
Approximate solution of system of equations arising in interior-point methods for bound-constrained optimization

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April 8, 2020

Abstract The focus in this paper is interior-point methods for bound-constrained nonlinear optimization where the system of nonlinear equations that arise are solved with Newton's method. There is a trade-off between solving Newton systems directly, which give high quality solutions, and solving many approximate Newton systems which are computationally less expensive but give lower quality solutions. We propose partial and full approximate solutions to the Newton systems, which in general involves solving a reduced system of linear equations. The specific approximate solution and the size of the reduced system that needs to be solved at each iteration are determined by estimates of the active and inactive constraints at the solution. These sets are at each iteration estimated by a simple heuristic. In addition, we motivate and suggest two modified-Newton approaches which are based on an intermediate step that consists of the partial approximate solutions. The theoretical setting is introduced and asymptotic error bounds are given along with numerical results for bound-constrained convex quadratic optimization problems, both random and from the CUTEst test collection.

Keywords interior-point methods · bound-constrained optimization · approximate solution of system of linear equations · modified-Newton approaches

Research partially supported by the Swedish Research Council (VR).

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1 Introduction

This work is intended for bound-constrained nonlinear optimization problems on the form

$$\begin{aligned} \min \quad & f(x) && \text{(NLP)} \\ \text{s.t.} \quad & l \leq x \leq u, \end{aligned}$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is twice continuously differentiable, $\nabla^2 f(x)$ is locally Lipschitz continuous and $l, u \in \{\mathbb{R} \cup \{-\infty, \infty\}\}^n$ are such that $l < u$. However, to make the work and its ideas more comprehensible we initially describe the theoretical framework and the corresponding results for problems on the form

$$\begin{aligned} \min \quad & f(x) && \text{(P)} \\ \text{s.t.} \quad & x \geq 0. \end{aligned}$$

For completeness, analogous results for problems on the form of (NLP) together with complementary remarks are given in Appendix A.

Bound-constrained optimization problems appear in many different applications and are frequently subproblems in Augmented Lagrangian methods. For a general overview of solution methods see [13] and e.g. the introduction in [16] for a thorough review of previous work. Common solution techniques are; active-set methods, which aim to determine the active constraints and solve a reduced problem with the inactive variables, e.g. [8, 16]; Methods involving projections onto the feasible set such as projected-gradient methods, e.g. [1, 24], projected-Newton or trust-region methods, e.g. [2, 6, 7, 20] and projected quasi-Newton methods, e.g. [4, 19, 28]. We are not aware of any primal-dual interior-point methods specialized for bound-constrained optimization except for more general methods, e.g. [9, 11, 25–27]. Other techniques that are related to trust-region and interior methods are affine-scaling interior-point methods, which are based upon a reformulation of the first-order necessary optimality conditions combined with a Newton-like method, e.g. [5, 17, 18].

In contrast we consider the classical primal-dual interior-point framework. This means solving or approximately solving a sequence of systems of nonlinear equations for which we consider Newton's methods as the model method. As interior methods converge the Newton systems typically become increasingly ill-conditioned due to large diagonal elements in the Schur complement. This is not harmful for direct solvers but it may deteriorate the performance of iterative solvers. We propose a strategy for generating approximate solutions to Newton systems, which in general involves solving a smaller system of linear equations that do not become increasingly ill-conditioned due to the barrier parameter approaching zero. The specific approximate solutions and the size of the system that needs to be solved at each iteration are determined by estimates of the active and inactive constraints at the solution. However, in general these sets are unknown and have to be estimated as the iterations proceed. In this work we use a simple heuristic to determine the considered sets but other approaches may also be used, e.g. approaches similar to those in [8, 16].

In addition, we motivate and suggest two modified-Newton approaches which are based on an intermediate step that consists of the partial approximate solutions.

The work is meant to contribute to the theoretical and numerical understanding on approximate solutions to systems of linear equations arising in interior-point methods. Mainly for, but not limited to, bound-constrained problems. We envisage the use of the approximate solution procedure as an accelerator for a direct solver when solving a sequence of Newton systems for a given value of the barrier parameter μ . E.g., when the direct solver and the approximate solution procedure can be run in parallel. To give an indication of the potential we show numerical simulations on randomly generated problems as well as problems from the CUTEst test collection [14].

The manuscript is organized as follows; Section 2 contains a brief background to primal-dual interior-point methods and an introduction to the theoretical framework; in Section 3 we propose partial and full approximate solutions to Newton systems arising in interior-point methods, as well as motivate two modified-Newton approaches; Section 4 contains numerical results on convex bound-constrained quadratic optimization problems, both randomly generated and problems from the CUTEst test collection; finally in Section 5 we give some concluding remarks.

2 Background

We are interested in the asymptotic behavior of primal-dual interior-point methods in the vicinity of a local minimizer x^* and its corresponding multipliers λ^* . In particular as the iterates of the method converge to a vector $(x^{*T}, \lambda^{*T})^T \triangleq (x^*, \lambda^*)$ that satisfies

$$\nabla f(x^*) - \lambda^* = 0, \quad (\text{stationarity}) \quad (1a)$$

$$x^* \geq 0, \quad (\text{feasibility}) \quad (1b)$$

$$\lambda^* \geq 0, \quad (\text{non-negativity of multipliers}) \quad (1c)$$

$$x^* \cdot \lambda^* = 0, \quad (\text{complementarity}) \quad (1d)$$

$$Z(x^*)^T \nabla^2 f(x^*) Z(x^*) \succ 0, \quad (1e)$$

$$x^* + \lambda^* > 0, \quad (\text{strict complementarity}) \quad (1f)$$

where "·" is defined as the component-wise operator and $Z(x^*)$ is a matrix whose columns span the nullspace of the Jacobian corresponding to the constraints with a strictly positive multiplier, λ^* . Equations (1a)-(1d) constitute first-order necessary optimality conditions for a local minimizer of (P). These conditions together with (1e) form second-order sufficient conditions [15]. For the theoretical framework we also assume that (x^*, λ^*) satisfies (1f). Define the function $F_\mu : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ by

$$F_\mu(x, \lambda) = \begin{bmatrix} \nabla f(x) - \lambda \\ \Lambda X e - \mu e \end{bmatrix},$$

where $\mu \in \mathbb{R}$ is the barrier parameter, $X \in \mathbb{R}^{n \times n}$, $A \in \mathbb{R}^{n \times n}$, $X = \text{diag}(x)$, $A = \text{diag}(\lambda)$ and e is a vector of ones of appropriate size. A vector (x, λ) with $x \geq 0$, $\lambda \geq 0$ and $F_\mu(x, \lambda) = 0$ for $\mu = 0$ satisfies the first-order optimality conditions (1a)-(1d). Primal-dual interior-point methods aim to solve or approximately solve $F_\mu(x, \lambda) = 0$ for a decreasing sequence of $\mu > 0$ while maintaining $x > 0$ and $\lambda > 0$. This is typically done with Newton-like methods which means solving a sequence of systems of linear equations on the form

$$F'(x, \lambda) \begin{bmatrix} \Delta x^N \\ \Delta \lambda^N \end{bmatrix} = -F_\mu(x, \lambda), \quad (2)$$

where $F' : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ is the Jacobian of F_μ . The Jacobian is given by

$$F'(x, \lambda) = \begin{bmatrix} H & -I \\ A & X \end{bmatrix}, \quad (3)$$

where $H = \nabla^2 f(x)$ and the subscript μ is omitted since F' is independent of the barrier parameter. For each μ , iterations are performed until a specified measure of improvement is achieved, thereupon μ is decreased and the process is repeated. A natural measure in our setting is $\|F_\mu(x, \lambda)\|_2$ where $\|F_\mu(x, \lambda)\|_2 = 0$ gives the exact solution. To improve efficiency many algorithms seek approximate solutions, a basic condition for the reduction of μ is $\|F_\mu(x, \lambda)\|_2 < \mu$ [21, Ch. 17, pp. 572]. Herein we consider a possibly weaker version, namely $\|F_\mu(x, \lambda)\|_2 < C\mu$. Moreover, it will throughout be assumed that all considered vectors (x, λ) satisfy $x > 0$ and $\lambda > 0$. The subscript in the norms will hereafter be omitted since all considered norms in this work are of type 2-norm.

Definition 1 (Order-notation) Let $\alpha, \gamma \in \mathbb{R}$ be two positive related quantities. If there exists a constant $C_1 > 0$ such that $\gamma \geq C_1\alpha$ for sufficiently small α , then $\gamma = \Omega(\alpha)$. Similarly, if there exists a constant $C_1 > 0$ such that $\gamma \leq C_1\alpha$ for sufficiently small α , then $\gamma = \mathcal{O}(\alpha)$. If there exist constants $C_1, C_2 > 0$ such that $C_1\alpha \leq \gamma \leq C_2\alpha$ for sufficiently small α then, $\gamma = \Theta(\alpha)$.

Definition 2 (Neighborhood) Let the neighborhood around (x^*, λ^*) be defined by $\mathcal{B}((x^*, \lambda^*), \delta) = \{(x, \lambda) : \|(x, \lambda) - (x^*, \lambda^*)\| < \delta\}$.

Assumption 1 (Strict local minimizer) *The vector (x^*, λ^*) satisfies (1), i.e. second-order sufficient optimality conditions and strict complementarity.*

The following two results provide the theoretical framework and additional definitions of various quantities. In particular, the existence of a neighborhood where the Jacobian is nonsingular and there exists a Lipschitz continuous barrier trajectory which is parameterized by the barrier parameter μ . The results are well known and can be found in e.g. the work of Ortega and Rheinboldt [23] and Byrd, Liu and Nocedal [3] whose setting is similar to the one in this work.

Lemma 1 Under Assumption 1 there exists $\delta > 0$ such that $F'(x, \lambda)$ is continuous and nonsingular for $(x, \lambda) \in \mathcal{B}((x^*, \lambda^*), \delta)$ and

$$\|F'(x, \lambda)^{-1}\| \leq M,$$

for some constant $M > 0$.

Proof See [23, pp. 46]. \square

Lemma 2 Let Assumption 1 hold and let $\mathcal{B}((x^*, \lambda^*), \delta)$ be defined by Lemma 1. Then there exists $\hat{\mu} > 0$ such that for each $0 < \mu \leq \hat{\mu}$ there is a Lipschitz continuous function $(x^\mu, \lambda^\mu) \in \mathcal{B}((x^*, \lambda^*), \delta)$ that satisfies $F_\mu(x^\mu, \lambda^\mu) = 0$ and

$$\|(x^\mu, \lambda^\mu) - (x^*, \lambda^*)\| \leq C_3 \mu,$$

where $C_3 = \inf_{(x, \lambda) \in \mathcal{B}((x^*, \lambda^*), \delta)} \|F'(x, \lambda)^{-1} \frac{\partial F_\mu(x, \lambda)}{\partial \mu}\|$.

Proof The result follows from the implicit function theorem, see e.g. [23, pp. 128]. \square

The next lemma relates the measure $\|F_\mu(x, \lambda)\|$ to the distance between the barrier trajectory and vectors (x, λ) that are sufficiently close. An analogous result is given by Byrd, Liu and Nocedal in [3].

Lemma 3 Under Assumption 1 let $\mathcal{B}((x^*, \lambda^*), \delta)$ and $\hat{\mu}$ be defined by Lemma 1 respectively Lemma 2. For $0 < \mu \leq \hat{\mu}$ and (x, λ) sufficiently close to $(x^\mu, \lambda^\mu) \in \mathcal{B}((x^*, \lambda^*), \delta)$ there exist constants $C_4, C_5 > 0$ such that

$$C_4 \|(x, \lambda) - (x^\mu, \lambda^\mu)\| \leq \|F_\mu(x, \lambda)\| \leq C_5 \|(x, \lambda) - (x^\mu, \lambda^\mu)\|.$$

Proof See [3, pp. 43]. \square

Recall that the reduction of μ can be determined with the condition $\|F_\mu(x, \lambda)\| < C\mu$ for some constant $C > 0$. It can be shown that vectors (x, λ) , which satisfy this condition and are sufficiently close to the barrier trajectory, have their individual components bounded within certain intervals at sufficiently small μ . The individual components can be partitioned into two sets of indices which depend on how close the iterate is to its feasibility bound, see Definition 3. The order of magnitude of the individual components, which are given in Lemma 4 below, will be of importance in the derivation of various approximate solutions to (2).

Definition 3 (Active/inactive constraint). For a given $x^* \geq 0$ constraint $i \in \{1, \dots, n\}$ is defined as active if $x_i^* = 0$ and inactive if $x_i^* > 0$. The corresponding active and inactive set are defined as $\mathcal{A} = \{i \in \{1, \dots, n\} : x_i^* = 0\}$, respectively $\mathcal{I} = \{i \in \{1, \dots, n\} : x_i^* > 0\}$.

Lemma 4 *Under Assumption 1 let $\mathcal{B}((x^*, \lambda^*), \delta)$ and $\hat{\mu}$ be defined by Lemma 1 and Lemma 2 respectively. Then there exists $\bar{\mu}$, with $0 < \bar{\mu} \leq \hat{\mu}$, such that for $0 < \mu \leq \bar{\mu}$ and (x, λ) sufficiently close to $(x^\mu, \lambda^\mu) \in \mathcal{B}((x^*, \lambda^*), \delta)$ so that $\|F_\mu(x, \lambda)\| = \mathcal{O}(\mu)$ it holds that*

$$x_i = \begin{cases} \mathcal{O}(\mu) & i \in \mathcal{A}, \\ \Theta(1) & i \in \mathcal{I}, \end{cases} \quad \lambda_i = \begin{cases} \Theta(1) & i \in \mathcal{A}, \\ \mathcal{O}(\mu) & i \in \mathcal{I}. \end{cases} \quad (4)$$

Proof Under Assumption 1 it holds that

$$x_i^* = \begin{cases} 0 & i \in \mathcal{A}, \\ c_i & i \in \mathcal{I}, \end{cases} \quad \lambda_i^* = \begin{cases} c_i & i \in \mathcal{A}, \\ 0 & i \in \mathcal{I}, \end{cases}$$

where $c_i = \Theta(1)$, $i = 1, \dots, n$. The function (x^μ, λ^μ) is Lipschitz continuous and hence for each $\mu \leq \hat{\mu}$ it holds that $(x^\mu, \lambda^\mu) \in \mathcal{B}((x^*, \lambda^*), L_{F'}\mu)$, where $L_{F'}$ is the Lipschitz constant of F' on $\mathcal{B}((x^*, \lambda^*), \delta)$. There exist $\bar{\mu}_1$, with $0 < \bar{\mu}_1 \leq \hat{\mu}$, such that for $0 < \mu \leq \bar{\mu}_1$ it holds that

$$x_i^\mu = \begin{cases} \mathcal{O}(\mu) & i \in \mathcal{A}, \\ \Theta(1) & i \in \mathcal{I}, \end{cases} \quad \lambda_i^\mu = \begin{cases} \Theta(1) & i \in \mathcal{A}, \\ \mathcal{O}(\mu) & i \in \mathcal{I}. \end{cases}$$

The condition $\|F_\mu(x, \lambda)\| = \mathcal{O}(\mu)$ implies that there exist a constant $C > 0$ such that $\|F_\mu(x, \lambda)\| \leq C\mu$. Which in combination with Lemma 3 give

$$\|(x, \lambda) - (x^\mu, \lambda^\mu)\| \leq \frac{1}{C_4} \|F_\mu(x, \lambda)\| \leq \frac{C}{C_4} \mu,$$

which implies that $(x, \lambda) \in \mathcal{B}((x^\mu, \lambda^\mu), \frac{C}{C_4}\mu)$. Similarly here, there exists $\bar{\mu}_2$, with $0 < \bar{\mu}_2 \leq \hat{\mu}$, such that the result follows for $0 < \mu \leq \bar{\mu}$ with $\bar{\mu} = \min\{\bar{\mu}_1, \bar{\mu}_2\}$. \square

The result of Lemma 4 shows two regions which depend on μ . The first region, $0 < \mu \leq \hat{\mu}$, defines where the barrier trajectory (x^μ, λ^μ) exists and the second region, $0 < \mu \leq \bar{\mu} \leq \hat{\mu}$, defines where asymptotic behavior occurs.

3 Approximate solutions

This section initially contains an introduction to the groundwork of the ideas which precede the results. It is followed by a subsection that contains approximate solutions for specific components of the solution of (2) together with related results. The last subsection contains procedures for approximating the full solution of (2), as well as related results. Under Assumption 1 it holds that

$$\lim_{\mu \rightarrow 0} x_i^\mu = 0, \quad i \in \mathcal{A}, \quad \text{and} \quad \lim_{\mu \rightarrow 0} \lambda_i^\mu = 0, \quad i \in \mathcal{I},$$

in consequence the Schur complement of X in (3) becomes increasingly ill-conditioned as $\mu \rightarrow 0$. These properties have been utilized by several authors

before, e.g. in the development of preconditioners [10,12]. The idea in this work is to exploit them and that (P) only has bound constraints to obtain partial or full approximate solutions of (2). In particular by utilization of structure and the asymptotic behavior of coefficients in the arising systems of linear equations. With the partition $(\Delta x^N, \Delta \lambda^N) = (\Delta x_{\mathcal{A}}^N, \Delta x_{\mathcal{I}}^N, \Delta \lambda_{\mathcal{A}}^N, \Delta \lambda_{\mathcal{I}}^N)$, (2) can be written as

$$\begin{bmatrix} H_{AA} & H_{AI} & -I_{AA} & \\ H_{IA} & H_{II} & & -I_{II} \\ \Lambda_{AA} & & X_{AA} & \\ & \Lambda_{II} & & X_{II} \end{bmatrix} \begin{bmatrix} \Delta x_{\mathcal{A}}^N \\ \Delta x_{\mathcal{I}}^N \\ \Delta \lambda_{\mathcal{A}}^N \\ \Delta \lambda_{\mathcal{I}}^N \end{bmatrix} = - \begin{bmatrix} \nabla f(x)_{\mathcal{A}} - \lambda_{\mathcal{A}} \\ \nabla f(x)_{\mathcal{I}} - \lambda_{\mathcal{I}} \\ \Lambda_{AA} X_{AA} e - \mu e \\ \Lambda_{II} X_{II} e - \mu e \end{bmatrix}, \quad (5)$$

where the first and second letter in the matrix subscripts indicate rows respectively columns corresponding to indices in the respective set. The Schur complement of X_{AA} and X_{II} in (5) is

$$\begin{bmatrix} H_{AA} + X_{AA}^{-1} \Lambda_{AA} & H_{AI} \\ H_{IA} & H_{II} + X_{II}^{-1} \Lambda_{II} \end{bmatrix} \begin{bmatrix} \Delta x_{\mathcal{A}}^N \\ \Delta x_{\mathcal{I}}^N \end{bmatrix} = - \begin{bmatrix} \nabla f(x)_{\mathcal{A}} - \mu X_{AA}^{-1} e \\ \nabla f(x)_{\mathcal{I}} - \mu X_{II}^{-1} e \end{bmatrix}. \quad (6)$$

By continuity of (x^μ, λ^μ) it follows that $x_i \rightarrow 0$, $i \in \mathcal{A}$ and $\lambda_i \rightarrow 0$, $i \in \mathcal{I}$ as $\mu \rightarrow 0$. In consequence, X_{II} and Λ_{AA} dominate the coefficients of the third and fourth block of (5) for sufficiently small μ . Similarly $X_{AA}^{-1} \Lambda_{AA}$ dominates the coefficients of the first block of (6). Consequently approximate solutions of $\Delta x_{\mathcal{A}}^N$ and $\Delta \lambda_{\mathcal{I}}^N$ can be obtained from the third and fourth block of (5) and of $\Delta x_{\mathcal{I}}^N$ from the first block of (6). These approximates can then be inserted into (5), or (6), to obtain a reduced system of size $|\mathcal{I}| \times |\mathcal{I}|$ that involves H_{II} whose solution gives an approximation of $\Delta x_{\mathcal{I}}^N$. These observations together with Lemma 4 and Lemma 5 below provide the foundation for the results. The essence of Lemma 5 is that the norm of the solution of (2) is bounded by a constant times μ .

Lemma 5 *Under Assumption 1 let $\mathcal{B}((x^*, \lambda^*), \delta)$ and $\hat{\mu}$ be defined by Lemma 1 and Lemma 2 respectively. For $0 < \mu \leq \hat{\mu}$ and $(x, \lambda) \in \mathcal{B}((x^*, \lambda^*), \delta)$, let $(\Delta x^N, \Delta \lambda^N)$ be the solution of (2) with $\mu^+ = \sigma \mu$, where $0 < \sigma < 1$. If (x, λ) is sufficiently close to $(x^\mu, \lambda^\mu) \in \mathcal{B}((x^*, \lambda^*), \delta)$ such that $\|F_\mu(x, \lambda)\| = \mathcal{O}(\mu)$ then*

$$\|(\Delta x^N, \Delta \lambda^N)\| = \mathcal{O}(\mu).$$

Proof By (2) it holds that

$$\begin{aligned} \|(\Delta x^N, \Delta \lambda^N)\| &= \|F'(x, \lambda)^{-1} F_{\mu^+}(x, \lambda)\| \\ &= \|F'(x, \lambda)^{-1} [F_{\mu^+}(x, \lambda) - F_{\mu^+}(x^{\mu^+}, \lambda^{\mu^+})]\|. \end{aligned}$$

Both (x, λ) and $(x^{\mu^+}, \lambda^{\mu^+})$ belong to $\mathcal{B}((x^*, \lambda^*), \delta)$. Continuity of F' on \mathcal{B} implies that F_{μ^+} is Lipschitz continuous. Which together with Lemma 1 yield

$$\begin{aligned} \|(\Delta x^N, \Delta \lambda^N)\| &\leq ML_{F'} \|(x, \lambda) - (x^{\mu^+}, \lambda^{\mu^+})\| \\ &= ML_{F'} \|(x, \lambda) - (x^\mu, \lambda^\mu) + (x^\mu, \lambda^\mu) - (x^{\mu^+}, \lambda^{\mu^+})\| \\ &\leq ML_{F'} \left(\|(x, \lambda) - (x^\mu, \lambda^\mu)\| + \|(x^\mu, \lambda^\mu) - (x^{\mu^+}, \lambda^{\mu^+})\| \right) \\ &\leq ML_{F'} \left(\frac{1}{C_4} \|F_\mu(x, \lambda)\| + C_3(1 - \sigma)\mu \right) \\ &\leq ML_{F'} \left(\frac{C}{C_4} + C_3(1 - \sigma) \right) \mu, \end{aligned}$$

where the second last inequality follows from Lemma 3 and Lipschitz continuity of (x^μ, λ^μ) . The last inequality follows from $\|F_\mu(x, \lambda)\| = \mathcal{O}(\mu)$, i.e. there exists a constant $C > 0$ such that $\|F_\mu(x, \lambda)\| \leq C\mu$. \square

3.1 Partial approximate solutions

In this section we initially propose an approximate solution of $\Delta x_{\mathcal{A}}^N$ which originates from the Schur complement form (6). As $\mu \rightarrow 0$ the diagonal elements of the (1,1)-block become large and dominate the coefficients of the matrix. In Proposition 1 we show that an approximate solution of $\Delta x_{\mathcal{A}}^N$ can be obtained by neglecting all off-diagonal coefficients in the the first block of (6). Thereafter we propose another approximate solution of $\Delta x_{\mathcal{A}}^N$ as well as an approximate solution of $\Delta \lambda_{\mathcal{I}}^N$ which originate from the complementarity blocks of (5). These approximate solutions are obtained by neglecting the coefficients in the complementarity blocks which approach zero as $\mu \rightarrow 0$, i.e. those given by $X_{\mathcal{A}\mathcal{A}}$ and $A_{\mathcal{I}\mathcal{I}}$. The resulting partial approximate solutions are given below in Proposition 2. The essence of both results is that, under certain conditions, the asymptotic component-error bounds are in the order of μ^2 . Finally we motivate and propose two modified-Newton methods which we later on investigate numerically.

Proposition 1 *Under Assumption 1 let $\mathcal{B}((x^*, \lambda^*), \delta)$ and $\hat{\mu}$ be defined by Lemma 1 and Lemma 2 respectively. For $(x, \lambda) \in \mathcal{B}((x^*, \lambda^*), \delta)$, let $(\Delta x^N, \Delta \lambda^N)$ be the solution of (2) with $\mu^+ = \sigma\mu$, where $0 < \sigma < 1$. If the search direction components are defined as*

$$\Delta x_i = -\frac{x_i [\nabla f(x)]_i - \mu^+}{x_i [\nabla^2 f(x)]_{ii} + \lambda_i}, \quad i = 1, \dots, n, \quad (7)$$

then

$$\Delta x_i - \Delta x_i^N = \frac{x_i}{x_i [\nabla^2 f(x)]_{ii} + \lambda_i} \sum_{i \neq j} [\nabla^2 f(x)]_{ij} \Delta x_j^N, \quad i = 1, \dots, n. \quad (8)$$

If in addition, $0 < \mu \leq \hat{\mu}$ and (x, λ) is sufficiently close to $(x^\mu, \lambda^\mu) \in \mathcal{B}((x^*, \lambda^*), \delta)$ such that $\|F_\mu(x, \lambda)\| = \mathcal{O}(\mu)$. Then there exists $\bar{\mu}$, with $0 < \bar{\mu} \leq \hat{\mu}$, such that for $0 < \mu \leq \bar{\mu}$ it holds that

$$\frac{1}{x_i [\nabla^2 f(x)]_{ii} + \lambda_i} = \Theta(1), \quad i = 1, \dots, n. \quad (9)$$

and

$$|\Delta x_i - \Delta x_i^N| = \mathcal{O}(\mu^2), \quad i \in \mathcal{A}. \quad (10)$$

Proof The solution of (2) is equivalent to the solution of (6) where the i 'th, $i = 1, \dots, n$, row is

$$\sum_{j \neq i}^n [\nabla^2 f(x)]_{ij} \Delta x_j^N + \left([\nabla^2 f(x)]_{ii} + \frac{\lambda_i}{x_i} \right) \Delta x_i^N = - \left([\nabla f(x)]_i - \frac{\mu^+}{x_i} \right). \quad (11)$$

If $x_i [\nabla^2 f(x)]_{ii} + \lambda_i \neq 0$ then (11) can be written as

$$\begin{aligned} \Delta x_i^N &= \frac{x_i}{x_i [\nabla^2 f(x)]_{ii} + \lambda_i} \left(- \left([\nabla f(x)]_i - \frac{\mu^+}{x_i} \right) - \sum_{j \neq i}^n [\nabla^2 f(x)]_{ij} \Delta x_j^N \right) \\ &= - \frac{x_i [\nabla f(x)]_i - \mu^+}{x_i [\nabla^2 f(x)]_{ii} + \lambda_i} - \frac{x_i}{x_i [\nabla^2 f(x)]_{ii} + \lambda_i} \sum_{j \neq i}^n [\nabla^2 f(x)]_{ij} \Delta x_j^N. \end{aligned} \quad (12)$$

Subtraction of (12) from (7) gives (8). By Lemma 4 there exists $\bar{\mu}_3$, with $0 < \bar{\mu}_3 \leq \hat{\mu}$ such that the components of (x, λ) satisfy (4). Due to the boundedness of f on $\mathcal{B}((x^*, \lambda^*), \delta)$ there exists $\bar{\mu}_4$, with $0 < \bar{\mu}_4 \leq \hat{\mu}$, such that (9) holds for $0 < \mu \leq \bar{\mu}$ with $\bar{\mu} = \min\{\bar{\mu}_3, \bar{\mu}_4\}$. The result of (10) follows from application of Lemma 4 and Lemma 5 to (8) while taking (9) into account. \square

The approximate solution of Δx_A^N in (7) of Proposition 1 and its corresponding error (8) may be undefined for certain components. However, as shown by (9), the expressions are well-defined sufficiently close to the barrier trajectory for sufficiently small μ . An approximate solution that is guaranteed to have all its components well-defined can be obtained from the complementarity blocks of (5). This approximate solution, and in addition an approximate solution of $\Delta \lambda_T^N$, are given in the proposition below.

Proposition 2 *Under Assumption 1 let $\mathcal{B}((x^*, \lambda^*), \delta)$ and $\hat{\mu}$ be defined by Lemma 1 and Lemma 2 respectively. For $(x, \lambda) \in \mathcal{B}((x^*, \lambda^*), \delta)$, let $(\Delta x^N, \Delta \lambda^N)$ be the solution of (2) with $\mu^+ = \sigma \mu$, where $0 < \sigma < 1$. If the search direction components are defined as*

$$\Delta x_i = -x_i + \frac{\mu^+}{\lambda_i}, \quad i = 1, \dots, n, \quad (13a)$$

$$\Delta \lambda_i = -\lambda_i + \frac{\mu^+}{x_i}, \quad i = 1, \dots, n, \quad (13b)$$

then

$$\Delta x_i - \Delta x_i^N = \frac{x_i}{\lambda_i} \Delta \lambda_i^N, \quad i = 1, \dots, n, \quad (14a)$$

$$\Delta \lambda_i - \Delta \lambda_i^N = \frac{\lambda_i}{x_i} \Delta x_i^N, \quad i = 1, \dots, n, \quad (14b)$$

If in addition, $0 < \mu \leq \hat{\mu}$ and (x, λ) is sufficiently close to $(x^\mu, \lambda^\mu) \in \mathcal{B}((x^*, \lambda^*), \delta)$ such that $\|F_\mu(x, \lambda)\| = \mathcal{O}(\mu)$. Then there exists $\bar{\mu}$, with $0 < \bar{\mu} \leq \hat{\mu}$, such that for $0 < \mu \leq \bar{\mu}$ it holds that

$$|\Delta x_i - \Delta x_i^N| = \mathcal{O}(\mu^2), \quad i \in \mathcal{A}, \quad (15a)$$

$$|\Delta \lambda_i - \Delta \lambda_i^N| = \mathcal{O}(\mu^2), \quad i \in \mathcal{I}. \quad (15b)$$

Proof The i 'th, $i = 1, \dots, n$, row in the second block of (2) is

$$\lambda_i \Delta x_i^N + x_i \Delta \lambda_i^N = -\lambda_i x_i + \mu^+,$$

For $x_i > 0$, $\lambda_i > 0$, $i = 1, \dots, n$, it holds that

$$\Delta x_i^N = -x_i + \frac{\mu^+}{\lambda_i} - \frac{x_i}{\lambda_i} \Delta \lambda_i^N, \quad (16a)$$

$$\Delta \lambda_i^N = -\lambda_i + \frac{\mu^+}{x_i} - \frac{\lambda_i}{x_i} \Delta x_i^N. \quad (16b)$$

Subtraction of (16a) from (13a) and subtraction of (16b) from (13b) gives (14a) respectively (14b). By Lemma 4 there exists $\bar{\mu}$, with $0 < \bar{\mu} \leq \hat{\mu}$ such that the components of (x, λ) satisfy (4) for $0 < \mu \leq \bar{\mu}$. The result of (15) then follows from application of Lemma 5 to (16) while taking (4) into account. \square

Both (7) and (13a) provide approximate solutions of Δx_i^N , $i \in \mathcal{A}$, with asymptotically similar error bounds. Note that the order of the approximation error, $\|\Delta x_{\mathcal{A}} - \Delta x_{\mathcal{A}}^N\|$, is maintained even if some components $i \in \mathcal{A}$ are updated with (7) and others with (13a). Which expression to use can hence be chosen individually for each index $i \in \mathcal{A}$. The factors in front of Δx_i^N and $\Delta \lambda_i^N$, $i = 1, \dots, n$, in (8) respectively (14) may be used as an indicator for which of the approximations to use, and also whether either expression is likely to provide an accurate approximation. Note also that the approximate solution of (13a) does not take into account any information from the first block of equations in (2) approximations will whereas (7) includes information from both blocks.

Provided that the norm of the combined steps $\Delta x_{\mathcal{A}}^N$ and $\Delta \lambda_{\mathcal{I}}^N$ is not smaller than the approximation error, then stepping in these components with (7) or (13) give a vector which is closer to the Newton iterate. This is formalized in Proposition 6 below.

Proposition 3 Under Assumption 1 let $\mathcal{B}((x^*, \lambda^*), \delta)$ and $\hat{\mu}$ be defined by Lemma 1 and Lemma 2 respectively. For $(x, \lambda) \in \mathcal{B}((x^*, \lambda^*), \delta)$, define $(x_+^N, \lambda_+^N) = (x, \lambda) + (\Delta x^N, \Delta \lambda^N)$ where $(\Delta x^N, \Delta \lambda^N)$ is the solution of (2) with $\mu^+ = \sigma\mu$, where $0 < \sigma < 1$. Moreover, let $(x_+, \lambda_+) = (x, \lambda) + (\Delta x, \Delta \lambda)$ where

$$\Delta x_i = \begin{cases} (13a) \text{ or } (7) & i \in \mathcal{A}, \\ 0 & i \in \mathcal{I}, \end{cases} \quad \Delta \lambda_i = \begin{cases} 0 & i \in \mathcal{A}, \\ (13b) & i \in \mathcal{I}. \end{cases} \quad (17)$$

If $0 < \mu \leq \hat{\mu}$, $\|(\Delta x_{\mathcal{A}}^N, \Delta \lambda_{\mathcal{I}}^N)\| = \Omega(\mu^\gamma)$ for $0 \leq \gamma < 2$, and (x, λ) is sufficiently close to $(x^\mu, \lambda^\mu) \in \mathcal{B}((x^*, \lambda^*), \delta)$ such that $\|F_\mu(x, \lambda)\| = \mathcal{O}(\mu)$. Then there exists $\bar{\mu}$, with $0 < \bar{\mu} \leq \hat{\mu}$, such that for $0 < \mu \leq \bar{\mu}$ it holds that

$$\|(x_+^N, \lambda_+^N) - (x_+, \lambda_+)\| \leq \|(x_+^N, \lambda_+^N) - (x, \lambda)\|. \quad (18)$$

Proof With $(\Delta x, \Delta \lambda)$ defined as in (17) of the proposition it holds that

$$\begin{aligned} & \|(x_+^N, \lambda_+^N) - (x_+, \lambda_+)\|^2 - \|(x_+^N, \lambda_+^N) - (x, \lambda)\|^2 = \\ & \|(\Delta x^N - \Delta x, \Delta \lambda^N - \Delta \lambda)\|^2 - \|(\Delta x^N, \Delta \lambda^N)\|^2 = \\ & \|(\Delta x_{\mathcal{A}}^N - \Delta x_{\mathcal{A}}, \Delta \lambda_{\mathcal{I}}^N - \Delta \lambda_{\mathcal{I}})\|^2 - \|(\Delta x_{\mathcal{A}}^N, \Delta \lambda_{\mathcal{I}}^N)\|^2. \end{aligned} \quad (19)$$

By Proposition 1 and Proposition 2 there exists $\bar{\mu}_5$, with $0 < \bar{\mu}_5 \leq \hat{\mu}$, such that for $0 < \mu \leq \bar{\mu}_5$ it holds that $|\Delta x_i - \Delta x_i^N| = \mathcal{O}(\mu^2)$, $i \in \mathcal{A}$ and $|\Delta \lambda_i - \Delta \lambda_i^N| = \mathcal{O}(\mu^2)$, $i \in \mathcal{I}$. Hence, for $0 < \mu \leq \bar{\mu}_5$, there exist constants $C_6 > 0$ and $C_7 > 0$, where C_7 comes from the condition $\|(\Delta x_{\mathcal{A}}^N, \Delta \lambda_{\mathcal{I}}^N)\| = \Omega(\mu^\gamma)$, $0 \leq \gamma < 2$, such that

$$\|(\Delta x_{\mathcal{A}}^N - \Delta x_{\mathcal{A}}, \Delta \lambda_{\mathcal{I}}^N - \Delta \lambda_{\mathcal{I}})\|^2 - \|(\Delta x_{\mathcal{A}}^N, \Delta \lambda_{\mathcal{I}}^N)\|^2 \leq C_6^2 \mu^4 - C_7^2 \mu^{2\gamma}, \quad (20)$$

which is non-positive for $0 < \mu \leq (C_7/C_6)^{\frac{1}{2-\gamma}}$, $0 \leq \gamma < 2$. Combining (19)-(20) and letting $\bar{\mu} = \min\{\bar{\mu}_5, C_7/C_6\}$ gives the result. \square

The partial approximate solution (17) of Proposition 6 is computationally inexpensive compared to solving (2). The result of (18) motivates the study of modified-Newton approaches which make use of the partial approximate solution. Provided that the iterate $(x^E, \lambda^E) = (x + \Delta x^E, \lambda + \Delta \lambda^E)$, where $(\Delta x^E, \Delta \lambda^E)$ is given by (17), is strictly feasible and lies in $\mathcal{B}((x^*, \lambda^*), \delta)$, solving a Newton system from this iterate provides potential improvement. Moreover, only updating the factors of the matrix in (2) gives an approximate higher-order Newton method under strict complementarity. The two strategies involve a partial approximate update as well as the solution of

$$F'(x^E, \lambda^E) \begin{bmatrix} \Delta x \\ \Delta \lambda \end{bmatrix} = -F_\mu(x^E, \lambda^E), \quad (\text{Approximate intermediate step}) \quad (21)$$

respectively,

$$F'(x^E, \lambda^E) \begin{bmatrix} \Delta x \\ \Delta \lambda \end{bmatrix} = -F_\mu(x, \lambda), \quad (\text{Approximate higher-order}) \quad (22)$$

with the corresponding update. Numerical results for these strategies are shown in Section 4, also for problems where strict complementarity does not hold.

3.2 Full approximate solution

In this section we propose an approximate solution of (2) that, in the considered framework, has an asymptotic error bound in the order of μ^2 . The approximate solution is obtained by insertion of any of the partial approximate solutions of Δx_A^N in Proposition 1 or Proposition 2 and utilization of the structure in the systems that arise. Specifically, suppose that an approximate Δx_A is given, e.g. (7) of Proposition 1 or (13a) of Proposition 2. Insertion of the approximate Δx_A into (5) yields

$$\begin{bmatrix} H_{AI} & -I_{AA} \\ H_{II} & -I_{II} \\ & X_{AA} \\ \Lambda_{II} & X_{II} \end{bmatrix} \begin{bmatrix} \Delta x_I \\ \Delta \lambda_A \\ \Delta \lambda_I \end{bmatrix} = - \begin{bmatrix} \nabla f(x)_A - \lambda_A + H_{AA}\Delta x_A \\ \nabla f(x)_I - \lambda_I + H_{IA}\Delta x_A \\ \Lambda_{AA}X_{AA}e - \mu e + \Lambda_{AA}\Delta x_A \\ \Lambda_{II}X_{II}e - \mu e \end{bmatrix}. \quad (23)$$

The second and fourth block of (23) provide unique solutions of Δx_I and $\Delta \lambda_I$ which satisfy

$$\begin{bmatrix} H_{II} & -I_{II} \\ \Lambda_{II} & X_{II} \end{bmatrix} \begin{bmatrix} \Delta x_I \\ \Delta \lambda_I \end{bmatrix} = - \begin{bmatrix} \nabla f(x)_I - \lambda_I + H_{IA}\Delta x_A \\ \Lambda_{II}X_{II}e - \mu e \end{bmatrix}. \quad (24)$$

The solution of (24) can be obtained by first solving with the Schur complement of X_{II}

$$(H_{II} + X_{II}^{-1}\Lambda_{II}) \Delta x_I = -(\nabla f(x)_I + H_{IA}\Delta x_A) + \mu X_{II}^{-1}e, \quad (25)$$

and then

$$\Delta \lambda_I = -X_{II}^{-1}(\Lambda_{II}X_{II}e - \mu e) - X_{II}^{-1}\Lambda_{II}\Delta x_I. \quad (26)$$

Note that (25) can also be obtained by insertion of the given Δx_A into the second block of (6). The matrix of (25) is by Assumption 1 a symmetric positive definite ($|I| \times |I|$)-matrix. Moreover, the matrix does not become increasingly ill-conditioned due to large elements in $X^{-1}\Lambda$, as $\mu \rightarrow 0$, in contrast to the matrix of (6). The remaining part of the solution of (23), that is $\Delta \lambda_A$, is then given by

$$\begin{bmatrix} -I_{AA} \\ X_{AA} \end{bmatrix} \Delta \lambda_A = - \begin{bmatrix} \nabla f(x)_A + \lambda_A + H_{AA}\Delta x_A + H_{AI}\Delta x_I \\ \Lambda_{AA}X_{AA}e - \mu e + \Lambda_{AA}\Delta x_A \end{bmatrix}. \quad (27)$$

If the approximate Δx_A is exact, i.e. if $\Delta x_A = \Delta x_A^N$, then so is Δx_I by (25). In consequence, the over-determined system (27) has a unique solution that satisfies all equations, i.e. $\Delta \lambda_A$ is the corresponding part of the solution to (2). This solution or an approximate can then be obtained from

$$\Delta \lambda_A = \nabla f(x)_A + \lambda_A + H_{AA}\Delta x_A + H_{AI}\Delta x_I, \quad (28a)$$

or

$$\Delta \lambda_A = -\lambda_A + \mu X_{AA}^{-1}e + X_{AA}^{-1}\Lambda_{AA}\Delta x_A. \quad (28b)$$

Alternatively, $\Delta\lambda_{\mathcal{A}}$ can be obtained as the least squares solution of (27) that is

$$\begin{aligned} \Delta\lambda_{\mathcal{A}} = & (I_{\mathcal{A}\mathcal{A}} + X_{\mathcal{A}\mathcal{A}}^2)^{-1} \left[\nabla f(x)_{\mathcal{A}} + \lambda_{\mathcal{A}} + H_{\mathcal{A}\mathcal{A}}\Delta x_{\mathcal{A}} + H_{\mathcal{A}\mathcal{I}}\Delta x_{\mathcal{I}} \right. \\ & \left. - X_{\mathcal{A}\mathcal{A}}(\Lambda_{\mathcal{A}\mathcal{A}}X_{\mathcal{A}\mathcal{A}}e - \mu e + \Lambda_{\mathcal{A}\mathcal{A}}\Delta x_{\mathcal{A}}) \right]. \end{aligned} \quad (29)$$

In Theorem 1 it is shown that under certain conditions both (28a) and (29) can be used to update $\Delta\lambda_{\mathcal{A}}$ without affecting the order of the asymptotic error. Note however that this is not true for (28b) due to the last term that contains $X_{\mathcal{A}\mathcal{A}}^{-1}$ in combination with approximation error.

Theorem 1 *Under Assumption 1 let $\mathcal{B}((x^*, \lambda^*), \delta)$ and $\hat{\mu}$ be defined by Lemma 1 and Lemma 2 respectively. For $0 < \mu \leq \hat{\mu}$ and $(x, \lambda) \in \mathcal{B}((x^*, \lambda^*), \delta)$, let $(\Delta x^N, \Delta\lambda^N)$ be the solution of (2) with $\mu^+ = \sigma\mu$, where $0 < \sigma < 1$. Moreover, let the search direction components be defined as*

$$\Delta x_i = \begin{cases} (7) \text{ or } (13a) & i \in \mathcal{A}, \\ (25) & i \in \mathcal{I}, \end{cases} \quad \Delta\lambda_i = \begin{cases} (29) \text{ or } (28a) & i \in \mathcal{A}, \\ (13b) \text{ or } (26) & i \in \mathcal{I}. \end{cases}$$

If $0 < \mu \leq \hat{\mu}$ and (x, λ) is sufficiently close to $(x^\mu, \lambda^\mu) \in \mathcal{B}((x^*, \lambda^*), \delta)$ such that $\|F_\mu(x, \lambda)\| = \mathcal{O}(\mu)$. Then there exists $\bar{\mu}$, with $0 < \bar{\mu} \leq \hat{\mu}$, such that for $0 < \mu \leq \bar{\mu}$ it holds that

$$\|(\Delta x, \Delta\lambda) - (\Delta x^N, \Delta\lambda^N)\| = \mathcal{O}(\mu^2).$$

Proof By Proposition 1 and Proposition 2 there exists $\bar{\mu}_6$ respectively $\bar{\mu}_7$, with $0 < \bar{\mu}_i \leq \hat{\mu}$, $i = 6, 7$, such that for $0 < \mu \leq \min\{\bar{\mu}_6, \bar{\mu}_7\}$ it holds that $\|\Delta x_{\mathcal{A}} - \Delta x_{\mathcal{A}}^N\| = \mathcal{O}(\mu^2)$ with (7) respectively (13a). Proposition 2 also gives $\|\Delta\lambda_{\mathcal{I}} - \Delta\lambda_{\mathcal{I}}^N\| = \mathcal{O}(\mu^2)$, $0 < \mu \leq \bar{\mu}_7$, with (13b). The backward error with (25) is

$$\Delta x_{\mathcal{I}} - \Delta x_{\mathcal{I}}^N = -(H_{\mathcal{I}\mathcal{I}} + X_{\mathcal{I}\mathcal{I}}^{-1}\Lambda_{\mathcal{I}\mathcal{I}})^{-1} H_{\mathcal{I}\mathcal{A}} (\Delta x_{\mathcal{A}} - \Delta x_{\mathcal{A}}^N),$$

which gives

$$\begin{aligned} \|\Delta x_{\mathcal{I}} - \Delta x_{\mathcal{I}}^N\| & \leq \| (H_{\mathcal{I}\mathcal{I}} + X_{\mathcal{I}\mathcal{I}}^{-1}\Lambda_{\mathcal{I}\mathcal{I}})^{-1} \| \|H_{\mathcal{I}\mathcal{A}}\| \|\Delta x_{\mathcal{A}} - \Delta x_{\mathcal{A}}^N\| \\ & \leq \frac{1}{\sigma_{\min}(H_{\mathcal{I}\mathcal{I}} + X_{\mathcal{I}\mathcal{I}}^{-1}\Lambda_{\mathcal{I}\mathcal{I}})} \|H_{\mathcal{I}\mathcal{A}}\| \|\Delta x_{\mathcal{A}} - \Delta x_{\mathcal{A}}^N\|. \end{aligned}$$

Due to the assumption on f the elements of $H_{\mathcal{I}\mathcal{A}}$ are bounded. Moreover, the smallest singular value of $H_{\mathcal{I}\mathcal{I}} + X_{\mathcal{I}\mathcal{I}}^{-1}\Lambda_{\mathcal{I}\mathcal{I}}$ is bounded away from zero since the matrix is positive definite by Assumption 1. Hence it follows that $\|\Delta x_{\mathcal{I}} - \Delta x_{\mathcal{I}}^N\| = \mathcal{O}(\mu^2)$, $0 < \mu \leq \min\{\bar{\mu}_6, \bar{\mu}_7\}$. Note that $\Delta\lambda_{\mathcal{A}}^N$ is the solution to (29) with $\Delta x_{\mathcal{A}}^N$ and $\Delta x_{\mathcal{I}}^N$. Subtraction of (29), with $\Delta x_{\mathcal{A}}^N$ and $\Delta x_{\mathcal{I}}^N$, from (29) with the approximated solutions gives

$$\begin{aligned} (I_{\mathcal{A}\mathcal{A}} + X_{\mathcal{A}\mathcal{A}}^2) (\Delta\lambda_{\mathcal{A}} - \Delta\lambda_{\mathcal{A}}^N) & = (H_{\mathcal{A}\mathcal{A}} - X_{\mathcal{A}\mathcal{A}}\Lambda_{\mathcal{A}\mathcal{A}}) (\Delta x_{\mathcal{A}} - \Delta x_{\mathcal{A}}^N) \\ & \quad + H_{\mathcal{A}\mathcal{I}} (\Delta x_{\mathcal{I}} - \Delta x_{\mathcal{I}}^N). \end{aligned}$$

The the largest singular value of $(I_{\mathcal{A}\mathcal{A}} + X_{\mathcal{A}\mathcal{A}}^2)^{-1}$ is bounded by 1 and hence

$$\|\Delta\lambda_{\mathcal{A}} - \Delta\lambda_{\mathcal{A}}^N\| \leq (\|H_{\mathcal{A}\mathcal{A}}\| + \|X_{\mathcal{A}\mathcal{A}}\Lambda_{\mathcal{A}\mathcal{A}}\|) \|\Delta x_{\mathcal{A}} - \Delta x_{\mathcal{A}}^N\| + \|H_{\mathcal{A}\mathcal{I}}\| \|\Delta x_{\mathcal{I}} - \Delta x_{\mathcal{I}}^N\|.$$

The elements of $H_{\mathcal{A}\mathcal{A}}$ and $H_{\mathcal{A}\mathcal{I}}$ are bounded and by Lemma 4 it holds that $\|X_{\mathcal{A}\mathcal{A}}\Lambda_{\mathcal{A}\mathcal{A}}\| = \mathcal{O}(\mu)$, $0 < \mu \leq \max\{\bar{\mu}_6, \bar{\mu}_7\}$. Thus it follows that $\|\Delta\lambda_{\mathcal{A}} - \Delta\lambda_{\mathcal{A}}^N\| = \mathcal{O}(\mu^2)$, $0 < \mu \leq \min\{\bar{\mu}_6, \bar{\mu}_7\}$. Similarly, (28a) gives the backward error

$$\Delta\lambda_{\mathcal{A}} - \Delta\lambda_{\mathcal{A}}^N = H_{\mathcal{A}\mathcal{A}} (\Delta x_{\mathcal{A}} - \Delta x_{\mathcal{A}}^N) + H_{\mathcal{A}\mathcal{I}} (\Delta x_{\mathcal{I}} - \Delta x_{\mathcal{I}}^N).$$

Hence

$$\|\Delta\lambda_{\mathcal{A}} - \Delta\lambda_{\mathcal{A}}^N\| \leq \|H_{\mathcal{A}\mathcal{A}}\| \|\Delta x_{\mathcal{A}} - \Delta x_{\mathcal{A}}^N\| + \|H_{\mathcal{A}\mathcal{I}}\| \|\Delta x_{\mathcal{I}} - \Delta x_{\mathcal{I}}^N\|,$$

from which it follows that $\|\Delta\lambda_{\mathcal{A}} - \Delta\lambda_{\mathcal{A}}^N\| = \mathcal{O}(\mu^2)$, $0 < \mu \leq \min\{\bar{\mu}_6, \bar{\mu}_7\}$. Thus the result holds for $\bar{\mu} = \min\{\bar{\mu}_6, \bar{\mu}_7\}$. \square

The equations for the approximate solution in Theorem 1 show that it is essential to obtain a good approximate solution of $\Delta x_{\mathcal{A}}^N$. It is in the calculation of $\Delta x_{\mathcal{A}}$ with either (7) or (13a), and of $\Delta\lambda_{\mathcal{I}}$ with (13b), where information is discarded and hence introduce errors. This error then propagates through the suggested approximate solution. In general the active and inactive sets at the optimal solution are unknown and have to be estimated as the iterations proceed. There is a trade-off when estimating the set of active constraints. A restrictive strategy may lead to more accurate approximations of $\Delta x_{\mathcal{A}}^N$ however it increases the size of the system (25) that needs to be solved. It may also increase the size of some coefficients in the diagonal of the matrix of (25), or (42) in the general case, which may increase the condition number. A generous strategy on the other hand decreases the size of the system that has to be solved but may increase the error in the approximate solution $\Delta x_{\mathcal{A}}$, which then propagates through the rest of the solution.

To increase the comprehensibility of the work we have described the theoretical foundation for problems on the form (P). Analogous results for problems on the more general form (NLP) together with complementary remarks are given in Appendix A. Quantities associated with the approximate solutions which are related to Schur complement, i.e. those of Proposition 1 and in the general case Proposition 4, will henceforth be labeled with superscript "S". Similarly, quantities associated with the approximate solutions which are related to complementarity blocks, i.e. those of Proposition 2 and in the general case Proposition 5, will be labeled with "C".

4 Numerical results

As an initial numerical study we consider convex quadratic optimization problems with lower and upper bounds. In particular randomly generated problems

and a selection from the corresponding class in the CUTEst test collection [14]. The minimizers of the the randomly generated problems satisfy strict complementarity whereas the minimizers of the CUTEst problems typically do not. The benchmark problems were initially processed using the Julia packages `CUTEst.jl` and `NLPmodels.jl` by Orban and Siqueira [22].

The purpose of the first part of this section is to compare the proposed approximations of Theorem 2 and to give a rough indication of how the approximation errors develop for practical values of μ . A setting is considered where the vector (x, λ) , that satisfies $\|F_\mu(x, \lambda)\| < \mu$, is found by an interior-point method. Thereafter μ is decreased by a factor $\sigma = 0.1$ to $\mu^+ = \sigma\mu$ and the approximate solution of (2) is calculated. This procedure was then repeated for different values of μ . Mean errors with one standard deviation error bars for the proposed approximate solutions are shown in Figure 1. The equations that were used and how they are denoted in the figure are shown in Table 1. The figure also shows a measure of the mean improvement for two new iterates (x_+^S, λ_+^S) and (x_+^C, λ_+^C) , see Table 2, compared to the Newton iterate (x_+^N, λ_+^N) which is defined analogously. The results are for 10^2 randomly generated problems, with 10^3 variables, whose minimizers satisfy (30). For each problem, both the specific bounds as well as the specific active and inactive constraints were chosen by random. Moreover, the elements of the Hessian were uniformly distributed around zero with a sparsity level corresponding to approximately 40 percent non-zero elements. The conditions numbers were in the order of magnitude 10^7 - 10^{10} and the largest singular values in the order of magnitude of 10^3 .

Table 1 Equations for the approximations in Figure 1.

Quantity	$\Delta x_{\mathcal{A}}^S$	$\Delta x_{\mathcal{A}}^C$	$\Delta x_{\mathcal{I}}^{ls}$	$\Delta \lambda_{\mathcal{A}}^{ls}$	$\Delta \lambda_{\mathcal{A}}^b$	$\Delta \lambda_{\mathcal{I}}^C$	$\Delta \lambda_{\mathcal{I}}^{ls}$
Equation	(34)	(35)	(42)	(46)	(45)	(36)	(43)

As mentioned, the approximate solutions $\Delta x_{\mathcal{A}}^S$ respectively $\Delta x_{\mathcal{A}}^C$ and $\Delta \lambda_{\mathcal{I}}^C$ were given their superscripts since they were obtained from a Schur complement block respectively complementarity blocks. The approximation $\Delta \lambda_{\mathcal{A}}^b$ originates from one block and was given the superscript b . The remaining approximate solutions in Table 1 were obtained in the elimination procedure that ended up with a least-squares problem and were therefore given the superscript ls . Furthermore, $\Delta x_{\mathcal{A}}^S$ was used in the equations which require an initial approximation of the active components of Δx .

Table 2 Definitions of (x_+^S, λ_+^S) and (x_+^C, λ_+^C) . The step lengths α^P and α^D are chosen as in Algorithm 1.

$(x_+^S, \lambda_+^S) = (x + \alpha^P \Delta x, \lambda + \alpha^D \Delta \lambda)$				$(x_+^C, \lambda_+^C) = (x + \alpha^P \Delta x, \lambda + \alpha^D \Delta \lambda)$			
$\Delta x_{\mathcal{A}}$	$\Delta x_{\mathcal{I}}$	$\Delta \lambda_{\mathcal{A}}$	$\Delta \lambda_{\mathcal{I}}$	$\Delta x_{\mathcal{A}}$	$\Delta x_{\mathcal{I}}$	$\Delta \lambda_{\mathcal{A}}$	$\Delta \lambda_{\mathcal{I}}$
(34)	(42)	(46)	(43)	(35)	(42)	(46)	(43)

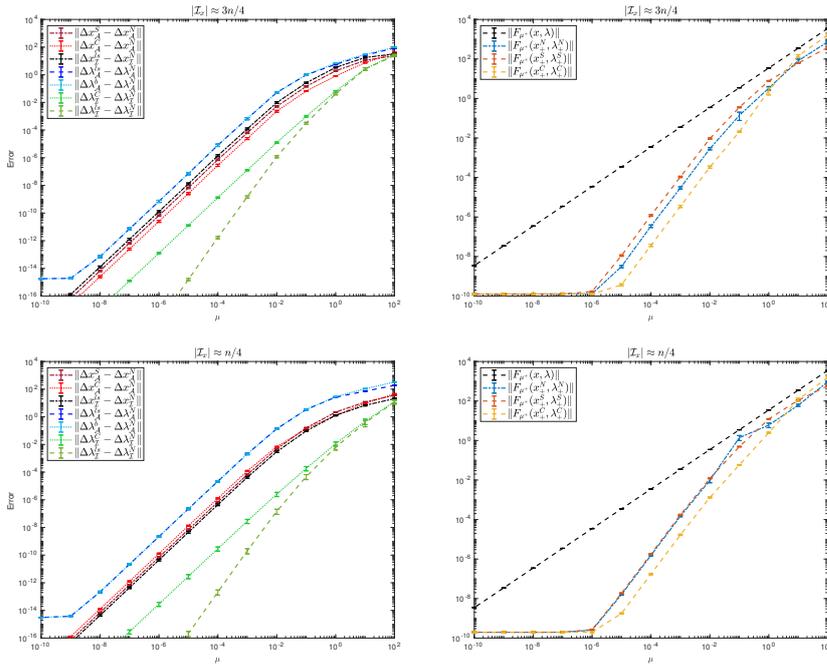


Fig. 1 Mean approximation error and mean progress with measure $\|F_{\mu+}(x, \lambda)\|$ with one standard deviation error bars for randomly generated quadratic problems. The top respectively the bottom correspond to problems where approximately 3/4 respectively 1/4 of the variables are inactive at the solution.

The least accurate approximate solutions in Figure 1 are those corresponding to active λ and inactive x . This is anticipated as their error bounds rely more heavily on the size of the elements of H . Moreover, it can be seen that $\Delta \lambda_{\mathcal{T}}^{ls}$ is favorable over $\Delta \lambda_{\mathcal{T}}^C$ for the problems considered. In general Figure 1 gives an indication of what equation that is favorable for each partial approximate solution if one is to be chosen. However, as mentioned, more sophisticated choices can be made by carefully considering the known quantities in the individual error terms for specific components. The right side of Figure 1 shows that the iterates (x_+^S, λ_+^S) and (x_+^C, λ_+^C) perform similar to (x_+^N, λ_+^N) in terms of the measure $\|F_{\mu+}(x, \lambda)\|$ for a wide range of μ . The error bars show that the results are not sensitive to changes in specific bounds, which of the constraints are active/inactive or different initial solutions. Numerical simulations have shown, as the theory also predicts, that the results can be improved or dis-improved by increasing respectively decreasing the size of the coefficients of the matrix H as well as its sparsity level.

Next we show results for a selection of problems in the CUTEst test collection in the analogous setting. The number of primal variables, n_x , in each considered problem is shown in Table 3. In the problems with variable options, n_x was typically chosen as approximately $5 \cdot 10^3$ resulting in a total number of variables in the order of 10^4 . Each problem was initially solved by an interior-

point method with stopping criterion $\|F_0(x, \lambda)\| < 10^{-14}$, i.e., the first-order optimality conditions given by (31) for $\mu = 0$. This was to determine the selection of problems as well as estimates of the active and inactive sets. Problems with an unconstrained optimal solution or an optimal solution with only degenerate active constraints were not considered. For the first case the proposed approximate solutions are equivalent to the true solution and for the second it is not clear how to deduce active/inactive sets. A constraint was considered as active if the corresponding variable was closer than 10^{-10} to its bound and degenerate if the corresponding multiplier value was below 10^{-6} . An exception was made for problem ODNAMUR, due to its larger size, for which the tolerances above were increased by a factor of $10^1 - 10^2$. Figure 2 shows mean errors with the approximate solutions of Table 1 on each CUTEst problem. The results are for three different values of μ with 10 different random initial solutions. The figure also shows the measure $\|F_{\mu+}(x, \lambda)\|$ for (x, λ) , (x_+^S, λ_+^S) , (x_+^C, λ_+^C) and (x_+^N, λ_+^N) . Simulations with the set estimation heuristic above have shown that the behavior of the approximate solution varies in three different regions depending on μ . These regions are approximately, $[10^2, 10^{-2})$, $[10^{-2}, 10^{-6}]$ and $(10^{-6}, 0)$. The μ -values in Figure 2 correspond to representative behavior in their respective region. The problems are ordered such that the fraction of estimated active constraints at the solution decreases from left to right.

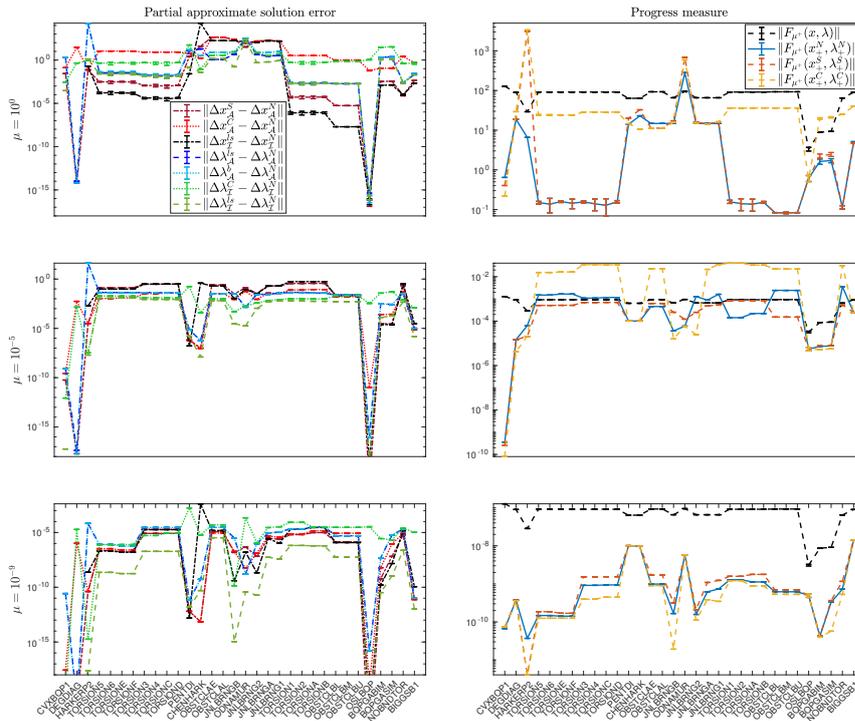


Fig. 2 Mean approximation error and mean progress with measure $\|F_{\mu+}(x, \lambda)\|$ with one standard deviation error bars for a collection of CUTEst test problems.

The partial approximate solution errors in Figure 2 are significantly larger compared to those of Figure 1. This is expected since the optimal solutions of the CUTEst test problems typically do not satisfy strict complementarity. Moreover, with the above strategy for determining the active and inactive sets, the smallest active multipliers may be in the order of 10^{-5} which is likely to cause inaccurate approximations of the components in $\Delta x_{\mathcal{A}}^N$. Nevertheless, the approximate solutions perform asymptotically similar to the Newton solution in terms of the measure $\|F_{\mu^+}(x, \lambda)\|$, as shown in Figure 2. The figure also shows that the approximation error and progress measure are not particularly sensitive to different initial solutions for smaller μ whereas some effects can be seen for larger μ . The results may be improved and dis-improved depending on how the estimation of the active constraints at the solution is made. We chose to give the results for the strategy described above which gives a potentially significant reduction in the computational iteration cost. In practice the active constraints at the optimal solution are unknown and have to be estimated as the iterations proceed. The purpose of the following simulations is to give an initial indication of the performance of the purposed approximate solutions within a primal-dual interior-point framework. Mainly on the behavior on problems that do not satisfy the assumptions for which the theoretical results are valid. But also on the robustness in regards to how the set of active constraints is estimated. Algorithm 1 and 2 were considered with the aim of not drowning or combining approximation effects with other effects from more advanced features of more sophisticated methods. Algorithm 1 should here be seen as the reference method as it only contains Newton steps.

Algorithm 1 Reference interior-point method for convex (NLP).

```

 $k \leftarrow 0, \quad \mu \leftarrow 10^2, \quad (x_k, \lambda_k) \leftarrow \text{Feasible point such that } \|F_{\mu}(x_k, \lambda_k)\| < \mu/\sigma.$ 
While  $\|F_0(x_k, \lambda_k)\| < \epsilon$  do
   $(\Delta x_k, \Delta \lambda_k) \leftarrow (2)$ 
   $(\alpha^P, \alpha^D) \leftarrow (\min\{1, 0.98\alpha_{max}^P\}, \min\{1, 0.98\alpha_{max}^D\})$ 
   $(x_{k+1}, \lambda_{k+1}) \leftarrow (x_k + \alpha^P \Delta x_k, \lambda_k + \alpha^D \Delta \lambda_k)$ 
  If  $\|F_{\mu}(x_{k+1}, \lambda_{k+1})\| < \mu$ 
     $\mu \leftarrow \sigma\mu$ 
  End
   $k \leftarrow k + 1$ 
End

```

Algorithm 2 Simple interior-point method for convex (NLP).

```

 $k \leftarrow 0, \quad \mu \leftarrow 10^2, \quad (x_k, \lambda_k) \leftarrow \text{Feasible point such that } \|F_{\mu}(x_k, \lambda_k)\| < \mu/\sigma.$ 
While  $\|F_0(x_k, \lambda_k)\| < \epsilon$  do
  Estimate active constraints to obtain active/inactive-sets
   $(\Delta x_k, \Delta \lambda_k) \leftarrow (34)$  or  $(35)$  combined with  $(42), (43)$  and  $(46)$ 
   $(\alpha^P, \alpha^D) \leftarrow (\min\{1, 0.98\alpha_{max}^P\}, \min\{1, 0.98\alpha_{max}^D\})$ 
   $(x_{k+1}, \lambda_{k+1}) \leftarrow (x_k + \alpha^P \Delta x_k, \lambda_k + \alpha^D \Delta \lambda_k)$ 
  If  $\|F_{\mu}(x_{k+1}, \lambda_{k+1})\| < \mu$ 
     $\mu \leftarrow \sigma\mu$ 
  End
   $k \leftarrow k + 1$ 
End

```

In Algorithm 1 and Algorithm 2, α_{max}^P and α_{max}^D are the maximum feasible step lengths with respect to x and λ variables respectively. Table 3 contains a comparison of Algorithm 1 and two versions of Algorithm 2 which differ in how $\Delta x_{\mathcal{A}}$ is computed. The versions are denoted by \mathbf{aN}^S respectively \mathbf{aN}^C and use the approximates $\Delta x_{\mathcal{A}}^S$ respectively $\Delta x_{\mathcal{A}}^C$. In Algorithm 2 a constraint was considered active if the distance to its bound was smaller than the value of the corresponding multiplier and a threshold $\tau_{\mathcal{A}}$. The procedure is thus a simple heuristic to attempt to determine the non-degenerate active constraints resulting in the set \mathcal{A}_x , compare to Definition 4 in the theoretical setting. The thresholds of the two versions \mathbf{aN}^S and \mathbf{aN}^C were chosen to $\tau_{\mathcal{A}} = \mu^{2/3}$ respectively the more restrictive $\tau_{\mathcal{A}} = \mu^{3/4}$. This was done to show the effects of two different thresholds $\tau_{\mathcal{A}}$ but also because numerical experiments have shown that steps with Schur-based approximate, see Figure 2, are more robust at larger μ . Table 3 compares the number of iterations for different μ as well as the average cardinality of the set of indices corresponding to inactive x , \mathcal{I}_x , i.e., the size of the systems that has to be solved in every iteration. The symbol "-" marks when the method failed to converge within 50 iterations for the corresponding μ . If this happened Newton steps was performed instead until $\|F_{\mu}(x, \lambda)\| < \mu$. The order of the problems is the same as in Figure 2.

Table 3 Comparison of Algorithm 1, (N), and two versions of Algorithm 2, (\mathbf{aN}^S and \mathbf{aN}^C) on a selection of CUTEst test problems.

	μ	10^1	10^0	10^{-2}	10^{-3}	10^{-5}	10^{-6}	10^{-8}	10^{-9}	10^{-10}
CVXBP1 $n_x=10000$	N	3	3	2	1	1	1	1	1	1
	\mathbf{aN}^S	3	2	2	1	1	1	1	1	1
	\mathbf{aN}^C	4	2	1	1	1	1	1	1	1
	$ \mathcal{I}_x^S $	0	0	0	0	0	0	0	0	0
	$ \mathcal{I}_x^C $	0	0	0	0	0	0	0	0	0
DEGDIAG $n_x=10001$	N	4	4	3	3	3	3	2	2	2
	\mathbf{aN}^S	4	4	3	3	3	3	2	2	2
	\mathbf{aN}^C	12	7	42	5	3	3	2	2	2
	$ \mathcal{I}_x^S $	1	1	830	635	195	93	28	13	6
	$ \mathcal{I}_x^C $	1	142	93	709	428	237	100	56	32
HARKER2 $n_x=1000$	N	3	3	4	3	3	3	2	2	1
	\mathbf{aN}^S	-	-	-	-	15	6	2	1	1
	\mathbf{aN}^C	-	-	-	2	15	6	2	1	1
	$ \mathcal{I}_x^S $	-	-	-	-	1	1	1	1	1
	$ \mathcal{I}_x^C $	-	-	-	1	1	1	1	1	1
TORSION5* $n_x=5184$	N	1	1	2	3	3	3	3	2	2
	\mathbf{aN}^S	1	1	2	3	3	4	3	2	2
	\mathbf{aN}^C	29	-	-	-	3	3	3	2	2
	$ \mathcal{I}_x^S $	0	0	2564	4535	5083	2802	2277	968	960
	$ \mathcal{I}_x^C $	0	-	-	-	5101	5064	2376	2944	2936
TORSION6* $n_x=5184$	N	1	1	2	3	3	3	3	2	2
	\mathbf{aN}^S	1	1	2	3	3	4	3	2	2
	\mathbf{aN}^C	29	-	-	-	3	3	3	2	2
	$ \mathcal{I}_x^S $	0	0	2564	4535	5171	2872	2379	984	976
	$ \mathcal{I}_x^C $	0	-	-	-	5184	5171	2387	3080	3080

Table 3 continued:

	μ	10^1	10^0	10^{-2}	10^{-3}	10^{-5}	10^{-6}	10^{-8}	10^{-9}	10^{-10}
TORSION3* $n_x=5184$	N	1	1	2	2	4	3	3	3	3
	aN ^S	1	1	2	3	4	3	3	3	3
	aN ^C	29	-	-	-	4	3	3	3	3
	$ \bar{I}_x^S $	0	0	2564	4535	5062	4933	2931	2867	1872
	$ \bar{I}_x^C $	0	-	-	-	5184	5069	4008	3035	2931
TORSIONC* $n_x=5184$	N	1	1	2	2	3	3	3	3	3
	aN ^S	1	1	2	3	3	3	3	3	3
	aN ^C	29	-	-	-	3	3	3	3	3
	$ \bar{I}_x^S $	0	0	2564	4535	5184	5104	3043	2976	1907
	$ \bar{I}_x^C $	0	-	-	-	5184	5179	4109	3059	3040
PENTDI $n_x=5000$	N	4	4	4	4	4	4	4	4	4
	aN ^S	6	7	7	7	5	4	4	4	4
	aN ^C	-	-	-	-	4	4	4	4	4
	$ \bar{I}_x^S $	0	0	2	2	1000	1873	2498	2498	2498
	$ \bar{I}_x^C $	-	-	-	-	2498	2498	2498	2498	2498
CHENHARK $n_x=5000$	N	4	4	4	4	4	4	4	3	3
	aN ^S	4	4	4	4	4	4	4	3	3
	aN ^C	4	-	4	4	4	4	4	4	3
	$ \bar{I}_x^S $	4999	4567	2502	2502	2502	2502	2502	2502	2502
	$ \bar{I}_x^C $	4999	-	2502	2502	2502	2502	2502	2502	2502
JNLBRNGB $n_x=5329$	N	4	4	4	3	3	3	3	2	2
	aN ^S	6	15	16	19	20	9	3	2	2
	aN ^C	-	-	-	-	3	3	3	2	2
	$ \bar{I}_x^S $	5196	5171	5157	5190	3220	3291	3111	3063	3026
	$ \bar{I}_x^C $	-	-	-	-	4899	4182	3843	3758	3283
OBSTCLAB* $n_x=5329$	N	4	4	4	4	3	3	3	3	2
	aN ^S	4	4	4	4	5	4	3	3	2
	aN ^C	4	4	4	4	3	3	3	3	2
	$ \bar{I}_x^S $	5329	5329	5329	5329	5063	4153	3766	3268	2978
	$ \bar{I}_x^C $	5329	5329	5329	5329	5290	5313	4539	3786	4158
JNLBRNG2 $n_x=5329$	N	4	4	4	3	3	3	3	2	2
	aN ^S	5	9	11	9	11	6	3	2	2
	aN ^C	-	-	-	-	3	3	3	2	2
	$ \bar{I}_x^S $	5206	5177	5187	5217	3561	3622	3315	3270	3232
	$ \bar{I}_x^C $	-	-	-	-	4736	4297	3981	3861	3450
OBSTCLBL* $n_x=5329$	N	1	1	2	3	3	3	3	3	2
	aN ^S	1	1	3	6	4	3	3	3	2
	aN ^C	24	-	-	-	3	3	3	3	2
	$ \bar{I}_x^S $	0	0	1121	3190	4469	4318	3950	3903	3880
	$ \bar{I}_x^C $	0	-	-	-	4862	4515	4212	4077	4002
JNLBRNGA $n_x=5329$	N	4	4	4	4	3	3	3	3	2
	aN ^S	4	4	4	4	3	3	3	3	2
	aN ^C	4	4	4	4	3	3	3	3	2
	$ \bar{I}_x^S $	5329	5329	5329	5329	5329	4991	4257	3985	3718
	$ \bar{I}_x^C $	5329	5329	5329	5329	5329	5279	4728	4381	4506
TORSION1* $n_x=5184$	N	1	1	1	2	4	3	3	3	3
	aN ^S	1	1	1	2	4	3	3	3	3
	aN ^C	29	-	-	-	4	3	3	3	3
	$ \bar{I}_x^S $	0	0	1764	4490	5098	5032	4133	4040	4024
	$ \bar{I}_x^C $	0	-	-	-	5184	5061	5011	4968	4080
JNLBRNG1 $n_x=5329$	N	4	4	4	4	3	3	3	3	2
	aN ^S	4	5	6	5	3	4	3	3	2
	aN ^C	10	-	39	-	3	3	3	3	2
	$ \bar{I}_x^S $	5322	5319	5319	5320	5329	4972	4331	4028	3800
	$ \bar{I}_x^C $	5312	-	5319	-	5329	5283	4993	4446	4536

Table 3 continued:

	μ	10^1	10^0	10^{-2}	10^{-3}	10^{-5}	10^{-6}	10^{-8}	10^{-9}	10^{-10}
TORSION* $n_x=5184$	N	1	1	1	2	4	3	3	3	2
	aN ^S	1	1	1	2	4	3	3	3	2
	aN ^C	29	-	-	-	4	3	3	3	2
	$ \bar{I}_x^S $	0	0	1764	4490	5184	5184	4261	4173	4416
	$ \bar{I}_x^C $	0	-	-	-	5184	5184	5168	5125	4444
OSLBQP $n_x=8$	N	2	3	3	2	2	3	3	2	3
	aN ^S	2	3	3	2	2	3	3	2	3
	aN ^C	4	5	2	3	3	2	2	3	2
	$ \bar{I}_x^S $	4	2	4	6	6	6	6	6	6
	$ \bar{I}_x^C $	1	2	6	6	6	6	6	6	6
BQPGABIM $n_x=46$	N	1	2	3	3	3	3	2	2	1
	aN ^S	1	3	9	7	3	3	2	2	1
	aN ^C	-	-	-	-	-	-	2	2	1
	$ \bar{I}_x^S $	0	0	4	24	29	31	37	36	36
	$ \bar{I}_x^C $	-	-	-	-	-	-	39	38	38
BQPGASIM $n_x=50$	N	1	2	3	3	3	3	2	2	2
	aN ^S	1	3	10	7	3	3	2	2	2
	aN ^C	-	-	-	-	-	-	2	2	2
	$ \bar{I}_x^S $	0	0	4	27	32	34	41	42	42
	$ \bar{I}_x^C $	-	-	-	-	-	-	43	42	42
NOBNDTOR $n_x=5184$	N	1	1	2	2	4	4	3	3	2
	aN ^S	1	1	2	2	4	4	3	3	2
	aN ^C	21	-	-	-	4	4	3	3	2
	$ \bar{I}_x^S $	2592	2592	3874	4837	5132	5107	4621	4570	4681
	$ \bar{I}_x^C $	2592	-	-	-	5183	5148	5077	5050	4704
BIGGSB1 $n_x=5000$	N	1	1	2	3	3	3	4	4	4
	aN ^S	1	1	4	3	3	3	4	4	4
	aN ^C	11	13	2	3	3	3	4	4	4
	$ \bar{I}_x^S $	1	1	4999	4999	4998	4998	4998	4998	4998
	$ \bar{I}_x^C $	1	1	5000	4999	4998	4998	4998	4998	4998

The results in Table 3 display similar characteristics as the results in Figure 2. The version associated with the Schur-based approximate solution, aN^S of Algorithm 2, makes sufficient progress at $\mu \in [10^2, 10^{-2})$, often at a relatively low computational cost. aN^S Converges at $\mu \in [10^{-2}, 10^{-6}]$ however often while solving relatively large systems due to the difficulty of estimating \mathcal{A}_x . At $\mu \in (10^{-6}, 0)$ the asymptotic behavior becomes more pronounced and aN^S does similar in terms of iteration count to Algorithm 1 while solving systems of reduced size. Version aN^S converges at all considered μ in all problems of Table 3, except on HARKKERP2 for larger μ . The version associated with the complementarity-based approximate solution, aN^C of Algorithm 2, tend to perform poorly overall for $\mu \in [10^2, 10^{-2})$ and parts of $[10^{-2}, 10^{-6}]$, although it manages to converge for large μ this is often at the expense of either solving relatively large systems or performing many iterations. In general, aN^C performs similar to Algorithm 1 for μ in the approximate region $[10^{-5}, 0)$ while

*The tables would be identical for other versions of the same problem and are therefore omitted

solving systems of reduced size. The versions \mathbf{aN}^S and \mathbf{aN}^C have similar asymptotic performance, however in general, \mathbf{aN}^S performs better for larger values of μ , as also indicated by previous results in Figure 2.

Finally we show results for the two modified-Newton approaches, mentioned in Section 3.1, in a simple primal-dual interior-point setting. Both methods contain the approximate intermediate iterate $(x^E, \lambda^E) = (x + \alpha^P \Delta x^E, \lambda + \alpha^D \Delta \lambda^E)$, where $(\Delta x^E, \Delta \lambda^E)$ is defined by (17) and α^P, α^D are chosen as in Algorithm 1. Figure 3 shows the total number of iterations required at different intervals of μ for the two modified-Newton methods described in Algorithm 3. The figure shows results for three different choices of $(\Delta x^E, \Delta \lambda^E)$. Moreover, the selection of which components to update was done as the iterations proceeded similarly as above. Note however that it is not necessary to label each constraint and each component of λ as active respectively inactive in this case, some may be defined as neither. The set of indices corresponding to active constraints, \mathcal{A}_x , was estimated as above and the sets of indices corresponding to inactive λ , \mathcal{I}_l and \mathcal{I}_u , see Definition 4, were estimated analogously. I.e. a multiplier was considered inactive if its value was smaller than the distance of the corresponding x to its feasibility bound and a threshold $\tau_{\mathcal{I}}$. Table 4 shows how the nonzero components of $(\Delta x^E, \Delta \lambda^E)$ were chosen in the different versions of the methods as well as the different thresholds $\tau_{\mathcal{A}}$ and $\tau_{\mathcal{I}}$.

Table 4 Thresholds and nonzero components of the steps in (x^E, λ^E) for the three versions compared in Figure 3.

Nonzero components in $(\Delta x^E, \Delta \lambda^E)$	$\tau_{\mathcal{A}}$	$\tau_{\mathcal{I}}$
$\Delta x_{\mathcal{A}}^S$	$\mu^{1/2}$	
$\Delta x_{\mathcal{A}}^S, \Delta \lambda_{\mathcal{I}}^C$	$\mu^{1/2}$	$\mu^{3/4}$
$\Delta x_{\mathcal{A}}^C, \Delta \lambda_{\mathcal{I}}^C$	$\mu^{3/4}$	$\mu^{3/4}$

Algorithm 3 Simple interior-point method with modified-Newton solve for convex (NLP).

```

k ← 0, μ ← 101, (xk, λk) ← Feasible point such that ||Fμ(xk, λk)|| < μ/σ.
While ||F0(xk, λk)|| < ε do
  Estimate  $\mathcal{A}$  and  $\mathcal{I}$ 
  (ΔxE, ΔλE) ← (38)
  (Δxk, Δλk) ← (21) or (22)
  (αP, αD) ← (min{1, 0.98αPmax}, min{1, 0.98αDmax})
  (xk+1, λk+1) ← (xk + αPΔxk, λk + αDΔλk)
  If ||Fμ(xk+1, λk+1)|| < μ
    μ ← σμ
  End
  k ← k + 1
End

```

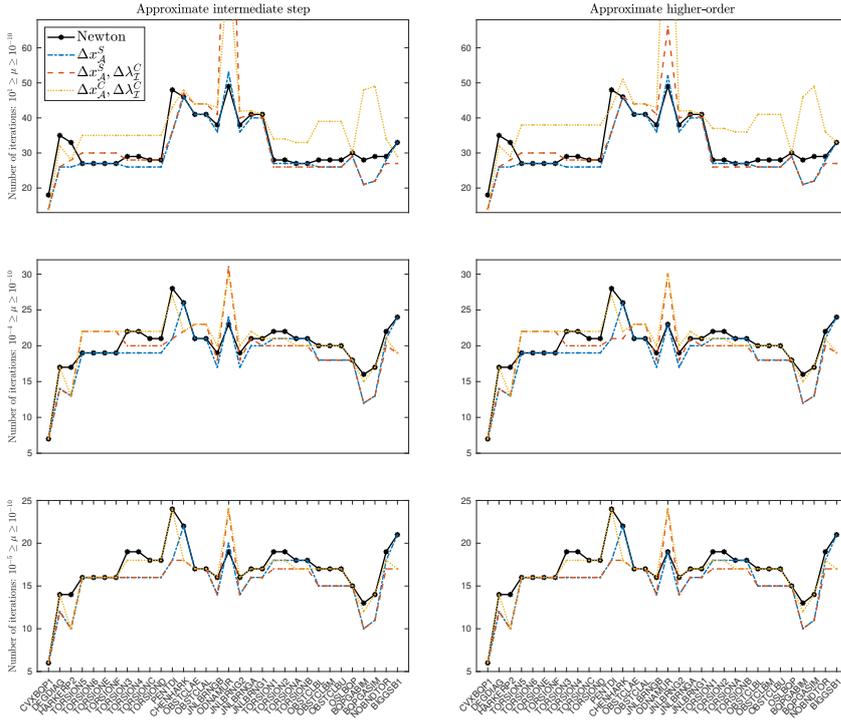


Fig. 3 Number of iterations required at different intervals of μ for three versions of the modified-Newton methods.

The total iteration count for $\mu \in [10^1, 10^{-10}]$ in Figure 3 shows that the approximate higher order-method requires the same or fewer iterations compared to the method with the approximate intermediate step. The iteration count for the approximate higher-order method is similar to that of Algorithm 1 for this range. Also here, numerical experiments show indications of three regions. For μ in the approximate region $[10^2, 10^{-2})$ the versions with the Schur-based approximate yield potentially reduced number of iterations. In the region of intermediate sized μ the performance varies however it can not be discarded that this is an effect of the relatively simple set estimation heuristic. On all problems, with the exception of ODNAMUR, in Figure 3 for $\mu \in [10^{-5}, 10^{-10}]$ all versions of both methods give an iteration count less or equal to Algorithm 1, hence providing potential savings in computation cost. The results may be improved with a flexible set estimation heuristics, e.g. more restrictive thresholds for intermediate sized μ . However, we chose not to include another layer of detail and instead give the results for a relatively simple setting to obtain an initial evaluation of the potential performance.

5 Conclusions

In this work we have given approximate solutions to systems of linear equations that arise in interior point methods for bound-constrained optimization. In particular partial approximate solutions, where the asymptotic component-error bounds are in the order of μ^2 , and full approximate solutions with asymptotic error bounds in the order of μ^2 . Numerical simulations on randomly generated bound-constrained convex quadratic optimization problems, whose minimizers satisfy strict complementarity, have shown that the approximate solutions does similar to Newton solutions for sufficiently small μ . Simulations on convex bound-constrained quadratic problems from the CUTEst test collection, whose minimizers typically do not satisfy strict complementarity, has shown that the predicted asymptotic behavior still occurs, however at significantly smaller values of μ .

We have performed numerical simulations in a simple yet more realistic setting, namely a primal-dual interior point framework where the active and inactive sets were estimated with a simple heuristic as the iterations proceeded. These simulations were done on a selection of CUTEst benchmark problems and showed that the behavior roughly varied with three regions determined by the size of μ . The Schur-based approximate solutions showed potential in the region for larger value μ , in the region of intermediate sized μ the performance varied, partly due to difficulties in determining the active and inactive sets. For sufficiently small μ the approximate solutions show performance similar to our reference method while solving systems of reduced size.

Finally we showed numerical results on the considered CUTEst benchmark for two modified-Newton approaches which include an approximate intermediate step consisting of partial approximate solutions. The simulations showed similar characteristics as the previous results and also a potential for reducing the overall iteration count of interior-point methods.

The results of this work are meant to contribute to the theoretical and numerical understanding for approximate solutions to linear systems of equations that arise in interior-point methods. We hope that the work can lead to further research on approximate solutions and approximate higher-order methods for optimization problems with linear inequality constraints.

A General case

Consider problems on the form of (NLP). We are interested in the asymptotic behavior of primal-dual interior-point methods in the vicinity of a local minimizer x^* and its corresponding multipliers $\lambda^* = \begin{pmatrix} \lambda^{l*} \\ \lambda^{u*} \end{pmatrix}$. In particular as the iterates of the method converge to a

vector (x^*, λ^*) that satisfies

$$\nabla f(x^*) - \lambda^{l*} + \lambda^{u*} = 0, \quad (30a)$$

$$x^* - l \geq 0, \quad u - x^* \geq 0, \quad (30b)$$

$$\lambda^{l*} \geq 0, \quad \lambda^{u*} \geq 0, \quad (30c)$$

$$(x^* - l) \cdot \lambda^{l*} = 0, \quad (u - x^*) \cdot \lambda^{u*} = 0, \quad (30d)$$

$$Z(x^*)^T \nabla^2 f(x^*) Z(x^*) \succ 0. \quad (30e)$$

$$x + \lambda^{l*} > 0, \quad x + \lambda^{u*} > 0. \quad (30f)$$

Similarly as in Section 2, define the function $F_\mu : \mathbb{R}^{3n} \rightarrow \mathbb{R}^{3n}$ by

$$F_\mu(x, \lambda) = \begin{bmatrix} \nabla f(x) - \lambda^l + \lambda^u \\ \Lambda^l (X - L)e - \mu e \\ \Lambda^u (U - X)e - \mu e \end{bmatrix}, \quad (31)$$

where $L = \text{diag}(l)$, $U = \text{diag}(u)$, $\Lambda^l = \text{diag}(\lambda^l)$ and $\Lambda^u = \text{diag}(\lambda^u)$. The corresponding Jacobian $F' : \mathbb{R}^{3n} \rightarrow \mathbb{R}^{3n}$ is

$$F'(x, \lambda) = \begin{bmatrix} H & -I & I \\ \Lambda^l & (X - L) & \\ -\Lambda^u & & (U - X) \end{bmatrix}. \quad (32)$$

For the case with upper and lower bounds it is useful to distinguish whether a specific component of x^* is active with respect to an upper or a lower bound.

Definition 4 (Active/inactive sets). For given $x^* \geq 0$ define the sets

$$\begin{aligned} \mathcal{A}_l &= \{i \in \{1, \dots, n\} : x_i^* - l_i = 0\}, & \mathcal{I}_l &= \{1, \dots, n\} \setminus \mathcal{A}_l, \\ \mathcal{A}_u &= \{i \in \{1, \dots, n\} : u_i - x_i^* = 0\}, & \mathcal{I}_u &= \{1, \dots, n\} \setminus \mathcal{A}_u, \\ \mathcal{A}_x &= \mathcal{A}_l \cup \mathcal{A}_u, & \mathcal{I}_x &= \{1, \dots, n\} \setminus \mathcal{A}_x. \end{aligned}$$

Bounds on individual components of the solution (x, λ) in the region of asymptotical behavior is given the lemma below.

Lemma 6 Under Assumption 1 let $\mathcal{B}((x^*, \lambda^*), \delta)$ and $\hat{\mu}$ be defined by Lemma 1 respectively Lemma 2. Then there exists $\bar{\mu}$, with $0 < \bar{\mu} \leq \hat{\mu}$, such that for $0 < \mu \leq \bar{\mu}$ and (x, λ) sufficiently close to $(x^\mu, \lambda^\mu) \in \mathcal{B}((x^*, \lambda^*), \delta)$ so that $\|F_\mu(x, \lambda)\| = \mathcal{O}(\mu)$ it holds that

$$\begin{aligned} x_i - l_i &= \begin{cases} \mathcal{O}(\mu) & i \in \mathcal{A}_l, \\ \Theta(1) & i \in \mathcal{I}_l, \end{cases} & \lambda_i^l &= \begin{cases} \Theta(1) & i \in \mathcal{A}_l, \\ \mathcal{O}(\mu) & i \in \mathcal{I}_l, \end{cases} \\ u_i - x_i &= \begin{cases} \mathcal{O}(\mu) & i \in \mathcal{A}_u, \\ \Theta(1) & i \in \mathcal{I}_u, \end{cases} & \lambda_i^u &= \begin{cases} \Theta(1) & i \in \mathcal{A}_u, \\ \mathcal{O}(\mu) & i \in \mathcal{I}_u. \end{cases} \end{aligned}$$

A.1 Partial approximate solutions

In this section we give results analogous to those given in Section 3.1 together with some complementary remarks. With $F'(x, \lambda)$ and $F_\mu(x, \lambda)$ defined as in (32) respectively (31) the Schur complement of X in (2) is

$$\begin{aligned} \left(H + (X - L)^{-1} \Lambda^l + (U - X)^{-1} \Lambda^u \right) \Delta x^N &= -\nabla f(x) \\ &+ \mu [(X - L)^{-1} - (U - X)^{-1}] e. \end{aligned} \quad (33)$$

For $i \in \mathcal{A}_x$ either $(u_i - x_i) \rightarrow 0$ or $(l_i - x_i) \rightarrow 0$ as $\mu \rightarrow 0$. In consequence, approximates of Δx_i^N , $i \in \mathcal{A}_x$, can be obtained from the Schur complement (33). These approximate solutions are given below in Proposition 4 which is the result analogous to Proposition 1.

Proposition 4 Under Assumption 1 let $\mathcal{B}((x^*, \lambda^*), \delta)$ and $\hat{\mu}$ be defined by Lemma 1 and Lemma 2 respectively. For $(x, \lambda) \in \mathcal{B}((x^*, \lambda^*), \delta)$, let $(\Delta x^N, \Delta \lambda^N)$ be the solution of (2) with $\mu^+ = \sigma\mu$, where $0 < \sigma < 1$. If the search direction components are defined as

$$\Delta x_i = \frac{-1}{[\nabla^2 f(x)]_{ii} + \frac{\lambda_i^l}{x_i - l_i} + \frac{\lambda_i^u}{u_i - x_i}} \left([\nabla f(x)]_i - \mu^+ \left[\frac{1}{x_i - l_i} - \frac{1}{u_i - x_i} \right] \right), \quad (34)$$

for $i = 1, \dots, n$, then

$$\Delta x_i - \Delta x_i^N = \frac{1}{[\nabla^2 f(x)]_{ii} + \frac{\lambda_i^l}{x_i - l_i} + \frac{\lambda_i^u}{u_i - x_i}} \sum_{i \neq j} [\nabla^2 f(x)]_{ij} \Delta x_j^N, \quad i = 1, \dots, n.$$

If in addition, $0 < \mu \leq \hat{\mu}$ and (x, λ) is sufficiently close to $(x^\mu, \lambda^\mu) \in \mathcal{B}((x^*, \lambda^*), \delta)$ such that $\|F_\mu(x, \lambda)\| = \mathcal{O}(\mu)$. Then there exists $\bar{\mu}$, with $0 < \bar{\mu} \leq \hat{\mu}$, such that for $0 < \mu \leq \bar{\mu}$ it holds that

$$\frac{1}{[\nabla^2 f(x)]_{ii} + \frac{\lambda_i^l}{x_i - l_i} + \frac{\lambda_i^u}{u_i - x_i}} = \begin{cases} \mathcal{O}(\mu) & i \in \mathcal{A}_x, \\ \Theta(1) & i \in \mathcal{I}_x, \end{cases}$$

and

$$|\Delta x_i - \Delta x_i^N| = \mathcal{O}(\mu^2), \quad i \in \mathcal{A}_x.$$

Next we give results corresponding to those in Proposition 2. As $\mu \rightarrow 0$ then $\lambda_i^l \rightarrow 0$ for $i \in \mathcal{I}_l$ and $\lambda_i^u \rightarrow 0$ for $i \in \mathcal{I}_u$. Consequently, approximations based on the complementarity blocks of $F'(x, \lambda)(\Delta x^N, \Delta \lambda^N) = -F_\mu(x, \lambda)$ can be formed for $\Delta x_i^N, i \in \mathcal{A}_x, \Delta \lambda_i^{Nl}, i \in \mathcal{I}_l$ and $\Delta \lambda_i^{Nu}, i \in \mathcal{I}_u$.

Proposition 5 Under Assumption 1 let $\mathcal{B}((x^*, \lambda^*), \delta)$ and $\hat{\mu}$ be defined by Lemma 1 and Lemma 2 respectively. For $(x, \lambda) \in \mathcal{B}((x^*, \lambda^*), \delta)$, let $(\Delta x^N, \Delta \lambda^N)$ be the solution of (2) with $\mu^+ = \sigma\mu$, where $0 < \sigma < 1$. If the search direction components are defined as

$$\Delta x_i = -(x_i - l_i) + \frac{\mu^+}{\lambda_i^l}, \quad i \in \mathcal{A}_l, \quad (35a)$$

$$\Delta x_i = (u_i - x_i) - \frac{\mu^+}{\lambda_i^u}, \quad i \in \mathcal{A}_u, \quad (35b)$$

$$\Delta \lambda_i^l = -\lambda_i^l + \frac{\mu^+}{x_i - l_i}, \quad i \in \mathcal{I}_l, \quad (36a)$$

$$\Delta \lambda_i^u = -\lambda_i^u + \frac{\mu^+}{u_i - x_i}, \quad i \in \mathcal{I}_u, \quad (36b)$$

then

$$\Delta x_i - \Delta x_i^N = \frac{x_i - l_i}{\lambda_i^l} \Delta \lambda_i^{lN}, \quad i \in \mathcal{A}_l,$$

$$\Delta x_i - \Delta x_i^N = \frac{u_i - x_i}{\lambda_i^u} \Delta \lambda_i^{uN}, \quad i \in \mathcal{A}_u,$$

$$\Delta \lambda_i^l - \Delta \lambda_i^{lN} = \frac{\lambda_i^l}{x_i - l_i} \Delta x_i^N, \quad i \in \mathcal{I}_l,$$

$$\Delta \lambda_i^u - \Delta \lambda_i^{uN} = -\frac{\lambda_i^u}{u_i - x_i} \Delta x_i^N, \quad i \in \mathcal{I}_u.$$

If in addition, $0 < \mu \leq \hat{\mu}$ and (x, λ) is sufficiently close to $(x^\mu, \lambda^\mu) \in \mathcal{B}((x^*, \lambda^*), \delta)$ such that $\|F_\mu(x, \lambda)\| = \mathcal{O}(\mu)$. Then there exists $\bar{\mu}$, with $0 < \bar{\mu} \leq \hat{\mu}$, such that for $0 < \mu \leq \bar{\mu}$ it holds that

$$\begin{aligned} |\Delta x_i - \Delta x_i^N| &= \mathcal{O}(\mu^2), & i \in \mathcal{A}_x, \\ |\Delta \lambda_i^l - \Delta \lambda_i^{lN}| &= \mathcal{O}(\mu^2), & i \in \mathcal{I}_l, \\ |\Delta \lambda_i^u - \Delta \lambda_i^{uN}| &= \mathcal{O}(\mu^2), & i \in \mathcal{I}_u. \end{aligned}$$

Finally we give the general result for the approximate intermediate step, i.e. for the case with lower and upper bounds.

Proposition 6 Under Assumption 1 let $\mathcal{B}((x^*, \lambda^*), \delta)$ and $\hat{\mu}$ be defined by Lemma 1 and Lemma 2 respectively. For $(x, \lambda) \in \mathcal{B}((x^*, \lambda^*), \delta)$, define $(x_+^N, \lambda_+^N) = (x, \lambda) + (\Delta x^N, \Delta \lambda^N)$ where $(\Delta x^N, \Delta \lambda^N)$ is the solution of (2) with $\mu^+ = \sigma\mu$, where $0 < \sigma < 1$. Moreover, let $(x_+, \lambda_+) = (x, \lambda) + (\Delta x, \Delta \lambda)$ where

$$\Delta x_i = \begin{cases} (35a) \text{ or } (34) & i \in \mathcal{A}_l, \\ (35b) \text{ or } (34) & i \in \mathcal{A}_l, \\ 0 & i \in \mathcal{I}_x, \end{cases} \quad \Delta \lambda_i = \begin{cases} 0 & i \in \mathcal{A}_x, \\ (36a) & i \in \mathcal{I}_l, \\ (36b) & i \in \mathcal{I}_u. \end{cases} \quad (38)$$

If $0 < \mu \leq \bar{\mu}$ and (x, λ) is sufficiently close to $(x^\mu, \lambda^\mu) \in \mathcal{B}((x^*, \lambda^*), \delta)$ such that $\|F_\mu(x, \lambda)\| = \mathcal{O}(\mu)$ and $\|(\Delta x_{\mathcal{A}_l}^N, \Delta \lambda_{\mathcal{I}_l}^N)\| = \Omega(\mu^\gamma)$ for $\gamma < 2$. Then there exists $\bar{\mu}$, with $0 < \bar{\mu} \leq \hat{\mu}$, such that for $0 < \mu \leq \bar{\mu}$ it holds that

$$\|(x_+^N, \lambda_+^N) - (x_+, \lambda_+)\| \leq \|(x_+^N, \lambda_+^N) - (x, \lambda)\|.$$

A.2 Full approximate solutions

In this section we give results analogous to those given in Section 3.2 together with some complementary remarks. Note that $\mathcal{I}_x \cap \mathcal{A}_l = \emptyset$ and $\mathcal{I}_x \cap \mathcal{A}_u = \emptyset$. By partitioning $(\Delta x^N, \Delta \lambda^N) = (\Delta x_{\mathcal{A}_x}^N, \Delta x_{\mathcal{I}_x}^N, \Delta \lambda_{\mathcal{A}_l}^{Nl}, \Delta \lambda_{\mathcal{I}_l}^{Nl}, \Delta \lambda_{\mathcal{A}_u}^{Nu}, \Delta \lambda_{\mathcal{I}_u}^{Nu})$, (2) can be written as

$$\begin{aligned} & \begin{bmatrix} H_{\mathcal{A}_x \mathcal{A}_x} & H_{\mathcal{A}_x \mathcal{I}_x} & -I_{\mathcal{A}_x \mathcal{A}_l} & -I_{\mathcal{A}_x \mathcal{I}_l} & I_{\mathcal{A}_x \mathcal{A}_u} & I_{\mathcal{A}_x \mathcal{I}_u} \\ H_{\mathcal{I}_x \mathcal{A}_x} & H_{\mathcal{I}_x \mathcal{I}_x} & & -I_{\mathcal{I}_x \mathcal{I}_l} & & I_{\mathcal{I}_x \mathcal{I}_u} \\ \Lambda_{\mathcal{A}_l \mathcal{A}_x}^l & & (X-L)_{\mathcal{A}_l \mathcal{A}_l} & & & \\ \Lambda_{\mathcal{I}_l \mathcal{A}_x}^l & \Lambda_{\mathcal{I}_l \mathcal{I}_x}^l & & (X-L)_{\mathcal{I}_l \mathcal{I}_l} & & \\ -\Lambda_{\mathcal{A}_u \mathcal{A}_x}^u & & & & (U-X)_{\mathcal{A}_u \mathcal{A}_u} & \\ -\Lambda_{\mathcal{I}_u \mathcal{A}_x}^u & -\Lambda_{\mathcal{I}_u \mathcal{I}_x}^u & & & & (U-X)_{\mathcal{I}_u \mathcal{I}_u} \end{bmatrix} \begin{bmatrix} \Delta x_{\mathcal{A}_x}^N \\ \Delta x_{\mathcal{I}_x}^N \\ \Delta \lambda_{\mathcal{A}_l}^{Nl} \\ \Delta \lambda_{\mathcal{I}_l}^{Nl} \\ \Delta \lambda_{\mathcal{A}_u}^{Nu} \\ \Delta \lambda_{\mathcal{I}_u}^{Nu} \end{bmatrix} \\ & = - \begin{bmatrix} \nabla f(x)_{\mathcal{A}_x} - \lambda_{\mathcal{A}_x}^l + \lambda_{\mathcal{A}_x}^u \\ \nabla f(x)_{\mathcal{I}_x} - \lambda_{\mathcal{I}_x}^l + \lambda_{\mathcal{I}_x}^u \\ \Lambda_{\mathcal{A}_l \mathcal{A}_l}^l (X-L)_{\mathcal{A}_l \mathcal{A}_l} e - \mu e \\ \Lambda_{\mathcal{I}_l \mathcal{I}_l}^l (X-L)_{\mathcal{I}_l \mathcal{I}_l} e - \mu e \\ \Lambda_{\mathcal{A}_u \mathcal{A}_u}^u (U-X)_{\mathcal{A}_u \mathcal{A}_u} e - \mu e \\ \Lambda_{\mathcal{I}_u \mathcal{I}_u}^u (U-X)_{\mathcal{I}_u \mathcal{I}_u} e - \mu e \end{bmatrix}, \quad (39) \end{aligned}$$

Suppose that an approximate solution of $\Delta x_{\mathcal{A}_x}^N$ is given, e.g. (34) or (35a) and (35b) of Proposition 4 respectively Proposition 5. Insertion of an approximate $\Delta x_{\mathcal{A}_x}$ into (39) yields

$$\begin{aligned}
& \begin{bmatrix} H_{\mathcal{A}_x \mathcal{I}_x} & -I_{\mathcal{A}_x \mathcal{A}_l} & -I_{\mathcal{A}_x \mathcal{I}_l} & I_{\mathcal{A}_x \mathcal{A}_u} & I_{\mathcal{A}_x \mathcal{I}_u} \\ H_{\mathcal{I}_x \mathcal{I}_x} & (X-L)_{\mathcal{A}_l \mathcal{A}_l} & -I_{\mathcal{I}_x \mathcal{I}_l} & & I_{\mathcal{I}_x \mathcal{I}_u} \\ \Lambda_{\mathcal{I}_l \mathcal{I}_x}^l & & (X-L)_{\mathcal{I}_l \mathcal{I}_l} & & \\ & & & (U-X)_{\mathcal{A}_u \mathcal{A}_u} & \\ -\Lambda_{\mathcal{I}_u \mathcal{I}_x}^u & & & & (U-X)_{\mathcal{I}_u \mathcal{I}_u} \end{bmatrix} \begin{bmatrix} \Delta x_{\mathcal{I}_x} \\ \Delta \lambda_{\mathcal{A}_l}^l \\ \Delta \lambda_{\mathcal{I}_l}^l \\ \Delta \lambda_{\mathcal{A}_u}^u \\ \Delta \lambda_{\mathcal{I}_u}^u \end{bmatrix} \\
& = - \begin{bmatrix} \nabla f(x)_{\mathcal{A}_x} - \lambda_{\mathcal{A}_x}^l + \lambda_{\mathcal{A}_x}^u + H_{\mathcal{A}_x \mathcal{A}_x} \Delta x_{\mathcal{A}_x} \\ \nabla f(x)_{\mathcal{I}_x} - \lambda_{\mathcal{I}_x}^l + \lambda_{\mathcal{I}_x}^u + H_{\mathcal{I}_x \mathcal{A}_x} \Delta x_{\mathcal{A}_x} \\ \Lambda_{\mathcal{A}_l \mathcal{A}_l}^l (X-L)_{\mathcal{A}_l \mathcal{A}_l} e - \mu e + \Lambda_{\mathcal{A}_l \mathcal{A}_x}^l \Delta x_{\mathcal{A}_x} \\ \Lambda_{\mathcal{I}_l \mathcal{I}_l}^l (X-L)_{\mathcal{I}_l \mathcal{I}_l} e - \mu e + \Lambda_{\mathcal{I}_l \mathcal{A}_x}^l \Delta x_{\mathcal{A}_x} \\ \Lambda_{\mathcal{A}_u \mathcal{A}_u}^u (U-X)_{\mathcal{A}_u \mathcal{A}_u} e - \mu e - \Lambda_{\mathcal{A}_u \mathcal{A}_x}^u \Delta x_{\mathcal{A}_x} \\ \Lambda_{\mathcal{I}_u \mathcal{I}_u}^u (U-X)_{\mathcal{I}_u \mathcal{I}_u} e - \mu e - \Lambda_{\mathcal{I}_u \mathcal{A}_x}^u \Delta x_{\mathcal{A}_x} \end{bmatrix} \quad (40)
\end{aligned}$$

The second, fourth and sixth block of (40) provide unique solutions of $\Delta x_{\mathcal{I}_x}$, $\Delta \lambda_{\mathcal{I}_l}^l$ and $\Delta \lambda_{\mathcal{I}_u}^u$ which satisfy

$$\begin{aligned}
& \begin{bmatrix} H_{\mathcal{I}_x \mathcal{I}_x} & -I_{\mathcal{I}_x \mathcal{I}_l} & I_{\mathcal{I}_x \mathcal{I}_u} \\ \Lambda_{\mathcal{I}_l \mathcal{I}_x}^l & (X-L)_{\mathcal{I}_l \mathcal{I}_l} & \\ -\Lambda_{\mathcal{I}_u \mathcal{I}_x}^u & & (U-X)_{\mathcal{I}_u \mathcal{I}_u} \end{bmatrix} \begin{bmatrix} \Delta x_{\mathcal{I}_x} \\ \Delta \lambda_{\mathcal{I}_l}^l \\ \Delta \lambda_{\mathcal{I}_u}^u \end{bmatrix} \\
& = - \begin{bmatrix} \nabla f(x)_{\mathcal{I}_x} - \lambda_{\mathcal{I}_x}^l + \lambda_{\mathcal{I}_x}^u + H_{\mathcal{I}_x \mathcal{A}_x} \Delta x_{\mathcal{A}_x} \\ \Lambda_{\mathcal{I}_l \mathcal{I}_l}^l (X-L)_{\mathcal{I}_l \mathcal{I}_l} e - \mu e + \Lambda_{\mathcal{I}_l \mathcal{A}_x}^l \Delta x_{\mathcal{A}_x} \\ \Lambda_{\mathcal{I}_u \mathcal{I}_u}^u (U-X)_{\mathcal{I}_u \mathcal{I}_u} e - \mu e - \Lambda_{\mathcal{I}_u \mathcal{A}_x}^u \Delta x_{\mathcal{A}_x} \end{bmatrix} \quad (41)
\end{aligned}$$

The solution of (41) can be obtained by first solving with the Schur complement of $(X-L)_{\mathcal{I}_l \mathcal{I}_l}$ and $(U-X)_{\mathcal{I}_u \mathcal{I}_u}$

$$\begin{aligned}
& \left(H_{\mathcal{I}_x \mathcal{I}_x} + I_{\mathcal{I}_x \mathcal{I}_l} (X-L)_{\mathcal{I}_l \mathcal{I}_l}^{-1} \Lambda_{\mathcal{I}_l \mathcal{I}_x}^l + I_{\mathcal{I}_x \mathcal{I}_u} (U-X)_{\mathcal{I}_u \mathcal{I}_u}^{-1} \Lambda_{\mathcal{I}_u \mathcal{I}_x}^u \right) \Delta x_{\mathcal{I}} \\
& = - \left(\nabla f(x)_{\mathcal{I}_x} + H_{\mathcal{I}_x \mathcal{A}_x} \Delta x_{\mathcal{A}_x} \right) + I_{\mathcal{I}_x \mathcal{I}_l} (X-L)_{\mathcal{I}_l \mathcal{I}_l}^{-1} \left(\mu e - \Lambda_{\mathcal{I}_l \mathcal{A}_x}^l \Delta x_{\mathcal{A}_x} \right) \\
& \quad - I_{\mathcal{I}_x \mathcal{I}_u} (U-X)_{\mathcal{I}_u \mathcal{I}_u}^{-1} \left(\mu e + \Lambda_{\mathcal{I}_u \mathcal{A}_x}^u \Delta x_{\mathcal{A}_x} \right), \quad (42)
\end{aligned}$$

and then

$$\Delta \lambda_{\mathcal{I}_l}^l = -\lambda_{\mathcal{I}_l}^l + (X-L)_{\mathcal{I}_l \mathcal{I}_l}^{-1} \left(\mu e - \Lambda_{\mathcal{I}_l \mathcal{A}_x}^l \Delta x_{\mathcal{A}_x} - \Lambda_{\mathcal{I}_l \mathcal{I}_x}^l \Delta x_{\mathcal{I}_x} \right), \quad (43a)$$

$$\Delta \lambda_{\mathcal{I}_u}^u = -\lambda_{\mathcal{I}_u}^u + (U-X)_{\mathcal{I}_u \mathcal{I}_u}^{-1} \left(\mu e - \Lambda_{\mathcal{I}_u \mathcal{A}_x}^u \Delta x_{\mathcal{A}_x} + \Lambda_{\mathcal{I}_u \mathcal{I}_x}^u \Delta x_{\mathcal{I}_x} \right). \quad (43b)$$

Note that the matrix of (42) is by Assumption 1 a symmetric positive definite $(|\mathcal{I}_x| \times |\mathcal{I}_x|)$ -matrix. The remaining part of the solution of (40), that is $\Delta \lambda_{\mathcal{A}_l}^l$ and $\Delta \lambda_{\mathcal{A}_u}^u$ are then given by

$$\begin{aligned}
& \begin{bmatrix} -I_{\mathcal{A}_x \mathcal{A}_l} & I_{\mathcal{A}_x \mathcal{A}_u} \\ (X-L)_{\mathcal{A}_l \mathcal{A}_l} & (U-X)_{\mathcal{A}_u \mathcal{A}_u} \end{bmatrix} \begin{bmatrix} \Delta \lambda_{\mathcal{A}_l}^l \\ \Delta \lambda_{\mathcal{A}_u}^u \end{bmatrix} \\
& = - \begin{bmatrix} \nabla_x \mathcal{L}(x, \lambda)_{\mathcal{A}_x} + H_{\mathcal{A}_x \mathcal{A}_x} \Delta x_{\mathcal{A}_x} + H_{\mathcal{A}_x \mathcal{I}_x} \Delta x_{\mathcal{I}_x} + I_{\mathcal{A}_x \mathcal{I}_l} \Delta \lambda_{\mathcal{I}_l}^l - I_{\mathcal{A}_x \mathcal{I}_u} \Delta \lambda_{\mathcal{I}_u}^u \\ \Lambda_{\mathcal{A}_l \mathcal{A}_l}^l (X-L)_{\mathcal{A}_l \mathcal{A}_l} e - \mu e + \Lambda_{\mathcal{A}_l \mathcal{A}_x}^l \Delta x_{\mathcal{A}_x} \\ \Lambda_{\mathcal{A}_u \mathcal{A}_u}^u (U-X)_{\mathcal{A}_u \mathcal{A}_u} e - \mu e - \Lambda_{\mathcal{A}_u \mathcal{A}_x}^u \Delta x_{\mathcal{A}_x} \end{bmatrix}, \quad (44)
\end{aligned}$$

where $\nabla_x \mathcal{L}(x, \lambda)_{\mathcal{A}_x} = \nabla f(x)_{\mathcal{A}_x} - \lambda_{\mathcal{A}_x}^l + \lambda_{\mathcal{A}_x}^u$. If the approximate $\Delta x_{\mathcal{A}_x}$ is exact then so is $\Delta x_{\mathcal{I}_x}$ by (42). In consequence, the over-determined system (44) has a unique solution that

satisfies all equations, i.e. $\Delta\lambda_{\mathcal{A}_x}$, or equivalently $\Delta\lambda_{\mathcal{A}_l}^l$ and $\Delta\lambda_{\mathcal{A}_u}^u$ since $\mathcal{A}_x = \mathcal{A}_l \cup \mathcal{A}_u$, are the corresponding parts of the solution to (2). These solutions or approximates can then be obtained from

$$\begin{aligned} [-I_{\mathcal{A}_x\mathcal{A}_l} \ I_{\mathcal{A}_x\mathcal{A}_u}] \begin{bmatrix} \Delta\lambda_{\mathcal{A}_l}^l \\ \Delta\lambda_{\mathcal{A}_u}^u \end{bmatrix} &= - \left[\nabla f(x)_{\mathcal{A}_x} - \lambda_{\mathcal{A}_x}^l + \lambda_{\mathcal{A}_x}^u + H_{\mathcal{A}_x\mathcal{A}_x} \Delta x_{\mathcal{A}_x} + H_{\mathcal{A}_x\mathcal{I}_x} \Delta x_{\mathcal{I}_x} \right. \\ &\quad \left. + I_{\mathcal{A}_x\mathcal{I}_l} \Delta\lambda_{\mathcal{I}_l}^l - I_{\mathcal{A}_x\mathcal{I}_u} \Delta\lambda_{\mathcal{I}_u}^u \right], \end{aligned} \quad (45)$$

or

$$\begin{bmatrix} (X-L)_{\mathcal{A}_l\mathcal{A}_l} & \\ & (U-X)_{\mathcal{A}_u\mathcal{A}_u} \end{bmatrix} \begin{bmatrix} \Delta\lambda_{\mathcal{A}_l}^l \\ \Delta\lambda_{\mathcal{A}_u}^u \end{bmatrix} = - \begin{bmatrix} A_{\mathcal{A}_l\mathcal{A}_l}^l (X-L)_{\mathcal{A}_l\mathcal{A}_l} e - \mu e + A_{\mathcal{A}_l\mathcal{A}_x}^l \Delta x_{\mathcal{A}_x} \\ A_{\mathcal{A}_u\mathcal{A}_u}^u (U-X)_{\mathcal{A}_u\mathcal{A}_u} e - \mu e - A_{\mathcal{A}_u\mathcal{A}_x}^u \Delta x_{\mathcal{A}_x} \end{bmatrix}$$

Alternatively, $\Delta\lambda_{\mathcal{A}_l}^l$ and $\Delta\lambda_{\mathcal{A}_u}^u$ can be obtained as the least squares solution of (44) that is

$$\begin{aligned} &\begin{bmatrix} I_{\mathcal{A}_l\mathcal{A}_l} + (X-L)_{\mathcal{A}_l\mathcal{A}_l}^2 & \\ & I_{\mathcal{A}_u\mathcal{A}_u} + (U-X)_{\mathcal{A}_u\mathcal{A}_u}^2 \end{bmatrix} \begin{bmatrix} \Delta\lambda_{\mathcal{A}_l}^l \\ \Delta\lambda_{\mathcal{A}_u}^u \end{bmatrix} \\ &= \begin{bmatrix} I_{\mathcal{A}_x\mathcal{A}_l}^T (\nabla f(x)_{\mathcal{A}_x} - \lambda_{\mathcal{A}_x}^l + \lambda_{\mathcal{A}_x}^u + H_{\mathcal{A}_x\mathcal{A}_x} \Delta x_{\mathcal{A}_x} + H_{\mathcal{A}_x\mathcal{I}_x} \Delta x_{\mathcal{I}_x} + I_{\mathcal{A}_x\mathcal{I}_l} \Delta\lambda_{\mathcal{I}_l}^l \\ -I_{\mathcal{A}_x\mathcal{A}_u}^T (\nabla f(x)_{\mathcal{A}_x} - \lambda_{\mathcal{A}_x}^l + \lambda_{\mathcal{A}_x}^u + H_{\mathcal{A}_x\mathcal{A}_x} \Delta x_{\mathcal{A}_x} + H_{\mathcal{A}_x\mathcal{I}_x} \Delta x_{\mathcal{I}_x} + I_{\mathcal{A}_x\mathcal{I}_l} \Delta\lambda_{\mathcal{I}_l}^l \\ -I_{\mathcal{A}_x\mathcal{I}_u} \Delta\lambda_{\mathcal{I}_u}^u) - (X-L)_{\mathcal{A}_l\mathcal{A}_l} (A_{\mathcal{A}_l\mathcal{A}_l}^l (X-L)_{\mathcal{A}_l\mathcal{A}_l} e - \mu e + A_{\mathcal{A}_l\mathcal{A}_x}^l \Delta x_{\mathcal{A}_x}) \\ -I_{\mathcal{A}_x\mathcal{I}_u} \Delta\lambda_{\mathcal{I}_u}^u) - (U-X)_{\mathcal{A}_u\mathcal{A}_u} (A_{\mathcal{A}_u\mathcal{A}_u}^u (U-X)_{\mathcal{A}_u\mathcal{A}_u} e - \mu e - A_{\mathcal{A}_u\mathcal{A}_x}^u \Delta x_{\mathcal{A}_x}) \end{bmatrix} \end{aligned}$$

since $I_{\mathcal{A}_x\mathcal{A}_l}^T I_{\mathcal{A}_x\mathcal{A}_l} = I_{\mathcal{A}_l\mathcal{A}_l}$, $I_{\mathcal{A}_x\mathcal{A}_u}^T I_{\mathcal{A}_x\mathcal{A}_u} = I_{\mathcal{A}_u\mathcal{A}_u}$ and $I_{\mathcal{A}_x\mathcal{A}_l}^T I_{\mathcal{A}_x\mathcal{A}_u} = I_{\mathcal{A}_x\mathcal{A}_u}^T I_{\mathcal{A}_x\mathcal{A}_l} = 0$. The equations can also be written as

$$\begin{aligned} \Delta\lambda_{\mathcal{A}_l}^l &= \left(I_{\mathcal{A}_l\mathcal{A}_l} + (X-L)_{\mathcal{A}_l\mathcal{A}_l}^2 \right)^{-1} \left[I_{\mathcal{A}_x\mathcal{A}_l}^T (\nabla f(x)_{\mathcal{A}_x} - \lambda_{\mathcal{A}_x}^l + \lambda_{\mathcal{A}_x}^u \right. \\ &\quad \left. + H_{\mathcal{A}_x\mathcal{A}_x} \Delta x_{\mathcal{A}_x} + H_{\mathcal{A}_x\mathcal{I}_x} \Delta x_{\mathcal{I}_x} + I_{\mathcal{A}_x\mathcal{I}_l} \Delta\lambda_{\mathcal{I}_l}^l - I_{\mathcal{A}_x\mathcal{I}_u} \Delta\lambda_{\mathcal{I}_u}^u \right) \\ &\quad \left. - (X-L)_{\mathcal{A}_l\mathcal{A}_l} (A_{\mathcal{A}_l\mathcal{A}_l}^l (X-L)_{\mathcal{A}_l\mathcal{A}_l} e - \mu e + A_{\mathcal{A}_l\mathcal{A}_x}^l \Delta x_{\mathcal{A}_x}) \right] \end{aligned} \quad (46a)$$

$$\begin{aligned} \Delta\lambda_{\mathcal{A}_u}^u &= - \left(I_{\mathcal{A}_u\mathcal{A}_u} + (U-X)_{\mathcal{A}_u\mathcal{A}_u} \right)^{-1} \left[I_{\mathcal{A}_x\mathcal{A}_u}^T (\nabla f(x)_{\mathcal{A}_x} - \lambda_{\mathcal{A}_x}^l + \lambda_{\mathcal{A}_x}^u \right. \\ &\quad \left. + H_{\mathcal{A}_x\mathcal{A}_x} \Delta x_{\mathcal{A}_x} + H_{\mathcal{A}_x\mathcal{I}_x} \Delta x_{\mathcal{I}_x} + I_{\mathcal{A}_x\mathcal{I}_l} \Delta\lambda_{\mathcal{I}_l}^l - I_{\mathcal{A}_x\mathcal{I}_u} \Delta\lambda_{\mathcal{I}_u}^u \right) \\ &\quad \left. + (U-X)_{\mathcal{A}_u\mathcal{A}_u} (A_{\mathcal{A}_u\mathcal{A}_u}^u (U-X)_{\mathcal{A}_u\mathcal{A}_u} e - \mu e - A_{\mathcal{A}_u\mathcal{A}_x}^u \Delta x_{\mathcal{A}_x}) \right] \end{aligned} \quad (46b)$$

Finally, we state the main result which is analogous to the result of Theorem 1.

Theorem 2 Under Assumption 1 let $\mathcal{B}((x^*, \lambda^*), \delta)$ and $\hat{\mu}$ be defined by Lemma 1 and Lemma 2 respectively. For $0 < \mu \leq \hat{\mu}$ and $(x, \lambda) \in \mathcal{B}((x^*, \lambda^*), \delta)$, let $(\Delta x^N, \Delta\lambda^N)$ be the solution of (2) with $\mu^+ = \sigma\mu$, where $0 < \sigma < 1$. Moreover, let the search direction components be defined as

$$\Delta x_i = \begin{cases} (34) \text{ or } (35a) & i \in \mathcal{A}_l, \\ (34) \text{ or } (35b) & i \in \mathcal{A}_u, \\ (42) & i \in \mathcal{I}_x, \end{cases}$$

$$\Delta\lambda_i^l = \begin{cases} (46a) \text{ or } (45) & i \in \mathcal{A}_l, \\ (36a) \text{ or } (43a) & i \in \mathcal{I}_l, \end{cases}$$

$$\Delta\lambda_i^u = \begin{cases} (46b) \text{ or } (45) & i \in \mathcal{A}_u, \\ (36b) \text{ or } (43b) & i \in \mathcal{I}_u. \end{cases}$$

If $0 < \mu \leq \hat{\mu}$ and (x, λ) is sufficiently close to $(x^\mu, \lambda^\mu) \in \mathcal{B}((x^*, \lambda^*), \delta)$ such that $\|F_\mu(x, \lambda)\| = \mathcal{O}(\mu)$. Then there exists $\bar{\mu}$, with $0 < \bar{\mu} \leq \hat{\mu}$, such that for $0 < \mu \leq \bar{\mu}$ it holds that

$$\|(\Delta x, \Delta \lambda) - (\Delta x^N, \Delta \lambda^N)\| = \mathcal{O}(\mu^2).$$

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