A mixed-integer programming formulation of the double row layout problem based on a linear extension of a partial order

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Abstract

The Double Row Layout Problem (DRLP) occurs in automated manufacturing environments, where machines arranged in a double-row layout, i.e. the machines are located on either side of a straight line corridor. The DRLP is how to minimize the total cost of transporting materials between machines. The problem is NP-Hard. In this paper, we give a new mixed-integer programming formulation of the DRLP, which is based on a linear extension of a partial order.

1 Introduction

A double-row layout of machines in a flexible manufacturing system is a layout in which the machines are located on either side of a straight line corridor. The problem of allocating machines according to a double row layout in such a way that the total cost of transporting materials between machines is minimized is called the double row layout problem (DRLP).

The following notation will be utilized throughout:

n	number of machines
$N = \{1, \dots, n\}$	set of machines
$R =$\{upper row, lower row\}	set of rows that define a corridor
c_{ij}	amount of flow between machines i e j $(1 \leq i < j \leq n)$
ℓ_i	length of machine i $(1 \leq i \leq n)$
$L := \sum_{k=1}^{n} \ell_i$	

Assumptions:

i. The corridor lies with its length along the x-axis on the interval $[0,\ L]$;

ii. The width of the corridor is negligible;

iii. The distance between two machines is taken as the x-distance between their centroids.

iv. The total cost of transporting materials between machines to be minimized is given by $\sum_{1\le i<j\le n} c_{ij} d_{ij}^{\varphi}$

where d_{ij}^{φ} is the distance between machines i and j with reference to a double row layout φ.

We also assume that clearances have been included in the dimensions of the machines.

As an illustrative example, Figure 1 presents data for the instance S5 from [1], for which we have found an optimal double row layout $\varphi*$ of the five machines.

$$
\begin{array}{cccccccccccc}
n & & & & & = & 5 & & & & \\
(\ell_1 & \ell_2 & \ell_3 & \ell_4 & \ell_5) & = & (1 & 3 & 4 & 6 & 7) \\
 & (c_{12} & c_{13} & c_{14} & c_{15}) & = & & (2 & 1 & 0 & 1) \\
 & & (c_{23} & c_{24} & c_{25}) & = & & & (0 & 2 & 2) \\
 & & & (c_{34} & c_{35}) & = & & & & (6 & 3) \\
 & & & & c_{45} & = & & & & & 4
\end{array}
$$

Figure 1: Data for a DRLP instance with $n = 5$

Inter-machine distances relative to the optimal layout $\varphi*$ are shown in Figure 2(a). The cost of $\varphi*$ is thus: $(2\times 2)+(1\times 3.5)+(1\times 2)+(2\times 5.5)+(3\times 5.5)+(4\times 5.5) = 59$. The optimal layout is shown in Figure 2(b).

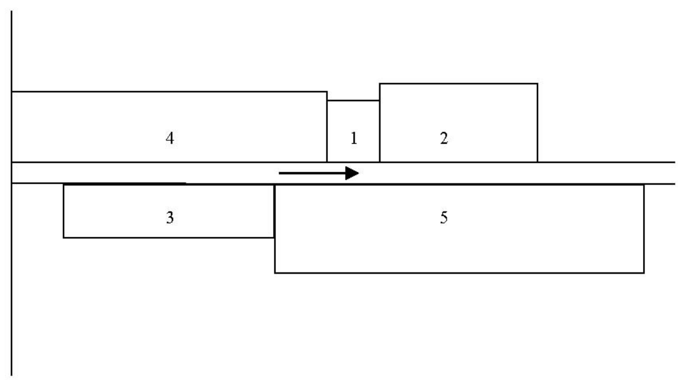
In this paper, we propose a new mixed-integer programming formulation of the DRLP, which is based on a linear extension of a partial order on the set of machines. The new formulation has the least number of 0-1 variables in comparison with previous ones.

2 Terminology

A *strict partial order* $<$ on a set S is an irreflexive, asymmetric, transitive binary relation on S. For distinct $i,\ j \in S$, i and j are said to be *comparable* in $<$ if either $i < j$ or $i > j$. Otherwise, if $i \not< j$ and $i \not> j$, then i and j are said to be *incomparable* in $<$. A pair $(S, <)$ defined by a set S together with the strict partial order $<$ is called a *strict partially ordered set* or *strict poset*. If every two distinct elements of S are comparable in the strict partial order $<$, the strict poset $(S, <)$ is called a *chain*. If $(S, <)$ is a chain, then $<$ is called a *linear order*

$$
\begin{array}{cccccccc}
(d_{12}^{\varphi*} & d_{13}^{\varphi*} & d_{14}^{\varphi*} & d_{15}^{\varphi*}) & = & (2 & 3.5 & 17.5 & 2) \\
 & (d_{23}^{\varphi*} & d_{24}^{\varphi*} & d_{25}^{\varphi*}) & = & & (5.5 & 5.5 & 0) \\
 & & (d_{34}^{\varphi*} & d_{35}^{\varphi*}) & = & & & (0 & 5.5) \\
 & & & d_{45}^{\varphi*} & = & & & & 5.5
\end{array}
$$

(a) Inter-machine distances in an optimal layout



(b) Optimal layout

Figure 2: Inter-machine distances in (a) are relative to the optimal DRLP layout $\varphi*$ shown in (b)

(or *total order*) on S. A *linear extension* of a strict poset $(S, <)$ is a poset $(S, \prec)$ such that: *(i)* $(S, \prec)$ is a chain and *(ii)* for $i,\ j \in S$, $i \prec j$ whenever $i < j$.

Let $(P, <)$ and $(Q, <)$ be disjoint strict posets. The *sum* of $(P, <)$ and $(Q, <)$ is the strict poset $(P \cup Q, <_{P+Q})$ such that $i <_{P+Q} \ j$ if and only if $(i,\ j \in P$ and $i < j)$ or $(i,\ j \in Q$ and $i < j)$; and the *ordinal sum* of $(P, <)$ and $(Q, <)$ is the strict poset $(P \cup Q, <_{P \oplus Q})$ such that $i <_{P \oplus Q} \ j$ if and only if either $(i \in P,\ j \in Q)$ or $(i,\ j \in P$ and $i < j)$ or $(i,\ j \in Q$ and $i < j)$.

3 The new proposed mixed-integer programming model

In a given double row layout, let P be the set of machines allocated at the lower row; let Q be the set of machines allocated at the upper row (clearly, $P \cup Q = N$); and let $<$ be the partial order such that $i < j$ means that machine

i is to the left of machine j. Then we can associate with each row the disjoint chains $(P,<)$ and $(Q,<)$. Recall that the MIP model of [2] with the corrections given in [3] as well as the MIP models in [4, 5] present binary 0-1 variables so as to maintain $(P \cup Q, <_{P+Q})$, where $i <_{P+Q} j$ means that machine i is to the left of machine j and both at the same row. In this paper, we shall take a different approach. We shall present a formulation of the DRLP based on the linear order $<_{P\oplus Q}$ on the set $P \cup Q$, where $i <_{P\oplus Q} j$ means that i is to the left of j both at the same row; or i is below j.

In the example of Figure 2, $(P,<) = \{(3,5)\}$ and $(Q,<) = \{(1,2),\ (4,1),\ (4,2)\}$. Then, the previous models in the literature maintain:

$$(P \cup Q, <_{P+Q}) \quad = \quad \{(1,\ 2),\ (3,5),\ (4,\ 1),\ (4,2)\},$$

while the proposed model will maintain:

$$\begin{aligned}(P \cup Q, <_{P\oplus Q}) \quad = \quad & \{(1,\ 2),\ (3,\ 1),\ (3,\ 2),\ (3,\ 4),\ (3,\ 5),\ (4,\ 1), \\ & (4,\ 2),\ (5,\ 1),\ (5,\ 2),\ (5,\ 4)\}.\end{aligned}$$

Note that $(P,<)$ and $(Q,<)$ are disjoint chains. The strict poset $(P \cup Q, <_{P+Q})$ is not a chain because $\forall i \in P,\ j \in Q\ ,\ i \not< j$ and $i \not> j$. Moreover, $(P \cup Q, <_{P\oplus Q})$ is a chain and is a linear extension of $(P \cup Q, <_{P+Q})$.

3.1 Binary 0-1 variables

For $1 \le i < j \le n$, define the following 0-1 variables:

$$\sigma_{ij} = \begin{cases} 1, & \text{if machine } i \text{ precedes machine } j; \\ 0, & \text{otherwise;} \end{cases}$$

$$y_i = \begin{cases} 1, & \text{if machine } i \text{ is at the upper row;} \\ 0, & \text{if machine } i \text{ is at the lower row;} \end{cases}$$

3.2 Continuous variables

x_i: abscissa of the centroid of machine i, $(1 \le i \le n)$, in the x-axis;

d_{ij}: x-distance between (the centroids of) machines i and j, $(1 \le i < j \le n)$.

3.3 Constraints for determining inter-machine distances

Given the definition of the x_i variables and since we have a minimization problem, the inter-machine distances d_{ij} are determined by Constraints (1).

$$d_{ij} \quad \ge \quad x_i - x_j; \quad d_{ij} \ge x_j - x_i, \quad (1 \le i < j \le n). \tag{1}$$

3.4 Non-overlapping constraints

In order to avoid overlap among machines placed at the same row, the following constraints are used:

$$\begin{aligned}
x_j + \left(\frac{\ell_i + \ell_j}{2}\right) &\leq x_i + L(y_i + y_j + \sigma_{ij}) && (2)\\
x_i + \left(\frac{\ell_i + \ell_j}{2}\right) &\leq x_j + L(1 + y_i + y_j - \sigma_{ij}), && (3)\\
x_j + \left(\frac{\ell_i + \ell_j}{2}\right) &\leq x_i + L(2 - y_i - y_j + \sigma_{ij}), && (4)\\
x_i + \left(\frac{\ell_i + \ell_j}{2}\right) &\leq x_j + L(3 - y_i - y_j - \sigma_{ij}),\ (1 \leq i < j \leq n). && (5)
\end{aligned}$$

If machines i and j are at the same row, then these constraints ensure that machine j is either to the right or to the left of machine i, which avoids overlap. When $y_i = y_j = 0$, the pair (2)-(3) is active; while when $y_i = y_j = 1$, the pair (4)-(5) is active. Note that all of the inequalities (2)-(5) are redundant if machines i and j are at different rows (i.e. $y_i \neq y_j$).

3.5 Constraints on σ_{ij} variables

For $\sigma = (\sigma_{ij})_{1 \leq i < j \leq n}$ to be an incidence vector of a linear ordering, we need to impose (e.g. [6]):

$$\begin{aligned}
-\sigma_{ij} - \sigma_{jk} + \sigma_{ik} &\leq 0 && (6)\\
\sigma_{ij} + \sigma_{jk} - \sigma_{ik} &\leq 1 && (7)
\end{aligned}$$

3.6 Implementing $(P \cup Q, <_{P \oplus Q})$

For implementing $(P \cup Q, <_{P \oplus Q})$, the following inequalities are in place:

$$\begin{aligned}
\sigma_{ij} &\leq 1 - y_i + y_j && (8)\\
\sigma_{ij} &\geq -y_i + y_j,\ (1 \leq i < j \leq n). && (9)
\end{aligned}$$

When ($y_i = 1$, $y_j = 0$), (8) implies $\sigma_{ij} = 0$, while (9) is redundant; However, if ($y_i = 0$, $y_j = 1$), (9) implies $\sigma_{ij} = 1$, while (8) is redundant. Both (8) and (9) are redundant if $y_i = y_j$.

3.7 Strengthened lower bound on distance variables

If machine i is placed at the same row as machine j, the distance between their centroids is at least $(\ell_i + \ell_j)/2$. Then:

$$d_{ij} \geq \left(\frac{\ell_i+\ell_j}{2}\right)(1-y_i-y_j) \tag{10}$$

$$d_{ij} \geq \left(\frac{\ell_i+\ell_j}{2}\right)(-1+y_i+y_j),\ (1 \leq i < j \leq n). \tag{11}$$

Here, when $y_i = y_j = 0$, (10) is active and (11) is redundant; and if $y_i = y_j = 1$, (11) is active and (10) is redundant. Both (10) and (11) are redundant if $y_i \neq y_j$.

3.8 Symmetry-breaking constraints

For two specific machines, say $i*$ and $j*$, there could be symmetric optimal solutions, which we can try to eliminate by imposing:

$$\sigma_{i*,\,j*} = 1 \tag{12}$$

$$y_{i*} = 0 \tag{13}$$

In this paper, we have adopted $i*$ and $j*$ as the first two machines listed in the instance data. More precisely: $i* = 1$ and $j* = 2$.

3.9 Bounds on continuous variables

Within the interval $[0,\, L]$, the leftmost position for the center of machine i is $\frac{\ell_i}{2}$ and the rigthmost position is $L - \frac{\ell_i}{2}$, then:

$$\frac{\ell_i}{2} \leq x_i \leq L - \frac{\ell_i}{2} \tag{14}$$

As for d_{ij} variables, we can state:

$$d_{ij} \geq 0,\ (1 \leq i < j \leq n) \tag{15}$$

3.10 Proposed formulation

Therefore, a mixed-integer programming formulation of the DRLP is given by:

$$Minimize \left\{ \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} c_{ij} d_{ij}\ (1) - (15), \right.$$

$$\left. \begin{array}{rcl} \sigma_{ij} & \in & \{0,\,1\},\ (1 \leq i < j \leq n) \quad (16) \\ y_i & \in & \{0,\,1\}, \quad (1 \leq i \leq n) \quad (17) \end{array} \right\}.$$

Model	Number of Constraints*	Number of variables	
		Binary	Continuous
M_{CT}	$\frac{5}{2}n(n-1)+3n$	$n(n-1)+2n$	$n(n-1)+2n$
$M1$	$\frac{5}{2}n(n-1)+n(n-1)(n-2)+1$	$n(n-1)$	$\frac{1}{2}n(n-1)+n$
$M2$	$\frac{5}{2}n(n-1)+\frac{7}{6}n(n-1)(n-2)+1$	$n(n-1)$	$\frac{1}{2}n(n-1)+n$
$M3$	$5n(n-1)+\frac{1}{3}n(n-1)(n-2)+2$	$\frac{1}{2}n(n-1)+n-2$	$\frac{1}{2}n(n-1)+n$

*Excluding non-negativity constraints and bounds on variables.

Table 1: Comparison of times of MIP formulations of the DRLP.

In Table 1, the model in [2] is denoted by M_{CT}; the model in [5] by $M1$; the model in [4] by $M2$; and the new model is denoted by $M3$. The table shows a comparison among these models in relation to number of variables and constraints. It can be seen that the proposed model $M3$ presents the least number of zero-one variables. $M1$, $M2$ and $M3$ have the same number of continuous variables ($\frac{1}{2}n(n-1)+n$), which is half the number of continuous variables in M_{CT} . The number of constraints of M_{CT} is in $O(n^2)$, while the number of constraints of $M1$, $M2$ or $M3$ is in $O(n^3)$. The larger number of constraints is due to the inclusion of many valid inequalities, which strengthen these models.

4 Computational experiments

The computational results presented in this paper were obtained by solving the MIP models by means of the CPLEX 12.7.1.0 solver. Data for all DRLP instances used in this paper are available from the author.

Initial tests with ten instances of size $n = 10$ were run on an Intel® Core i35005U CPU 2GHz with 8 GB of RAM with the Windows 10 operating system.

The comparison of models in Table 2 shows that Model M3 presents the smallest computational times for all of the instances with $n = 10$. Moreover, the speed up provided by $M3$ is formidable.

5 Conclusions

This paper presented a new MIP formulation of the DRLP. The new formulation is based on a linear extension of a partial order. Such a construct allowed encoding a larger amount of information about the relative position of machines, while at the same time reducing the number of binary variables by half, as compared to the formulations in the literature.

Initial tests with ten instances of size $n = 10$ showed that with the new formulation, solutions for the considered instances can be computed faster than using previous formulations.

Instance	Optimal value	Times (s)		
		$M3$	$M2$	$M1$
Small.10-1	1,385.0	15.09	33.40	34.89
Small.10-2	1,437.0	17.47	38.78	49.56
Small.10-3	1,452.5	20.15	50.10	40.88
Small.10-4	1,313.5	14.40	44.46	35.15
Small.10-5	722.5	9.55	25.45	21.46
Small.10-6	792.0	13.74	57.21	37.72
Small.10-7	607.5	11.21	40.94	20.13
Small.10-8	529.0	6.68	17.39	11.36
Small.10-9	929.5	14.44	32.25	33.83
Small.10-10	828.0	16.79	42.37	50.93

Table 2: Comparison of Model M3 with M2 and M1 on instances with $n = 10$.

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