An inexact scalarized proximal algorithm with quasi-distance for convex and quasiconvex multi-objective minimization

E. A. Papa Quiroz · Rogério A. Rocha ·
Paulo R. Oliveira · Ronaldo M. Gregório

Abstract In the paper of Rocha et al., J Optim Theory Appl (2016) 171:964979, the authors introduced a proximal point algorithm with quasi-distances to solve unconstrained convex multi-objective minimization problems. They proved that all accumulation points are efficient solutions of the problem. In this paper we analyze an inexact proximal point algorithm to solve convex and quasiconvex unconstrained multi-objective minimization problems using quasi-distances. For the convex case, we extend the result obtained by the exact algorithm of Rocha et al. and for the quasiconvex case we prove that all accumulation points are Pareto-Clarke critical points of the problem. Finally, to show the practicality of the introduced algorithm, we present numerical examples that confirm the convergence of our algorithm.

Erik Alex Papa Quiroz - Corresponding author
Universidad Nacional Mayor de San Marcos and Universidad Privada del Norte
Lima, Perú,
telephone : +51 (1) 938257115
erikpapa@gmail.com

Rogério Azevedo Rocha
Federal University of Tocantins
Tocantins, Brazil,
azevedo@uft.edu.br

Paulo Roberto Oliveira
Federal University of Rio de Janeiro
Rio de Janeiro, Brazil
poliveir@cos.ufrj.br

Ronaldo Malheiros Gregório
Rural Federal University of Rio de Janeiro
Rio de Janeiro, Brazil
rgregor@ufrrj.br

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1 Introduction

The classical proximal point algorithm (PPA) to minimize a convex function of scalar value $f : \mathbb{R}^n \to \mathbb{R}$ was originally introduced by Martinet [15] and deeply studied by Rochafellar [22]. This PPA generates a sequence $\{x^k\}$ via the following iterative procedure: Given a starting point $x^0 \in \mathbb{R}^n$, then

$$x^{k+1} = \operatorname{argmin}\{f(x) + (\lambda_k/2)\|x - x^k\|^2 \mid x \in \mathbb{R}^n\}, \tag{1}$$

where for each k, λ_k is a positive real number and $\|.\|$ is the usual Euclidean norm.

In recent decades the convergence analysis of this proximal point algorithm has been extensively studied and, consequently, the literature of proximal point methods has come to include a large number of papers presenting generalizations of this PPA for several classes of problems: variational inequality problems, equilibrium problems, etc.

In the paper of Bonnel et al. [3], Rochafellar's results are extended from the scalar case to the vectorial optimization case. More specifically, these authors proposed a vector-valued PPA (in exact and inexact versions) to solve unconstrained convex vector/multi-objective optimization problems.

Motivated by the results of Bonnel et al., several studies have emerged in the literature presenting variations/generalizations of the vector-valued PPA, including: Ceng and Yao [5], Villacorta and Oliveira [25], Chuong [6], Chen et al. [7], Bento et al. [2], Rocha et al. [21], Apolinário et al [1] and Papa Quiroz et al. [19].

This paper proposes an inexact version of the proximal point method to find efficient solutions of unconstrained convex multi-objective minimization problems and Pareto-Clarke critical points of unconstrained quasiconvex multi-objective minimization problems. One of its main contributions is the introduction of quasi-distances into subproblems of this proximal point algorithm, which has important applications in computer theory [4,12], economics [18, 24], and other fields. According to Rockafellar [22] and many researches, the importance of inexact versions is that for a proximal method to be practical it is important that it also work with approximate solutions of the subproblems.

The organization of the paper is the following: Section 2 presents concepts and results related to quasi-distances, subdifferential theory and multi-objective optimization. Section 3 describes the inexact proximal algorithm to solve convex multi-objective minimization problems, proving the existence of iterations, establishing some properties and proposing a convergence analysis for efficient solutions. Section 4 the inexact proximal algorithm is extended to solve quasiconvex multi-objective minimization problems and we prove that each accumulation point is a Pareto-Clarke critical point of the problem. Finally, in Section 5, the algorithm is tested and numerical examples are offered.

2 Basic definitions and preliminary results

Throughout this paper $\mathbb{R}_+^m$ and $\mathbb{R}_{++}^m$ represent, respectively, the non-negative orthant of $\mathbb{R}^m$ and its topological interior and we will use the following conventions: If $x, y \in \mathbb{R}^m$, then (i) $x \ll y$, iff $x_i < y_i$, $\forall i = 1, .., m$; (ii) $x \leq y$, iff $x_i \leq y_i, \forall i = 1, .., m$ and (iii) $x < y$, iff $x \leq y$ and $x \neq y$.

Consider a map $G : \mathbb{R}^n \to \mathbb{R}^m$ and the following multi-objective optimization problem (MOP)

$$\min_{\mathbb{R}_+^m} \{G(x) \mid x \in \mathbb{R}^n\}. \tag{2}$$

Definition 2.1 (a) $a \in \mathbb{R}^n$ *is a* **local efficient solution** *to the problem* (2) *if there is a disc* $B_\delta(a) \subset \mathbb{R}^n$, *with* $\delta > 0$, *such that there is no* $x \in B_\delta(a)$ *that satisfies* $G(x) < G(a)$; **(b)** $a \in \mathbb{R}^n$ *is a* **weak local efficient solution** *to the problem* (2) *if there is a disc* $B_\delta(a) \subset \mathbb{R}^n$, *with* $\delta > 0$, *such that there is no* $x \in B_\delta(a)$ *that satisfies* $G(x) \ll G(a)$.

If G is convex in (2), then every local solution (efficient and weak efficient) is also a global solution. Let $\mathrm{argmin}\{G(x) \mid x \in \mathbb{R}^n\}$ and $\mathrm{argmin}_w\{G(x) \mid x \in \mathbb{R}^n\}$ denote the local efficient solution set and the weak local efficient solution set to the problem (2), respectively. It is easy to see that

$$\mathrm{argmin}\{G(x) \mid x \in \mathbb{R}^n\} \subset \mathrm{argmin}_w\{G(x) \mid x \in \mathbb{R}^n\}.$$

Remark 2.1 *Given an map* $G : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, *consider the multi-objective optimization problem*

$$\min_{\mathbb{R}_+^m} \{exp(G(x)) \mid x \in \mathbb{R}^n\}, \tag{3}$$

where $exp(G(x)) = (exp(G_1(x)), \ldots, exp(G_m(x)))$. *Then,*

$$\mathrm{argmin}\{G(x) \mid x \in \mathbb{R}^n\} = \mathrm{argmin}\{exp(G(x)) \mid x \in \mathbb{R}^n\}.$$

Note also that, if the problem (2) *is a convex problem, then the problem* (3) *is also convex. Hence, without loss of generality, we can assume that for all* $z \in \mathbb{R}_+^m \setminus \{0\}$, $\langle G(x), z \rangle \geq 0$, $\forall x \in \mathbb{R}^n$.

Next, we will recall the concepts of Fréchet and limiting subdifferential.

Definition 2.2 *Let* $h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *be a proper function and* $x \in \mathbb{R}^n$.

1. *O subdiferencial Fréchet (also known as regular-subdifferential) de* h *em* $x \in \mathbb{R}^n, \hat{\partial}h(x)$, *is defined as follows:*

$$\hat{\partial}h(x) := \begin{cases} \left\{ x^* \in \mathbb{R}^n : \liminf_{y \neq x, y \to x} \dfrac{h(y) - h(x) - \langle x^*, y - x \rangle}{\|x - y\|} \geq 0 \right\}, & \text{se } x \in dom(h) \\ \varnothing, & \text{se } x \notin dom(h) \end{cases}$$

2. *O subdiferencial-limite (also known as Mordukhovich basic subdifferential) de* h *em* $x \in \mathbb{R}^n, \partial h(x)$, *is defined as follows:*

$$\partial h(x) := \left\{ x^* \in \mathbb{R}^n : \exists x_n \to x, \quad h(x_n) \to h(x), \quad \hat{x}_n^* \in \hat{\partial} h(x_n) \to x^* \right\}$$

Remark 2.2 a) *If $\bar{x} \in \mathrm{dom(h)}$, then the sets $\hat{\partial} h(\bar{x})$ e $\partial h(\bar{x})$ are closed, with $\hat{\partial} h(\bar{x})$ convex, and $\hat{\partial} h(x) \subset \partial h(x)$;* **b)** *Let $C \subseteq \mathbb{R}^n$. If a proper function $h : C \to \mathbb{R} \cup \{+\infty\}$ has a local minimum at $\bar{x} \in C$, then $0 \in \hat{\partial}(h + \delta_C)(\bar{x})$ and $0 \in \partial(h + \delta_C)(\bar{x})$, where $\delta_C : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is the indicator function of C.*

Proposition 2.1 *(Mordukhovich, [16], Theorem 2.33) Let $f, g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ proper functions such that f is locally Lipschitz at $\overline{x} \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$, and g is lower semicontinuous function at this point. Then*

$$\partial(f + g)(\overline{x}) \subset \partial f(\overline{x}) + \partial g(\overline{x}).$$

The subproblems of the our inexact PPA that we will present in the next section involve quasi-distances.

Definition 2.3 *A function $q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ is called a quasi-distance if, for all $x, y, z \in \mathbb{R}^n$,*
(i) $q(x, y) = q(y, x) = 0$ *iff* $x = y$ *and*   **(ii)** $q(x, z) \leq q(x, y) + q(y, z)$.

Remark 2.3 a) *Various examples of quasi-distance can be found in Moreno et al. [18];* **(b)** *If a quasi-distance q is also symmetric, that is, for all $x, y \in \mathbb{R}^n$, $q(x, y) = q(y, x)$, then q is a distance;* **(c)** *A quasi-distance is not necessarily a convex function, nor continuously differentiable, nor even a coercive function in any of its arguments (see [18, Remarks 3 and 4]).*

Throughout our paper we will consider quasi-distances that are locally Lipschitz continuous and coercive in any of their arguments. The following proposition provides us a class of quasi-distances that satisfy these properties.

Proposition 2.2 *Let $q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ be a quasi-distance. Suppose there are positive constants α and β such that*

$$\alpha \|x - y\| \leq q(x, y) \leq \beta \|x - y\|, \quad \forall x, y \in \mathbb{R}^n. \tag{4}$$

Then, for each $z \in \mathbb{R}^n$, $q(z, .)$ and $q(., z)$ are Lipschitz continuous functions on $\mathbb{R}^n$, $q^2(z, .)$ and $q^2(., z)$ are locally Lipschitz continuous functions on $\mathbb{R}^n$, and $q(z, .)$, $q(., z)$, $q^2(z, .)$ and $q^2(., z)$ are coercive functions.

Proof See [18, Propositions 3.6 and 3.7 and Remark 5]. $\square$

Remark 2.4 a) *If $q(x, y) = \|x - y\|$ then q satisfies property* (4)*;* b) *The quasi-distance of example 3.3 of Moreno et al.[18] is non-symmetric and satisfies the property* (4)*.*

3 A inexact Proximal Algorithm

In this section, we propose an inexact PPA with quasi-distances, denoted by QDIP, to solve the unconstrained multi-objective minimization problem:

$$\min_{\mathbb{R}^m_+} \{F(x) \; : \; x \in \mathbb{R}^n\}, \qquad (5)$$

where $F = (F_1, F_2, ..., F_m)^T : \mathbb{R}^n \to \mathbb{R}^m$ is convex , i.e., $F_i : \mathbb{R}^n \to \mathbb{R}$ is convex, for all $i \in \{1, ..., m\}$.

For the definition of QDIP algorithm we will consider the following assumptions:

(H1) $q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ is a quasi-distance satisfying the property (4);
(H2) $\{\beta_k\} \subset \mathbb{R}$ a sequence of parameters satisfying $0 < c_1 < \beta_k < c_2$; $\forall k \in \mathbb{N}$;
(H3) $\{z^k\} \subset \mathbb{R}^m_+ \setminus \{0\}$ a bounded sequence whose points of accumulation belong to $\mathbb{R}^m_{++}$.

QDIP Algorithm generates a sequence $\{x^k\} \subset \mathbb{R}^n$ as follows:

QDIP

1. Choose $x^0 \in \mathbb{R}^n$.
2. Given x^k, if $x^k \in \text{argmin}_w \{F(x) \mid x \in \mathbb{R}^n\}$, then $x^{k+p} = x^k, \forall p \geq 1$.
3. Given x^k, if $x^k \notin \text{argmin}_w \{F(x) \mid x \in \mathbb{R}^n\}$, then consider as x^{k+1} any vector $x \in \Omega^k$ such that it exists $\varepsilon_k \in \mathbb{R}_+$ satisfying

$$0 \in \partial_{\varepsilon_k}(\langle F(.), z^k \rangle)(x) + \beta_k q(x, x^k) \partial(q(., x^k))(x) + N_{\Omega^k}(x), \qquad (6)$$

$$q^2(x, x^k) \leq \bar{c} \, \|F(x) - F(x^k)\| \quad \text{and} \quad \lim_{k \to \infty} \varepsilon_k = 0. \qquad (7)$$

where $N_{\Omega^k}(x)$ denotes the normal cone at the point x related to the set Ω^k, $\partial_{\varepsilon_k}(\langle F(.), z^k \rangle)(x)$ is the ε_k-subdifferential of convex function $\langle F(.), z^k \rangle$ at point x, $\bar{c} \in \mathbb{R}_{++}$ and $\Omega^k = \{x \in \mathbb{R}^n \mid F(x) \leq F(x^k)\}$.

Remark 3.1 a) *If $\{x^k\}$ is a sequence generated by QDIP Algorithm, then $\Omega^k \supseteq \Omega^{k+1}$, $\forall k$. Thus, $x^k \in \Omega^{k-1} \subseteq \Omega^0, \forall k \geq 1$. Therefore, Ω^0 bounded implies that $\{x^k\}$ is bounded,* **b)** *If $\lim_{\|x\| \to \infty} F_r(x) = \infty$ for same $r \in \{1, ..., m\}$, then Ω^0 is bounded (see [20, Lemma 4.1]),* **c)** *The hypothesis (H3) implies convergence for efficient solution (see proof of Theorem 3.1). If it were not guaranteed that the accumulation points of $\{z^k\}$ belong to R^m_{++}, then the convergence would be for weak efficient solution (Remark 3.3),* **d)** *The importance of efficient solutions is due to the fact that, in real-world applications, it is often the case that only efficient solutions (instead of weak efficient) are of interest (see, e.g., [11, Section 2.3]) and* **e)** *Any sequence generated by the exact proximal algorithm of Rocha et al. [21], satisfies (6) and (7) to $\varepsilon_k \equiv 0$ (see [21, proof of Theorem 3.1]). In this sense, the QDIP algorithm generalizes the proximal algorithm of Rocha et al. [21].*

Proposition 3.1 (Well-posedness of QDIP algorithm) *If $F : \mathbb{R}^n \to \mathbb{R}^m$ is a convex map, $q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ is a quasi-distance satisfying the conditions (H1)-(H3) and Ω^0 is bounded, then the sequence generates by the QDIP algorithm is well defined.*

Proof $x^0 \in \mathbb{R}^n$ is chosen in the initialization step. Assuming that the algorithm reaches iteration k, we show next that an appropriate x^{k+1} exists. By the stop criterion, if $x^k \in \operatorname{argmin}_w \{F(x), \ x \in \mathbb{R}^n\}$ then $x^{k+p} = x^k$, $\forall p \geq 1$. Otherwise, define $\varphi^k : \mathbb{R}^n \to \mathbb{R}$ such that $\varphi^k(x) = \langle F(x), z^k \rangle + \frac{\beta_k}{2} q^2(x, x^k)$. The convexity of F in $\mathbb{R}^n$ implies the convexity of $\langle F(.), z^k \rangle$ in $\mathbb{R}^n$ and then that $\langle F(.), z^k \rangle$ is a continuous map in $\mathbb{R}^n$. From assumption (H1) and Proposition 2.2, $q^2(., x^k) : \mathbb{R}^n \to \mathbb{R}_+$ is a continuous map, then $\varphi^k : \mathbb{R}^n \to \mathbb{R}$ is also a continuous map in $\mathbb{R}^n$. Since F is convex in $\mathbb{R}^n$, $\Omega^k \supseteq \Omega^{k+1}, \forall k$ and Ω^0 is bounded (hypothesis), we have that the set $\Omega^k, \forall k$ is convex and compact. Then, the set $\operatorname{argmin}\{\varphi^k(x) \mid x \in \Omega^k\}$ is nonempty. Let $x^{k+1} \in \operatorname{argmin}\{\varphi^k(x) \mid x \in \Omega^k\}$. So, by Remark 2.2, **b**,

$$0 \in \partial(\langle F(.), z^k \rangle + \frac{\beta_k}{2} \ q^2(., x^k) + \delta_{\Omega^k})(x^{k+1}). \tag{8}$$

Since $\langle F(.), z^k \rangle : \mathbb{R}^n \to \mathbb{R}$ is convex, $\langle F(.), z^k \rangle$ is locally Lispschitzian in $\mathbb{R}^n$; by Proposition 2.2, $\frac{\beta_k}{2} q^2(., x^k)$ is locally Lispschitzian in $\mathbb{R}^n$; Ω^k closed implies δ_{Ω^k} is lower semi-continuous (see [23, page 11]). Then, using the Theorem 2.33 of [16] in (8), we obtain

$$0 \in \partial(\langle F(.), z^k \rangle)(x^{k+1}) + \partial(\frac{\beta_k}{2} q^2(., x^k))(x^{k+1}) + \partial(\delta_{\Omega^k})(x^{k+1}). \tag{9}$$

Since Ω^k is convex and $x^{k+1} \in \Omega^k$ we have $\partial(\delta_{\Omega^k})(x^{k+1}) = N_{\Omega^k}(x^{k+1})$, where $N_{\Omega^k}(x^{k+1})$ denotes the normal cone at the point x^{k+1} related to the set Ω^k. From Proposition 2.2, $q(., x^k)$ is Lipschitzian on $\mathbb{R}^n$. Therefore, as $q(x, y) \geq 0$; $\forall(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, taking $\varphi_1 = \varphi_2 = q(., x^k)$ in the Theorem 7.1 of [17], we obtain $\partial\left(\frac{\beta_k}{2} q^2(., x^k)\right)(x^{k+1}) = \beta_k q(x^{k+1}, x^k)\partial\left(q(., x^k)\right)(x^{k+1})$, and then from (9),

$$0 \in \partial\left(\langle F(.), z^k \rangle\right)(x^{k+1}) + \beta_k q(x^{k+1}, x^k)\partial\left(q(., x^k)\right)(x^{k+1}) + N_{\Omega^k}(x^{k+1}).$$

Thus, since $\partial f(x) = \partial_0 f(x) \subset \partial_\varepsilon f(x)$ for any convex function $f : \mathbb{R}^n \to \mathbb{R}$, all $x \in \mathbb{R}^n$ and all $\varepsilon \in \mathbb{R}_+$, x^{k+1} satisfies (6) for $\varepsilon_k \equiv 0$. As $x^k \in \Omega^k$; $\forall k$ and $x^{k+1} \in \operatorname{argmin}\{\varphi^k(x) \mid x \in \Omega^k\}$ we have $\varphi^k(x^{k+1}) \leq \varphi^k(x^k)$; $\forall k$. Therefore, since $q(x^k, x^k) = 0$,

$$\langle F(x^{k+1}), z^k \rangle + \frac{\beta_k}{2} \ q^2(x^{k+1}, x^k) \leq \langle F(x^k), z^k \rangle.$$

Then, from assumption (H2) and (H3) we have $0 < c_1 < \beta_k$, $\|z^k\| \leq m$, and thus we obtain

$$0 \leq q^2(x^{k+1}, x^k) \ \leq \ \frac{2}{c_1}\langle F(x^k) - F(x^{k+1}), z^k \rangle \ \leq \ \frac{2m}{c_1}\|F(x^k) - F(x^{k+1})\|.$$

Therefore x^{k+1} satisfies (6) and (7). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 3.2 *We are interested in the asymptotic convergence of the QDIP algorithm, so we assume that $x^{k+1} \neq x^k, \forall k$. Otherwise, it is easy to prove that x^k is a efficent solution of the problem, that is, $x^k \in \text{argmin}\{F(x) \mid x \in \mathbb{R}^n\}$.*

The following properties are fundamental for proof of the convergence of QDIP algorithm.

Proposition 3.2 (Properties) *Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a convex map, $q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ be a quasi-distance satisfying the condition (H1), Ω^0 is bounded and the conditions (H2)-(H3) are satisfied. Let $\{x^k\}$ be a sequence generated by QDIP algorithm, then we have:*

(a) *$\{x^k\}$ is bounded;*
(b) *$\forall z \in \mathbb{R}^m_+ \setminus \{0\}, \{\langle F(x^k), z \rangle\}_{k \in N}$ is nonincreasing and convergent;*
(c) *$\lim_{k \to \infty} \|F(x^{k+1}) - F(x^k)\| = 0;$*
(d) *$\lim_{k \to \infty} q(x^{k+1}, x^k) = 0;$*
(e) *$\lim_{k \to \infty} \|x^{k+1} - x^k\| = 0.$*

Proof **(a)** Since $\Omega^k \supseteq \Omega^{k+1}, k = 0, 1, ...,$ we have $x^k \in \Omega^{k-1} \subseteq \Omega^0, \forall k \geq 1$. Then, as Ω^0 is limited (by hypothesis), $\{x^k\}$ is limited.
(b) Since $F(x^{k+1}) \leq F(x^k)$ $(x^{k+1} \in \Omega^k)$ and $z \in \mathbb{R}^m_+ \setminus \{0\}$ we have

$$\langle F(x^{k+1}), z \rangle \leq \langle F(x^k), z \rangle \ \forall k \in \mathbb{N},$$

i.e., $\{\langle F(x^k), z \rangle\}_{k \in \mathbb{N}}$ is nonincreasing. By Remark 2.1, $\{\langle F(x^k), z \rangle\}_{k \in \mathbb{N}}$ is bounded lower and therefore convergent.
(c) Let $\bar{z} \in \mathbb{R}^m_{++}$ be fixed. From (b), $\{\langle F(x^k), \bar{z} \rangle\}_{k \in \mathbb{N}}$ is convergent. Then,

$$\lim_{k \to \infty} \langle F(x^k) - F(x^{k+1}), \bar{z} \rangle = \lim_{k \to \infty} \sum_{i=1}^{m} (F_i(x^k) - F_i(x^{k+1}))\bar{z}_i = 0. \qquad (10)$$

As $x^{k+1} \in \Omega^k$, $F(x^{k+1}) \leq F(x^k)$, i.e., $F_i(x^{k+1}) \leq F_i(x^k), \forall i = 1, ..., m$. Thus $(F_i(x^k) - F_i(x^{k+1}))\bar{z}_i \geq 0, \forall i = 1, ..., m$. Then, of (10),

$$\lim_{k \to \infty} (F_i(x^k) - F_i(x^{k+1}))\bar{z}_i = 0, \ \forall i = 1, ..., m.$$

Since that $\bar{z}_i > 0, i = 1, ..., m$, we have $\lim_{k \to \infty} (F_i(x^k) - F_i(x^{k+1})) = 0, \forall i = 1, ..., m$, and then $\lim_{k \to \infty} (F(x^k) - F(x^{k+1})) = 0 \in \mathbb{R}^m$. Therefore

$$\lim_{k \to \infty} \|F(x^{k+1}) - F(x^k)\| = 0.$$

(d) By (7), $q^2(x^{k+1}, x^k) \leq \bar{c} \|F(x^{k+1}) - F(x^k)\|$. Then, since that $q(x, y) \geq 0$, the result is a consequence of the previous item.
(e) From assumptiom (H1), $0 \leq \alpha \|x^{k+1} - x^k\| \leq q(x^{k+1}, x^k), \forall k \in N$. Therefore, the result is a consequence of the previous item. $\qquad \square$

Next we will enunciate and prove our main result. Assuming that the stopping criterion of the QDIP algorithm never applies, we demonstrate that any sequence generated converges to a efficient solution of the problem (5).

Theorem 3.1 (Convergence) *Suppose that $F : \mathbb{R}^n \to \mathbb{R}^m$ is a convex map and $q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ be a quasi-distance satisfying the condition (H1). If the set Ω^0 is bounded and the assumptions (H2) and (H3) are satisfied, then every sequence generated by QDIP algorithm is bounded and its accumulation points are efficient solutions to the problem (5).*

Proof Let $\{x^k\}$ be a sequence generated by the QDIP algorithm. By Proposition 3.2(a), there are $x^* \in \mathbb{R}^n$ and $\{x^{k_j}\}_{j\in\mathbb{N}}$, a subsequence of $\{x^k\}$, such that $\lim_{j\to\infty} x^{k_j} = x^*$. By (H3) there are $\bar{z} \in \mathbb{R}^m_{++}$ and $\{z^{k_l}\}_{l\in\mathbb{N}}$, a subsequence of $\{z^k\}$, such that $\lim_{l\to\infty} z^{k_l} = \bar{z}$. Fix $z \in \mathbb{R}^m_+ \setminus \{0\}$. The convexity of application $\langle F(.), z \rangle$ in $\mathbb{R}^n$ implies it is continuous. $\forall z \in \mathbb{R}^m_+ \setminus \{0\}$; $\{\langle F(x^k), z \rangle\}_{k\in\mathbb{N}}$ is nonincreasing and convergent (Proposition 3.2(b)). Therefore, $\forall z \in \mathbb{R}^m_+ \setminus \{0\}$; $\lim_{j\to\infty}\langle F(x^{k_j}), z \rangle = \langle F(x^*), z \rangle = \inf_{k\in N}\{\langle F(x^k), z \rangle\}$. Then,

$$\langle F(x^*), z \rangle \leq \langle F(x^k), z \rangle, \ \forall z \in \mathbb{R}^m_+ \setminus \{0\} \text{ and } k \in \mathbb{N}. \tag{11}$$

By (6), there are $\zeta^{k_l+1} \in \partial(q(., x^{k_l}))(x^{k_l+1})$ e $v^{k_l+1} \in N_{\Omega^{k_l}}(x^{k_l+1})$, such that

$$-\beta_{k_l} q(x^{k_l+1}, x^{k_l})\zeta^{k_l+1} - v^{k_l+1} \in \partial_{\varepsilon_{k_l}}(\langle F(.), z^{k_l}\rangle)(x^{k_l+1}).$$

Therefore, from the ε_{k_l}-subgradient inequality for convex function $\langle F(.), z^{k_l}\rangle$, we have: $\forall x \in \mathbb{R}^n$,

$$\langle F(x), z^{k_l} \rangle \geq \langle F(x^{k_l+1}), z^{k_l} \rangle - \beta_{k_l} q(x^{k_l+1}, x^{k_l})\langle \zeta^{k_l+1}, x - x^{k_l+1} \rangle$$
$$- \langle v^{k_l+1}, x - x^{k_l+1} \rangle - \varepsilon_{k_l}.$$

As $v^{k_l+1} \in N_{\Omega^{k_l}}(x^{k_l+1})$, we have $-\langle v^{k_l+1}, x - x^{k_l+1}\rangle \geq 0 \ \forall x \in \Omega^{k_l}$. Since Ω^0 is limited, $\Omega^k \subseteq \Omega^0, \forall k$ and F is continuous, we have $\Omega^k, \forall k \in \mathbb{N}$ is a compact set. Therefore, as $\Omega^{k+1} \subseteq \Omega^k, \forall k \in \mathbb{N}$ we have $\Omega = \bigcap_{k=0}^{\infty} \Omega^k \neq \emptyset$. Then, in particular,

$$\langle F(x), z^{k_l} \rangle \geq \langle F(x^{k_l+1}), z^{k_l} \rangle$$
$$- \beta_{k_l} q(x^{k_l+1}, x^{k_l})\langle \zeta^{k_l+1}, x - x^{k_l+1} \rangle - \varepsilon_{k_l}, \ \forall x \in \Omega. \tag{12}$$

By (11), $\langle F(x^{k_l+1}), z^{k_l} \rangle \geq \langle F(x^*), z^{k_l} \rangle$. Then, of (12),

$$\langle F(x), z^{k_l} \rangle \geq \langle F(x^*), z^{k_l} \rangle$$
$$- \beta_{k_l} q(x^{k_l+1}, x^{k_l})\langle \zeta^{k_l+1}, x - x^{k_l+1} \rangle - \varepsilon_{k_l}, \ \forall x \in \Omega. \tag{13}$$

By [21, Proposition 2.3], $\|\zeta^{k_l+1}\| \leq M$. Therefore, as the sequences $\{x^k\}$ and $\{\beta_k\}$ are bounded, using the Cauchy-Swartz inequality, we conclude that

$$\mid \beta_{k_j} < \zeta^{k_l+1}, x - x^{k_l+1} > \mid \leq M_1.$$

Then, provided that, from the Proposition 3.2(d), $\lim_{k \to \infty} q(x^{k+1}, x^k) = 0$, we have

$$| \beta_{k_l} q(x^{k_l+1}, x^{k_l}) < \zeta^{k_l+1}, x - x^{k_l+1} >| \to 0 \text{ when } l \to \infty.$$

Therefore, recalling that $\lim_{l \to \infty} z^{k_l} = \bar{z} \in \mathbb{R}^m_{++}$ and $\lim_{k \to \infty} \varepsilon_k = 0$, from (13)

$$\langle F(x), \bar{z} \rangle \geq \langle F(x^*), \bar{z} \rangle, \ \forall x \in \Omega. \tag{14}$$

We will demonstrate that $x^* \in \operatorname{argmin}\{F(x) \mid x \in \mathbb{R}^n\}$. Suppose, by contradiction, that there exists $\bar{x} \in \mathbb{R}^n$ such that

$$F(\bar{x}) \leq F(x^*) \text{ and } F_r(\bar{x}) < F_r(x^*) \text{ for some } r \in \{1, ..., m\}. \tag{15}$$

As $\bar{z} \in \mathbb{R}^m_{++}$ we have:

$$\langle F(\bar{x}), \bar{z} \rangle < \langle F(x^*), \bar{z} \rangle. \tag{16}$$

Since, $\Omega^{k+1} \subseteq \Omega^k$, $\forall k \geq 0$ and $x^{k_j} \in \Omega^{k_j-1}$, $\forall j$ with $x^{k_j} \to x^*$; $j \to \infty$, we have $x^* \in \Omega$, i.e., $F(x^*) \leq F(x^k), \forall k \in \mathbb{N}$. Therefore, from (15),

$$F(\bar{x}) \ll F(x^k), \forall k \in \mathbb{N},$$

i.e., $\bar{x} \in \Omega$, which contradicts (14) and (16). □

Remark 3.3 *If in the Theorem 3.1 we were not guaranteed that $\bar{z} \in \mathbb{R}^m_{++}$, i.e., if we only had $\bar{z} \in \mathbb{R}^m_+ \setminus \{0\}$ then, we would conclude that x^* is weak efficient solution (instead of efficient) of the problem (5). Indeed, suppose by contradiction that x^* is not a weak efficient solution of problem (5). Then, there is $\bar{x} \in \mathbb{R}^n$ such that*

$$F(\bar{x}) \ll F(x^*). \tag{17}$$

$x^* \in \Omega$, $\bar{z} \in \mathbb{R}^m_+ \setminus \{0\}$ and (17) imply $\bar{x} \in \Omega$ and $\langle F(\bar{x}), \bar{z} \rangle < \langle F(x^*), \bar{z} \rangle$. This is a contradiction because (14) is valid.

4 Quasiconvex case

In this section we extend the algorithm introduced in the previous section to solve quasiconvex multi-objective minimization problems. We start with some facts as Clarke subdifferential, descent direction and Pareto Clarke-critical point. Then we present the algorithm and analyze the convergence of the generated sequence.

4.1 Clarke subdifferential

Definition 4.1 Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper locally Lipschitz function at $x \in \text{dom}(f)$, and $d \in \mathbb{R}^n$.

1. The Clarke directional derivative of f at x in the direction d, denoted by $f^o(x, d)$, is defined as:

$$f^o(x, d) = \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y + td) - f(y)}{t}.$$

2. The Clarke subdifferential of f at x, denoted by $\partial^o f(x)$ is defined as

$$\partial^o f(x) = \{w \in \mathbb{R}^n : \langle w, d \rangle \leq f^o(x, d), \quad \forall d \in \mathbb{R}^n\}.$$

3. The $\epsilon-$Clarke subdifferential of f at x, denoted by $\partial^o_\epsilon f(x)$ is defined as

$$\partial^o_\epsilon f(x) = \{w \in \mathbb{R}^n : \langle w, d \rangle \leq f^o(x, d) - \epsilon, \quad \forall d \in \mathbb{R}^n\}.$$

Lemma 4.1 *Let $f, g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be proper locally Lipschitz functions at $x \in \text{dom}(f) \cap \text{dom}(g)$. Then, for all $d \in \mathbb{R}^n$:*

i) $(f + g)^o(x, d) \leq f^o(x, d) + g^o(x, d)$;
ii) $(\lambda f)^o(x, d) = \lambda f^o(x, d), \forall \lambda \geq 0$;
iii) $f^o(x, \lambda d) = \lambda f^o(x, d), \forall \lambda \geq 0$.

Proof. It is immediate from Clarke directional derivative.

Lemma 4.2 *(Clarke [8], Proposition 2.3.1.)Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be proper locally Lipschitz function at $x \in \text{dom}(f)$, and λ an arbitrary scalar, then*

$$\partial^o(\lambda f)(x) = \lambda \partial^o f(x). \tag{18}$$

Lemma 4.3 *Let $f_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, $i = 1, \cdots, p$, be proper locally Lipschitz functions at $x \in \cap_{i=1}^p \text{dom}(f_i)$, then*

$$\partial^o \left(\sum_{i=1}^p f_i \right)(x) \subset \sum_{i=1}^p \partial^o f_i(x). \tag{19}$$

Proof. It is immediately followed from Lemma 4.1 (i).

Proposition 4.1 *(Clarke [8], Proposition 2.1.1, (b)) Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper locally Lipschitz function on $\mathbb{R}^n$, then f^o is upper semicontinuous, i.e, if $(x, d) \in \mathbb{R}^n \times \mathbb{R}^n$ and $\{(x^k, d^k)\}$ is a sequence in $\mathbb{R}^n \times \mathbb{R}^n$ such that $\lim_{k \to +\infty} (x^k, d^k) = (x, d)$, then*

$$\limsup_{k \to +\infty} f^o(x^k, d^k) \leq f^o(x, d). \tag{20}$$

4.2 Descent direction

We are now able to introduce the definition of Pareto-Clarke critical point for locally Lipschitz functions on $\mathbb{R}^n$ which will play a key role in the paper.

Definition 4.2 (Custdio et al. [9], Definition 4.6) Let $F : \mathbb{R}^n \to \mathbb{R}^p$ be a locally Lipschitz function on $\mathbb{R}^n$. We say that $\hat{x} \in \mathbb{R}^n$ is a Pareto-Clarke critical point of F, if for all directions $d \in \mathbb{R}^n$, there exists $i_0 = i_0(d) \in \{1, \cdots, p\}$ such that $F_{i_0}^o(\hat{x}, d) \geq 0$.

The previous definition essentially tells us that there is no direction in $\mathbb{R}^n$ that is descent for all the objective functions F_i. If a point is a Pareto solution, then it is necessarily a Pareto-Clarke critical point.

Remark 4.1 *It follows from the previous definition that, if a point x is not a Pareto-Clarke critical point, then there exists a direction $d \in \mathbb{R}^n$ satisfying*

$$F_i^o(x, d) < 0, \quad \forall\, i \in \{1, \cdots, p\};$$

this implies that, for each $i \in \{1, \cdots, p\}$, d is a descent direction, for each function F_i, i.e, there exists $\epsilon > 0$, such that

$$F_i(x + td) < F_i(x), \quad \forall\, t \in (0, \epsilon] \ and \ \forall\, i \in \{1, \cdots, p\}.$$

In other words, d is a descent direction for the multiobjective function F at x, i.e, exist $\epsilon > 0$ such that

$$F(x + td) \prec F(x), \quad \forall\, t \in (0, \epsilon]. \tag{21}$$

4.3 A inexact proximal algorithm for quasiconvex minimization

We consider the function $F = (F_1, F_2, ..., F_m)^T : \mathbb{R}^n \to \mathbb{R}^m$ as a locally Lipschitz quasiconvex map, that is, each $F_i : \mathbb{R}^n \to \mathbb{R}$ is a locally Lipschitz and quasiconvex function on $\mathbb{R}^n$, where $i \in \{1, ..., m\}$. CQDIP Algorithm generates a sequence $\{x^k\} \subset \mathbb{R}^n$ as follows:

<u>CQDIP</u>

1. Choose $x^0 \in \mathbb{R}^n$.
2. Given x^k, if x^k is a Pareto-Clarke critical point, then $x^{k+p} = x^k, \forall p \geq 1$.
3. Given x^k, if x^k is not a Pareto-Clarke critical point, then consider as x^{k+1} any vector $x \in \Omega^k$ such that it exists $\varepsilon_k \in \mathbb{R}_+$ satisfying

$$0 \in \partial_{\varepsilon_k}^0(\langle F(.), z^k \rangle)(x) + \beta_k q(x, x^k)\partial(q(., x^k))(x) + N_{\Omega^k}(x), \tag{22}$$

$$q^2(x, x^k) \leq \bar{c}\, \|F(x) - F(x^k)\| \quad \text{and} \quad \lim_{k \to \infty} \varepsilon_k = 0. \tag{23}$$

where $N_{\Omega^k}(x)$ denotes the normal cone at the point x related to the set Ω^k, $\partial_{\varepsilon_k}^0(\langle F(.), z^k \rangle)(x)$ is the ε_k-Clarke subdifferential of $\langle F(.), z^k \rangle$ at point x, $\bar{c} \in \mathbb{R}_{++}$ and $\Omega^k = \{x \in \mathbb{R}^n \mid F(x) \leq F(x^k)\}$.

Proposition 4.2 (Well-posedness of CQDIP algorithm) *If $F : \mathbb{R}^n \to \mathbb{R}^m$ is a locally Lipschitz quasiconvex map, $q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ is a quasi-distance satisfying the conditions (H1)-(H3) and Ω^0 is bounded, then the sequence generates by the CQDIP algorithm is well defined.*

Proof Define $\varphi^k(x) = \langle F(x), z^k \rangle + \frac{\beta_k}{2} q^2(x, x^k)$. This function is continuous and Ω^k is compact, then, from Remark 2.2, **b**, there exists x^{k+1} such that

$$0 \in \partial \left(\varphi^k + \delta_{\Omega^k}(.) \right)(x^{k+1})$$

Using Proposition 2.1 we have

$$0 \in \partial(\langle F(.), z^k \rangle)(x^{k+1}) + \partial(\frac{\beta_k}{2} q^2(., x^k))(x^{k+1}) + \partial(\delta_{\Omega^k})(x^{k+1})$$

Since $\delta_{\Omega^k}(.)$ is convex and from [18, Proposition 3.15], then

$$0 \in \partial \left(\langle F(.), z^k \rangle \right)(x^{k+1}) + \beta_k q(x^{k+1}, x^k) \partial \left(q(., x^k) \right)(x^{k+1}) + N_{\Omega^k}(x^{k+1}).$$

From [1, Remark 2.51] we obtain

$$0 \in \partial^0 \left(\langle F(.), z^k \rangle \right)(x^{k+1}) + \beta_k q(x^{k+1}, x^k) \partial \left(q(., x^k) \right)(x^{k+1}) + N_{\Omega^k}(x^{k+1}),$$

that is, the condition (22) is satisfied with $\epsilon_k = 0$. The proof of the condition (23) is simmilar to the convex case. $\square$

Remark 4.2 *We are interested in the asymptotic convergence of the CQDIP algorithm, so we assume that $x^{k+1} \neq x^k, \forall k$. Otherwise, it is easy to prove that x^k is a Pareto-Clarke critical point of the problem (5).*

Proposition 4.3 (Properties) *Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a locally quasiconvex map, $q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ be a quasi-distance that satisfies the condition (H1), Ω^0 is bounded and the conditions (H2)-(H3) are satisfied. Let $\{x^k\}$ be a sequence generated by CQDIP algorithm, then we have:*

(a) *$\{x^k\}$ is bounded;*
(b) *$\forall z \in \mathbb{R}_+^m \setminus \{0\}, \{\langle F(x^k), z \rangle\}_{k \in N}$ is nonincreasing and convergent;*
(c) *$\lim\limits_{k \to \infty} \|F(x^{k+1}) - F(x^k)\| = 0$;*
(d) *$\lim\limits_{k \to \infty} q(x^{k+1}, x^k) = 0$;*
(e) *$\lim\limits_{k \to \infty} \|x^{k+1} - x^k\| = 0$.*

Proof Simmilar to Proposition 3.2 $\square$

Theorem 4.1 (Convergence) *Suppose that $F : \mathbb{R}^n \to \mathbb{R}^m$ is a locally Lipscitz quasiconvex map and $q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ is a quasi-distance that satisfies the condition (H1). If the set Ω^0 is bounded and the assumptions (H2) and (H3) are satisfied, then every sequence generated by CQDIP algorithm is bounded and its accumulation points are Pareto-Clarke critical points of the problem (5).*

Proof Let $\{x^k\}$ be a sequence generated by the CQDIP algorithm. By Proposition 4.3(a), there are $\hat{x} \in \mathbb{R}^n$ and $\{x^{k_j}\}_{j \in \mathbb{N}}$, a subsequence of $\{x^k\}$, such that $\lim_{j \to \infty} x^{k_j} = \hat{x}$. We will prove that $\hat{x}$ is a Pareto-Clarke critical point. By contradiction, suppose that there is some vector $d \in \mathbb{R}^n$, such that

$$F_i^o(\hat{x}, d) < 0, \quad \forall i \in \{1, \cdots, m\}. \tag{24}$$

Therefore d is a descent direction for the multiobjective function F at $\hat{x}$, so, $\exists \delta > 0$ such that $F(\hat{x} + \lambda d) \prec F(\hat{x})$ for all $\lambda \in (0, \delta]$ and thus $(\hat{x} + \lambda d) \in \Omega_k$. On the other hand, taking $v_{k+1} \in N_{\Omega_k}(x^{k+1})$ and $\zeta^{k+1} \in \partial(q(., x^k))(x^{k+1})$ in (22), we have

$$-\beta_k q(x^{k+1}, x^k)\zeta^{k+1} - v_{k+1} \in \partial_{\epsilon_k}^o(\langle F(.), z^k \rangle)(x^{k+1}).$$

Using Definition 4.1, (2), in the previous expression we have

$$\langle -\beta_k q(x^{k+1}, x^k)\zeta^{k+1} - v_{k+1}, q \rangle \leq \langle F(.), z^k \rangle^o(x^{k+1}, q) - \epsilon_k, \quad \forall q \in \mathbb{R}^n,$$

where $\langle F(.), z^k \rangle^o(x^{k+1}, q)$ represents the directional derivative of Clarke. Using properties of inner product and taking into account that $\beta_k > 0$, the previous expression implies

$$-\beta_k q(x^{k+1}, x^k)\langle \zeta^{k+1}, q \rangle - \langle v_{k+1}, q \rangle \leq \langle F(.), z^k \rangle^o(x^{k+1}, q) - \epsilon_k, \quad \forall q \in \mathbb{R}^n.$$

Considering $q = (\hat{x} + \lambda d) - x^{k+1}$ and taking into account that $v_{k+1} \in N_{\Omega_k}(x^{k+1})$ in the previous expression we have

$$-\beta_k q(x^{k+1}, x^k)\langle \zeta^{k+1}, q \rangle \leq \left(\sum_{i=1}^m z_i^k F_i(.) \right)^o (x^{k+1}, q) - \epsilon_k. \tag{25}$$

Applying respectively the items (i), (ii) and (iii) of Lemma 4.1, in the term of the right side of the previous inequality, we have

$$\left(\sum_{i=1}^m z_i^k F_i(.) \right)^o (x^{k+1}, q) \leq \sum_{i=1}^m \left(z_i^k F_i(x^{k+1}, q) \right)^o = \sum_{i=1}^m F_i^o(x^{k+1}, z_i^k q),$$

where z_i^k are the components of vector z_k. Replacing the previous expression in (25), we have to

$$-\beta_k q(x^{k+1}, x^k)\langle \zeta^{k+1}, q \rangle \leq \sum_{i=1}^m F_i^o(x^{k+1}, z_i^k q) - \epsilon_k, \tag{26}$$

Considering $k = k_j$ in (26), using $0 < \beta_k < c_2$, considering that $\{\zeta^{k_j}\}$ is bounded, Proposition 4.3,**d**, and taking $\limsup$ we have

$$0 \leq \limsup_{j \to +\infty} \sum_{i=1}^m F_i^o(x^{k_j+1}, z_{k_j}^i q') = \sum_{i=1}^m \limsup_{j \to +\infty} F_i^o(x^{k_j+1}, z_{k_j}^i q').$$

Since F is locally Lipschitz then from Proposition 4.1, F_i^0 is semicontinuous superior and $\lim_{j \to +\infty} \sup F_i^o(x^{k_j+1}, z_i^{k_j} q') \leq F_i^o(\hat{x}, \overline{z}_i \lambda d)$; so the above expression becomes

$$0 \leq \sum_{i=1}^{m} \limsup_{j \to +\infty} F_i^o(x^{k_j+1}, z_i^{k_j} q') \leq \sum_{i=1}^{m} F_i^o(\hat{x}, \overline{z}_i \lambda d). \qquad (27)$$

Making use of Lemma 4.1 (iii), and taking into account that $\lambda > 0$, in the previous expression we have

$$\sum_{i=1}^{m} \overline{z}_i F_i^0(\hat{x}, d) = \overline{z}^1 F_1^0(\hat{x}, d) + \cdots + \overline{z}^p F_p^0(\hat{x}, d) \geq 0. \qquad (28)$$

As $\overline{z} \in \mathbb{R}_{++}^m$, then in (28) there exists $i_0 \in \{1, 2, ..., m\}$ such that $F_{i_0}^0(\hat{x}, d) \geq 0$, which contradicts (24). Thus $\hat{x}$ is a Pareto-Clarke critical point of the problem (5). $\qquad \square$

5 Numerical experiments

Consider $F : \mathbb{R}^n \to \mathbb{R}^m$ a convex map, $q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ a quasi-distance map satisfying the condition (4), $\{z^k\} \subset \mathbb{R}_+^m \setminus \{0\}$ a limited sequence whose points of accumulation belong to $\mathbb{R}_{++}^m$, $\{\beta_k\} \subset \mathbb{R}$ a sequence of parameters satisfying $0 < c_1 < \beta_k < c_2$; $\forall k \in \mathbb{N}$ and $\{x^k\}$ a sequence generated by the QDIP algorithm.

It follows from the proof of proposition 3.1 that, for each fixed k, the critical points of the problem

$$\min \langle F(x), z^k \rangle + \frac{\beta_k}{2} q^2(x, x^k)$$
$$\text{s.a.} \quad F(x) \leq F(x^k) \qquad (29)$$

are solutions of (6) and (7), that is, satisfy

$$0 \in \partial(\langle F(.), z^k \rangle)(x) + \beta_k q(x, x^k) \partial(q(., x^k))(x) + N_{\Omega^k}(x) \quad \text{and}$$

$$q^2(x, x^k) \leq \overline{c} \, \|F(x) - F(x^k)\|.$$

The iterative procedure of Problem (29) was implemented in Matlab and tested in three groups (G1,G2 and G3) of multi-objective continuous test instants. An important feature of the instances is that the efficient solution sets are known and that the number n of variables can be chosen. The instances of groups $G1$ and $G2$ have two objective functions and, the those of group $G3$ have three objective functions. Table 1 presents the definition of the objectives and the respective efficient solution sets of instances.

In all tests, we considered the quasi-distance $q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ of Moreno et al. [18] given by: $q(x, y) = \sum_{i=1}^{n} q_i(x_i, y_i)$, where

Table 1 Objectives and efficient sets of the test instances.

Group	Objectives and efficient sets
G1 [10]	$f_1 = 1 - \exp(-((x_1 - 3)^2 + (x_2 - 3)^2 + ... + (x_n - 3)^2)),$ $f_2 = (x_1 - 3)^2 + (x_2 - 3)^2 + ... + (x_n - 3)^2.$ Its efficient set is $\{(3, 3, ..., 3)\}.$
G2 [13]	$f_1 = 1 + \frac{1}{2}\left[(x_1 - 1)^2 + x_2^2 + ... + x_n^2\right],$ $f_2 = 1 + \frac{1}{2}\left[x_1^2 + (x_2 - 1)^2 + x_3^2 + ... + x_n^2\right].$ Its efficient set is the line segment joining the points $(1, 0, ..., 0)$ and $(0, 1, 0, ..., 0).$
G3 [14]	$f_1 = \cos(x_1\pi/2)\cos(x_2\pi/2) + \frac{2}{|J_1|}\sum_{j \in J_1}\left(x_j - 2x_2\sin\left(2\pi x_1 + \frac{j\pi}{n}\right)\right)^2,$ $f_2 = \cos(x_1\pi/2)\sin(x_2\pi/2) + \frac{2}{|J_2|}\sum_{j \in J_2}\left(x_j - 2x_2\sin\left(2\pi x_1 + \frac{j\pi}{n}\right)\right)^2,$ $f_3 = \sin(x_1\pi/2) + \frac{2}{|J_3|}\sum_{j \in J_1}\left(x_j - 2x_2\sin\left(2\pi x_1 + \frac{j\pi}{n}\right)\right)^2,$ where $J_i = \{j\,|\,3 \le j \le n, \text{ and } j - 1 \text{ is a multiple of } 3\}, i = 1, 2.$ Its efficient set is $x_j = 2x_2\sin\left(2\pi x_1 + \frac{j\pi}{n}\right), j = 3, \ldots, n$

$$q_i(x_i, y_i) = \begin{cases} 2(y_i - x_i) \; if \; y_i - x_i > 0 \\ 3(x_i - y_i) \; if \; y_i - x_i \le 0. \end{cases}$$

All numerical experiments were performed using an Intel Core 2 Quad CPU Q9550 2.83 GHz, 4GB of RAM, running a 32-bit Linux. In the experiments we analyzed the convergence of the QDIP algorithm in different instances of each group. For each instance, we execute the QDIP algorithm with 100 initial iterations chosen randomly at $[0, 1]^n$. We represent the results in the table 2, where the ITER, TIME and DIST columns represent, respectively, the mean number of iterations, the mean CPU time and the mean Euclidean distance between the 100 solutions obtained by the QDIP algorithm and the respective efficient set. The results show that the algorithm converges for the efficient set.

Table 2 Mean number of iterations (ITER), mean CPU time in seconds (TIME) and mean Euclidean distance (DIST) between solutions obtained by the QDIP algorithm and the efficient set.

Group	n	m	ITER	TIME	DIST
G1	04	02	11.46	1.25	2.89E-05
G1	08	02	23.86	6.36	8.53E-05
G1	12	02	46.37	17.35	3.15E-04
G2	04	02	10.92	1.36	1.87E-05
G2	08	02	27.76	5.98	6.89E-05
G2	12	02	51.27	18.76	1.16E-05
G3	04	03	17.29	1.87	8.23E-05
G3	08	03	35.83	7.12	1.46E-05
G3	12	03	56.58	19.64	2.86E-04

6 Conclusions

This paper proposes a generalized inexact proximal point algorithm to solve unconstrained convex and quasiconvex multi-objective minimization problems.

Based on results of convex analysis, multi-objective optimization, techniques of convex and non-convex variational analysis and generalized differentiation, we prove that the proposed algorithm generates a sequence where each accumulation point is an efficient solution for the convex case and a Pareto-Clarke critical point for the quasiconvex ones. Numerical examples are also offered that confirm the convergence of the algorithm.

As future research, the authors are interested in studying the behavior of the (CQDIP) algorithm to solve quasiconvex vectorial minimization problems.

References

1. Apolinario, H.C.F., Papa Quiroz, E.A. and Oliveira, P.R., A scalarization proximal point method for quasiconvex multiobjective minimization, J. of Global Optim. 64(1), 79-96 (2016)
2. Bento, G. C., Cruz Neto, J. X., Soubeyran, A.: A proximal point-type method for multicriteria optimization. Set-Valued Var. Anal. **22**, 557-573 (2014)
3. Bonnel, H., Iusem, A.N., Svaiter, B.F.: Proximal methods in vector optimization. SIAM J. Optimiz. **15**, 953-970 (2005)
4. Brattka, V.: Recursive quasi-metric spaces. Theor. Comput. Sci. **305**, 17-42 (2003)
5. Ceng, L.C., Yao, J.C.: Approximate proximal methods in vector optimization. Eur. J. Oper. Res. **183**, 1-19 (2007)
6. Chuong, T.D.: Generalized proximal method for efficient solutions in vector optimization. Numer. Func. Anal. Opt. **32**, 843-857 (2011)
7. Chen, Z., Huang, X.X., Yang, X.Q.: Generalized proximal point algoritms for multiobjective optimization. Appl. Anal. **90**, 935-949 (2011)
8. Clarke, H.F. Optimization and nonsmooth analysis. Classics in applied mathematics, SIAM, New York (1983).
9. Custodio, A.L., Madeira, J.F.A., Vaz, A.I.F and Vicente, L.N. Direct multisearch for multiobjective optimization. SIAM Journal on Optimization. 21, 1109-1140 (2011).
10. Fonseca, C.M., Fleming, P.J.: An overview of evolutionary algorithms in multiobjective optimization, Evol. Comput. **3**(1), 1-16 (1995)
11. J. Janh, Theory of Vector Maximization: Various Concepts of Efficient Solutions, in Multicriteria Decision Making, Advances in MCDM Models, Algorithms, Theory, and Applications, T. Gal, T. J. Stewart, and T. Hanne, eds., vol 21, pp. 37-68, Kluwer, Boston (1999)
12. Kunzi, H.P.A., Pajoohesh, H., Schellekens, M.P.: Partial quasi-metrics. Theor. Comput. Sci. **365**, 237-246 (2006)
13. Laumanns, M.: Self-adaptation and Convergence of Multiobjective Evolutionary Algorithms in Continuous Search Spaces. In: Abraham, A., Jain, L., Goldberg, R. (Eds.) Evolutionary Multiobjective Optimization: Theoretical Advances and Applications, pp. 33-52. Springer, Berlim (2005)
14. Li, H., Zhang, Q.: Multiobjective optimization problems with complicated Pareto sets, MOEA/D and NSGA-II. IEEE Trans. Evol. Comput. **13**, 284-302 (2011)
15. Martinet, B.: Régularization d'inéquations variationelles par approximations successives. Revue Fracaise d'informatique et Recherche Operationelle **4**, 154-159 (1970)
16. Mordukhovich, B.S.: Variational Analysis and Generalized Differentiation I, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 330. Springer, Berlin (2006)
17. Mordukhovich, B.S., Shao, Y.: Nonsmooth sequential analysis in asplund spaces. Trans. Amer. Math. Soc. **348**, 1235-1280 (1996)

18. Moreno, F.G, Oliveira, P.R., Soubeyran, A.: A proximal algorithm with quasi distance. Application to habit's formation. Optim. **61**, 1383-1403 (2012)
19. E. A. P. Quiroz, H. Apolinario, K. Villacorta, and P. R. Oliveira, A linear scalarization proximal point method for quasiconvex multiobjective minimization J. Optimiz. Theory Appl., vol. 183, no. 3, pp. 1028-52, 2019.
20. Rocha, R.A., Oliveira, P.R., Gregório, R.M., Souza, M.: Logarithmic quasi-distance proximal point scalarization method for multi-objective programming, Appl. Math. Comput. **273**, 856-867 (2016)
21. Rocha, R.A., Oliveira, P.R., Gregório, R.M., Souza, M.: A proximal point algorithm with quasi-distance in multi-objective optimization. J. Optim. Theory Appl. **171**, 964-979 (2016)
22. Rockafellar, R.T.: Monotone operators and the proximal point algorithm. SIAM J. Control. Optim. **14**(5), 877-898 (1976)
23. Rockafellar, R.T., Wets, R.J-B.: Variational Analysis, Vol. 317. Springer, Berlin (1998).
24. Romaguera, S., Sanchis, M.: Applications of utility functions defined on quasi-metric spaces. J Math. Anal. Appl. **283** (1), 219-235 (2003)
25. Villacorta, K.D.V., Oliveira, P.R.: An interior proximal method in vector optimization. Eur. J. Oper. Res. **214**, 485-492 (2011)