

A golden ratio primal-dual algorithm for structured convex optimization

Xiaokai Chang¹ · Junfeng Yang²

Received: date / Revised: date / Accepted: date

Abstract We design, analyze and test a golden ratio primal-dual algorithm (GRPDA) for solving structured convex optimization problem, where the objective function is the sum of two closed proper convex functions, one of which involves a composition with a linear transform. GRPDA preserves all the favorable features of the classical primal-dual algorithm (PDA), i.e., the primal and the dual variables are updated in a Gauss-Seidel manner, and the per iteration cost is dominated by the evaluation of the proximal point mappings of the two component functions and two matrix-vector multiplications. Compared with the classical PDA, which takes an extrapolation step, the novelty of GRPDA is that it is constructed based on a convex combination of essentially the whole iteration trajectory. We show that GRPDA converges within a broader range of parameters than the classical PDA, provided that the reciprocal of the convex combination parameter is bounded above by the golden ratio, which explains the name of the algorithm. An $\mathcal{O}(1/N)$ ergodic convergence rate result is also established based on the primal-dual gap function, where N denotes the number of iterations. When either the primal or the dual problem is strongly convex, an accelerated GRPDA is constructed to improve the ergodic convergence rate from $\mathcal{O}(1/N)$ to $\mathcal{O}(1/N^2)$. Moreover, we show for regularized least-squares and linear equality constrained problems that the reciprocal of the convex combination parameter can be extended from the golden ratio to 2 and meanwhile a relaxation step can be taken. Our preliminary numerical results on LASSO, nonnegative least-squares and minimax matrix game problems, with comparisons to some state-of-the-art relative algorithms, demonstrate the efficiency of the proposed algorithms.

Keywords Structured convex optimization · saddle point problem · primal-dual algorithm · golden ratio · acceleration · convergence rate · fixed point iteration

Mathematics Subject Classification (2000) 49M29 · 65K10 · 65Y20 · 90C25

1 Introduction

Let $\mathbb{R}^p$ and $\mathbb{R}^q$ be finite-dimensional Euclidean spaces, each endowed with an inner product and the induced norm denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \| = \sqrt{\langle \cdot, \cdot \rangle}$, respectively. In this paper, we consider structured convex optimization problem of the form

$$\min_{x \in \mathbb{R}^q} f(Kx) + g(x), \quad (1)$$

¹ School of Science, Lanzhou University of Technology, Lanzhou, Gansu, P. R. China. Email: xkchang@lut.cn.

² Department of Mathematics, Nanjing University, Nanjing, P. R. China. This author was supported by the NSFC grants 11922111 and 11771208. Email: jfyang@nju.edu.cn.

where $f : \mathbb{R}^p \rightarrow (-\infty, +\infty]$ and $g : \mathbb{R}^q \rightarrow (-\infty, +\infty]$ are extended real-valued closed proper convex functions [30], and $K \in \mathbb{R}^{p \times q}$ is a linear operator from $\mathbb{R}^q$ to $\mathbb{R}^p$. Problem (1) arises from numerous applications, including signal and image processing, machine learning, statistics, mechanics and economics, to name a few, see, e.g., [4, 5, 8, 17, 34] and the references therein. The saddle point or primal-dual form of (1) reads

$$\min_{x \in \mathbb{R}^q} \max_{y \in \mathbb{R}^p} \mathcal{L}(x, y) := g(x) + \langle Kx, y \rangle - f^*(y), \quad (2)$$

where $f^*(y) = \sup_{u \in \mathbb{R}^p} \langle y, u \rangle - f(u)$, $y \in \mathbb{R}^p$, is the Legendre-Fenchel conjugate of f . Saddle point problems are ubiquitous in optimization as they provide a very convenient way to represent many nonsmooth problems. In particular, (2) is intrinsically related to the Fenchel dual problem of (1), which is given by

$$\max_{y \in \mathbb{R}^p} -f^*(y) - g^*(-K^\top y), \quad (3)$$

where $K^\top$ denotes the matrix transpose or adjoint operator of K .

We note that in this paper we restrict the decision variable x in $\mathbb{R}^q$, yet all the analysis can be extended to any finite dimensional real Euclidean spaces, e.g., the matrix space $\mathbb{R}^{m \times n}$ with trace inner product and the induced Frobenius norm. In the rest of this section, we define some notation, make our assumptions, review some algorithms for solving (1)-(3) that are closely related to this work, and summarize our contributions and the organization of this paper.

1.1 Notation and assumptions

The matrix and vector transpose operation is denoted by superscript “ $\top$ ”. The operator norm of K is denoted by L , i.e., $L := \|K\| = \|K^\top\| = \sup\{\langle Kx, y \rangle : \|x\| = \|y\| = 1, x \in \mathbb{R}^q, y \in \mathbb{R}^p\}$. Throughout this paper, we denote the golden ratio by ϕ , i.e., $\phi = \frac{\sqrt{5}+1}{2}$. Let h be any extended real-valued closed proper convex function defined on a finite dimensional Euclidean space $\mathbb{R}^m$. The effective domain of h is denoted by $\text{dom}(h) := \{x \in \mathbb{R}^m : h(x) < +\infty\}$, and the subdifferential of h at $x \in \mathbb{R}^m$ is denoted by $\partial h(x) := \{\xi \in \mathbb{R}^m : h(y) \geq h(x) + \langle \xi, y - x \rangle \text{ for all } y \in \mathbb{R}^m\}$. Furthermore, for $\lambda > 0$, the proximal point mapping of λh is given by

$$\text{Prox}_{\lambda h}(x) := \arg \min_{y \in \mathbb{R}^m} \left\{ h(y) + \frac{1}{2\lambda} \|y - x\|^2 \right\}, \quad x \in \mathbb{R}^m.$$

Since h is closed proper and convex, for any $\lambda > 0$, $\text{Prox}_{\lambda h}$ is uniquely well defined everywhere. The indicator function of a set C is denoted by $\iota_C(x)$, i.e., $\iota_C(x) = 0$ if $x \in C$ and $+\infty$ if otherwise. The relative interior of C is denoted by $\text{ri}(C)$. The identity operator or identity matrix is denoted by I , whose domain or order is clear from the context. The zero vector or matrix is simply denoted by 0 . The composition of two operators is denoted by “ $\circ$ ”. The stagnation of two column vectors u and v is also denoted by $(u; v)$, i.e., $(u; v) = (u^\top, v^\top)^\top$. The sequence of positive natural numbers is denoted by $\mathbb{N} = \{1, 2, 3, \dots\}$. Other notation will be specified later.

Throughout the paper, we make the following blanket assumptions.

Assumption 1.1 Assume that the set of solutions of (1) is nonempty and, in addition, there exists $\tilde{x} \in \text{ri}(\text{dom}(g))$ such that $K\tilde{x} \in \text{ri}(\text{dom}(f))$.

Under Assumption 1.1, it follows from [30, Corollaries 28.2.2 and 28.3.1] that $\bar{x} \in \mathbb{R}^q$ is a solution of (1) if and only if there exists $\bar{y} \in \mathbb{R}^p$ such that $(\bar{x}, \bar{y})$ is a saddle point of $\mathcal{L}(x, y)$, i.e., $\mathcal{L}(\bar{x}, y) \leq \mathcal{L}(\bar{x}, \bar{y}) \leq \mathcal{L}(x, \bar{y})$ for all $(x, y) \in \mathbb{R}^q \times \mathbb{R}^p$, and furthermore, such $\bar{y}$ is an optimal solution of the dual problem (3). Throughout this paper, we denote the set of solutions of (2) by $\mathcal{S}$, which is nonempty under Assumption 1.1 and characterized by

$$\mathcal{S} := \{(\bar{x}, \bar{y}) \in \mathbb{R}^q \times \mathbb{R}^p : 0 \in \partial g(\bar{x}) + K^\top \bar{y} \text{ and } 0 \in \partial f^*(\bar{y}) - K\bar{x}\}. \quad (4)$$

In many applications including signal and image processing and machine learning, the two component functions in (1) enforce, respectively, data fitting and regularization. In such cases, f and g usually preserve simple structures so that their proximal point mappings can be evaluated efficiently. Examples of such functions are abundant, see, e.g., [2, Chapter 6]. We therefore make the following assumption.

Assumption 1.2 *Assume that the proximal point mappings of the component functions f and g either have closed form formulas or can be evaluated efficiently.*

1.2 Related algorithms

To solve (1)-(3) simultaneously, one may resort to the well known alternating direction method of multipliers (ADMM) [15, 16, 19, 20], the primal-dual algorithm (PDA) [8, 13, 18, 29] and their accelerated and generalized variants [21, 23]. Since the literature on numerical algorithms for solving (1)-(3) has become so vast, a thorough overview is far beyond the focus of this paper. Instead, we next review only some primal-dual type algorithms that are most closely related to our work.

A main feature of primal-dual type algorithms is that both the primal and the dual variables are updated at each iteration, and thus the primal and the dual problems are solved simultaneously. Among others, ADMM [15, 16] has been well studied in the literature [14, 19, 20] and widely used in practice [6]. The main difficulty encountered by ADMM when applied to (a reformulation of) (1) is that a subproblem of the form $\min_{x \in \mathbb{R}^q} \frac{1}{2} \|Kx - b_n\|^2 + g(x)$ needs to be solved at each iteration for some $b_n \in \mathbb{R}^p$ varying with the iteration counter n . We note that even though Prox_g is easy to evaluate, this problem, which involves a linear operator K , has to be solved iteratively in general. On the other hand, when $f(Kx)$ has a least-squares structure, i.e., $f(Kx) = \frac{1}{2} \|Kx - b\|^2$ for some $b \in \mathbb{R}^p$, ADMM can be applied to the equivalent problem $\min_{x,y} \{f(Kx) + g(y) : x = y\}$. In this case, the x -subproblem appears as a least-squares problem and is equivalent to solving a linear system of equations with coefficient matrix of the form $I + \rho K^\top K$ for some $\rho > 0$, which could be expensive or even prohibitive for large scale problems.

The simplest primal-dual type algorithm for solving (1)-(3), which does not require to solve any subproblem iteratively or any linear system of equations, is probably the classical Arrow-Hurwicz method [33], which, started at $(x_0, y_0) \in \mathbb{R}^q \times \mathbb{R}^p$, iterates as

$$\begin{cases} x_n = \text{Prox}_{\tau g}(x_{n-1} - \tau K^\top y_{n-1}), \\ y_n = \text{Prox}_{\sigma f^*}(y_{n-1} + \sigma K x_n), \end{cases} \quad (5)$$

for $n \geq 1$. Here $\tau > 0$ and $\sigma > 0$ are step size parameters. As a conjugate function, f^* is always closed and convex. Furthermore, f^* is also proper since f is proper and convex. Thus, $\text{Prox}_{\sigma f^*}$ is uniquely well defined everywhere. The iterative scheme (5) is also known as primal-dual hybrid gradient method in image processing community, see [8, 13, 35]. The main computational cost per iteration of (5) is the evaluation of two proximal point mappings and two matrix-vector multiplications. The convergence of the Arrow-Hurwicz method was studied in [13] with very small step sizes, and $\mathcal{O}(1/\sqrt{N})$ rate of convergence, measured by primal-dual gap, was obtained in [8, 25] when the domain of f^* is assumed to be bounded. However, the Arrow-Hurwicz method does not converge in general, see [18] for a divergent example.

Chambolle and Pock [8, 29] adopted an extrapolation step after obtaining x_n in (5) to obtain $\bar{x}_n = x_n + \delta(x_n - x_{n-1})$ for some $\delta \in [0, 1]$, which is then used to replace x_n in (5) in the update of y_n . The resulting scheme is nowadays widely accepted as PDA and appears as

$$\begin{cases} x_n = \text{Prox}_{\tau g}(x_{n-1} - \tau K^\top y_{n-1}), \\ \bar{x}_n = x_n + \delta(x_n - x_{n-1}), \\ y_n = \text{Prox}_{\sigma f^*}(y_{n-1} + \sigma K \bar{x}_n). \end{cases} \quad (6)$$

In the case $\delta = 1$, the convergence of (6) was established in [8] under the condition $\tau\sigma L^2 < 1$, where $L = \|K\|$. Note that, for $\delta = 1$ the scheme (6) is referred to as a split inexact Uzawa method by Esser, Zhang and Chan [13], where the connection of PDA with preconditioned or linearized ADMM has been revealed, see also [8, 31]. Later, it was shown in [18] that PDA (6) can be viewed as a weighted proximal point method applied to the variational inequality (VI) representation of the optimality conditions of (2). The overrelaxed, inertial and accelerated versions of (6) were investigated in [9]. See also [7] for a stochastic variant of PDA.

The effectiveness of the PDA scheme (6) is essentially guaranteed by the extrapolation or inertial step $\bar{x}_n = x_n + \delta(x_n - x_{n-1})$ since the Arrow-Hurwicz method (5), which corresponds to $\delta = 0$, fails to converge in general. For a fixed $\delta \in [0, 1]$, the scheme (6) can be written abstractly as $(x_n, y_n) = T(x_{n-1}, y_{n-1})$ for some suitably defined mapping T . Therefore, PDA can be viewed as a one-step fixed point iterative method. Recently, Malitsky [22] proposed a fully adaptive forward-backward type splitting algorithm, called golden ratio algorithm, for solving mixed VI problem, where a novel convex combination technique is introduced. It is shown that the golden ratio algorithm preserves $\mathcal{O}(1/N)$ ergodic convergence and R-linear convergence under an error bound condition. The mixed VI problem is to find $z^* \in \mathbb{R}^m$ such that

$$\theta(z) - \theta(z^*) + \langle z - z^*, F(z^*) \rangle \geq 0, \quad \forall z \in \mathbb{R}^m, \quad (7)$$

where $\theta : \mathbb{R}^m \rightarrow (-\infty, +\infty]$ is a closed proper convex function, and $F : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is a monotone mapping. Given an initial point $z_0 \in \mathbb{R}^m$ and let $\bar{z}_0 = z_0$, a basic version of the golden ratio algorithm [22] applied to (7) iterates for $n \geq 1$ as

$$\begin{cases} \bar{z}_n = \frac{\phi-1}{\phi} z_{n-1} + \frac{1}{\phi} \bar{z}_{n-1}, \\ z_n = \text{Prox}_{\tau\theta}(\bar{z}_n - \tau F(z_{n-1})), \end{cases} \quad (8)$$

where $\tau > 0$ is a step size parameter and ϕ is the golden ratio. By induction, $\bar{z}_n$ is a convex combination of $\{z_i : i = 0, 1, \dots, n-1\}$. As a result, the update from z_{n-1} to z_n in (8) is essentially dependent on the whole iteration trajectory, which is vastly different from the one-step iterative scheme (6). Note that the primal-dual problem (2) is equivalent to (7) with $\mathbb{R}^m = \mathbb{R}^q \times \mathbb{R}^p$, endowed with the natural inner product $\langle (x, y), (u, v) \rangle := \langle x, u \rangle + \langle y, v \rangle$ for $(x, y), (u, v) \in \mathbb{R}^q \times \mathbb{R}^p$,

$$\theta(z) := g(x) + f^*(y) \quad \text{and} \quad F(z) := (K^\top y, -Kx). \quad (9)$$

Given the special structure of θ and F in (9), the golden ratio algorithm (8) breaks into

$$\begin{cases} \bar{x}_n = \frac{\phi-1}{\phi} x_{n-1} + \frac{1}{\phi} \bar{x}_{n-1}, & x_n = \text{Prox}_{\tau g}(\bar{x}_n - \tau K^\top y_{n-1}), \\ \bar{y}_n = \frac{\phi-1}{\phi} y_{n-1} + \frac{1}{\phi} \bar{y}_{n-1}, & y_n = \text{Prox}_{\tau f^*}(\bar{y}_n + \tau K x_{n-1}). \end{cases} \quad (10)$$

Apparently, (10) is a Jacobian type algorithm, which fails to fully utilize the latest available information. Furthermore, the primal and dual step sizes are identically τ , which offers less flexibility. Given the nice convergence properties and the promising numerical results of golden ratio algorithm [22], it is desirable to construct a Gauss-Seidel type golden ratio PDA for the saddle point problem (2), which is able to fully take advantage of the problem structure. This motivated the current work.

1.3 Contributions

We adapt the convex combination technique [22] into the Arrow-Hurwicz scheme (5) to construct a Gauss-Seidel type golden ratio PDA (abbreviated as GRPDA), which not only preserves nice convergence properties but also performs favorably in practice. Our main contributions are summarized below.

- Using the convex combination technique [22], we construct a GRPDA with fixed step size parameters $\tau > 0$ and $\sigma > 0$. Under the condition $\tau\sigma L^2 < \psi \in (1, \phi]$, global convergence and $\mathcal{O}(1/N)$ ergodic rate of convergence measured by primal-dual gap function are established. Since $\tau\sigma L^2 > 1$ is permitted, GRPDA converges in a broader range of parameters compared to PDA, which typically requires $\tau\sigma L^2 < 1$. It is further explained that GRPDA is an ADMM-like method with a proximal term $\frac{1}{2}\|x - x_{n-1}\|_M^2$, where $M = \frac{1}{\tau}I - \sigma K^\top K$, plus an extra linear term. As a consequence of the extra linear term, the weighting matrix M is allowed to be indefinite, leading to a broader range of parameters.
- When either g or f^* is strongly convex, an accelerated GRPDA is constructed to improve the ergodic convergence rate from $\mathcal{O}(1/N)$ to $\mathcal{O}(1/N^2)$.
- Let $b \in \mathbb{R}^p$. For regularized least-squares problem, i.e., $f(\cdot) = \frac{1}{2}\|\cdot - b\|^2$, or linear equality constrained problem, i.e., $f(\cdot) = \iota_{\{b\}}(\cdot)$, we show via spectral analysis that the permitted range of ψ can be extended from $(1, \phi]$ to $(1, 2]$ and meanwhile a relaxation step can be taken as well, where the relaxation parameter lies in $(0, 3/2)$. The analysis for these two special cases is based on the fixed point theory of averaged operators [1, Chapter 5].
- We carry out numerical experiments on LASSO, nonnegative least-squares and minimax matrix game problems, with comparisons to some state-of-the-art algorithms, to demonstrate the favorable performance of the proposed algorithms.

1.4 Organization

The rest of this paper is organized as follows. In Section 2, we summarize some useful facts and identities and define the primal-dual gap function used in subsequent analysis. The GRPDA with fixed step sizes is presented in Section 3, where its convergence and $\mathcal{O}(1/N)$ ergodic rate of convergence are established as well. In Section 4, we present an accelerated GRPDA that enjoys a faster $\mathcal{O}(1/N^2)$ ergodic rate of convergence under the assumption that either g or f^* is strongly convex. Section 5 is devoted to the analysis of two special cases, i.e., regularized least-squares problem and linear equality constrained problem. Numerical results in comparison with some state-of-the-art relative algorithms on LASSO, nonnegative least-squares and minimax matrix game problems are given in Section 6 to demonstrate the efficiency of the proposed algorithms. Finally, some concluding remarks are drawn in Section 7.

2 Preliminaries

In this section, we summarize some useful facts and identities and define the primal-dual gap function, which will be useful in our analysis.

Fact 2.1 *For any extended real-valued closed proper convex function h defined on an Euclidean space $\mathbb{R}^m$, $\lambda > 0$ and $x \in \mathbb{R}^m$, it holds that $p = \text{Prox}_{\lambda h}(x)$ if and only if*

$$\langle p - x, y - p \rangle \geq \lambda(h(p) - h(y)), \quad \forall y \in \mathbb{R}^m.$$

Fact 2.2 *Let $\{a_n\}_{n \in \mathbb{N}}$ and $\{b_n\}_{n \in \mathbb{N}}$ be two nonnegative real sequences. If there exists an integer $N > 0$ such that $a_{n+1} \leq a_n - b_n$ for all $n > N$, then $\lim_{n \rightarrow \infty} a_n$ exists and $\lim_{n \rightarrow \infty} b_n = 0$.*

The proofs of Facts 2.1 and 2.2 are easily derived and thus are omitted. The following elementary identities will be used in our analysis. For any $x, y, z \in \mathbb{R}^m$ and $\alpha \in \mathbb{R}$, there hold

$$\langle x - y, x - z \rangle = \frac{1}{2}\|x - y\|^2 + \frac{1}{2}\|x - z\|^2 - \frac{1}{2}\|y - z\|^2, \quad (11)$$

$$\|\alpha x + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2. \quad (12)$$

We next define the primal-dual gap function that will be used in our analysis. Let $(\bar{x}, \bar{y}) \in \mathcal{S}$ be any saddle point of the primal-dual problem (2). Then, there hold $K\bar{x} \in \partial f^*(\bar{y})$ and $-K^\top \bar{y} \in \partial g(\bar{x})$, which are equivalent to

$$\begin{cases} P(x) := g(x) - g(\bar{x}) + \langle K^\top \bar{y}, x - \bar{x} \rangle \geq 0, & \forall x \in \mathbb{R}^q, \\ D(y) := f^*(y) - f^*(\bar{y}) - \langle K\bar{x}, y - \bar{y} \rangle \geq 0, & \forall y \in \mathbb{R}^p. \end{cases}$$

The primal-dual gap function is defined by

$$G(x, y) := P(x) + D(y) \geq 0, \quad \forall (x, y) \in \mathbb{R}^q \times \mathbb{R}^p. \quad (13)$$

This primal-dual gap function is also used in, e.g., [9, 23]. Note that, for fixed $(\bar{x}, \bar{y}) \in \mathcal{S}$, $P(x)$ and $D(y)$, and thus $G(x, y)$, are convex. Although $P(\cdot)$, $D(\cdot)$ and $G(\cdot, \cdot)$ depend on $(\bar{x}, \bar{y})$, we do not indicate this dependence in our notation since it is always clear from the context.

3 Golden ratio primal-dual algorithm

In this section, we present our GRPDA with fixed step sizes and establish its convergence and $\mathcal{O}(1/N)$ ergodic convergence rate.

3.1 GRPDA with fixed step sizes

Recall that ϕ represents the golden ratio and $L = \|K\|$ is the operator norm of K . Below, we introduce our GRPDA with fixed step sizes $\tau > 0$ and $\sigma > 0$.

Algorithm 3.1 (GRPDA with fixed step sizes)

Step 0. Let $\tau, \sigma > 0$ and $\psi \in (1, \phi]$ be such that $\tau\sigma L^2 < \psi$. Choose $x_0 \in \mathbb{R}^q$, $y_0 \in \mathbb{R}^p$. Set $z_0 = x_0$ and $n = 1$.

Step 1. Compute

$$\begin{cases} z_n = \frac{\psi-1}{\psi} x_{n-1} + \frac{1}{\psi} z_{n-1}, \\ x_n = \text{Prox}_{\tau g}(z_n - \tau K^\top y_{n-1}), \\ y_n = \text{Prox}_{\sigma f^*}(y_{n-1} + \sigma K x_n). \end{cases} \quad (14)$$

Step 2. Set $n \leftarrow n + 1$ and return to Step 1.

The GRPDA (14) and the PDA (6) are quite similar, both of which are modifications of the Arrow-Hurwicz scheme (5). The difference is that PDA (6) adopts an inertial or extrapolation technique, while GRPDA (14) uses a convex combination $z_n = \frac{\psi-1}{\psi} x_{n-1} + \frac{1}{\psi} z_{n-1}$. To our knowledge, this convex combination technique with ψ upper bounded by the golden ratio ϕ was initially introduced by Malitsky [22] to solve monotone mixed VI problem (7). A direct adaptation of [22, Eq. (10)] to the primal-dual problem (2) would give the algorithm (10), which is a Jacobian type algorithm and has less flexibility since the primal and the dual step sizes are required to be identical. In contrast, GRPDA (14) is a combination of [22] with the Arrow-Hurwicz scheme (5), and it is of Gauss-Seidel type since it utilizes the latest available information. Compared with the direct adaptation (10), the primal and the dual step sizes in (14) are not necessarily identical, which offers more flexibility in practice. Moreover, compared with PDA (6), which requires $\tau\sigma L^2 < 1$, the condition $\tau\sigma L^2 < \psi \in (1, \phi]$ required by GRPDA (14) permits larger step sizes. Apparently, GRPDA has the same per iteration cost as those of the Arrow-Hurwicz method and the PDA.

3.2 Connection with ADMM and PDA

It is well known that the classical PDA can be interpreted as a preconditioned or linearized ADMM with positive definition proximal term, see [8, 13, 31]. As the main difference between GRPDA and PDA is that the extrapolation step in (6) is replaced by the convex combination in (14), it is natural to connect GRPDA with the ADMM. To be specific, we define the augmented Lagrangian function of $\min_{x,w} \{g(x) + f(w) : Kx = w\}$, a reformulation of (1), as

$$\mathcal{L}_\sigma(x, w, y) := g(x) + f(w) + \langle y, Kx - w \rangle + \frac{\sigma}{2} \|Kx - w\|^2,$$

where $y \in \mathbb{R}^p$ is the Lagrange multiplier and $\sigma > 0$ is the penalty parameter. Then, by using the Moreau decomposition $y = \text{Prox}_{\lambda f}(y) + \lambda \text{Prox}_{\frac{1}{\lambda} f^*}(\frac{y}{\lambda})$ for any $\lambda > 0$ and $y \in \mathbb{R}^p$ and following [8, 13, 31], it is easy to show that GRPDA (14) is equivalent to

$$\begin{cases} z_n = \frac{\psi-1}{\psi} x_{n-1} + \frac{1}{\psi} z_{n-1}, \\ x_n = \arg \min_x \{ \mathcal{L}_\sigma(x, w_{n-1}, y_{n-1}) + \ell(x, z_n, x_{n-1}, w_{n-1}) + \frac{1}{2} \|x - x_{n-1}\|_M^2 \}, \\ w_n = \arg \min_w \mathcal{L}_\sigma(x_n, w, y_{n-1}), \\ y_n = y_{n-1} + \sigma(Kx_n - w_n), \end{cases} \quad (15)$$

where $\ell(x, z_n, x_{n-1}, w_{n-1}) = \frac{1}{\tau} \langle x, x_{n-1} - z_n \rangle - \sigma \langle Kx, Kx_{n-1} - w_{n-1} \rangle$, $M = \frac{1}{\tau} I - \sigma K^\top K$ and $w_0 \in \mathbb{R}^p$ is arbitrary. Here the proximal term $\frac{1}{2} \|x - x_{n-1}\|_M^2$ is used to cancel out $\frac{\sigma}{2} \|Kx\|^2$ in the x -subproblem. Since, by following [8, 13, 31], the derivation of (15) from (14) is standard, we omit the details. By discarding the linear term $\ell(x, z_n, x_{n-1}, w_{n-1})$ in (15) and using the Moreau decomposition to carry out a similar reduction, we obtain

$$\begin{cases} \bar{y}_{n-1} = 2y_{n-1} - y_{n-2}, \\ x_n = \text{Prox}_{\tau g}(x_{n-1} - \tau K^\top \bar{y}_{n-1}), \\ y_n = \text{Prox}_{\sigma f^*}(y_n + \sigma Kx_n), \end{cases}$$

which is a form of PDA with $\delta = 1$. Note that PDA requires $\tau\sigma L^2 < 1$ to guarantee the positive definiteness of M . In comparison, by introducing the linear term $\ell(x, z_n, x_{n-1}, w_{n-1})$, GRPDA (15) allows larger step sizes since $\tau\sigma L^2 < \psi \in (1, \phi]$ suffices for global convergence. In this case, M is permitted to be indefinite. This broader convergence region of GRPDA is beneficial in practice.

3.3 Convergence results

We next establish the convergence and $\mathcal{O}(1/N)$ ergodic convergence rate of GRPDA (14). For convenience, we let $\beta := \sigma/\tau$ in the rest of this section. First, we present a useful lemma.

Lemma 3.1 *Let $\{(x_n, y_n, z_n)\}_{n \in \mathbb{N}}$ be the sequence generated by Algorithm 3.1 from any initial point $(x_0, y_0) \in \mathbb{R}^q \times \mathbb{R}^p$ and $z_0 = x_0$, and let $G(\cdot, \cdot)$ be defined as in (13) with any fixed $(\bar{x}, \bar{y}) \in \mathcal{S}$. Then, it holds for any $(x, y) \in \mathbb{R}^q \times \mathbb{R}^p$ that*

$$\begin{aligned} \tau G(x_n, y_n) &\leq \langle x_{n+1} - z_{n+1}, \bar{x} - x_{n+1} \rangle + \frac{1}{\beta} \langle y_n - y_{n-1}, \bar{y} - y_n \rangle + \psi \langle x_n - z_{n+1}, x_{n+1} - x_n \rangle \\ &\quad + \tau \langle K^\top (y_n - y_{n-1}), x_n - x_{n+1} \rangle. \end{aligned} \quad (16)$$

Proof. It follows from (14) and Fact 2.1 that

$$\langle x_{n+1} - z_{n+1} + \tau K^\top y_n, \bar{x} - x_{n+1} \rangle \geq \tau (g(x_{n+1}) - g(\bar{x})), \quad (17)$$

$$\left\langle \frac{1}{\beta} (y_n - y_{n-1}) - \tau Kx_n, \bar{y} - y_n \right\rangle \geq \tau (f^*(y_n) - f^*(\bar{y})). \quad (18)$$

Similar to (17), it holds that

$$\langle x_n - z_n + \tau K^\top y_{n-1}, x_{n+1} - x_n \rangle \geq \tau(g(x_n) - g(x_{n+1})). \quad (19)$$

Elementary calculations show that the addition of (17), (18) and (19) gives

$$\begin{aligned} & \langle x_{n+1} - z_{n+1}, \bar{x} - x_{n+1} \rangle + \frac{1}{\beta} \langle y_n - y_{n-1}, \bar{y} - y_n \rangle + \langle x_n - z_n, x_{n+1} - x_n \rangle \\ & + \tau \langle K^\top (y_n - y_{n-1}), x_n - x_{n+1} \rangle - \tau \langle K^\top \bar{y}, x_n - \bar{x} \rangle + \tau \langle K \bar{x}, y_n - \bar{y} \rangle \\ & \geq \tau(f^*(y_n) - f^*(\bar{y})) + \tau(g(x_n) - g(\bar{x})), \end{aligned}$$

which is the same as (16) by considering $x_n - z_n = \psi(x_n - z_{n+1})$ (implied by the first equality in (14)) and following the definition of $G(\cdot, \cdot)$ in (13). This completes the proof. $\square$

Now, we are ready to establish the convergence of GRPDA (14).

Theorem 3.1 *Let $\{(x_n, y_n, z_n)\}_{n \in \mathbb{N}}$ be the sequence generated by Algorithm 3.1 from any initial point $(x_0, y_0) \in \mathbb{R}^q \times \mathbb{R}^p$ and $z_0 = x_0$, and let $G(\cdot, \cdot)$ be defined as in (13) with any fixed $(\bar{x}, \bar{y}) \in \mathcal{S}$. Then, $\{(x_n, y_n)\}_{n \in \mathbb{N}}$ converges to a solution of (2), i.e., an element of $\mathcal{S}$.*

Proof. From Lemma 3.1 and (11), we obtain

$$\begin{aligned} & \|x_{n+1} - \bar{x}\|^2 + \frac{1}{\beta} \|y_n - \bar{y}\|^2 + 2\tau G(x_n, y_n) \\ & \leq \|z_{n+1} - \bar{x}\|^2 + \frac{1}{\beta} \|y_{n-1} - \bar{y}\|^2 + 2\tau \langle K^\top (y_n - y_{n-1}), x_n - x_{n+1} \rangle \\ & \quad - \psi \|z_{n+1} - x_n\|^2 + (\psi - 1) \|x_{n+1} - z_{n+1}\|^2 - \psi \|x_{n+1} - x_n\|^2 - \frac{1}{\beta} \|y_n - y_{n-1}\|^2. \end{aligned} \quad (20)$$

Since $x_{n+1} = \frac{\psi}{\psi-1} z_{n+2} - \frac{1}{\psi-1} z_{n+1}$, it follows from (12) that

$$\begin{aligned} \|x_{n+1} - \bar{x}\|^2 &= \frac{\psi}{\psi-1} \|z_{n+2} - \bar{x}\|^2 - \frac{1}{\psi-1} \|z_{n+1} - \bar{x}\|^2 + \frac{\psi}{(\psi-1)^2} \|z_{n+2} - z_{n+1}\|^2 \\ &= \frac{\psi}{\psi-1} \|z_{n+2} - \bar{x}\|^2 - \frac{1}{\psi-1} \|z_{n+1} - \bar{x}\|^2 + \frac{1}{\psi} \|x_{n+1} - z_{n+1}\|^2, \end{aligned} \quad (21)$$

where the second equality is due to $z_{n+2} - z_{n+1} = \frac{\psi-1}{\psi} (x_{n+1} - z_{n+1})$. By plugging (21) into (20) and combining terms, we obtain

$$\begin{aligned} & \frac{\psi}{\psi-1} \|z_{n+2} - \bar{x}\|^2 + \frac{1}{\beta} \|y_n - \bar{y}\|^2 + 2\tau G(x_n, y_n) \\ & \leq \frac{\psi}{\psi-1} \|z_{n+1} - \bar{x}\|^2 + \frac{1}{\beta} \|y_{n-1} - \bar{y}\|^2 + 2\tau \langle K^\top (y_n - y_{n-1}), x_n - x_{n+1} \rangle \\ & \quad - \psi \|z_{n+1} - x_n\|^2 + (\psi - 1 - \frac{1}{\psi}) \|x_{n+1} - z_{n+1}\|^2 - \psi \|x_{n+1} - x_n\|^2 - \frac{1}{\beta} \|y_n - y_{n-1}\|^2 \\ & \leq \frac{\psi}{\psi-1} \|z_{n+1} - \bar{x}\|^2 + \frac{1}{\beta} \|y_{n-1} - \bar{y}\|^2 + 2\tau \langle K^\top (y_n - y_{n-1}), x_n - x_{n+1} \rangle \\ & \quad - \psi \|z_{n+1} - x_n\|^2 - \psi \|x_{n+1} - x_n\|^2 - \frac{1}{\beta} \|y_n - y_{n-1}\|^2, \end{aligned} \quad (22)$$

where the second “ $\leq$ ” follows from $\psi - 1 - \frac{1}{\psi} \leq 0$ as $\psi \leq \phi$. Recall that $\beta = \sigma/\tau$. Then, $\tau\sigma L^2 < \psi$ is equivalent to $\tau < \sqrt{\psi/\beta}/L$. Let $\tau = \zeta \sqrt{\psi/\beta}/L$ with $0 < \zeta < 1$. It follows from Cauchy-Schwarz inequality and $L = \|K^\top\|$ that

$$\begin{aligned} 2\tau \langle K^\top (y_n - y_{n-1}), x_n - x_{n+1} \rangle &\leq 2\zeta \sqrt{\psi/\beta} \|y_n - y_{n-1}\| \|x_{n+1} - x_n\| \\ &\leq \zeta \left(\psi \|x_{n+1} - x_n\|^2 + \frac{1}{\beta} \|y_n - y_{n-1}\|^2 \right), \end{aligned}$$

which together with (22) leads to

$$2\tau G(x_n, y_n) + a_{n+1} \leq a_n - b_n, \quad (23)$$

with

$$\begin{cases} a_n = \frac{\psi}{\psi-1} \|z_{n+1} - \bar{x}\|^2 + \frac{1}{\beta} \|y_{n-1} - \bar{y}\|^2, \\ b_n = \psi \|z_{n+1} - x_n\|^2 + (1 - \zeta) \left(\psi \|x_{n+1} - x_n\|^2 + \frac{1}{\beta} \|y_n - y_{n-1}\|^2 \right). \end{cases} \quad (24)$$

Since $G(x_n, y_n) \geq 0$, $a_n \geq 0$ and $b_n \geq 0$, it follows from Fact 2.2 that $\lim_{n \rightarrow \infty} a_n$ exists and $\lim_{n \rightarrow \infty} b_n = 0$. By the definition of b_n , this implies

$$\lim_{n \rightarrow \infty} \|z_{n+1} - x_n\| = \lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = \lim_{n \rightarrow \infty} \|y_{n+1} - y_n\| = 0.$$

Furthermore, we also have $\lim_{n \rightarrow \infty} \|x_n - z_n\| = 0$ since $x_n - z_n = \psi(x_n - z_{n+1})$.

It follows from (24) and the existence of $\lim_{n \rightarrow \infty} a_n$ that the sequences $\{z_{n+1}\}_{n \in \mathbb{N}}$ and $\{y_n\}_{n \in \mathbb{N}}$, and thus $\{(z_{n+1}, y_n)\}_{n \in \mathbb{N}}$, are bounded. Since $\lim_{n \rightarrow \infty} \|x_n - z_n\| = 0$, $\{x_n\}_{n \in \mathbb{N}}$ is also bounded. Let $\{(x_{n_k+1}, y_{n_k})\}_{k \in \mathbb{N}}$ be a subsequence of $\{(z_{n+1}, y_n)\}_{n \in \mathbb{N}}$, which is convergent to (x^*, y^*) . Then $\lim_{n \rightarrow \infty} x_{n_k} = \lim_{n \rightarrow \infty} x_{n_k+1} = x^*$. Similar to (17) and (18), for any $(x, y) \in \mathbb{R}^q \times \mathbb{R}^p$, there hold

$$\begin{cases} \langle x_{n_k+1} - z_{n_k+1} + \tau K^\top y_{n_k}, x - x_{n_k+1} \rangle \geq \tau (g(x_{n_k+1}) - g(x)), \\ \langle \frac{1}{\beta} (y_{n_k} - y_{n_k-1}) - \tau K x_{n_k}, y - y_{n_k} \rangle \geq \tau (f^*(y_{n_k}) - f^*(y)). \end{cases}$$

By driving $k \rightarrow \infty$, taking into account that both g and f^* are closed (and thus lower semicontinuous) and canceling out $\tau > 0$, we obtain

$$\langle K^\top y^*, x - x^* \rangle \geq g(x^*) - g(x) \quad \text{and} \quad -\langle K x^*, y - y^* \rangle \geq f^*(y^*) - f^*(y),$$

which hold for any $(x, y) \in \mathbb{R}^q \times \mathbb{R}^p$. This implies that $(x^*, y^*) \in \mathcal{S}$. Since all the discussions remain valid for any $(\bar{x}, \bar{y}) \in \mathcal{S}$, $(\bar{x}, \bar{y})$ can be replaced by (x^*, y^*) in the definition of $\{a_n\}_{n \in \mathbb{N}}$ in the first place. As such, we have $\lim_{k \rightarrow \infty} a_{n_k} = 0$ since

$$\lim_{k \rightarrow \infty} \|z_{n_k+1} - x^*\| = \lim_{k \rightarrow \infty} \|x_{n_k} - x^*\| = 0 \quad \text{and} \quad \lim_{k \rightarrow \infty} \|y_{n_k-1} - x^*\| = \lim_{k \rightarrow \infty} \|y_{n_k} - x^*\| = 0.$$

Since $\{a_n\}_{n \in \mathbb{N}}$ is monotonically nonincreasing, it follows that $\lim_{n \rightarrow \infty} a_n = 0$ and thus

$$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} z_{n+1} = x^* \quad \text{and} \quad \lim_{n \rightarrow \infty} y_n = y^*.$$

This completes the proof. $\square$

Remark 3.1 Our analysis is motivated by Malitsky [22]. In particular, to get rid of the term $\|x_{n+1} - z_{n+1}\|^2$ in (22), it is required that $\psi - 1 - \frac{1}{\psi} \leq 0$, and apparently the golden ratio is an upper bound of such ψ . As long as $\tau\sigma L^2 < \psi$ is satisfied, larger ψ allows larger step sizes τ and σ , which is helpful to improve numerical performance. On the other hand, $\psi > 1$ is necessary to guarantee the nonnegativity of $\{a_n\}_{n \in \mathbb{N}}$.

3.4 Ergodic convergence rate

For convex-concave saddle point problems, many algorithms exhibit $\mathcal{O}(1/N)$ ergodic rate of convergence, see [9, 24, 27]. It is shown by Nemirovski [27] that this rate is in fact tight. We next establish the same ergodic rate of convergence for GRPDA, where the measure is the primal-dual gap function defined in (13).

Theorem 3.2 *Let $\{(x_n, y_n, z_n)\}_{n \in \mathbb{N}}$ be the sequence generated by Algorithm 3.1 from any initial point $(x_0, y_0) \in \mathbb{R}^q \times \mathbb{R}^p$ and $z_0 = x_0$, and let $G(\cdot, \cdot)$ be defined as in (19) with any fixed $(\bar{x}, \bar{y}) \in \mathcal{S}$. Let $N \geq 1$ be any integer and define*

$$X_N = \frac{1}{N} \sum_{n=1}^N x_n \quad \text{and} \quad Y_N = \frac{1}{N} \sum_{n=1}^N y_n. \quad (25)$$

Then, it holds that

$$G(X_N, Y_N) \leq \frac{1}{2\tau N} \left(\frac{\psi}{\psi-1} \|z_2 - \bar{x}\|^2 + \frac{1}{\beta} \|y_0 - \bar{y}\|^2 \right).$$

Proof. It follows from (23) and (24) that $2\tau G(x_n, y_n) \leq a_n - a_{n+1}$ for all $n \geq 1$. Sum this up for $n = 1, \dots, N$, we obtain $2\tau \sum_{n=1}^N G(x_n, y_n) \leq a_1 - a_{N+1} \leq a_1$. Since $G(x, y) = P(x) + D(y)$ and $P(x)$ and $D(y)$ are, respectively, convex in x and y , it follows from (25) that

$$G(X_N, Y_N) = P(X_N) + D(Y_N) \leq \frac{1}{N} \sum_{n=1}^N P(x_n) + \frac{1}{N} \sum_{n=1}^N D(y_n) = \frac{1}{N} \sum_{n=1}^N G(x_n, y_n).$$

The conclusion follows since $a_1 = \frac{\psi}{\psi-1} \|z_2 - \bar{x}\|^2 + \frac{1}{\beta} \|y_0 - \bar{y}\|^2$. $\square$

4 Accelerated GRPDA

When either g or f^* is strongly convex, it was shown in [8] that one can adaptively choose the primal and the dual step sizes, as well as the inertial parameter, so that PDA achieves a faster $\mathcal{O}(1/N^2)$ convergence rate. Similar results have been achieved in [23] for PDA with variable step sizes determined by linesearch. In this section, we show that the same is true for GRPDA provided that some algorithmic parameters are chosen adaptively. Similar to [8, 23], the ratio $\beta = \sigma/\tau$ plays an important role.

Under the regularity condition specified in Assumption 1.1, the “min” and the “max” in the primal-dual problem (2) can be switched [30, Corollary 31.2.1], and thus (2) is equivalent to

$$\min_{y \in \mathbb{R}^p} \max_{x \in \mathbb{R}^q} f^*(y) - \langle K^\top y, x \rangle - g(x), \quad (26)$$

As such, by swapping the roles of (g, K, x, q) and $(f^*, -K^\top, y, p)$, (26) is reducible to (2). Therefore, we will only treat the case when g is strongly convex for succinctness, and it is completely analogous when f^* is strongly convex. In this section, we thus make the following assumption.

Assumption 4.1 *Assume that g is γ -strongly convex, i.e., for some $\gamma > 0$ it holds that*

$$g(y) \geq g(x) + \langle u, y - x \rangle + \frac{\gamma}{2} \|y - x\|^2, \quad \forall x, y \in \mathbb{R}^q, \quad \forall u \in \partial g(x).$$

Our accelerated GRPDA is presented below. Recall that ϕ denotes the golden ratio.

Algorithm 4.1 (Accelerated GRPDA when g is γ -strongly convex)

Step 0. Let $\psi_0 = 1.3247\dots$ be the unique real root of $\psi^3 - \psi - 1 = 0$. Choose $\psi \in (\psi_0, \phi)$, $\beta_0 > 0$, $x_0 \in \mathbb{R}^q$ and $y_0 \in \mathbb{R}^p$, and set $z_0 = x_0$, $\varphi = \frac{1+\psi}{\psi^2}$, $\tau_0 = \frac{1}{L}\sqrt{\frac{\psi}{\beta_0}}$ and $n = 1$.

Step 1. Compute

$$z_n = \frac{\psi - 1}{\psi} x_{n-1} + \frac{1}{\psi} z_{n-1}, \quad (27)$$

$$x_n = \text{Prox}_{\tau_{n-1}g}(z_n - \tau_{n-1}K^\top y_{n-1}). \quad (28)$$

Step 2. Compute

$$\omega_n = \frac{\psi - \varphi}{\psi + \varphi\tau_{n-1}}, \quad (29)$$

$$\beta_n = \beta_{n-1}(1 + \omega_n\gamma\tau_{n-1}), \quad (30)$$

$$\tau_n = \min \left\{ \varphi\tau_{n-1}, \frac{\psi}{\tau_{n-1}\beta_n L^2} \right\}, \quad (31)$$

$$y_n = \text{Prox}_{\beta_n\tau_n f^*}(y_{n-1} + \beta_n\tau_n Kx_n). \quad (32)$$

Step 3. Set $n \leftarrow n + 1$ and return to Step 1.

Remark 4.1 We present the following remarks on Algorithm 4.1.

1. In Algorithm 4.1, ω_n is used to update β_n , which plays a key role in establishing the $\mathcal{O}(1/N^2)$ convergence rate. The condition $\psi > \psi_0$, where ψ_0 is the unique real root of $\psi^3 - \psi - 1 = 0$, ensures that $\omega_n > 0$.
2. To achieve the $\mathcal{O}(1/N^2)$ convergence rate, a balance between the increase rate τ_n/τ_{n-1} and the size of τ_n is maintained in (31). In particular, the ratio $\tau_n/\tau_{n-1} \leq \varphi \in (1, \psi)$ for $\psi \in (\psi_0, \phi)$. It follows from $\varphi = \frac{1+\psi}{\psi^2}$ that larger ψ gives smaller bound φ , and vice versa.
3. Assume $\gamma = 0$. Then $\beta_n = \beta_0$ for all $n \geq 1$. Furthermore, it is easy to show by induction that, for all $n \geq 1$, the “min” in (31) is always attained by the second term on the right hand side and thus $\tau_n = \tau_0$ for all $n \geq 1$. Therefore, Algorithm 4.1 reduces to Algorithm 3.1, i.e., GRPDA with fixed step sizes $\tau = \tau_0$ and $\sigma = \sigma_0 := \beta_0\tau_0$, in which case $\tau_0\sigma_0 L^2 = \psi < \phi$ meets the requirement of Algorithm 3.1 as we require $\psi < \phi$ in Algorithm 4.1.

We next establish some useful properties of τ_n , β_n and ω_n generated in Algorithm 4.1. For convenience, we let $\delta_n = \frac{\tau_n}{\tau_{n-1}}$ in the rest of this section.

Lemma 4.1 Let $\{(\omega_n, \beta_n, \tau_n)\}_{n \in \mathbb{N}}$ be generated by Algorithm 4.1 and define $\delta_n = \frac{\tau_n}{\tau_{n-1}}$ for $n \geq 1$. Then, for all $n \geq 1$, there hold $0 < \beta_{n-1} \leq \beta_n$, $\frac{\psi - \varphi}{\psi + \varphi\gamma\sqrt{\varphi}\tau_0} \leq \omega_n < 1$ and

$$\frac{1}{\sqrt{\varphi}(1 + \gamma\sqrt{\varphi}\tau_0)} \frac{\sqrt{\psi}}{L\sqrt{\beta_n}} \leq \tau_n \leq \frac{\sqrt{\delta_n\psi}}{L\sqrt{\beta_n}} \leq \frac{\sqrt{\varphi\psi}}{L\sqrt{\beta_n}} \leq \frac{\sqrt{\varphi\psi}}{L\sqrt{\beta_0}} = \sqrt{\varphi}\tau_0. \quad (33)$$

Moreover, there exists a constant $c > 0$ such that $\beta_n \geq cn^2$ for all $n \geq 1$.

Proof. Since $\beta_0 > 0$, $\omega_n > 0$ for $n \geq 1$ and $\tau_n > 0$ for $n \geq 0$, it is clear from (30) that $0 < \beta_{n-1} \leq \beta_n$ for all $n \geq 1$. Recall that $\delta_n = \frac{\tau_n}{\tau_{n-1}}$. It follows from (31) that $\tau_n^2\beta_n \leq \frac{\delta_n\psi}{L^2} \leq \frac{\varphi\psi}{L^2}$. Further considering $\beta_n \geq \beta_0 > 0$ for all $n \geq 1$ and $\tau_0 = \sqrt{\psi/\beta_0}/L$, we see that all the relations in (33), except the first one, follow. Now, the relation $\frac{\psi - \varphi}{\psi + \varphi\gamma\sqrt{\varphi}\tau_0} \leq \omega_n$ follows from (29) and $\tau_{n-1} \leq \sqrt{\varphi}\tau_0$ for all $n \geq 1$. On the other hand, $\omega_n < 1$ is obvious. We next prove the first inequality in (33) by induction.

It follows from $\tau_0 = \frac{\sqrt{\psi}}{\sqrt{\beta_0}L}$ and $\varphi > 1$ that $\tau_0 \geq \frac{1}{\sqrt{\varphi(1+\gamma\sqrt{\varphi}\tau_0)}} \frac{\sqrt{\psi}}{L\sqrt{\beta_0}}$. Now, suppose that $\tau_k \geq \frac{1}{\sqrt{\varphi(1+\gamma\sqrt{\varphi}\tau_0)}} \frac{\sqrt{\psi}}{L\sqrt{\beta_k}}$ for $0 \leq k \leq n$. Then, (31) implies that either

$$\tau_{n+1} = \varphi\tau_n \geq \tau_n \geq \frac{1}{\sqrt{\varphi(1+\gamma\sqrt{\varphi}\tau_0)}} \frac{\sqrt{\psi}}{L\sqrt{\beta_n}} \geq \frac{1}{\sqrt{\varphi(1+\gamma\sqrt{\varphi}\tau_0)}} \frac{\sqrt{\psi}}{L\sqrt{\beta_{n+1}}}$$

since $\beta_{n+1} \geq \beta_n$ and $\varphi > 1$, or

$$\begin{aligned} \tau_{n+1} &= \frac{\psi}{\tau_n\beta_{n+1}L^2} \geq \frac{L\sqrt{\beta_n}}{\sqrt{\varphi\psi}} \frac{\psi}{\beta_{n+1}L^2} = \sqrt{\frac{\beta_n}{\varphi\beta_{n+1}}} \frac{\sqrt{\psi}}{\sqrt{\beta_{n+1}}L} \\ &= \frac{1}{\sqrt{\varphi(1+\omega_{n+1}\gamma\tau_n)}} \frac{\sqrt{\psi}}{\sqrt{\beta_{n+1}}L} \\ &\geq \frac{1}{\sqrt{\varphi(1+\gamma\sqrt{\varphi}\tau_0)}} \frac{\sqrt{\psi}}{\sqrt{\beta_{n+1}}L}, \end{aligned}$$

where the first inequality follows from $\tau_n \leq \frac{\sqrt{\varphi\psi}}{L\sqrt{\beta_n}}$ and the second is due to $\omega_{n+1} < 1$ and $\tau_n \leq \sqrt{\varphi}\tau_0$ for all $n \geq 0$. Therefore, the first inequality in (33) also holds for all $n \geq 1$.

To complete the proof, it only remains to show that there exists $c > 0$ such that $\beta_n \geq cn^2$ for all $n \geq 1$. For simplicity, we let $\omega := \frac{\psi-\varphi}{\psi+\varphi\gamma\sqrt{\varphi}\tau_0}$. Then, $\omega_n \geq \omega$ for all $n \geq 1$ and thus

$$\beta_{n+1} = \beta_n(1 + \omega_{n+1}\gamma\tau_n) \geq \beta_n \left(1 + \frac{\omega\gamma}{\sqrt{\varphi(1+\gamma\sqrt{\varphi}\tau_0)}} \frac{\sqrt{\psi}}{L\sqrt{\beta_n}}\right) = \beta_n + c_0\sqrt{\beta_n}, \quad (34)$$

where $c_0 := \frac{\omega\gamma\sqrt{\psi}}{\sqrt{\varphi(1+\gamma\sqrt{\varphi}\tau_0)}L} > 0$. By induction, it is easy to show that $\beta_n \geq cn^2$ for all $n \geq 1$ with $c := \min(c_0^2/9, \beta_1) > 0$. This completes the proof. $\square$

Now, we are ready to establish the promised $\mathcal{O}(1/N^2)$ ergodic convergence rate.

Theorem 4.1 *Let $\{(z_n, x_n, y_n, \omega_n, \beta_n, \tau_n)\}_{n \in \mathbb{N}}$ be the sequence generated by Algorithm 4.1 and $(\bar{x}, \bar{y}) \in \mathcal{S}$ be any solution of (2). For any integer $N \geq 1$, define*

$$S_N = \sum_{n=1}^N \beta_n \tau_n, \quad X_N = \frac{1}{S_N} \sum_{n=1}^N \beta_n \tau_n x_n \quad \text{and} \quad Y_N = \frac{1}{S_N} \sum_{n=1}^N \beta_n \tau_n y_n.$$

Then, there hold $\|z_{N+2} - \bar{x}\| = \mathcal{O}(1/N)$ and $G(X_N, Y_N) = \mathcal{O}(1/N^2)$.

Proof. Fix $n \geq 1$. Since g is γ -strongly convex, an inequality stronger than the one stated in Fact 2.1 can be used. It follows from (28) that $z_n - \tau_{n-1}K^\top y_{n-1} - x_n \in \tau_{n-1}\partial g(x_n)$, and thus

$$\langle x_n - z_n + \tau_{n-1}K^\top y_{n-1}, x - x_n \rangle \geq \tau_{n-1} \left(g(x_n) - g(x) + \frac{\gamma}{2} \|x_n - x\|^2 \right), \quad \forall x. \quad (35)$$

By passing $n+1$ to n and $\bar{x}$ to x in (35), we obtain

$$\langle x_{n+1} - z_{n+1} + \tau_n K^\top y_n, \bar{x} - x_{n+1} \rangle \geq \tau_n \left(g(x_{n+1}) - g(\bar{x}) + \frac{\gamma}{2} \|x_{n+1} - \bar{x}\|^2 \right). \quad (36)$$

Similarly, by passing x_{n+1} to x in (35) and multiplying both sides by $\delta_n = \frac{\tau_n}{\tau_{n-1}}$, we obtain

$$\langle \delta_n(x_n - z_n) + \tau_n K^\top y_{n-1}, x_{n+1} - x_n \rangle \geq \tau_n \left(g(x_n) - g(x_{n+1}) + \frac{\gamma}{2} \|x_{n+1} - x_n\|^2 \right). \quad (37)$$

Similar to (18), it follows from (32) that

$$\left\langle \frac{1}{\beta_n}(y_n - y_{n-1}) - \tau_n K x_n, \bar{y} - y_n \right\rangle \geq \tau_n (f^*(y_n) - f^*(\bar{y})) \quad (38)$$

From (27), it is easy to derive $x_n - z_n = \psi(x_n - z_{n+1})$. Then, by adding (36)-(38) and using similar arguments as in Lemma 3.1, we obtain

$$\begin{aligned} \tau_n G(x_n, y_n) &\leq \langle x_{n+1} - z_{n+1}, \bar{x} - x_{n+1} \rangle + \frac{1}{\beta_n} \langle y_n - y_{n-1}, \bar{y} - y_n \rangle + \psi \delta_n \langle x_n - z_{n+1}, x_{n+1} - x_n \rangle \\ &\quad + \tau_n \langle K^\top (y_n - y_{n-1}), x_n - x_{n+1} \rangle - \frac{\gamma \tau_n}{2} \|x_{n+1} - \bar{x}\|^2 - \frac{\gamma \tau_n}{2} \|x_{n+1} - x_n\|^2. \end{aligned} \quad (39)$$

By using (11), removing $-\frac{\gamma \tau_n}{2} \|x_{n+1} - x_n\|^2 \leq 0$ and using Cauchy-Schwartz inequality, we obtain from (39) that

$$\begin{aligned} &(1 + \gamma \tau_n) \|x_{n+1} - \bar{x}\|^2 + \frac{1}{\beta_n} \|y_n - \bar{y}\|^2 + 2\tau_n G(x_n, y_n) \\ &\leq \|z_{n+1} - \bar{x}\|^2 + \frac{1}{\beta_n} \|y_{n-1} - \bar{y}\|^2 - \psi \delta_n \|z_{n+1} - x_n\|^2 + (\psi \delta_n - 1) \|x_{n+1} - z_{n+1}\|^2 \\ &\quad - \psi \delta_n \|x_{n+1} - x_n\|^2 - \frac{1}{\beta_n} \|y_n - y_{n-1}\|^2 + 2\tau_n \|K^\top (y_n - y_{n-1})\| \|x_{n+1} - x_n\|. \end{aligned}$$

Plugging in (21) and recalling $L = \|K\| = \|K^\top\|$, we obtain

$$\begin{aligned} &(1 + \gamma \tau_n) \frac{\psi}{\psi - 1} \|z_{n+2} - \bar{x}\|^2 + \frac{1}{\beta_n} \|y_n - \bar{y}\|^2 + 2\tau_n G(x_n, y_n) \\ &\leq \frac{\psi + \gamma \tau_n}{\psi - 1} \|z_{n+1} - \bar{x}\|^2 + \frac{1}{\beta_n} \|y_{n-1} - \bar{y}\|^2 + \left(\psi \delta_n - 1 - \frac{1 + \gamma \tau_n}{\psi} \right) \|x_{n+1} - z_{n+1}\|^2 \\ &\quad - \psi \delta_n \|z_{n+1} - x_n\|^2 - \psi \delta_n \|x_{n+1} - x_n\|^2 - \frac{1}{\beta_n} \|y_n - y_{n-1}\|^2 \\ &\quad + 2\tau_n L \|y_n - y_{n-1}\| \|x_{n+1} - x_n\|. \end{aligned} \quad (40)$$

It follows from $\tau_n \leq \frac{\sqrt{\psi \delta_n}}{L \sqrt{\beta_n}}$ (the second inequality in (33)) that

$$2\tau_n L \|y_n - y_{n-1}\| \|x_{n+1} - x_n\| \leq \psi \delta_n \|x_{n+1} - x_n\|^2 + \frac{1}{\beta_n} \|y_n - y_{n-1}\|^2.$$

Furthermore, $\psi \delta_n - 1 - \frac{1 + \gamma \tau_n}{\psi} \leq \psi \varphi - 1 - \frac{1}{\psi} = 0$ since $\delta_n \leq \varphi$. Therefore, (40) implies

$$\begin{aligned} &(1 + \gamma \tau_n) \frac{\psi}{\psi - 1} \|z_{n+2} - \bar{x}\|^2 + \frac{1}{\beta_n} \|y_n - \bar{y}\|^2 + 2\tau_n G(x_n, y_n) \\ &\leq \frac{\psi + \gamma \tau_n}{\psi - 1} \|z_{n+1} - \bar{x}\|^2 + \frac{1}{\beta_n} \|y_{n-1} - \bar{y}\|^2. \end{aligned} \quad (41)$$

Since $(1 + \gamma \tau_n) \frac{\psi}{\psi - 1} = \frac{\psi(1 + \gamma \tau_n)}{\psi + \gamma \tau_{n+1}} \frac{\psi + \gamma \tau_{n+1}}{\psi - 1}$ and

$$\frac{\psi(1 + \gamma \tau_n)}{\psi + \gamma \tau_{n+1}} \geq \frac{\psi(1 + \gamma \tau_n)}{\psi + \gamma \varphi \tau_n} = 1 + \frac{\psi - \varphi}{\psi + \gamma \varphi \tau_n} \gamma \tau_n = 1 + \omega_{n+1} \gamma \tau_n,$$

where the inequality follows from $\tau_{n+1} \leq \varphi \tau_n$, it follows that

$$(1 + \gamma \tau_n) \frac{\psi}{\psi - 1} \geq (1 + \omega_{n+1} \gamma \tau_n) \frac{\psi + \gamma \tau_{n+1}}{\psi - 1} = \frac{\beta_{n+1}}{\beta_n} \frac{\psi + \gamma \tau_{n+1}}{\psi - 1}. \quad (42)$$

Define $A_n := \frac{\psi + \gamma\tau_n}{2(\psi - 1)} \|z_{n+1} - \bar{x}\|^2 + \frac{1}{2\beta_n} \|y_{n-1} - \bar{y}\|^2$. Combining (41) and (42), we deduce

$$\beta_{n+1}A_{n+1} + \beta_n\tau_n G(x_n, y_n) \leq \beta_n A_n. \quad (43)$$

By summing (43) for $n = 1, \dots, N$, we obtain

$$\beta_{N+1}A_{N+1} + \sum_{n=1}^N \beta_n\tau_n G(x_n, y_n) \leq \beta_1 A_1. \quad (44)$$

The convexity of $G(x, y)$, (44) and the definition of A_n imply that

$$G(X_N, Y_N) \leq \frac{1}{S_N} \sum_{n=1}^N \beta_n\tau_n G(x_n, y_n) \leq \frac{\beta_1 A_1}{S_N}, \quad (45)$$

$$\|z_{N+2} - \bar{x}\|^2 \leq \frac{2(\psi - 1)}{\psi + \gamma\tau_{N+1}} \frac{\beta_1 A_1}{\beta_{N+1}} \leq \frac{2\beta_1 A_1}{\beta_{N+1}}. \quad (46)$$

From Lemma 4.1, there exists $c > 0$ such that $\beta_n \geq cn^2$ for all $n \geq 1$. Then, (46) gives

$$\|z_{N+2} - \bar{x}\| \leq \frac{\sqrt{2\beta_1 A_1/c}}{N+1}.$$

It follows from (34) that $\beta_{n+1} - \beta_n \geq c_0\sqrt{cn}$, which together with (30) and $\omega_{n+1} < 1$ implies

$$\beta_n\tau_n = \frac{\beta_{n+1} - \beta_n}{\omega_{n+1}\gamma} \geq \frac{c_0\sqrt{cn}}{\gamma}.$$

As a result, $S_N = \sum_{n=1}^N \beta_n\tau_n = \mathcal{O}(N^2)$, and thus $G(X_N, Y_N) = \mathcal{O}(1/N^2)$ follows from (45). $\square$

5 GRPDA for two special cases

This section is devoted to GRPDA for two special cases of the structured convex optimization problem (1), which are specified in the following assumption.

Assumption 5.1 *Let $b \in \mathbb{R}^p$ be given. Assume that either $f(\cdot) = \iota_{\{b\}}(\cdot)$, the indicator function of the singleton $\{b\}$, or $f(\cdot) = \frac{1}{2}\|\cdot - b\|^2$.*

The two cases specified in Assumption 5.1 correspond respectively to

$$\min_{x \in \mathbb{R}^q} \{g(x) : Kx = b\} \quad \text{and} \quad \min_{x \in \mathbb{R}^q} \frac{1}{2}\|Kx - b\|^2 + g(x),$$

which are abundant in practice. For example, linear inverse problems, which include many signal and image processing applications, usually enforce data fitting via $f(Kx)$, which takes the form $\frac{1}{2}\|Kx - b\|^2$ for noisy data and $\iota_{\{b\}}(Kx)$, or equivalently $Kx = b$, for ideal noiseless data. A well known application is compressive signal/image sensing [11, 26], which recovers sparse or compressible signals from a small number of linear measurements via solving the basis pursuit and/or the LASSO problems [10, 32].

The aim of this section is to propose a relaxed GRPDA when f satisfies Assumption 5.1, which is a modification of Algorithm 3.1 in two aspects: (i) extending the permitted range of ψ from $(1, \phi]$ to $(1, 2]$, and (ii) introducing a relaxation step. In the rest of this section, we always assume that f satisfies Assumption 5.1. Then, for any $u \in \mathbb{R}^p$ and $\sigma > 0$, it is easy to show that $\text{Prox}_{\sigma f^*}(u) = \eta u + \varrho b$, where

$$(\eta, \varrho) := (\eta(\sigma), \varrho(\sigma)) = \begin{cases} (1, -\sigma) & \text{if } f(\cdot) = \iota_{\{b\}}(\cdot), \\ (\frac{1, \sigma}{1+\sigma}) & \text{if } f(\cdot) = \frac{1}{2}\|\cdot - b\|^2. \end{cases} \quad (47)$$

In both cases, it holds that $\eta = \eta(\sigma) \in (0, 1]$ for any $\sigma > 0$.

5.1 GRPDA as fixed point iteration

The iterative scheme GRPDA (14) applied to (1) appears as

$$\begin{cases} y_{n-1} = \eta(y_{n-2} + \sigma K x_{n-1}) + \varrho b, \\ z_n = \frac{\psi-1}{\psi} x_{n-1} + \frac{1}{\psi} z_{n-1}, \\ x_n = \text{Prox}_{\tau g}(z_n - \tau K^\top y_{n-1}), \end{cases} \quad (48)$$

where η and ϱ are determined by (47). Note that, since f satisfies Assumption 5.1, the schemes (48) and (14) are equivalent in the sense that they, if initialized properly, generalize exactly the same sequence of iterates. Our choice of (48), which states explicitly the computing formulas of (y_{n-1}, z_n, x_n) , instead of (z_n, x_n, y_n) as in (14), is mainly for convenience of analysis.

To present our relaxed GRPDA, we first represent (48) as a fixed point iterative scheme, which is summarized below.

Lemma 5.1 *Let $\xi_n = (z_n; x_n; y_{n-1})$. Then, the iterative scheme (48) can be represented as the fixed point iteration $\xi_n = \mathbb{S} \circ \mathbb{T}(\xi_{n-1})$, where*

$$\begin{cases} \mathbb{T}(\cdot) = T(\cdot) + \vartheta_1, & \vartheta_1 = (0; -\varrho \tau K^\top b; 0), \\ \mathbb{S}(\cdot) = S(\cdot) + \vartheta_2, & \vartheta_2 = (0; 0^\top; \varrho b), \\ S = \begin{bmatrix} I & 0 & 0 \\ 0 & \text{Prox}_{\tau g} & 0 \\ 0 & 0 & I \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} \frac{1}{\psi} I & (1 - \frac{1}{\psi}) I & 0 \\ \frac{1}{\psi} I & (1 - \frac{1}{\psi}) I - \eta \tau \sigma K^\top K & -\eta \tau K^\top \\ 0 & \eta \sigma K & \eta I \end{bmatrix}. \end{cases} \quad (49)$$

Proof. Direct verification from (48). $\square$

Let $\mathbb{C}$ be the set of complex numbers and $\Lambda(M)$ be the set of eigenvalues of a matrix M . The following theorem is our key to extend ψ from $(1, \phi]$ to $(1, 2]$ and to take a relaxation step.

Theorem 5.1 *Let $\tau, \sigma > 0$ and $\psi \in (1, 2]$ be such that $\tau \sigma L^2 < \psi$ and $\eta \in (0, 1]$. Then, the matrix T , and thus the affine mapping $\mathbb{T}$, defined in (49) is firmly nonexpansive.*

Proof. It follows from, e.g., [1, Proposition 4.2] that T is firmly nonexpansive if and only if $2T - I$ is nonexpansive, which is clearly equivalent to $|\lambda| \leq 1$ for any $\lambda \in \Lambda(2T - I)$. Since $\eta \in (0, 1]$, we have $2\eta - 1 \in (-1, 1]$. Therefore, to guarantee nonexpansiveness of $2T - I$, it is sufficient to show that $|\lambda| \leq 1$ for any $\lambda \in \Lambda(2T - I) \setminus \{\pm 1, 2\eta - 1\}$. The reason that we exclude ± 1 and $2\eta - 1$ from the spectrum $\Lambda(2T - I)$ will be clear below.

Let $\lambda \in \Lambda(2T - I) \setminus \{\pm 1, 2\eta - 1\}$ and $\xi = (z; x; y) \in \mathbb{R}^q \times \mathbb{R}^q \times \mathbb{R}^p$ be any eigen-pair of $2T - I$, i.e., $(2T - I)\xi = \lambda\xi$ and $\lambda \notin \{\pm 1, 2\eta - 1\}$. By the definition of T , $(2T - I)\xi = \lambda\xi$ appears as

$$\left(\frac{2}{\psi} - 1\right)z + \left(2 - \frac{2}{\psi}\right)x = \lambda z, \quad (50)$$

$$\frac{2}{\psi}z + \left(1 - \frac{2}{\psi}\right)x - 2\eta\tau\sigma K^\top Kx - 2\eta\tau K^\top y = \lambda x, \quad (51)$$

$$2\eta\sigma Kx + (2\eta - 1)y = \lambda y. \quad (52)$$

Combining (50) and (51), we obtain

$$(\lambda + 1)(z - x) - 2\eta\tau\sigma K^\top Kx - 2\eta\tau K^\top y = 0. \quad (53)$$

First, we show that $KK^\top y \neq 0$ by contradiction. Assume that $KK^\top y = 0$. Then, it holds that $KK^\top Kx = 0$ from (52), combining which with (53) gives $(\lambda + 1)K(z - x) = 0$. Since $\lambda \neq -1$, there must hold $Kz = Kx$. Then, multiplying both sides of (51) by K shows that $(1 - \lambda)Kx = 0$, which

further implies that $Kx = 0$ since $\lambda \neq 1$. As a result, it follows from (52) and $\lambda \neq 2\eta - 1$ that there must hold $y = 0$. Thus, along with $Kx = 0$ and $\lambda \neq -1$, (53) would lead to $x = z$. Since $\lambda \neq 1$, it then follows from (50) that $x = z = 0$, which contradicts to the fact that ξ is an eigenvector and has to be nonzero. Therefore, there must hold $KK^\top y \neq 0$.

Since $KK^\top y \neq 0$, we thus have $y \neq 0$. For simplicity, we let $\lambda_1 := \frac{\lambda+1}{2}$. Since $\lambda \notin \{\pm 1, 2\eta - 1\}$, we have $\lambda_1 \notin \{0, 1, \eta\}$. Then, (50) and (52) can be restated, respectively, as

$$\left(1 - \frac{1}{\psi}\right)x = \left(\lambda_1 - \frac{1}{\psi}\right)z \quad \text{and} \quad \sigma Kx = \frac{\lambda_1 - \eta}{\eta}y. \quad (54)$$

Multiplying both sides of (52) by $\tau K^\top$, adding to (51) and using (50), we obtain

$$(\lambda + 1)(z - x - \tau K^\top y) = 0.$$

Since $\lambda \neq -1$, we thus obtain $z - x = \tau K^\top y$, which together with (54) gives

$$\begin{aligned} -\left(\lambda_1 - \frac{1}{\psi}\right)\frac{\lambda_1 - \eta}{\eta}y &= -\left(\lambda_1 - \frac{1}{\psi}\right)\sigma Kx \\ &= \left(\lambda_1 - \frac{1}{\psi}\right)\sigma K(\tau K^\top y - z) \\ &= \left(\lambda_1 - \frac{1}{\psi}\right)\tau\sigma KK^\top y - \sigma\left(\lambda_1 - \frac{1}{\psi}\right)Kz \\ &= \left(\lambda_1 - \frac{1}{\psi}\right)\tau\sigma KK^\top y - \sigma\left(1 - \frac{1}{\psi}\right)Kx \\ &= \left(\lambda_1 - \frac{1}{\psi}\right)\tau\sigma KK^\top y - \left(1 - \frac{1}{\psi}\right)\frac{\lambda_1 - \eta}{\eta}y. \end{aligned}$$

By combining the two terms of y on both sides, we obtain

$$(\lambda_1 - 1)\frac{\eta - \lambda_1}{\eta}y = \left(\lambda_1 - \frac{1}{\psi}\right)\tau\sigma KK^\top y,$$

which simplifies to

$$\frac{\eta - \lambda_1}{\eta}y = \frac{(\overline{\lambda_1} - 1)\left(\lambda_1 - \frac{1}{\psi}\right)}{|\lambda_1 - 1|^2}\tau\sigma KK^\top y. \quad (55)$$

Here $\overline{\lambda_1}$ denotes the complex conjugate of λ_1 . Recall that $y \neq 0$. Then (55) implies that y is an eigenvector of $KK^\top$. Since $KK^\top$ is real symmetric, positive semidefinite and $\|KK^\top\| = L^2$, its eigenvalues must be real and lie in $[0, L^2]$. It is then implied by $KK^\top y \neq 0$ and $0 < \tau\sigma L^2 < \psi$ that there exists $\psi_1 \in (0, \psi)$ such that $\tau\sigma KK^\top y = \psi_1 y$.

Let $s, t \in \mathbb{R}$ and $\lambda = s + t\sqrt{-1} \in \mathbb{C}$. Then $\lambda_1 = \frac{1+s}{2} + \frac{t}{2}\sqrt{-1}$ and it is elementary to deduce from (55) that

$$\frac{4\eta - 2 - 2s}{\eta} = \frac{(s + 1 - \frac{2}{\psi})(s - 1) + t^2}{|\lambda_1 - 1|^2}\psi_1, \quad (56)$$

$$\frac{t}{\eta} = \frac{t\left(1 - \frac{1}{\psi}\right)\psi_1}{|\lambda_1 - 1|^2}. \quad (57)$$

We split the discussions in two cases, (i) $t = 0$, and (ii) $t \neq 0$.

- Case (i). If $t = 0$, we have $s^2 - 2\eta(1 - \psi_1)s - 1 + 2\eta\varsigma = 0$ from (56), where $\varsigma := 1 + \psi_1 - \frac{2\psi_1}{\psi}$. Let $u := \eta(1 - \psi_1)$ and $v := \eta^2(1 - \psi_1)^2 - 2\eta\varsigma + 1$. Then, we have $\lambda = s = u \pm \sqrt{v}$. Since $s \in \mathbb{R}$, there must hold $\eta^2(1 - \psi_1)^2 - 2\eta\varsigma + 1 \geq 0$, which, by further considering $\eta \leq 1$, implies that

$$0 < \eta \leq \begin{cases} \min \left\{ 1, \frac{\varsigma - \sqrt{\varsigma^2 - (1 - \psi_1)^2}}{(1 - \psi_1)^2} \right\}, & \psi_1 \neq 1, \\ \min \left\{ 1, \frac{1}{4(1 - \frac{1}{\psi})} \right\}, & \psi_1 = 1. \end{cases}$$

By $\psi_1 \in (0, \psi)$ and $\psi > 1$, it is elementary to show that $\sqrt{v} < 1 - u$ and $\sqrt{v} < 1 + u$. Consequently, we have $\lambda \in (-1, 1)$.

- Case (ii). If $t \neq 0$, we have $|\lambda_1 - 1|^2 = \eta\psi_1 \left(1 - \frac{1}{\psi}\right)$ from (57). Then, (56) implies

$$(4\eta - 2 - 2s) \left(1 - \frac{1}{\psi}\right) = \left(s + 1 - \frac{2}{\psi}\right)(s - 1) + t^2,$$

from which we obtain

$$\begin{aligned} \left(s + 1 - \frac{2}{\psi}\right)^2 + t^2 &= (4\eta - 2 - 2s) \left(1 - \frac{1}{\psi}\right) + \left(s + 1 - \frac{2}{\psi}\right) \left(2 - \frac{2}{\psi}\right) \\ &= 4 \left(1 - \frac{1}{\psi}\right) \left(\eta - \frac{1}{\psi}\right) \\ &= \left(1 - \frac{2}{\psi}\right)^2 + 4\eta \left(1 - \frac{1}{\psi}\right) - 1 \\ &\leq \left(1 - \frac{2}{\psi}\right)^2 + 4 \left(1 - \frac{1}{\psi}\right) - 1 \\ &= 4 \left(1 - \frac{1}{\psi}\right)^2, \end{aligned} \tag{58}$$

where the inequality follows from $\eta \in (0, 1]$ and $\psi > 1$. Since $\psi \in (1, 2]$, a simple geometric argument based on (58) shows that $|\lambda| = \sqrt{s^2 + t^2} \leq 1$.

Combining cases (i) and (ii), we have shown that $|\lambda| \leq 1$ for any $\tau, \sigma > 0$ and $\psi \in (1, 2]$ such that $\tau\sigma L^2 < \psi$. This completes the proof. $\square$

5.2 Relaxed GRPDA

Now, we are ready to present a relaxed GRPDA when f satisfies Assumption 5.1. The relaxation parameters lie in $(0, 2/3)$ and the parameter ψ , which is restricted to $(1, \phi]$ in Algorithm 3.1, can now be extended to $(1, 2]$. The relaxed GRPDA is summarized below.

Algorithm 5.1 (Relaxed GRPDA)

Step 0. Let $\tau, \sigma > 0$ and $\psi \in (1, 2]$ be such that $\tau\sigma L^2 < \psi$, $(\eta, \varrho) = (\eta(\sigma), \varrho(\sigma))$ be defined in (47), and $\{\rho_n\}_{n \in \mathbb{N}} \subseteq (0, \frac{3}{2})$ be such that $\sum_{n \in \mathbb{N}} \rho_n(1 - \frac{2\rho_n}{3}) = +\infty$. Choose $x_0 \in \mathbb{R}^q$ and $y_{-1} \in \mathbb{R}^p$. Set $z_0 = x_0$ and $n = 1$.

Step 1. Compute

$$\begin{cases} \tilde{y}_{n-1} = \eta(y_{n-2} + \sigma K x_{n-1}) + \varrho b, \\ \tilde{z}_n = \frac{\psi-1}{\psi} x_{n-1} + \frac{1}{\psi} z_{n-1}, \\ \tilde{x}_n = \text{Prox}_{\tau g}(\tilde{z}_n - \tau K^\top \tilde{y}_{n-1}). \end{cases}$$

Step 2. Take a relaxation step

$$\begin{cases} y_{n-1} = y_{n-2} + \rho_n(\tilde{y}_{n-1} - y_{n-2}), \\ z_n = z_{n-1} + \rho_n(\tilde{z}_n - z_{n-1}), \\ x_n = x_{n-1} + \rho_n(\tilde{x}_n - x_{n-1}). \end{cases}$$

Step 3. Set $n \leftarrow n + 1$ and return to Step 1.

The convergence of Algorithm 5.1 is established in the following theorem. The key of its proof is to observe, based on Theorem 5.1 and [1, Proposition 4.32], that the operator $\mathbb{S} \circ \mathbb{T}$ is 2/3-averaged¹, and thus the convergence result follows from [1, Proposition 5.15].

Theorem 5.2 *Let f satisfy Assumption 5.1 and $\{(z_n, x_n, y_{n-1})\}_{n \in \mathbb{N}}$ be the sequence generated by Algorithm 5.1 from any initial point $(x_0, y_{-1}) \in \mathbb{R}^q \times \mathbb{R}^p$ and $z_0 = x_0$. Then, $\{(x_n, y_n)\}_{n \in \mathbb{N}}$ converges to a solution of (2), i.e., an element in $\mathcal{S}$.*

Proof. Let $\mathbb{S}$ and $\mathbb{T}$ be defined as in (49), $\mathbb{G} := \mathbb{S} \circ \mathbb{T}$, and $\xi_n = (z_n; x_n; y_{n-1})$ for $n \geq 0$. Then, the sequence $\{(z_n, x_n, y_{n-1})\}_{n \in \mathbb{N}}$ generated by Algorithm 5.1 from (z_0, x_0, y_{-1}) satisfies $\xi_n = \xi_{n-1} + \rho_n(\mathbb{G}(\xi_{n-1}) - \xi_{n-1})$ for $n \geq 1$. The key of the proof is to show that $\mathbb{G}$ is 2/3-averaged, which we argue below.

Apparently, η defined in (47) lies in $(0, 1]$. It then follows from Theorem 5.1 that $\mathbb{T}$ defined in (49) is firmly nonexpansive under the condition of Algorithm 5.1, i.e., $\tau, \sigma > 0$ and $\tau\sigma L^2 < \psi \in (1, 2]$. On the other hand, it is well known that the proximal operator $\text{Prox}_{\tau g}$ is firmly nonexpansive for any $\tau > 0$. Then, by the definition in (49), $\mathbb{S}$ is also firmly nonexpansive. Consequently, it follows from [1, Proposition 4.32] that $\mathbb{G} = \mathbb{S} \circ \mathbb{T}$ is 2/3-averaged.

Denote by $\text{Fix}(\mathbb{G})$ the set of fixed points of $\mathbb{G}$, i.e., $\text{Fix}(\mathbb{G}) := \{\xi \in \mathbb{R}^q \times \mathbb{R}^q \times \mathbb{R}^p : \xi = \mathbb{G}(\xi)\}$. By the construction of $\mathbb{S}$ and $\mathbb{T}$, it is elementary to show that

$$\text{Fix}(\mathbb{G}) = \{\xi = (z; x; y) \mid z = x, (x, y) \in \mathcal{S}\},$$

where $\mathcal{S}$ is defined in (4). By Assumption 1.1, $\mathcal{S}$, and thus $\text{Fix}(\mathbb{G})$, is nonempty. Consequently, it follows from [1, Proposition 5.15] that the sequence $\{\xi_n\}_{n \in \mathbb{N}}$ is Fejér monotone with respect to $\text{Fix}(\mathbb{G})$ and converges to a point in $\text{Fix}(\mathbb{G})$. This completes the proof. $\square$

6 Numerical Experiments

In this section, we present numerical results on LASSO [32], nonnegative least-squares and minimax matrix game problems to demonstrate the performance of Algorithm 3.1 (GRPDA), the accelerated GRPDA given in Algorithm 4.1 (A-GRPDA), and the relaxed GRPDA described in Algorithm 5.1 (R-GRPDA). All the experiments were performed within Python 3.8 running on a 64-bit Windows PC with an Intel(R) Core(TM) i5-4590 CPU@3.30 GHz and 8GB of RAM. All the results presented in this section are reproducible by specifying the *seed* of the random number generator in our code, which is available at <https://github.com/cxk9369010/Golden-Ratio-PDA>.

In this section, we let $\|\cdot\| = \|\cdot\|_2$ be the ℓ_2 -norm induced by the dot inner product. For LASSO and nonnegative least-squares problems, the component function f has the form $f(\cdot) = \frac{1}{2}\|\cdot - b\|^2$. Thus, Assumption 5.1 is satisfied and the relaxed GRPDA described in Algorithm 5.1 can be applied. Furthermore, GRPDA with fixed step sizes (Algorithm 3.1) can adopt the larger value $\psi = 2$ as well (equivalent to setting $\psi = 2$ and $\rho_n \equiv 1$ in Algorithm 5.1). Since f^* (rather than g , as required by Algorithm 4.1) is strongly convex, we can switch (g, K, x, q) with $(f^*, -K^T, y, p)$

¹ An operator P is α -averaged for some $\alpha \in (0, 1)$ if there exists a nonexpansive operator Q such that $P = (1 - \alpha)I + \alpha Q$.

and reduce (26) to (2), so that g is strongly convex. As such, the accelerated GRPDA given in Algorithm 4.1 is also applicable.

The algorithmic parameters are specified as follows. For GRPDA, we used fixed step sizes $\tau = \frac{\sqrt{\psi}}{\sqrt{\beta}L}$ and $\sigma = \beta\tau$. For the minimax matrix game problem, we set $\psi = 1.618$, while for LASSO and nonnegative least-squares problems, which satisfy Assumption 5.1, we set $\psi = 2$. For A-GRPDA, we set $\psi = 1.5$ and $\beta_0 = 1$, while for R-GRPDA we set $\tau = \frac{\sqrt{2}}{\sqrt{\beta}L}$, $\sigma = \beta\tau$ and $\rho_n \equiv 1.49$. The following state-of-the-art algorithms are compared:

- PDA (as given in (6)) with $\tau = \frac{1}{\sqrt{\beta}L}$ and $\sigma = \sqrt{\beta}\frac{1}{L}$ and $\delta = 1$.
- PGM (proximal gradient method, e.g., [3]) with fixed step size $\alpha > 0$. For LASSO and non-negative least-squares problems, it holds that $f(Kx) = \frac{1}{2}\|Kx - b\|^2$, and the PGM appears as $x_n = \text{Prox}_{\alpha g}(x_{n-1} - \alpha K^\top(Kx_{n-1} - b))$. We set $\alpha = 1/\|K\|^2$.
- FISTA (fast iterative shrinkage thresholding algorithm [3]) with fixed step size $\alpha > 0$. For LASSO and nonnegative least-squares problems, FISTA appears as

$$\begin{cases} t_n = \left(1 + \sqrt{1 + 4t_{n-1}^2}\right)/2, \\ y_{n-1} = x_{n-1} + \frac{t_{n-1}-1}{t_n}(x_{n-1} - x_{n-2}), \\ x_n = \text{Prox}_{\alpha g}(y_{n-1} - \alpha K^\top(Ky_{n-1} - b)), \end{cases}$$

where $t_0 = 1$ and $x_{-1} = x_0$. The same as PGM, we set $\alpha = 1/\|K\|^2$.

- GRAAL (golden ratio algorithm [22], or as given in (10)) with $\tau = \frac{\phi}{2L}$, where $\phi = 1.618$.

The choice of β for GRPDA, R-GRPDA and PDA will be specified later. For all algorithms, we used the same initial points. For the three tested problems, the evaluation of the proximal point mappings is relatively cheap compared with matrix-vector multiplications. As such, all the compared algorithms have the same dominant per iteration cost, i.e., two matrix-vector multiplications, we only compare the their performance in terms of iteration numbers.

Problem 6.1 (LASSO) *The LASSO problem has the form $\min_x F(x) := \frac{1}{2}\|Kx - b\|^2 + \mu\|x\|_1$, where $K \in \mathbb{R}^{p \times q}$ and $b \in \mathbb{R}^p$ are given, and $x \in \mathbb{R}^q$ is an unknown signal. In the context of compressive sensing, p is much smaller than q .*

Apparently, the above LASSO problem corresponds to (1) with $f(u) = \frac{1}{2}\|u - b\|^2$ and $g(x) = \mu\|x\|_1$. Thus, Assumption 5.1 is satisfied and $\psi = 2$ can be adopted in GRPDA. The *seed* of the random number generator was set to 1. We generated $x^* \in \mathbb{R}^q$ randomly. Specifically, s nonzero components of x^* were determined uniformly at random, and their values were drawn from the uniform distribution in $[-10, 10]$. The matrix $K \in \mathbb{R}^{p \times q}$ is constructed as in [23] by one of the following ways:

- (i) All entries of K were generated independently from $\mathcal{N}(0, 1)$, the normal distribution with mean 0 and standard deviation 1.
- (ii) First, we generated a matrix $A \in \mathbb{R}^{p \times q}$, whose entries are independently drawn from $\mathcal{N}(0, 1)$. Then, for a scalar $v \in (0, 1)$ we constructed the matrix K column by column as follows: $K_1 = A_1/\sqrt{1-v^2}$ and $K_j = vK_{j-1} + A_j$, $j = 2, \dots, q$. Here K_j and A_j represent the j th column of K and A , respectively. As v increases, K becomes more ill-conditioned. In this experiment, we tested $v = 0.5$ and 0.9 .

The additive noise $\nu \in \mathbb{R}^p$ was generated from $\mathcal{N}(0, 0.1)$. Finally, we set $b = Kx^* + \nu$.

The step size ratio $\beta = \sigma/\tau$ was set to be 400 as in [23], which was determined based on numerical experience. All the algorithms were initialized at $x_0 = 0$ and $y_0 = -b$. For the LASSO problem, we first compared the performance of GRPDA with varying ψ . The results are given in Figure 1, where the decreasing behavior of function value errors $F(x_n) - F^*$ as the algorithm proceeded is presented. Here $F^* = \inf_x F(x)$ was approximately computed by running our algorithms for sufficiently many iterations. It can be seen from Figure 1 that GRPDA with larger ψ

converges faster for all the tested three cases. Thus, in our comparison with other algorithms, we set $\psi = 2$. Besides, instead of terminating the algorithms with some stopping criteria, we ran all algorithms for a fixed number of iterations and examined their convergence behavior, see Figure 2. The parameters p , q , s and v are given in the captions of the figures.

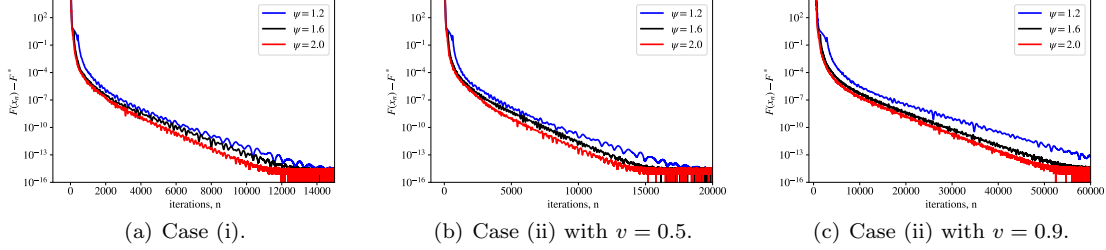


Fig. 1 Evolution of $F(x_n) - F^*$ for the LASSO problem with different ψ and $(p, q, s) = (200, 1000, 10)$.

It can be seen from Figure 2 that primal-dual type methods outperform the PGM, FISTA and GRAAL. For all the three tested cases, GRPDA performs better than PDA, R-GRPDA is even better and A-GRPDA is the best. In particular, the proposed Gauss-Seidel type golden ratio algorithms, including GRPDA, A-GRPDA and R-GRPDA, perform much better than the Jacobian type golden ratio algorithm GRAAL as given in (10).

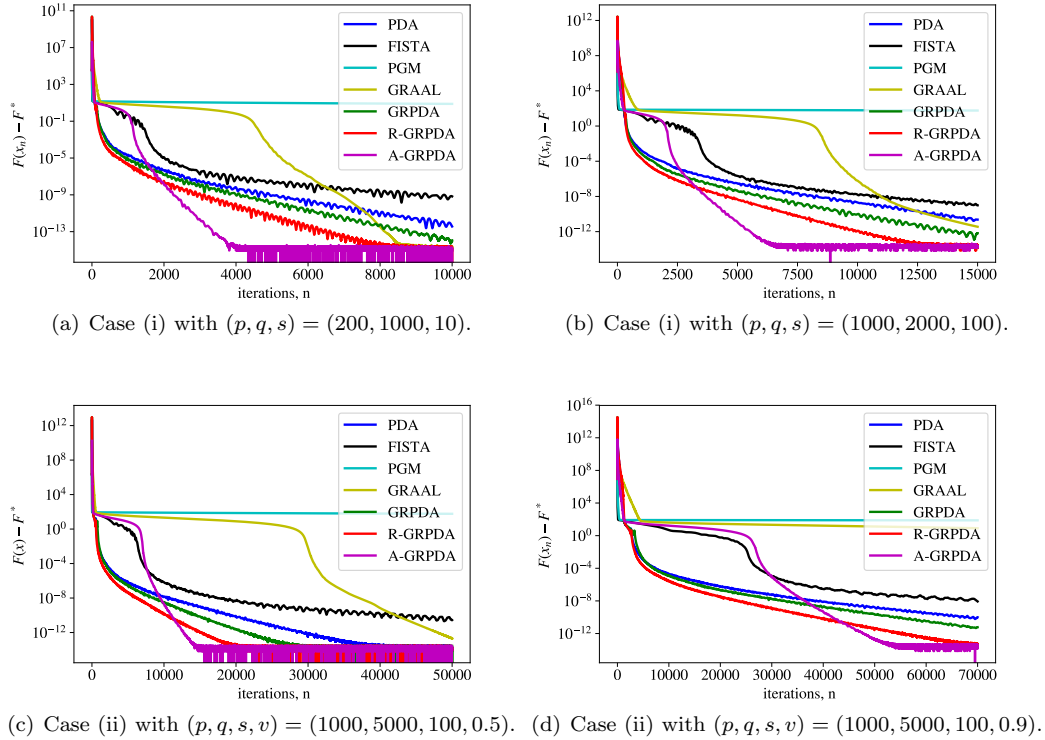


Fig. 2 Numerical results for the LASSO problem.

Problem 6.2 (Nonnegative least-squares problem) *The nonnegative least-squares problem aims to find $x \in \mathbb{R}_+^q$ (the nonnegative orthant) such that $\|Kx - b\|$ is minimized for given $K \in \mathbb{R}^{p \times q}$ and $b \in \mathbb{R}^p$, i.e., $\min_{x \in \mathbb{R}_+^q} F(x) := \frac{1}{2} \|Kx - b\|^2 + \iota_{\mathbb{R}_+^q}(x)$.*

Similar to the LASSO problem, the nonnegative least-squares problem can also be solved by A-GRPDA and R-GRPDA since the data fitting term $\frac{1}{2} \|Kx - b\|^2$ remains the same. The only difference is that the ℓ_1 -norm $\mu \|\cdot\|_1$ in LASSO is now replaced by the indicator function $\iota_{\mathbb{R}_+^q}(\cdot)$. Since the proximal mapping of the indicator function $\iota_{\mathbb{R}_+^q}(\cdot)$ is just the projection on $\mathbb{R}_+^q$, all the compared algorithms keep feasibility of the nonnegativity constraint. As such, $F(x_n) = \frac{1}{2} \|Kx_n - b\|^2$ since $x_n \geq 0$ is always satisfied. For this experiment, we consider two types of data described below.

- (i) Real data from the Matrix Market library². Two instances were tested, i.e., “illc1033” and “illc1850”, where $K \in \mathbb{R}^{p \times q}$ is sparse and has sizes $(p, q) = (1033, 320)$ and $(1850, 712)$, respectively. These matrices were used in the testing of the famous LSQR algorithm [28] and are much more ill-conditioned than the other two instances “well1033” and “well1850” available at the library. The entries of $b \in \mathbb{R}^p$ were drawn independently from $\mathcal{N}(0, 1)$.
- (ii) Random matrix. $K \in \mathbb{R}^{p \times q}$ has approximately dpq nonzeros, with $d \in (0, 1)$ as in [23], and x^* is a sparse vector, whose s nonzero entries were drawn uniformly from $[0, 100]$. We set $b = Kx^*$, which gives $F^* = 0$. The following two cases were tested.
 - (a) $p = 1000$, $q = 2000$, $d = 0.5$ and $s = 100$. The nonzero entries of K were generated independently from the uniform distribution in $[0, 1]$.
 - (b) $p = 10000$, $q = 20000$, $d = 0.01$ and $s = 500$. The nonzero entries of K were generated independently from the normal distribution $\mathcal{N}(0, 1)$.

For GRPDA and R-GRPDA, we set $\beta = 1$, except for the random data with $p = 1000$, for which we set $\beta = 25$ as in [23]. The same as the LASSO case, all the algorithms were initialized at $x_0 = 0$ and $y_0 = -b$.

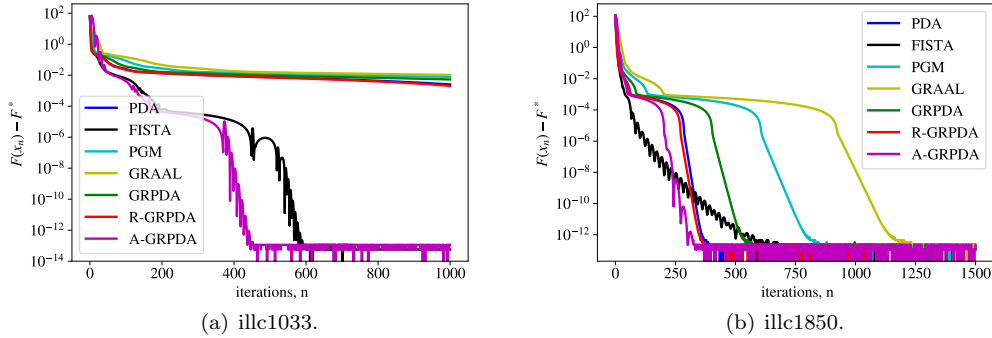


Fig. 3 Numerical results for nonnegative least-squares problem: Real data.

Similar to the LASSO case, the decreasing behavior of $F(x_n) - F^*$ as the algorithms proceeded is presented in Figure 3 for real data and Figure 4 for random data. It can be observed from Figure 3 that A-GRPDA performs the best, followed by FISTA. In comparison, for random data R-GRPDA performs the best, followed by GRPDA and PDA, as shown in Figure 4. Other algorithms seem to be less efficient. Nonetheless, all the compared algorithms perform favorably and have attained fairly high precision in a reasonable number of iterations.

² <https://math.nist.gov/MatrixMarket/data/Harwell-Boeing/lq/lq.html>.

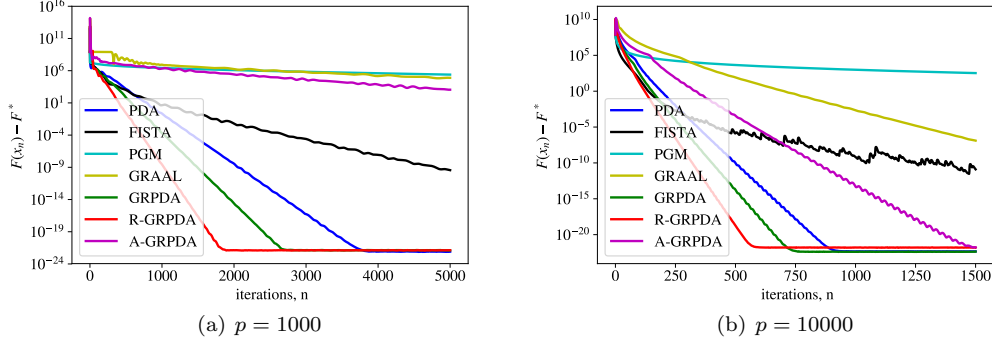


Fig. 4 Numerical results for nonnegative least-squares problem: Random data.

Problem 6.3 (Minimax matrix game) Let $\Delta_q = \{x \in \mathbb{R}^q : x \geq 0, \sum_{i=1}^q x_i = 1\}$ be the standard unit simplex in $\mathbb{R}^q$, and $K \in \mathbb{R}^{p \times q}$. The minimax matrix game problem is given by

$$\min_{x \in \Delta_q} \max_{y \in \Delta_p} \iota_{\Delta_q}(x) + \langle Kx, y \rangle - \iota_{\Delta_p}(y). \quad (59)$$

Apparently, (59) is a special case of (2) with $g = \iota_{\Delta_q}$ and $f^* = \iota_{\Delta_p}$. We note that the projection onto Δ_q can be computed efficiently, see, e.g., [2, Corollary 6.29]. Therefore, any algorithms depending on the proximal operators of ι_{Δ_q} can be implemented efficiently. In our implementation, we used the algorithm from [12] to compute the projection onto the standard unit simplex. To compare different algorithms, we used the primal-dual gap function as defined in (13), which can be easily computed for a feasible pair $(x, y) \in \Delta_q \times \Delta_p$ by $G(x, y) := \max_i (Kx)_i - \min_j (K^\top y)_j$. Here subscript i (or j) denotes the i th (or j th) component of the underlying vector.

For this minimax matrix game problem, the only relevant algorithms discussed at the beginning of this section are PDA, GRPDA and GRAAL. The rest are not applicable. In this experiment, we set $\beta = 1$ for PDA and GRPDA. The initial point for all algorithms was set to be $x_0 = \frac{1}{q}(1, \dots, 1)^\top$ and $y_0 = \frac{1}{p}(1, \dots, 1)^\top$. We tested two types of random matrix $K \in \mathbb{R}^{p \times q}$ with $seed = 50$, i.e., (i) $(p, q) = (100, 100)$ and all entries of K were generated independently from the uniform distribution in $[-1, 1]$, and (ii) $(p, q) = (100, 500)$ and all entries of K were generated independently from the normal distribution $\mathcal{N}(0, 1)$.

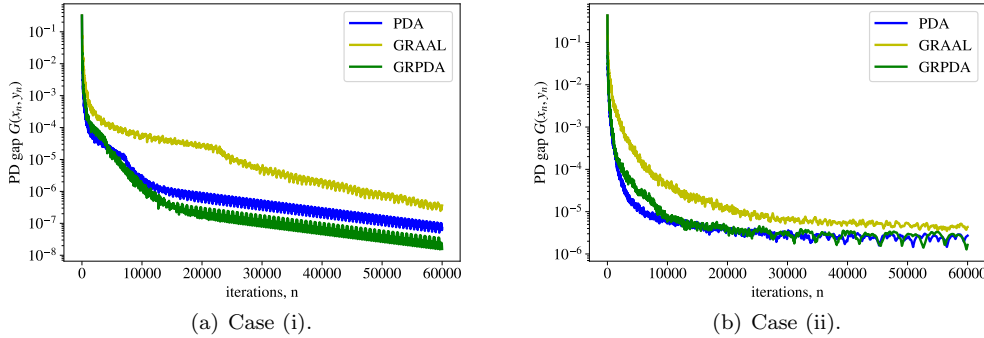


Fig. 5 Numerical results for minimax matrix game problem.

The decreasing behavior of the primal-dual gap function as the algorithms proceeded is given in Figure 5, from which it can be seen that in both tests GRPDA performs comparably with PDA and both GRPDA and PDA outperform GRAAL. Recall that in this experiment β was set to be 1 and thus the primal step size τ and the dual step size η are equal. We attribute the faster convergence of GRPDA and PDA, as compared to GRAAL, to the Gauss-Seidel nature of GRPDA (14) and PDA (6), as compared to the Jacobian type iteration GRAAL (10).

7 Concluding remarks

In this paper, motivated by the recent work [22], we proposed, analyzed and tested a golden ratio primal-dual algorithm (GRPDA), which can be viewed as a new remedy to the classical Arrow-Hurwicz method. Different from the PDA in [8], which uses an extrapolation or inertial step, the proposed GRPDA uses a convex combination technique originally introduced in [22]. Global iterate convergence and $\mathcal{O}(1/N)$ ergodic rate of convergence, measured by the primal-dual gap function, are established. When either g or f^* is strongly convex, we managed to modify the algorithm so that it enjoys a faster $\mathcal{O}(1/N^2)$ ergodic convergence rate. For two widely used special cases, i.e., regularized least-squares problem and linear equality constrained problem, we further show via spectral analysis that the algorithmic parameter ψ can be enlarged from $(1, \phi]$ to $(1, 2]$, which allows larger primal and dual step sizes. Moreover, in the fixed point perspective the iterative mapping is $2/3$ -averaged and thus a relaxation step can be taken with parameter $\rho \in (0, 2/3)$. Our preliminary numerical results on LASSO, nonnegative least-squares and minimax matrix game problems show that the proposed algorithms perform favorably. In particular, GRPDA with fixed step sizes is comparable with PDA in general, and the relaxed and accelerated variants can achieve superior performance.

Some interesting issues remain to be investigated. For example, how to modify the proposed algorithms so that they are suitable to the cases when the operator norm $\|K\|$ is hard to evaluate, how to adapt the proposed algorithms to more general settings, including the general linearly constrained separable convex optimization problem $\min_{x,y} \{f(x) + g(y) : Ax + By = b\}$, or when the coupling term is nonbilinear. Moreover, for large scale finite sum problem, which are abundant in machine learning, it is interesting to design and analyze their stochastic and/or incremental counterparts. We leave these issues for further investigations.

References

1. H. H. Bauschke and P. L. Combettes. *Convex analysis and monotone operator theory in Hilbert spaces*. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC. Springer, Cham, second edition, 2011.
2. A. Beck. *First-Order Methods in Optimization*. MOS-SIAM Series on Optimization. SIAM-Society for Industrial and Applied Mathematics, 2017.
3. A. Beck and M. Teboulle. A fast iterative shrinkage-thresholding algorithm for linear inverse problems. *SIAM Journal on Imaging Sciences*, 2(1):183–202, 2009.
4. D. P. Bertsekas and E. M. Gafni. Projection methods for variational inequalities with application to the traffic assignment problem. *Math. Programming Stud.*, 17:139–159, 1982.
5. T. Bouwmans, N. S. Aybat, and E. H. Zahzah. *Handbook of “Robust low-rank and sparse matrix decomposition: applications in image and video processing”*, volume 45. 2016.
6. S. Boyd, N. Parikh, E. Chu, B. Peleato, and J. Eckstein. Distributed optimization and statistical learning via the alternating direction method of multipliers. *Foundations & Trends in Machine Learning*, 3(1):1–122, 2010.
7. A. Chambolle, M. J. Ehrhardt, P. Richtarik, and C.-B. Schonlieb. Stochastic primal-dual hybrid gradient algorithm with arbitrary sampling and imaging application. *SIAM Journal on Optimization*, 28(4):2783–2808, 2018.
8. A. Chambolle and T. Pock. A first-order primal-dual algorithm for convex problems with applications to imaging. *Journal of Mathematical Imaging and Vision*, 40(1):120–145, 2011.
9. A. Chambolle and T. Pock. On the ergodic convergence rates of a first-order primal-dual algorithm. *Mathematical Programming*, 159(1–2):253–287, SEP 2016.
10. S. Chen, M. A. Saunders, and D. L. Donoho. Atomic decomposition by basis pursuit. *SIAM Review*, 43(1):129–159, 2001.

11. D. L. Donoho. Compressed sensing. *IEEE Trans. Inform. Theory*, 52(4):1289–1306, 2006.
12. J. Duchi, S. Shalev-Shwartz, Y. Singer, and T. Chandra. Efficient projections onto the l_1 -ball for learning in high dimensions. In *The 25th international conference on Machine learning*, pages 272–279, 2008.
13. E. Esser, X. Zhang, and T. F. Chan. A general framework for a class of first order primal-dual algorithms for convex optimization in imaging science. *SIAM Journal on Imaging Sciences*, 3(4):1015–1046, 2010.
14. M. Fazel, T. K. Pong, D. F. Sun, and P. Tseng. Hankel matrix rank minimization with applications in system identification and realization. *SIAM Journal on Matrix Analysis and Applications*, 34:946–977, 2013.
15. D. Gabay and B. Mercier. A dual algorithm for the solution of nonlinear variational problems via finite-element approximations. *Computers and Mathematics with Applications*, 2:17–40, 1976.
16. R. Glowinski and A. Marrocco. Sur l’approximation, par éléments finis d’ordre un, et la résolution, par pénalisation-dualité, d’une classe de problèmes de Dirichlet non linéaires. *R.A.I.R.O., R2*, 9(R-2):41–76, 1975.
17. S. Hayden and O. Stanley. A low patch-rank interpretation of texture. *SIAM Journal on Imaging Sciences*, 6(1):226–262, 2013.
18. B. He and X. Yuan. Convergence analysis of primal-dual algorithms for a saddle-point problem: from contraction perspective. *SIAM Journal on Imaging Sciences*, 5(1):119–149, 2012.
19. E. Jonathan and P. Dimitri. On the douglas-rachford splitting method and the proximal point algorithm for maximal monotone operators. *Mathematical Programming*, 55(1-3):293–318, 1992.
20. P. L. Lions and B. Mercier. Splitting algorithms for the sum of two nonlinear operators. *SIAM Journal on Numerical Analysis*, 16(6):964–979, 1979.
21. Y. Liu, Y. Xu, and W. Yin. Acceleration of primal-dual methods by preconditioning and simple subproblem procedures. *arXiv:1811.08937v2*, 2018.
22. Y. Malitsky. Golden ratio algorithms for variational inequalities. *Mathematical Programming*, To appear.
23. Y. Malitsky and T. Pock. A first-order primal-dual algorithm with linesearch. *SIAM Journal on Optimization*, 28(1):411–432, 2018.
24. R. D. C. Monteiro and B. F. Svaiter. Complexity of variants of Tseng’s modified FB splitting and Korpelevich’s methods for hemivariational inequalities with applications to saddle-point and convex optimization problems. *SIAM J. Optim.*, 21(4):1688–1720, 2011.
25. A. Nedic and A. Ozdaglar. Subgradient methods for saddle-point problems. *Journal of Optimization Theory & Applications*, 142(1):205–228, JUL 2009.
26. D. Needell and R. Ward. Near-optimal compressed sensing guarantees for anisotropic and isotropic total variation minimization. *IEEE Transactions on Image Processing*, 22(10):3941–3949, 2013.
27. A. Nemirovski. Prox-Method with Rate of Convergence $O(1/t)$ for Variational Inequalities with Lipschitz Continuous Monotone Operators and Smooth Convex-Concave Saddle Point Problems. *SIAM J. Optim.*, 15(1):229–251, 2006.
28. C. C. Paige and M. A. Saunders. Lsqqr: An algorithm for sparselinear equations and sparse least squares. *ACM Trans. Math. Soft.*, 8:43–71, 1982.
29. T. Pock and A. Chambolle. Diagonal preconditioning for first order primal-dual algorithms in convex optimization. *IEEE International Conference on Computer Vision*, pages 1762–1769, 2011.
30. R. T. Rockafellar. *Convex analysis*. Princeton University Press, 1970.
31. R. Shefi and M. Teboulle. Rate of convergence analysis of decomposition methods based on the proximal method of multipliers for convex minimization. *SIAM J. Optim.*, 24(1):269–297, 2014.
32. R. Tibshirani. Regression shrinkage and selection via the lasso. *Journal of the Royal Statistical Society*, 58(1):267–288, 1996.
33. H. Uzawa. *Iterative methods for concave programming*. Studies in Linear and Nonlinear Programming (K. J. Arrow, L. Hurwicz and H. Uzawa, eds). Stanford University Press, Stanford, CA, 1958.
34. J. Yang and Y. Zhang. Alternating direction algorithms for ℓ_1 -problems in compressive sensing. *SIAM Journal on Scientific Computing*, 33(1):250–278, 2011.
35. M. Zhu and T. F. Chan. An efficient primal-dual hybrid gradient algorithm for total variation image restoration. *CAM Report 08-34, UCLA, Los Angeles, CA*, 2008.