathbf{y} \in \mathcal{Y}} \sum_{i \in S^+ \cup S^-} w_i \log(y_i). \quad (8)$$

Observe that Problem (8) is basically maximizing a concave function over a mixed integer convex set (due to Assumption 2). So, the first approach calls a mixed integer convex programming solver to solve Problem (8). The following remark and proposition are helpful.

Remark 4. *Two advantages of the first approach are that (1) it can handle fractional weights directly and (2) it does not increase the size of the problem. However, a weakness of the first approach is that it involves log functions and (in practice) solvers are typically perform better when dealing with problems involving linear/quadratic functions than log functions (Vazirani, 2012a).*

Proposition 3. *Problem (7) is equivalent to*

$$\max_{\mathbf{y} \in \mathcal{Y}} \prod_{i \in S^+ \cup S^-} y_i^{\alpha w_i}$$

for any arbitrary $\alpha > 0$.

PROOF. This immediately follows from two observations: first, Problem (7) is equivalent to Problem (8).

Second, by multiplying the objective function of Problem (8) to a positive constant, an equivalent problem can be constructed. \square

The second approach applies piecewise linearization techniques to first linearize the log functions in Problem (8), and then call a (mixed) integer convex/linear programming solver to solve the linearized problem (Caragiannis et al., 2016). In general, the piecewise linear function is an approximation of $\sum_{i \in S^+ \cup S^-} w_i \log(y_i)$. However, if y_i can only take integer values in the interval $[l_i, u_i]$ for all $i \in S^+ \cup S^-$ where $l_i, u_i \in \mathbb{Z}$ and are non-negative then the piecewise linear function can be an exact representation of $\sum_{i \in S^+ \cup S^-} w_i \log(y_i)$. This is because one can set the breakpoints of the piecewise linear function for y_i where $i \in S^+ \cup S^-$ to be all integer values in the interval $[l_i, u_i]$. For example, if y_i can only take integer values for all $i \in S^+ \cup S^-$ then an exact piecewise linearized reformulation of Problem (8) can be stated as follows,

$$\begin{aligned}
& \max_{\mathbf{y} \in \mathcal{Y}} \sum_{i \in S^+ \cup S^-} w_i \sum_{j=1}^{u_i - l_i + 1} b_{ij} \log(l_i + j - 1) \\
& \text{s.t. } y_i = \sum_{j=1}^{u_i - l_i + 1} b_{ij} (l_i + j - 1) & \forall i \in S^+ \cup S^- \\
& \sum_{j=1}^{u_i - l_i + 1} b_{ij} = 1 & \forall i \in S^+ \cup S^- \\
& b_{ij} \in \{0, 1\} & \forall i \in S^+ \cup S^- \text{ and } \forall j \in \{1, \dots, u_i - l_i + 1\}
\end{aligned} \tag{9}$$

Observe that the interval $[l_i, u_i]$ consists of $u_i - l_i + 1$ integer values for any $i \in S^+ \cup S^-$. Hence, in Problem (9), b_{ij} is a binary decision variable that takes the value of one if the value of y_i is equal to j -th smallest integer value in the interval $[l_i, u_i]$, i.e, $l_i + j - 1$, and zero otherwise. The constraints of Problem (9) ensure that b_{ij} follows its definition for all $i \in S^+ \cup S^-$ and $j \in \{1, \dots, u_i - l_i + 1\}$. Also, the term

$$\sum_{j=1}^{u_i - l_i + 1} b_{ij} \log(l_i + j - 1)$$

in the objective function of Problem (9) captures the value of $\log(y_i)$ for all $i \in S^+ \cup S^-$. Note that the term $\log(l_i + j - 1)$ is a parameter and can be computed in advance. The following remark is helpful.

Remark 5. *Two advantages of the second approach is that it can handle fractional weights and it can use the power of commercial integer linear programming solvers such as CPLEX or Gurobi if \mathcal{Y} is represented by linear constraints involving continuous and/or integer decision variables. However, two weaknesses of the*

second approach includes: (1) it introduces additional binary decision variables which can increase the size of the problem, (2) it can be an approximation approach if y_i is continuous for $i \in S^+ \cup S^-$.

The third approach assumes that w_i is a positive integer value for all $i \in S^+ \cup S^-$. Of course this is not an unreasonable assumption because, by Proposition 3, one can multiply the objective function of Problem (8) by a sufficiently large (positive) number to change all fractional (rational) weights to integer values. So, we can assume that $w_i \in \mathbb{Z}$ for all $i \in S^+ \cup S^-$. The third approach is motivated by this observation that the NSWP can be reformulated as a geometric-mean optimization problem as follows,

$$\begin{aligned} \max_{\mathbf{y} \in \mathcal{Y}} \quad & \gamma \\ \text{s.t.} \quad & 0 \leq \gamma \leq \sqrt[W]{\prod_{i \in S^+ \cup S^-} y_i^{w_i}}, \end{aligned}$$

where $W := \sum_{i=1}^p w_i$ and γ is a nonnegative decision variable that captures the value of the geometric mean (for optimal solutions). Note that because the problem is in the form of maximization, we must have $\gamma^W = \prod_{i \in S^+ \cup S^-} y_i^{w_i}$ for any optimal solution. However, the main question is that how Problem (9) should be solved. The underlying idea of the third approach is that Problem (9) should be rewritten in a form that can be solved by commercial solvers. As stated by Charkhgard et al. (2018), based on the work of Ben-Tal and Nemirovski (2001), a geometric-mean constraint can be replaced by some linear constraints and second-order cone constraints (and introducing some additional continuous variables). In order to do so, in the geometric-mean constraint, $y_i^{w_i}$ should be first replaced by the multiplication of w_i copies of y_i for each $i \in S^+ \cup S^-$. In other words, for each $i \in S^+ \cup S^-$, we can replace $y_i^{w_i}$ by

$$\prod_{j=\mathcal{W}_{i-1}+1}^{\mathcal{W}_i} \tau_j^0,$$

$\mathcal{W}_i := \sum_{k=1}^i w_k$ and $\tau_j^0 = y_i$ for all $j = \mathcal{W}_{i-1} + 1, \dots, \mathcal{W}_i$. As an aside, in this article the superscript of variable τ is just an index and does not indicate its power. With this in mind, let K be the smallest positive integer value such that $2^K \geq W$. By introducing some additional non-negative variables and constraints,

Problem (9) can be reformulated as follows,

$$\begin{aligned}
& \max \gamma \\
& \text{s.t. } 0 \leq \gamma \leq \sqrt{\tau_1^{K-1} \tau_2^{K-1}} \\
& \quad 0 \leq \tau_j^l \leq \sqrt{\tau_{2j-1}^{l-1} \tau_{2j}^{l-1}} \quad \text{for } j = 1, \dots, 2^{K-l} \text{ and } l = 1, \dots, K-1 \\
& \quad 0 \leq \tau_j^0 = y_i \quad \text{for } j = \mathcal{W}_{i-1} + 1, \dots, \mathcal{W}_i \text{ and } \forall i \in S^+ \cup S^- \\
& \quad 0 \leq \tau_j^0 = \gamma \quad \text{for } j = W + 1, \dots, 2^K \\
& \quad \mathbf{y} \in \mathcal{Y}.
\end{aligned} \tag{10}$$

Interested readers may refer to Ben-Tal and Nemirovski (2001) for further details about this transformation. Note that any constraint of the form $\{u, v, z \geq 0 : u \leq \sqrt{vz}\}$ is equivalent to $\{u, v, z \geq 0 : \sqrt{u^2 + (\frac{v-z}{2})^2} \leq \frac{v+z}{2}\}$ which is a *second-order cone* constraint. The following remarks are helpful.

Remark 6. *An advantage of the third approach is that it can use the power of commercial integer linear/quadratic programming solvers such as CPLEX or Gurobi if \mathcal{Y} is represented by linear/quadratic constraints involving continuous and/or integer decision variables. However, a weakness of the third approach is that it requires all weights to be multiplied (at the same time) by a sufficiently large positive value to become integer and this can result in introducing many second-order cone constraints that can increase the size of the problem. For example, if $p = 2$ and $w_1 = 0.01$ and $w_2 = 1$ then the weights should be multiplied by 100 and we will have $w_1 = 1$ and $w_2 = 100$. So, in that case, we will have $K = 7$ and therefore 127 second-order cone constraint need to be added.*

Remark 7. *Based on Assumption 2, if \mathcal{Y} does not involve any integer decision variables then Problem (10) can be solved in polynomial time because it is a convex program.*

Finally, it is worth mentioning that some other effective (but sophisticated) approaches have been developed for solving the NSWP in recent years. Those approaches fall under the category of ‘solution methods for optimization over the Pareto-optimal solutions’ because they are based on this observation that the NSWP is trying to return a Pareto-optimal solution that maximizes the Nash social welfare function. In other words, those approaches try to find a Pareto-optimal point in each iteration and then check its optimality for the NSWP. If it is not optimal then they modify the search and find a different Pareto-optimal point. To the best of our knowledge, those approaches are shown to be competitive with (and even better than) all the

approaches mentioned above when \mathcal{Y} is represented by only linear constraints involving continuous and/or integer decision variables. Interested readers may refer to Charkhgard et al. (2018); Saghand et al. (2019); Saghand and Charkhgard (2019a,b) and Vazirani (2012a,b) to learn more about those approaches. Specifically, a detailed computational study comparing different solution approaches can be found in Saghand and Charkhgard (2019b).

5. Final Remarks

In this paper, we introduced four main challenges when dealing with single/multi objective optimization problems including computational barrier, confusion/nonexistence of decision makers, ignoring unsupported Pareto-optimal solutions, and fairness/equity. To overcome the challenges, we proposed to develop a game-theoretic mathematical program, named as the NSWP, for any single/multi objective optimization problem. We showed that there are several effective techniques to solve the NSWP. To see the effectiveness of the NSWP in different fields, interested readers may refer to Sierra-Altamiranda et al. (2020) in conservation planning/ecology, Mendoza-Alonzo et al. (2019); Acuna et al. (2019) in healthcare, and Melendez et al. (2019) in energy. Overall, we hope that the simplicity and advantageous of our proposed approach encourage practitioners/researchers to use the NSWP in practice.

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The first author of this paper has been working on the concept of the NSWP since 2013. He introduced this concept in his multi-objective optimization course in Spring 2017 and the technique became quite popular among many engineering PhD students/researchers at the University of South Florida from different fields in a short period of time. So, this motivated the authors to write this short article for disseminating the idea of the NSWP in the optimization community. Therefore, the authors would like to thank all the colleagues that employed the NSWP in their research because they were indeed the true motivation for writing this short article.

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