A note on the Lasserre hierarchy for different formulations of the maximum independent set problem

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ABSTRACT
In this note, we consider several polynomial optimization formulations of the maximum independent set problem and the use of the Lasserre hierarchy with these different formulations. We demonstrate using computational experiments that the choice of formulation may have a significant impact on the resulting bounds. We also provide theoretical justifications for the observed behavior.

KEYWORDS
Maximum Independent Set, Polynomial Optimization, Dual Bounds, Lasserre Hierarchy

1. Introduction

Polynomial optimization and its close connections with semidefinite and conic optimization have attracted a lot of attention in recent years [1]. It is well known that semidefinite optimization has had a tremendous impact on combinatorial optimization, particularly with the groundbreaking results of Lovász and Schrijver [2] and Goemans and Williamson [3]. This motivates the study of the application of polynomial optimization to combinatorial optimization problems, particularly through the use of the Lasserre hierarchy [4]. A comparison of the Lasserre hierarchy in this context with other hierarchies from the literature was carried out by Laurent [5].

Combinatorial problems can typically be formulated in different ways, and it is known that different formulations of the same combinatorial problem may lead to different semidefinite relaxations and hence to different global bounds; the example of maximum-cut is explored in [6]. Therefore it is of interest to study the impact of the choice of formulations when using relaxations from the Lasserre hierarchy.

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In this note, we consider several polynomial optimization formulations of the maximum independent set problem and the use of the Lasserre hierarchy with these different formulations. We demonstrate using computational experiments that the choice of formulation may have a significant impact on the resulting bounds. We also provide theoretical justifications for the observed behavior.

The paper is organized as follows. In Section 2, we give some preliminaries about polynomial optimization, while Section 3 introduces the maximum independent set problem and computationally motivates the interest of looking at different polynomial formulations for the problem. Section 4 provides the theoretical content of the paper and in Section 5 we draw some concluding remarks.

2. Preliminaries

Polynomial optimization is NP-hard in general, and the Lasserre hierarchy has great theoretical and practical appeal because it provides a sequence of tractable relaxations whose optimal objective values converge to the global optimum. We briefly review the construction of the Lasserre hierarchy (in the dual form). For more details about Lasserre hierarchy, see e.g. [1,4,7,8].

Given polynomials f, $g_1 \dots g_m$, we consider the following general polynomial optimization problem:

$$\begin{aligned} f_{min} = \min \quad & f(x) \\ \text{s.t.} \quad & g_j(x) \geq 0,\ \forall j = 1, \dots, m. \end{aligned} \tag{1}$$

Let $\{x^\alpha\}_{\alpha \in \mathbb{N}^n}$ be a canonical basis for $\mathbb{R}[x]$, and let $y = \{y_\alpha\}_{\alpha \in \mathbb{N}^n}$ be a given sequence indexed by that basis. Given $y = \{y_\alpha\}_{\alpha \in \mathbb{N}^n}$, let $L_y : \mathbb{R}[x] \to \mathbb{R}$ be the linear functional

$$f = \sum_{\alpha \in \mathbb{N}^n} f_\alpha x^\alpha \to L_y(f) = \sum_{\alpha \in \mathbb{N}^n} f_\alpha y_\alpha.$$

The *moment matrix* $M_d(y)$ is a matrix with rows and columns indexed in the basis $\{x^\alpha\}_{\alpha \in \mathbb{N}^n_d}$ of $\mathbb{R}[x]_d$, and it has entries

$$M_d(y)(\alpha, \beta) = y_{\alpha+\beta},\ \alpha,\ \beta\ \in\ \mathbb{N}^n,\ |\alpha|,\ |\beta| \leq d.$$

Given a polynomial $\theta(x) = \sum_{\gamma \in \mathbb{N}^n} \theta_\gamma x^\gamma$, the *localizing matrix* $M_d(\theta \star y)$ is also a matrix with rows and columns indexed in the basis $\{x^\alpha\}_{\alpha \in \mathbb{N}^n_d}$ of $\mathbb{R}[x]_d$, and it has entries

$$M_d(\theta \star y)(\alpha, \beta) = \sum_{\gamma \in \mathbb{N}^n} \theta_\gamma y_{\alpha+\beta+\gamma},\ \alpha,\ \beta\ \in\ \mathbb{N}^n,\ |\alpha|,\ |\beta| \leq d.$$

Let the degree of g_j is $2v_j$ or $2v_j - 1$. Then, for problem (1), the Lasserre relaxation of order d provides a lower bound ρ_d for f_{min}:

$$\begin{aligned} \rho_d = \min \quad & L_y(f) \\ \text{s.t.} \quad & M_d(y) \succeq 0, \\ & M_{d-v_j}(g_j^\star y) \succeq 0, \forall j = 1, \ldots, m, \\ & L_y(1) = 1. \end{aligned} \tag{2}$$

The sequence of Lasserre relaxations of increasing order $d = 1, 2, 3, \ldots$ forms the Lasserre hierarchy.

3. Maximum independent set

Given a graph $G = (V, E)$, the maximum independent set problem consists of determining the maximum cardinality of any subset of vertices such that no two vertices in that subset are connected by an edge of G. We consider four different formulations of this problem using quadratic polynomials. The formulations are:

$$\rho = \max \quad \sum_{i \in V} x_i$$

$$\text{s.t.} \quad \begin{aligned} x_i x_j = 0, \quad & \forall (i,j) \in E, \\ x_i^2 - x_i = 0, \quad & \forall i \in V. \end{aligned} \tag{3}$$

$$\text{or} \quad \text{s.t.} \quad \begin{aligned} x_i x_j \leq 0, \quad & \forall (i,j) \in E, \\ x_i^2 - x_i = 0, \quad & \forall i \in V. \end{aligned} \tag{4}$$

$$\text{or} \quad \text{s.t.} \quad \begin{aligned} x_i + x_j \leq 1, \quad & \forall (i,j) \in E, \\ x_i^2 - x_i = 0, \quad & \forall i \in V. \end{aligned} \tag{5}$$

$$\text{or} \quad \text{s.t.} \quad \begin{aligned} x_i x_j = 0, \quad & \forall (i,j) \in E, \\ 0 \leq x_i \leq 1, \quad & \forall i \in V. \end{aligned} \tag{6}$$

Let us compute the upper bounds arising from the Lasserre relaxation (2) of order $d = 1$ for each of the above four formulations. The bounds are reported in Table 1, where C_n denotes the cycle with n vertices and K_4 denotes the complete graph with 4 vertices.

We observe that the bounds obtained using (3) and (4) are always equal, and are the best for all of these graphs. On the other hand, the bounds from (6) are consistently the weakest; indeed the bound obtained using (6) is always equal to the trivial upper bound $|V|$, as we prove formally in Proposition 4.1 below. Finally, the bounds from (5) are equal or moderately weaker than those from (3) and (4).

Although these results are only for 7 small graphs, they clearly show that the choice of formulation dramatically impacts the quality of the resulting bound. The remainder of this note is concerned with providing some theoretical justification for the results in Table 1.

Table 1.: Bounds from the Lasserre relaxation of order $d = 1$ for different formulations

Graph	Optimal bound	Bound from (3)	Bound from (4)	Bound from (5)	Bound from (6)
C_3	1	1	1	1.5	3
C_4	2	2	2	2	4
K_4	1	1	1	2	4
C_5	2	2.236	2.236	2.5	5
C_6	3	3	3	3	6
C_7	3	3.318	3.318	3.5	7
Peterson graph	4	4	4	5	10

4. Independent Set Formulations and Lasserre Relaxations

4.1. Notation

Let us consider the following polynomials of $\mathbb{R}^{|V|}$

$$
\begin{aligned}
g_{ij}^{+}(x) &= x_i x_j, && \forall (i,j) \in E \\
g_{ij}^{-}(x) &= -x_i x_j, && \forall (i,j) \in E \\
h_i^{+}(x) &= x_i^2 - x_i, && \forall i \in V \\
h_i^{-}(x) &= -x_i^2 + x_i, && \forall i \in V \\
q_i^{+}(x) &= x_i, && \forall i \in V \\
q_i^{-}(x) &= 1 - x_i, && \forall i \in V \\
l_{ij}(x) &= 1 - x_i - x_j, && \forall (i,j) \in E \\
f(x) &= \sum_{i \in V} x_i.
\end{aligned}
$$

Let $(e_i)_{i \in V}$ be the canonical basis of $\mathbb{R}^{|V|}$.

For a fixed degree d, the Lasserre relaxations of order d of the above formulations are:

$$
\begin{aligned}
\rho_{d,3} = \max \quad & L_y(f) \\
\text{s.t.} \quad & M_d(y) \succeq 0, \\
& M_{d-1}(g_{ij}^{+} \star y) = 0, \quad \forall (i,j) \in E, \\
& M_{d-1}(h_i^{+} \star y) = 0, \quad \forall i \in V, \\
& L_y(1) = 1.
\end{aligned}
$$

$$
\begin{aligned}
\rho_{d,4} = \max \quad & L_y(f) \\
\text{s.t.} \quad & M_d(y) \succeq 0, \\
& M_{d-1}(g_{ij}^- \star y) \succeq 0, \quad \forall (i,j) \in E, \\
& M_{d-1}(h_i^+ \star y) = 0, \quad \forall i \in V, \\
& L_y(1) = 1.
\end{aligned}
$$

$$
\begin{aligned}
\rho_{d,5} = \max \quad & L_y(f) \\
\text{s.t.} \quad & M_d(y) \succeq 0, \\
& M_{d-1}(l_{ij} \star y) \succeq 0, \quad \forall (i,j) \in E, \\
& M_{d-1}(h_i^+ \star y) = 0, \quad \forall i \in V, \\
& L_y(1) = 1.
\end{aligned}
$$

$$
\begin{aligned}
\rho_{d,6} = \max \quad & L_y(f) \\
\text{s.t.} \quad & M_d(y) \succeq 0, \\
& M_{d-1}(g_{ij}^+ \star y) = 0, \quad \forall (i,j) \in E, \\
& M_{d-1}(q_i^+ \star y) \succeq 0, \quad \forall i \in V, \\
& M_{d-1}(q_i^- \star y) \succeq 0, \quad \forall i \in V, \\
& L_y(1) = 1.
\end{aligned}
$$

4.2. Value of $\rho_{1,6}$ for every graph G

Our first result is the proof that the optimal value of the Lasserre relaxation of order $d = 1$ using formulation (6) is equal to $|V|$ for every graph G.

Proposition 4.1. *For every graph G, $\rho_{1,6} = |V|$.*

Proof. For every feasible solution $\{y_\alpha\}$, we have $0 \leq y_{e_i} \leq 1, \ \forall i \in V$, therefore

$$\sum_{i \in V} y_{e_i} = L_y(f) \leq |V|.$$

To show attainment, consider $y^* = \{y^*_\alpha\}$ such that:

$$
\begin{cases}
y^*_0 = 1, & \\
y^*_{e_i} = 1, & \forall i \in V, \\
y^*_{2e_i} = |V| + 1, & \forall i \in V, \\
y^*_{e_i+e_j} = 0, & \forall (i,j) \in V^2, \quad i \neq j.
\end{cases}
$$

It is straightforward to check that y^* is a feasible solution of the Lasserre relaxation of order $d = 1$ for formulation (6), and that this solution achieves the objective value $|V|$. □

4.3. Relationship between $\rho_{d,3}$ and $\rho_{d,4}$

The next proposition shows that the set of feasible solutions of the d order Lasserre relaxation for formulation (3) is a subset of the set of feasible solutions the relaxation with the same order for formulation (4). Moreover, for $d \geq 2$, the two feasible sets are equal, and hence so are the bounds.

Proposition 4.2. *For every graph G and order $d \geq 1$, $\rho_{d,3} \leq \rho_{d,4}$. Moreover, if $d \geq 2$, then $\rho_{d,3} = \rho_{d,4}$.*

Proof. The first claim follows from the observation that $M_{d-1}(g_{ij}^{+} \star y) = 0$ implies $M_{d-1}(g_{ij}^{-} \star y) \succeq 0$.

To prove the second claim, let $y = \{y_\alpha\}$ be a feasible solution of the Lasserre hierarchy of order d for formulation (4). We know that for every $(\alpha, \beta) \in \mathbb{N}_{d-1}^{|V|} \times \mathbb{N}_{d-1}^{|V|}$

$$
\begin{aligned}
M_{d-1}(g_{ij}^{-} \star y)(\alpha, \beta) &= -y_{\alpha+\beta+e_i+e_j}, && \forall (i,j) \in E,\\
M_{d-1}(h_i^{+} \star y)(\alpha, \beta) &= y_{\alpha+\beta+2e_i} - y_{\alpha+\beta+e_i} = 0, && \forall i \in V.
\end{aligned}
$$

then for every $\alpha \in \mathbb{N}_{d-2}^{|V|}$ and $k \in V$,

$$
\begin{aligned}
M_{d-1}(g_{ij}^{-} \star y)(\alpha, \alpha) &= -y_{2\alpha+e_i+e_j}\\
&= -y_{2\alpha+2e_i+e_j}\\
&= -y_{2\alpha+2e_i+2e_j}\\
&= -M_d(y)(\alpha + e_i + e_j, \alpha + e_i + e_j).
\end{aligned}
$$

and

$$
\begin{aligned}
\det\left[M_{d-1}(g_{ij}^{-} \star y)_{\{\alpha,\alpha+e_k\},\{\alpha,\alpha+e_k\}}\right] &= \begin{vmatrix} -y_{2\alpha+e_i+e_j} & -y_{2\alpha+e_k+e_i+e_j} \\ -y_{2\alpha+e_k+e_i+e_j} & -y_{2\alpha+2e_k+e_i+e_j} \end{vmatrix}\\
&= \begin{vmatrix} -y_{2\alpha+e_i+e_j} & -y_{2\alpha+e_k+e_i+e_j} \\ -y_{2\alpha+e_k+e_i+e_j} & -y_{2\alpha+e_k+e_i+e_j} \end{vmatrix}\\
&= y_{2\alpha+e_i+e_j} y_{2\alpha+e_k+e_i+e_j} - y_{2\alpha+e_k+e_i+e_j}^2.
\end{aligned}
$$

Since $M_{d-1}(g_{ij}^{-} \star y)$ and $M_d(y)$ are semi-definite positive matrices then

$$
\begin{cases}
M_{d-1}(g_{ij}^{-} \star y)(\alpha, \alpha) \geq 0,\\
M_d(y)(\alpha + e_i + e_j, \alpha + e_i + e_j) \geq 0,\\
\det\left[M_{d-1}(g_{ij}^{-} \star y)_{\{\alpha,\alpha+e_k\},\{\alpha,\alpha+e_k\}}\right] \geq 0.
\end{cases}
$$

Which implies that

$$
\begin{cases}
y_{2\alpha+e_i+e_j} = 0,\\
-y_{2\alpha+e_k+e_i+e_j}^2 \geq 0,
\end{cases}
$$

and so

$$\begin{cases} y_{2\alpha+e_i+e_j} = 0, \\ y^2_{2\alpha+2e_k+e_i+e_j} = 0. \end{cases}$$

This proves that $M_{d-1}(g^-_{ij} \star y)(\alpha, \alpha) = 0$ for every $\alpha \in \mathbb{N}^{|V|}_{d-1}$. Therefore $M_{d-1}\left(g^-_{ij} \star y\right)$ is semi-definite positive matrix with zero on the diagonal. It is the zero matrix and this proves that y is also a feasible solution of the level d of the Lasserre hierarchy for formulation (3). □

4.4. Relationship between $\rho_{d,4}$ and $\rho_{d,5}$

The next result is that the feasible set of the Lasserre relaxation of order d using (4) is a subset of the feasible set of the relaxation of the same order for formulation (5). Hence, the bound $\rho_{d,5}$ is always dominated by the bound $\rho_{d,4}$.

Proposition 4.3. *For every graph G and order $d \geq 1$, $\rho_{d,4} \leq \rho_{d,5}$.*

Proof. Let $y = \{y_\alpha\}$ be a feasible solution of the relaxation of (4) of order d. We know that for every $(\alpha, \beta) \in \mathbb{N}^{|V|}_{d-1} \times \mathbb{N}^{|V|}_{d-1}$:

$$\begin{aligned} M_{d-1}(g^-_{ij} \star y)(\alpha, \beta) &= -y_{\alpha+\beta+e_i+e_j} && \forall (i,j) \in E, \\ M_{d-1}(h^+_i \star y)(\alpha, \beta) &= y_{\alpha+\beta+2e_i} - y_{\alpha+\beta+e_i} && \forall i \in V, \\ M_{d-1}(l_{ij} \star y)(\alpha, \beta) &= y_{\alpha+\beta} - y_{\alpha+\beta+e_i} - y_{\alpha+\beta+e_j} && \forall (i,j) \in E. \end{aligned}$$

Let A be the matrix indexed by $\mathbb{N}^{|V|}_{d-1} \times \mathbb{N}^{|V|}_{d}$ such that:

$$A(\alpha, \gamma) = \begin{cases} 1 & \text{if } \gamma = \alpha, \\ -1 & \text{if } \gamma = \alpha + e_i, \\ -1 & \text{if } \gamma = \alpha + e_j, \\ 0 & \text{otherwise.} \end{cases}$$

For every $(\alpha,\beta) \in \mathbb{N}^{|V|}_{d-1} \times \mathbb{N}^{|V|}_{d-1}$:

$$\begin{aligned}
\left[AM_d(y)A^T\right](\alpha,\beta) &= \sum_{\gamma\in\mathbb{N}^{|V|}_d}\sum_{\delta\in\mathbb{N}^{|V|}_d} A(\alpha,\gamma)M_d(y)(\gamma,\delta)A^T(\delta,\beta)\\
&= \sum_{\gamma\in\mathbb{N}^{|V|}_d}\sum_{\delta\in\mathbb{N}^{|V|}_d} A(\alpha,\gamma)M_d(y)(\gamma,\delta)A(\beta,\delta)\\
&= \sum_{\gamma\in\mathbb{N}^{|V|}_d}\sum_{\delta\in\mathbb{N}^{|V|}_d} A(\alpha,\gamma)y_{\gamma+\delta}A(\beta,\delta)\\
&= y_{\alpha+\beta} - y_{\alpha+\beta+e_i} - y_{\alpha+\beta+e_j}\\
&\quad - y_{\alpha+e_i+\beta} + y_{\alpha+e_i+\beta+e_j} + y_{\alpha+e_i+\beta+e_j}\\
&\quad - y_{\alpha+e_j+\beta} + y_{\alpha+e_j+\beta+e_i} + y_{\alpha+e_j+\beta+e_j}\\
&= \underbrace{y_{\alpha+\beta} - y_{\alpha+\beta+e_i} - y_{\alpha+\beta+e_j}}_{=M_{d-1}(l_{ij}\star y)(\alpha,\beta)} + \underbrace{y_{\alpha+\beta+2e_i} - y_{\alpha+\beta+e_i}}_{=M_{d-1}(h_i^+\star y)(\alpha,\beta)}\\
&\quad + \underbrace{y_{\alpha+\beta+2e_j} - y_{\alpha+\beta+e_j}}_{=M_{d-1}(h_j^+\star y)(\alpha,\beta)} + 2\underbrace{y_{\alpha+\beta+e_i+e_j}}_{=-M_{d-1}(g_{ij}^-\star y)(\alpha,\beta)},
\end{aligned}$$

which implies that

$$M_{d-1}(l_{ij}\star y) = AM_d(y)A^T + 2M_{d-1}(g_{ij}^-\star y) - M_{d-1}(h_j^+\star y) - M_{d-1}(h_i^+\star y).$$

Since $M_d(y) \succeq 0$ then $AM_d(y)A^T \succeq 0$. Moreover

$$\begin{cases} M_{d-1}(g_{ij}^-\star y) \succeq 0,\\ M_{d-1}(h_j^+\star y) = M_{d-1}(h_i^+\star y) = 0.\end{cases}$$

Then $M_{d-1}(l_{ij}\star y) \succeq 0$ for all $(i,j)\in E$, and thus y is a feasible solution of the d order Lasserre relaxation for formulation (5). □

4.5. Relationship between $\rho_{d,3}$ and $\rho_{d,6}$

The next result is that the feasible set of the Lasserre relaxation of order d using (3) is a subset of the feasible set of the relaxation of the same order for formulation (6). Hence, the bound $\rho_{d,3}$ is always dominated by the bound $\rho_{d,6}$.

Proposition 4.4. *For every graph G and order $d \geq 1$, $\rho_{d,3} \leq \rho_{d,6}$.*

Proof. Let $y = \{y_\alpha\}$ be a feasible solution of the relaxation of (3) of order d. We know that for every $(\alpha,\beta) \in \mathbb{N}^{|V|}_{d-1} \times \mathbb{N}^{|V|}_{d-1}$:

$$\begin{aligned}
M_{d-1}(g_{ij}^-\star y)(\alpha,\beta) &= -y_{\alpha+\beta+e_i+e_j} && \forall (i,j)\in E,\\
M_{d-1}(h_i^+\star y)(\alpha,\beta) &= y_{\alpha+\beta+2e_i} - y_{\alpha+\beta+e_i} && \forall i\in V,\\
M_{d-1}(q_i^+\star y)(\alpha,\beta) &= y_{\alpha+\beta+e_i} && \forall i\in V,\\
M_{d-1}(q_i^-\star y)(\alpha,\beta) &= y_{\alpha+\beta} - y_{\alpha+\beta+e_i} && \forall i\in V.
\end{aligned}$$

Let A be the matrix indexed by $\mathbb{N}^{|V|}_{d-1} \times \mathbb{N}^{|V|}_{d}$ such that:

$$A(\alpha,\gamma) = \begin{cases} 1 & \text{if } \gamma = \alpha + e_i, \\ 0 & \text{otherwise.} \end{cases}$$

For every $(\alpha,\beta) \in \mathbb{N}^{|V|}_{d-1} \times \mathbb{N}^{|V|}_{d-1}$:

$$\begin{aligned}
\left[AM_d(y)A^T\right](\alpha,\beta) &= \sum_{\gamma\in\mathbb{N}^{|V|}_d} \sum_{\delta\in\mathbb{N}^{|V|}_d} A(\alpha,\gamma)M_d(y)(\gamma,\delta)A^T(\delta,\beta) \\
&= \sum_{\gamma\in\mathbb{N}^{|V|}_d} \sum_{\delta\in\mathbb{N}^{|V|}_d} A(\alpha,\gamma)M_d(y)(\gamma,\delta)A(\beta,\delta) \\
&= \sum_{\gamma\in\mathbb{N}^{|V|}_d} \sum_{\delta\in\mathbb{N}^{|V|}_d} A(\alpha,\gamma)y_{\gamma+\delta}A(\beta,\delta) \\
&= y_{\alpha+e_i+\beta+e_i} \\
&= y_{\alpha+\beta+2e_i} - y_{\alpha+\beta+e_i} + y_{\alpha+\beta+e_i} \\
&= M_{d-1}(h_i^+ \star y)(\alpha,\beta) + M_{d-1}(q_i^+ \star y)(\alpha,\beta),
\end{aligned}$$

which implies that

$$M_{d-1}(q_i^+ \star y) = AM_d(y)A^T - M_{d-1}(h_i^+ \star y).$$

Since $M_d(y) \succeq 0$ then $AM_d(y)A^T \succeq 0$. Moreover $M_{d-1}(h_i^+ \star y) = 0$. Then $M_{d-1}(q_i^+ \star y) \succeq 0$ for all $i \in V$. On the other hand, let B be the matrix indexed by $\mathbb{N}^{|V|}_{d-1} \times \mathbb{N}^{|V|}_{d}$ such that:

$$B(\alpha,\gamma) = \begin{cases} 1 & \text{if } \gamma = \alpha, \\ -1 & \text{if } \gamma = \alpha + e_i, \\ 0 & \text{otherwise.} \end{cases}$$

For every $(\alpha,\beta) \in \mathbb{N}^{|V|}_{d-1} \times \mathbb{N}^{|V|}_{d-1}$:

$$\begin{aligned}
\left[BM_d(y)B^T\right](\alpha,\beta) &= \sum_{\gamma\in\mathbb{N}^{|V|}_d} \sum_{\delta\in\mathbb{N}^{|V|}_d} B(\alpha,\gamma)M_d(y)(\gamma,\delta)B^T(\delta,\beta) \\
&= \sum_{\gamma\in\mathbb{N}^{|V|}_d} \sum_{\delta\in\mathbb{N}^{|V|}_d} B(\alpha,\gamma)M_d(y)(\gamma,\delta)B(\beta,\delta) \\
&= \sum_{\gamma\in\mathbb{N}^{|V|}_d} \sum_{\delta\in\mathbb{N}^{|V|}_d} B(\alpha,\gamma)y_{\gamma+\delta}B(\beta,\delta) \\
&= y_{\alpha+\beta} - y_{\alpha+\beta+e_i} - y_{\alpha+e_i+\beta} + y_{\alpha+e_i+\beta+e_i} \\
&= y_{\alpha+\beta} - y_{\alpha+\beta+e_i} - y_{\alpha+\beta+e_i} + y_{\alpha+\beta+2e_i} \\
&= M_{d-1}(q_i^- \star y)(\alpha,\beta) + M_{d-1}(h_i^+ \star y)(\alpha,\beta),
\end{aligned}$$

which implies that

$$M_{d-1}(q_i^- \star y) = BM_d(y)B^T - M_{d-1}(h_i^+ \star y).$$

Since $M_d(y) \succeq 0$ then $BM_d(y)B^T \succeq 0$. Moreover $M_{d-1}(h_i^+ \star y) = 0$. Then $M_{d-1}(q_i^- \star y) \succeq 0$ for all $i \in V$. Since $M_{d-1}(g_{ij}^- \star y) = 0$, y is a feasible solution of the d order Lasserre relaxation for formulation (6). □

4.6. Relationships with the linear programming relaxations of maximum independent set

In this section, we establish the relationship among some of the formulations discussed above and two famous linear programming (LP) relaxations for the maximum independent set problem. More precisely, we consider two LP formulations and we refer to them as "weak" and "strong". The weak formulation is that with constraints

$$x_i + x_j \leq 1 \quad \forall (i,j) \in E \tag{7}$$

and nonnegativity, whereas the strong formulation replaces constraints (7) with constraints

$$\sum_{i \in C} x_i \leq 1 \quad \forall C \in \mathcal{C}, \tag{8}$$

where $\mathcal{C}$ is the set of all maximal cliques in G.

Proposition 4.5. *For every graph $G = (V, E)$, the optimal value of the order 1 of the Lasserre hierarchy for formulation* (5) *is equal to the value of the LP-relaxation of the weak formulation of the independent set problem.*

Proof. Let $y = \{y_\alpha\}_\alpha$ be a feasible solution of the order 1 of the Lasserre hierarchy of formulation (5), then

$$\begin{cases} y_0 = 1, & \\ y_{e_i} = y_{2e_i}, & \forall i \in V, \\ y_{e_i} + y_{e_j} \leq 1, & \forall (i,j) \in E \quad i \neq j. \end{cases}$$

This proves that

$$\begin{cases} y_0 & = 1, \\ 0 \leq y_{e_i} \leq 1, & \forall i \in V, \\ y_{e_i} + y_{e_j} \leq 1, & \forall (i,j) \in E \quad i \neq j, \end{cases}$$

and $(y_{e_i})_{i \in V}$ is a feasible solution of the LP-relaxation of the weak formulation of the independent set problem. Conversely, let $(x_i)_{i \in V}$ be a feasible solution of the LP-relaxation of the weak formulation of the independent set problem, let

$$X \in \mathbb{R}^{|V|+1} \ : \ \begin{cases} X_0 = 1, \\ X_i = x_i, \quad \forall i \in V, \end{cases}$$

$$A \in \mathbb{R}^{(|V|+1)\times(|V|+1)} \ : \ \begin{cases} A \text{ diagonal}, \\ A_{0,0} = 0, \\ A_{i,i} = x_i(1-x_i) \quad \forall i \in V, \end{cases}$$

and $y = \{y_\alpha\}_{\alpha \in \mathbb{N}_2^{|V|}}$ such that

$$\begin{cases} y_0 = 1, \\ y_{e_i} = y_{2e_i} = x_i, \quad \forall i \in V, \\ y_{e_i+e_j} = x_i \times x_j, \quad \forall (i,j) \in V^2 \quad i \neq j. \end{cases}$$

Then,

$$\begin{cases} L_y(1) = y_0 = 1, \\ M_0(l_{ij} \star y) = 1 - y_{e_i} - y_{e_j} = 1 - x_i - x_j \geq 0, \quad \forall (i,j) \in E, \\ M_0(h_i^+ \star y) = y_{2e_i} - y_{e_i} = 0, \quad \forall i \in V, \\ M_1(y) = XX^T + A \succeq 0. \end{cases}$$

This proves that the y is a feasible solution of the level 1 of the Lasserre hierarchy of formulation (5) with the value equal to $\sum_{i \in V} y_{e_i} = \sum_{i \in V} x_i$. □

Proposition 4.6. *For every graph $G = (V, E)$, the optimal value of the order 1 of the Lasserre hierarchy for formulation (4) is smaller than the value of the LP-relaxation of the strong formulation of the independent set problem.*

Proof. Let $y = \{y_\alpha\}_\alpha$ be a feasible solution of the level one of the Lasserre hierarchy of formulation (3), then

$$\begin{cases} y_0 = 1, \\ y_{e_i} = y_{2e_i}, \quad \forall i \in V, \\ y_{e_i+e_j} \leq 0, \quad \forall (i,j) \in E. \end{cases}$$

Let W be a clique of G and $X \in \mathbb{R}^{|V|+1}$ such that :

$$\begin{cases} X_0 = 1, \\ X_i = -1, \quad \text{if } i \in W, \\ X_i = 0, \quad \text{otherwise}. \end{cases}$$

Then,

$$
\begin{aligned}
0 \le X^T M_1(y) X &= y_0 - 2\sum_{i\in W} y_{e_i} + \sum_{i\in W} y_{2e_i} + \sum_{\substack{(i,j)\in W^2,\\ i\neq j}} y_{e_i+e_j} \\
&= y_0 - 2\sum_{i\in W} y_{e_i} + \sum_{i\in W} y_{e_i} \\
&\le 1 - \sum_{i\in W} y_{e_i}.
\end{aligned}
$$

This proves that $(y_{e_i})_{i\in V}$ is a feasible solution of the LP-relaxation of the strong formulation of the independent set problem. □

5. Summary of Results and Future Research

We have investigated the relationship among different Lasserre relaxations originated by different polynomial formulations of the maximum independent set problem. Using the notation previously defined, we summarize our results as follows:
For $d = 1$:

$$\rho_{1,3} \le \rho_{1,4} \le \mathrm{LP}_{\mathrm{strong}} \le \mathrm{LP}_{\mathrm{weak}} = \rho_{1,5} \le |V| = \rho_{1,6},$$

and for $d \ge 2$:

$$\rho_{d,3} = \rho_{d,4} \le \rho_{d,5} \quad \text{and} \quad \rho_{d,3} = \rho_{d,4} \le \rho_{d,6}.$$

We believe these results give an interesting, initial perspective on evaluating the quality of a formulation not only in terms of its relaxation but also with respect to the Lasserre relaxations originated by it.

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