An MISOCP-Based Solution Approach to the Reactive Optimal Power Flow Problem

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Abstract

In this letter, we present an alternative mixed-integer non-liner programming formulation of the reactive optimal power flow (ROPF) problem. We utilize a mixed-integer second-order cone programming (MISOCP) based approach to find global optimal solutions of the proposed ROPF problem formulation. We strengthen the MISOCP relaxation via the addition of convex envelopes and cutting planes. Computational experiments on challenging test cases show that the MISOCP-based approach yields promising results compared to a semidefinite programming based approach from the literature.

1 Introduction

The reactive optimal power flow (ROPF) problem is a variant of the well-known optimal power flow (OPF) problem in which additional discrete decisions, such as shunt susceptance and tap ratio, are considered. Due to the presence of these discrete variables in the ROPF problem, it can be formulated as a mixed-integer non-linear programming (MINLP) problem. This letter utilizes the recent developments in the OPF problem to propose an efficient way of solving the ROPF problem.

OPF is one of the most studied problems in the area of power systems and a variety of solution approaches have been proposed in the literature. Local methods such as the interior point method try to solve the OPF problem but they do not provide any assurances of global optimality. In recent years, convex relaxations of the OPF problem have drawn considerable research interest since the convexity property promises a globally optimal solution under certain conditions. Several approaches have been developed based on convex quadratic, semidefinite programming (SDP), second order cone programming (SOCP) and convex-distflow formulations. The ROPF problem has a similar structure with the OPF problem, except the inclusion of shunt susceptance and tap ratio variables, which are typically modelled as discrete variables. The resulting MINLP problem is difficult so solve and the literature has primarily focused on various heuristic methods [2]. The systematic treatment of the ROPF problem is limited to an SDP-based relaxation called *tight-and-cheap relaxation* (TCR) proposed in [1].

This letter proposes a new MINLP formulation for the ROPF problem along with its mixed-integer second-order cone programming (MISOCP) relaxation and an improved MISOCP relaxation with convex envelopes and cutting planes. We also test the accuracy and efficiency of our approach with the TCR method from the literature on difficult test cases and obtain promising results.

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2 Mathematical Model

2.1 MINLP Formulation

Consider a power network $\mathcal{N} = (\mathcal{B}, \mathcal{L})$, where $\mathcal{B}$ and $\mathcal{L}$ denote the set of buses and the set of transmission lines respectively. Let $\mathcal{G} \subseteq \mathcal{B}$, $\mathcal{S} \subseteq \mathcal{B}$ and $\mathcal{T} \subseteq \mathcal{L}$ respectively denote the set of generators connected to the grid, the buses with a variable shunt susceptance and the lines with a variable tap ratio. Rest of the parameters are given as follows:

- For each bus $i \in \mathcal{B}$; p_i^d and q_i^d are the real and reactive power load, $\underline{V}_i$ and $\overline{V}_i$ are the bounds on the voltage magnitude, $\delta(i)$ is the set of neighbors and $\{b_{ii}^k : k \in \mathcal{S}_i\}$ is the set of allowable shunt susceptances.

- For each generator located at bus $i \in \mathcal{G}$; active and reactive outputs must be in the intervals $[p_i^{\min}, p_i^{\max}]$ and $[q_i^{\min}, q_i^{\max}]$, and we have $p_i^{\min} = p_i^{\max} = q_i^{\min} = q_i^{\max} = 0$ for $i \in \mathcal{B} \setminus \mathcal{G}$.

- For each line $(i,j) \in \mathcal{L}$; G_{ij} and B_{ij} are conductance and succeptance, $\{\tau_{ij}^l : l \in \mathcal{T}_{ij}\}$ is the set of allowable tap ratios, $\overline{S}_{ij}$ is the apparent power flow limit and $\overline{\theta}_{ij}$ is the bound on the phase angle.

We define the following decision variables:

- For each bus $i \in \mathcal{B}$, $|V_i|$ and θ_i are the voltage magnitude and phase angle, b_{ii} is the shunt susceptance, α_i^k is one if $b_{ii} = b_{ii}^k$ and zero otherwise.

- For each generator located at bus $i \in \mathcal{G}$, p_i^g and q_i^g are the real and reactive power output.

- For each line $(i,j) \in \mathcal{L}$, p_{ij} and q_{ij} are the real and reactive power flow, τ_{ij} is the tap ratio, β_{ij}^l is one if $\tau_{ij} = \tau_{ij}^l$ and zero otherwise.

Then, the ROPF problem can be modeled as the following MINLP:

$$\min \sum_{i \in \mathcal{G}} f(p_i^g) \tag{1}$$

$$p_i^g - p_i^d = g_{ii}|V_i|^2 + \sum_{j \in \delta(i)} p_{ij} \qquad i \in \mathcal{B} \tag{2}$$

$$q_i^g - q_i^d = -b_{ii}|V_i|^2 + \sum_{j \in \delta(i)} q_{ij} \qquad i \in \mathcal{B} \tag{3}$$

$$\begin{aligned} p_{ij} = G_{ij}(|V_i|/\tau_{ij})^2 + (|V_i|/\tau_{ij})|V_j|[&G_{ij}\cos(\theta_i - \theta_j) \\ &-B_{ij}\sin(\theta_i - \theta_j)] \qquad (i,j) \in \mathcal{L} \end{aligned} \tag{4}$$

$$\begin{aligned} q_{ij} = -B_{ij}(|V_i|/\tau_{ij})^2 - (|V_i|/\tau_{ij})|V_j|[&B_{ij}\cos(\theta_i - \theta_j) \\ &+G_{ij}\sin(\theta_i - \theta_j)] \qquad (i,j) \in \mathcal{L} \end{aligned} \tag{5}$$

$$\underline{V}_i \leq |V_i| \leq \overline{V}_i \qquad i \in \mathcal{B} \tag{6}$$

$$\sum_{k \in \mathcal{S}_i} b_{ii}^k \alpha_i^k = b_{ii} \quad i \in \mathcal{B}, \quad \sum_{l \in \mathcal{T}_{ij}} \frac{\beta_{ij}^l}{\tau_{ij}^l} = \frac{1}{\tau_{ij}} \quad (i,j) \in \mathcal{L} \tag{7}$$

$$\sum_{k \in \mathcal{S}_i} \alpha_i^k = 1 \quad i \in \mathcal{B}, \quad \sum_{l \in \mathcal{T}_{ij}} \beta_{ij}^l = 1 \quad (i,j) \in \mathcal{L} \tag{8}$$

$$\alpha_i^k \in \{0,1\} \qquad i \in \mathcal{B}, \ \beta_{ij}^l \in \{0,1\} \qquad (i,j) \in \mathcal{L} \tag{9}$$

$$b_{ii} = 0 \qquad i \notin \mathcal{S}, \ \tau_{ij} = 1 \qquad (i,j) \notin \mathcal{T} \tag{10}$$

$$q_i^{\min} \le q_i^g \le q_i^{\max}, \qquad p_i^{\min} \le p_i^g \le p_i^{\max} \qquad i \in \mathcal{G} \tag{11}$$

$$p_{ij}^2 + q_{ij}^2 \le \overline{S}_{ij}^2, \qquad |\theta_i - \theta_j| \le \overline{\theta}_{ij} \qquad (i,j) \in \mathcal{L}. \tag{12}$$

Here, the objective function (1) minimizes the total real power generation cost subject to the following constraints: real and reactive power flow balance at bus i (2)–(3), real and reactive power flow from i to j (4)–(5), shunt susceptance selection for bus i and tap ratio selection for line (i,j) (7), voltage magnitude bounds at bus i (6), binary restrictions (8)–(9), reactive and active power output of generator i (11), apparent flow and phase angle limit for each line (i,j) (12).

2.2 An Alternative MINLP

In this section, we propose an alternative MINLP formulation of the ROPF problem motivated by [3]. Let us define a set of new decision variables c_{ii}, c_{ij} and s_{ij}, respectively representing the quantities $|V_i|^2$, $|V_i||V_j|\cos(\theta_i - \theta_j)$ and $s_{ij} := -|V_i||V_j|\sin(\theta_i - \theta_j)$ for $i \in \mathcal{B}$ and $(i,j) \in \mathcal{L}$. We denote the lower (upper) bounds of variables c_{ii}, c_{ij}, s_{ij} as $\underline{c}_{ii}, \underline{c}_{ij}, \underline{s}_{ij}$ $(\overline{c}_{ii}, \overline{c}_{ij}, \overline{s}_{ij})$ and set them as follows:

$$\begin{aligned}
&\underline{c}_{ii} := \underline{V}_i^2, \ \overline{c}_{ii} := \overline{V}_i^2 && i \in \mathcal{B} \\
&\underline{c}_{ij} := \overline{V}_i \overline{V}_j \cos(\overline{\theta}_{ij}), \ \overline{c}_{ij} := \overline{V}_i \overline{V}_j && (i,j) \in \mathcal{L} \\
&\underline{s}_{ij} := -\overline{V}_i \overline{V}_j \sin(\overline{\theta}_{ij}), \ \overline{s}_{ij} := \overline{V}_i \overline{V}_j \sin(\overline{\theta}_{ij}) && (i,j) \in \mathcal{L}.
\end{aligned}$$

We will now discuss the constraints in the alternative formulation and their relations with the MINLP in Section 2.1. The updated version of the real power flow balance constraint (2) is given as:

$$p_i^g - p_i^d = g_{ii} c_{ii} + \sum_{j \in \delta(i)} p_{ij} \qquad i \in \mathcal{B}. \tag{13}$$

Since the variable b_{ii} can be eliminated from the formulation by substituting $\sum_{k \in \mathcal{S}_i} b_{ii}^k \alpha_i^k$, the reactive power flow equation (3) is first rewritten as follows:

$$q_i^g - q_i^d = -\left(\sum_{k \in \mathcal{S}_i} b_{ii}^k \alpha_i^k \right) |V_i|^2 + \sum_{j \in \delta(i)} q_{ij} \qquad i \in \mathcal{B}. \tag{14}$$

Then, we define a new variable $\Gamma_i^k := c_{ii} \alpha_i^k$ to linearize (14) and include additional constraints as follows:

$$\begin{aligned}
&q_i^g - q_i^d = -\sum_{k \in \mathcal{S}_i} b_{ii}^k \Gamma_i^k + \sum_{j \in \delta(i)} q_{ij} && i \in \mathcal{B} \\
&\underline{c}_{ii} \alpha_i^k \le \Gamma_i^k \le \overline{c}_{ii} \alpha_i^k, \quad c_{ii} = \sum_{k \in \mathcal{S}_i} \Gamma_i^k && i \in \mathcal{B}.
\end{aligned} \tag{15}$$

We now update power flow constraints using a similar procedure. In particular, we substitute $1/\tau_{ij}$ with $\sum_{l \in \mathcal{T}_{ij}} \beta_{ij}^l / \tau_{ij}^l$ into constraints (4) and (5). After defining the new variables $\bar{\Phi}_{ij}^l := c_{ii} \beta_{ij}^l$,

$\Phi^l_{ij} := c_{ij}\beta^l_{ij}$ and $\Psi^l_{ij} := s_{ij}\beta^l_{ij}$, we rewrite the real and reactive power flow constraints (4)–(5) together with other equations necessary for the linearization as follows:

$$\begin{aligned}
&p_{ij} = \sum_{l\in\mathcal{T}_{ij}} G_{ij}\left(\frac{c_{ii}\beta^l_{ij}}{(\tau^l_{ij})^2} + \frac{\Phi^l_{ij}}{\tau^l_{ij}}\right) - B_{ij}\frac{\Psi^l_{ij}}{\tau^l_{ij}} && (i,j)\in\mathcal{L}\\
&q_{ij} = \sum_{l\in\mathcal{T}_{ij}} -B_{ij}\left(\frac{\bar{\Phi}^l_{ij}}{(\tau^l_{ij})^2} + \frac{\Phi^l_{ij}}{\tau^l_{ij}}\right) - G_{ij}\frac{\Psi^l_{ij}}{\tau^l_{ij}} && (i,j)\in\mathcal{L}\\
&\underline{c}_{ii}\beta^l_{ij} \le \bar{\Phi}^l_{ij} \le \bar{c}_{ii}\beta^l_{ij},\quad c_{ii} = \sum_{l\in\mathcal{T}_{i,j}} \bar{\Phi}^l_{ij} && (i,j)\in\mathcal{L}\\
&\underline{c}_{ij}\beta^l_{ij} \le \Phi^l_{ij} \le \bar{c}_{ij}\beta^l_{ij},\ c_{ij} = \sum_{l\in\mathcal{T}_{i,j}} \Phi^l_{ij} && (i,j)\in\mathcal{L}\\
&\underline{s}_{ij}\beta^l_{ij} \le \Psi^l_{ij} \le \bar{s}_{ij}\beta^l_{ij},\ s_{ij} = \sum_{l\in\mathcal{T}_{i,j}} \Psi^l_{ij} && (i,j)\in\mathcal{L}.
\end{aligned} \tag{16}$$

We also update the constraint on voltage magnitude bounds (6) as follows:

$$\underline{V}_i^2 \le c_{ii} \le \overline{V}_i^2 \qquad i\in\mathcal{B}. \tag{17}$$

Finally, we define the following consistency constraints for each line (i,j):

$$c_{ij}^2 + s_{ij}^2 = c_{ii}c_{jj} \qquad (i,j)\in\mathcal{L} \tag{18}$$

$$(\Phi^l_{ij})^2 + (\Psi^l_{ij})^2 = \bar{\Phi}^l_{ij}c_{jj} \qquad (i,j)\in\mathcal{L} \tag{19}$$

$$\theta_j - \theta_i = \arctan(s_{ij}/c_{ij}) \qquad (i,j)\in\mathcal{L}. \tag{20}$$

Equation (18) preserves the trigonometric relation between the variables c_{ii}, c_{ij} and s_{ij}. If we multiply (18) by β^l_{ij}, we can get a similar condition for the variables $\bar{\Phi}^l_{ij}, \Phi^l_{ij}$ and Ψ^l_{ij}.

The alternative formulation minimizes (1) subject to constraints (8)–(13) and (15)–(20).

2.3 MISOCP Relaxation

The feasible region of the alternative MINLP formulation is non-convex due to constraints (18)–(20). Let us relax these constraints as follows:

$$\begin{aligned}
c_{ij}^2 + s_{ij}^2 &\le c_{ii}c_{jj} && (i,j)\in\mathcal{L}\\
(\Phi^l_{ij})^2 + (\Psi^l_{ij})^2 &\le \bar{\Phi}^l_{ij}c_{jj} && (i,j)\in\mathcal{L}.
\end{aligned} \tag{21}$$

Then, an MISOCP relaxation is obtained as (1), (8)–(13), (15)–(17) and (21).

To strengthen the MISOCP relaxation, we also consider an outer-approximation of the arctangent constraint (20). This is achieved by the inclusion of four hyperplanes as described in [4]. We will use the abbreviation `MISOCPA` to refer to this stronger relaxation. Additionally, we generate cutting planes for each cycle in the cycle basis using a method called *SDP Separation*, more details can be found in [3]. We denote this further improved relaxation as `MISOCPA+`.

Table 1: Computational results (time is measured in seconds).

	TCR2				MISOCPA+			
Case	**LB**	**Time**	**UB**	**%Gap**	**LB**	**Time**	**UB**	**%Gap**
3lmbd	5769.87	0.65	5812.64	0.74	5783.94	0.53	5812.64	0.49
5pjm	15313.38	0.72	17551.89	12.75	16395.73	0.22	17551.89	6.59
30ieee	205.19	1.13	205.64	0.22	205.15	1.38	205.25	0.05
118ieee	3695.39	4.66	3720.08	0.66	3684.68	12.89	3714.91	0.81
Average		**1.79**		**3.59**		**3.75**		**1.99**
3lmbd_api	363.00	0.66	367.74	1.29	362.92	0.56	367.74	1.31
6ww_api	273.76	0.53	273.76	0.00	273.66	0.38	273.76	0.04
14ieee_api	319.12	0.93	323.29	1.29	318.65	0.76	321.09	0.76
30as_api	559.96	2.38	571.13	1.96	556.71	0.92	571.13	2.52
30fsr_api	213.93	2.16	372.14	42.51	227.57	0.95	372.11	38.84
39epri_api	7333.40	2.59	7466.25	1.78	7259.19	11.63	7480.45	2.96
118ieee_api	5932.26	4.47	10258.47	42.17	5910.20	14.23	10158.62	41.82
Average		**1.96**		**13.00**		**4.20**		**12.61**
3lmbd_sad	5831.07	0.57	5992.72	2.70	5867.46	0.53	5992.72	2.09
4gs_sad	321.55	0.58	324.02	0.76	323.65	0.16	324.02	0.12
5pjm_sad	25560.36	0.62	26423.33	3.27	26419.23	0.21	26423.32	0.02
9wscc_sad	5521.49	0.54	5590.09	1.23	5589.54	0.20	5590.09	0.01
29edin_sad	31173.80	3.19	46933.31	33.58	36270.50	2.77	45886.11	20.96
30as_sad	903.09	2.32	914.44	1.24	906.96	1.11	914.44	0.82
30ieee_sad	205.30	0.96	205.79	0.24	205.27	1.51	205.37	0.05
118ieee_sad	3869.62	4.66	4323.91	10.51	4003.35	9.30	4258.72	6.00
Average		**1.68**		**6.69**		**1.97**		**3.76**
Overall Average		**1.81**		**8.36**		**3.17**		**6.64**

3 Computational Experiments

3.1 Algorithm

We first solve the continuous relaxation of the `MISOCPA` formulation by relaxing the integrality of α_i^k and β_{ij}^l variables. Then, for each cycle in the cycle basis, we use the SDP separation method to generate cutting planes to separate this continuous relaxation solution from the feasible region of the SDP relaxation of the cycle. The separation process is parallelized over cycles. We repeat this procedure five times consecutively. Then, we solve the final `MISOCPA+` relaxation to obtain a lower (LB) bound, and then fix the binary variables in the MINLP formulation to obtain an upper bound (UB) from the remaining NLP using a local solver. The optimality gap is computed as %Gap $= 100 \times (1 - \text{LB}/\text{UB})$.

3.2 Results

We compare the percentage optimality gap and the computational time of the `MISOCPA+` approach with the publicly available implementation of TCR relaxation of Type 2 (`TCR2`) from [1]. All computational experiments have been carried out on a 64-bit desktop with Intel Core i7 CPU with 3.20GHz processor and 64 GB RAM. Our code is written in Python language using Spyder environment. The solvers Gurobi, IPOPT and MOSEK are used to solve the `MISOCPA+` relaxation, NLP and separation problems, respectively.

For the computational experiments, we use the OPF instances from the NESTA library; typical operating conditions, congested operating conditions (API) and small angle difference conditions (SAD). We only consider difficult instances in which the SOCP optimality gap is more than 1% [3].

The sets of the discrete values are determined as $b_{ii}^k \in \{0, 1\}$ for $i \in \mathcal{S}$ and $\tau_{ij}^l \in \{0.9, 0.95, 1, 1.05, 1.1\}$ for $(i, j) \in \mathcal{T}$, which represent the on/off status of the shunt susceptance and values of the tap ratio, respectively.

The results of our computational experiments are reported in Table 1. We observe that `MISOCPA+` has smaller optimality gap in fourteen out of nineteen instances, and better or same upper bound in eighteen of these instances (except 39epri_api). If we compare the averages of optimality gap, `MISOCPA+` outperforms `TCR2` in all types of NESTA instances. `MISOCPA+` has the best performance on SAD instances and dominates `TCR2` in all of them. Overall, we note that `MISOCPA+` relaxation has more accurate solutions with 6.64% optimality gap, on average, than `TCR` with 8.36%. In terms of computational time, `MISOCPA+` is slower with 3.17 seconds, on average, than `TCR2` with 1.81.

4 Conclusion

In this letter, we propose an MISOCP-based approach, namely `MISOCPA+`, to approximate globally optimal solutions of the ROPF problem. The accuracy and efficiency of this approach are compared with `TCR2` using difficult OPF instances from the NESTA library. The computational results indicate that `MISCOPA+` is quite promising to solve any type of instances accurately, especially the ones with small angle conditions.

References

[1] Christian Bingane, Miguel F Anjos, and Sebastien Le Digabel. Tight-and-cheap conic relaxation for the optimal reactive power dispatch problem. *IEEE Trans. on Power Syst.*, 34(6):4684–4693, 2019.

[2] Florin Capitanescu and Louis Wehenkel. Sensitivity-based approaches for handling discrete variables in optimal power flow computations. *IEEE Trans. on Power Syst.*, 25(4):1780–1789, 2010.

[3] Burak Kocuk, Santanu S Dey, and X Andy Sun. Strong SOCP relaxations for the optimal power flow problem. *Oper. Res.*, 64(6):1177–1196, 2016.

[4] Burak Kocuk, Santanu S Dey, and X Andy Sun. Matrix minor reformulation and SOCP-based spatial branch-and-cut method for the AC optimal power flow problem. *Math. Prog. Comp.*, 10(4):557–596, 2018.