A New Coherent Multivariate Average-Value-at-Risk

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Abstract

The problem of risk evaluation of multivariate risk sources has been studied, and a multivariate risk measure, so-called multivariate Average-Value-at-Risk, $\mathrm{mAVaR}_{\boldsymbol{\alpha}}$, is proposed to quantify the total risk. It is shown that the proposed operator satisfies the four axioms of a coherent risk measure while reducing to one variable Average-Value-at-Risk, AVaR_α, in case $N = 1$. In that respect, it is shown that $\mathrm{mAVaR}_{\boldsymbol{\alpha}}$ is the natural extension of AVaR_α to N-dimensional case maintaining its axiomatic properties. We further show $\mathrm{mAVaR}_{\boldsymbol{\alpha}}$ is flexible by giving the investor the option to choose the risk level α_i of each random loss i differently. This flexibility is novel and can not be achieved applying univariate AVaR_α with corresponding risk level α to the sum of the risk marginals. The framework is applicable for Gaussian mixture models with dependent risk factors that are naturally used in financial and actuarial modelling. A multivariate tail variance and its connection with $\mathrm{mAVaR}_{\boldsymbol{\alpha}}$ is also presented via Chebyshev inequality for tail events. Examples with numerical simulations are also illustrated throughout.

Keywords: Financial Risk Management, Multivariate Coherent Risk Measures, Average-Value-at-Risk.

1 Introduction

Coherent risk measures have been introduced in [1] to quantify univariate risk (loss) in an axiomatic framework. These axioms have been monotonicity, translation invariance, convexity (diversification of risk) and positive homogeneity. Since the introduction of this concept, they have seen huge development both in theory and practice. We refer the reader to the monographs [21], [23] and [5] for extensive surveys on coherent risk measures and their use in

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different settings. One of the fundamental coherent risk measures is the so-called Average-Value-at-Risk (also called Tail-Conditional-Expectation, Conditional-Tail-Expectation, Expected Shortfall, Tail-Value-at-Risk, Conditional-Value-at-Risk) of random loss X for a predetermined risk level α, denoted by $\mathsf{AVaR}_\alpha(X)$. Roughly speaking, $\mathsf{AVaR}_\alpha(X)$ evaluates random loss X by taking average of worst case scenarios from confidence level α onward. Beside its acceptance in Solvency II regulations, $\mathsf{AVaR}_\alpha(X)$ has been rigorously studied in many papers (see [12, 13, 14, 15, 16, 17, 22] among others).

Despite its wide use of AVaR_α, risk managers are not only dealing with one source of loss, but rather a portfolio of risk. In particular, they are dealing with random vectors representing different sources of risks. With possibly different risk levels defined by $\boldsymbol{\alpha}$, a risk manager has to evaluate the total sum of the risk sources. In this track, several extensions of AVaR to multivariate setting representing different risk sources have been introduced. In [3], a set of N sources of random losses that are mutually independent is considered. Here, the authors define a convex risk measure on N-dimensional random variables that maps them into $\mathbb{R}$. They further derive the corresponding duality result of that operator, while the requirement of independence does not answer the question of how to deal with dependent N-variate data that is frequent in real life applications. In [6], the authors study N-dimensional random source on a fixed probability space, whose marginals are not necessarily jointly independent. They define the convex risk measures on that probability space directly and also study these operators with respect to stochastic orderings. In [26], the authors define a set-valued average value at risk for multivariate risks by generalizing the certainty equivalent representation to the set-valued case. Similarly, in [27], the authors derive representations for set-valued shortfall risk measures, including the set-valued average-value-at-risk defined in [26] using set-valued duality theory. In parallel with [27] and [26], systemic risk measures should also be noted in this literature survey, since they are naturally multivariate risk measures. These operators measure the risk that the financial system is susceptible to failures due to the characteristics of the system itself. In this direction, [28] explains the conceptual framework, provides an algorithm for their computation and illustrates their applications in numerical case studies. In [29], the authors present dual representations for systemic risk measures as well as for the corresponding multivariate risk measures concerning capital allocations. In particular, the definition of multivariate average-value-at-risk in [27] can be seen as the scalarization of a systemic risk measure with properly selected aggregation function and acceptance set, and the authors obtain a dual representation using the results in [29]. Similarly, [7], [8], [25], [24] and [11] define vector-valued convex risk measures on N dimensions that depend on the set theory and use characterizations of the set-valued duality on these operators preserving risk diversification property of the proposed operators. In another attempt to define vector-valued tail conditional expectations, [9] defines them directly without requiring the necessity of the convexity axiom. In particular, the constructed operator is not necessarily convex. Hence, the diversification of the risk axiom is not necessarily satisfied by the constructed operator. In [9], various numerical studies on elliptical distributions are demonstrated, but the study does not include the mixture Gaussian distributions. In particular, they don't study non-symmetric mixture Gaussian distributions. In [3], the authors study multivariate tail conditional expectations, when the underlying multivariate distributions are of so-called phase type. Explicit representations are retrieved, but it is not checked whether the resulting operator satisfies the corresponding axioms of a coherent risk measure. Recently, [10] defined a vector-valued coherent risk measure that is peculiar for the discrete distributions.

This paper focuses on evaluating the sum of the risk in the portfolio with risk composed of N different risk/loss sources. In particular, our objective is to introduce a necessary extension of univariate AVaR to multivariate Average-Value-at-Risk $\mathrm{mAVaR}_{\boldsymbol{\alpha}}(\mathbf{X})$ that preserves the axioms of AVaR by taking N dimensional risk portfolio $\mathbf{X}$ and maps to $\mathbb{R}$ using N-dimensional risk level $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N)$. The introduced operator $\mathrm{mAVaR}_{\boldsymbol{\alpha}}$ reduces to univariate AVaR in case $N = 1$ by preserving the axioms of a coherent risk measure. The proposed operator $\mathrm{mAVaR}_{\boldsymbol{\alpha}}$ has the flexibility of choosing N-dimensional risk level $\boldsymbol{\alpha}$. This allows the risk manager to assign different risk levels α_i for each source $i = 1, \ldots, N$. Inspired by $\mathrm{mAVaR}_{\boldsymbol{\alpha}}(\mathbf{X})$, we introduce a multivariate Average-Value-at-Risk operator, denoted by $\mathrm{mAVaR}^G_{\boldsymbol{\alpha}}(\mathbf{X})$, that can handle dependent multivariate Gaussian mixtures $\mathbf{X}$. Modelling the random costs using mixture Gaussian distribution is prevalent in actuarial and financial mathematics. To the best of our knowledge, there is no construction of multidimensional AVaR satisfying both the axioms of a coherent risk measure and able to give explicit representations for multivariate mixture Gaussian models. To summarise, firstly, the main contribution of this paper is introducing an operator that quantifies the total risk of N sources by keeping the axioms of a coherent risk measure in insurance and finance. Secondly, we extend the operator to the setting, where the risk is modelled as a multivariate mixture Gaussian distribution. These are vital studies in risk management, since mixture Gaussian distributions play a key role in financial and actuarial mathematics. We refer the reader to Table 1, where we summarise the characteristics of the works in the references and compare them with the construction in this paper.

The rest of the paper is as follows. In Section 2, we introduce the framework and background theory regarding Average-Value-at-Risk, along with the frequent notation used in the rest of the paper. In Section 3, we define multivariate Average-Value-at-Risk and its regularity properties, along with the examples shedding light on its properties. In Section

4, we give examples of the use of $\mathrm{mAVaR}_{\boldsymbol{\alpha}}$ in Gaussian mixture models with possibly dependent marginals. In Section 5, we give a Chebyshev type stability result for $\mathrm{mAVaR}_{\boldsymbol{\alpha}}$ by introducing and using the corresponding tail variance of a loss portfolio and give an example in multivariate Gaussian setting. In Section 6, we summarise our results and give the outlook for future directions.

Table 1: **Literature Survey**

References	**Univariate/ Multivariate**	**Satisfies Risk Diversification**	**Vector valued/ Scalar valued/ Set Valued**	**Allows Dependent Marginals**
[9]	Multivariate	No	Vector Valued	Yes
[11]	Multivariate	Yes	Vector Valued	Yes
[12]	Univariate	Yes	Scalar Valued	Yes
[14]	Multivariate	No	Scalar Valued	Yes
[22]	Univariate	Yes	Scalar Valued	Yes
[18]	Multivariate	Yes	Scalar Valued	Yes
[3]	Multivariate	Yes	Scalar Valued	No
[6]	Multivariate	Yes	Scalar Valued	No
[8]	Multivariate	Yes	Set Valued	Yes
[10]	Multivariate	Yes	Vector Valued	Yes
[17]	Univariate	Yes	Scalar Valued	No
[23]	Univariate	Yes	Scalar Valued	No
[25]	Multivariate	Yes	Vector Valued	Yes
[26]	Multivariate	Yes	Vector Valued	Yes
[27]	Multivariate	Yes	Vector Valued	Yes
[28]	Multivariate	Yes	Vector Valued	Yes
[29]	Multivariate	Yes	Vector Valued	Yes
This Research	Multivariate	Yes	Scalar Valued	Only In Mixture Gaussian Case

2 Framework and Notation

We fix an integer $N \geq 1$, and let $(\Omega_i, \mathcal{F}_i, \mathbb{P}_i)$ for $i = 1, \ldots, N$ be the fixed atomless probability space. We denote $\mathcal{X}_i \triangleq L^1(\Omega_i, \mathcal{F}_i, \mathbb{P}_i)$, where an $\mathcal{F}_i$ measurable random variable $X_i \in \mathcal{X}_i$ means $\|X_i\| \triangleq \mathbb{E}^{\mathbb{P}_i}[|X_i|] < \infty$, and $\mathbb{E}^{\mathbb{P}_i}[\cdot]$ stands for the expectation with respect to $\mathbb{P}_i$. For $X_i, Y_i \in \mathcal{X}_i$, we identify X_i with Y_i if $\mathbb{P}_i(X_i = Y_i) = 1$. $\mathcal{X}_i$ stands for the riskiness of the financial random position. Namely, positive values of $X_i \in \mathcal{X}_i$ represent costs/loss, whereas negative values of it stand for profits/gain. We let $\mathcal{X}^N \triangleq \mathcal{X}_1 \times \dots \mathcal{X}_N$ with $\Omega \triangleq \otimes_{i=1}^N \Omega_i$, $\mathbb{P} \triangleq \prod_{i=1}^N \mathbb{P}_i$ and $\mathcal{F} \triangleq \otimes_{i=1}^N \mathcal{F}_i$. We note here that any $(X_1, \dots, X_N) \in \mathcal{X}^N$ is jointly independent with respect to $\mathbb{P}$. $\mathbb{E}[\cdot]$ ($\mathbb{E}[\cdot|\mathcal{F}]$) stands for the (conditional) expectation taken with respect to $\mathbb{P}$, whereas $\mathbb{E}^Q[\cdot]$ ($\mathbb{E}^Q[\cdot|\mathcal{F}]$) stands for the (conditional) expectation taken with respect to the specific probability measure Q. Moreover, let $\mathbf{X} \triangleq (X_1, \dots, X_N) \in \mathcal{X}^N$ be the corresponding portfolio of risky assets with $\|\boldsymbol{X}\| \triangleq \sum_{i=1}^N \|X_i\|$, and let $\mathcal{M}_1(\mathcal{X}_i)$ for $i = 1, \dots, N$ and $\mathcal{M}_1(\mathcal{X}^N)$ be the notation for the set of all probability measures that are absolutely continuous with respect to $\mathbb{P}_i$ and $\mathbb{P}$, denoted by $Q_i \prec \mathbb{P}_i$ and $Q \prec \mathbb{P}$, respectively. Namely, taking $\frac{1}{p} + \frac{1}{q} = 1$ with $q = \infty$ for $p = 1$,

$$\mathcal{M}_1^i(\mathcal{X}_i) \triangleq \left\{ Q_i \prec \mathbb{P}_i : \frac{dQ_i}{d\mathbb{P}_i} \in L^\infty(\Omega_i, \mathcal{F}_i, \mathbb{P}_i) \right\}$$
$$\mathcal{M}_1(\mathcal{X}^N) \triangleq \left\{ Q \prec \mathbb{P} : \frac{dQ}{d\mathbb{P}} \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \right\},$$

where $\frac{dQ_i}{d\mathbb{P}_i}$ and $\frac{dQ}{d\mathbb{P}}$ stand for the Radon-Nikodym density of Q_i and Q with respect to $\mathbb{P}_i$ and $\mathbb{P}$, respectively.

Further, let $\boldsymbol{\alpha} \triangleq (\alpha_1, \dots, \alpha_N)$, where $\alpha_i \in [0, 1)$ for $i = 1, \dots, N$ standing for the risk level of N-dimensional loss vector. Given $X_i \in \mathcal{X}_i$ for $i = 1, \dots, N$, we denote

$$x_{\alpha_i} \triangleq \inf\{t \in \mathbb{R} : \mathbb{P}_i(X_i \leq t) \geq \alpha_i\} \tag{2.1}$$

as the corresponding α_i quantile of X_i. Furthermore, letting $\mathbf{X}_1 \triangleq (X_1, \dots, X_N) \in \mathcal{X}^N$ and $\mathbf{X}_2 \triangleq (Y_1, \dots, Y_N) \in \mathcal{X}^N$, $\mathbf{X}_1 \leq \mathbf{X}_2$ stands for $X_i \leq Y_i$, $\mathbb{P}_i$ almost surely (a.s.) for $i = 1, 2, \dots, N$. Similarly, $\mathbf{X}_1 + \mathbf{X}_2$ stands for $(X_1 + Y_1, \dots, X_N + Y_N)$. For $\lambda \in \mathbb{R}$ and $\mathbf{X} \triangleq (X_1, \dots, X_N) \in \mathcal{X}^N$, $\lambda\mathbf{X} \triangleq (\lambda X_1, \dots, \lambda X_N)$. Moreover, it is said that $\mathbf{X}_n$ converges pointwise to $\mathbf{X}$ i.e. $\mathbf{X}_n \to \mathbf{X}$ $\mathbb{P}$-a.s if each $X_i^n \to X_i$ $\mathbb{P}_i$ a.s. as $n \to \infty$ for $i = 1, 2, \dots, N$. We summarise the frequent notations and symbols used in the rest of the paper in the Table 2 below.

Table 2: **List of Notation**

$\boldsymbol{X} \triangleq (X_1, \ldots, X_N)$	N-dimensional random vector representing the loss
$\boldsymbol{\alpha} \triangleq (\alpha_1, \ldots, \alpha_N)$	N-dimensional Risk Averseness Level
$x_{\alpha_i} \triangleq \inf\{t \in \mathbb{R} : \mathbb{P}_i(X_i \leq t) \geq \alpha_i\}$	α_i quantile of X_i.
$\mathcal{X}_i = L^1(\Omega_i, \mathcal{F}_i, \mathbb{P}_i)$	Integrable Space of Riskiness of the Random Position i
$\mathcal{X}^N \triangleq \mathcal{X}_1 \times \ldots \times \mathcal{X}_N$	N-dimensional product space of $\mathcal{X}_i$.
$\mathsf{AVaR}_{\alpha_i}(X_i)$	Average-Value-at-Risk of X_i at risk level α_i
$\mathbb{E}^{\mathbb{P}_i}[\cdot]$	Expectation with Respect to $\mathbb{P}_i$
$\mathbb{P} \triangleq \bigotimes_{i=1}^N \mathbb{P}_i$	Reference Probability Measure Composed of the Product of $\mathbb{P}_i$'s.
$\mathcal{F} \triangleq \bigotimes_{i=1}^N \mathcal{F}_i$	Product σ-algebra of N σ-algebras $\mathcal{F}_i$.
$\Omega \triangleq \bigotimes_{i=1}^N \Omega_i$	N-dimensional Product Sample Space of Ω_i
$\mathbb{E}[\cdot]$	Expectation with respect to $\mathbb{P}$.
$\text{mAVaR}_{\boldsymbol{\alpha}}(\boldsymbol{X})$	Multivariate-Average-Value-at-Risk of $\boldsymbol{X}$ at level $\boldsymbol{\alpha}$.
$\mathcal{M}_1^i(\mathcal{X}_i)$	Probability Measures on Ω_i abs. cont. to $\mathbb{P}_i$
$\mathcal{M}_1(\mathcal{X}^N)$	Probability Measures on Ω abs. cont. to $\mathbb{P}$
$\text{tVar}_{\alpha_i}(X_i)$	The tail variance of X_i with risk level α_i
$\text{mtVar}_{\boldsymbol{\alpha}}(\mathbf{X})$	The tail variance of total sum of risk $\mathbf{X}$ with risk ratio $\boldsymbol{\alpha}$
$\text{mAVaR}^G_{\boldsymbol{\alpha}}(\boldsymbol{X})$	Multivariate-AVaR of $\boldsymbol{X}$ at level $\boldsymbol{\alpha}$ for mixture Gaussian.
$\text{mtVar}^{\text{G}}_{\boldsymbol{\alpha}}(\mathbf{X})$	The tail variance for mixture Gaussian of risk $\mathbf{X}$ with $\boldsymbol{\alpha}$

Definition 2.1. *[3] A multivariate risk measure ρ for portfolio vectors is any map from $\mathcal{X}^N$ to $\mathbb{R}$. Given a portfolio $\mathbf{X} \in \mathcal{X}^N$, the expression $\rho(\mathbf{X})$ stands for the amount of risk capital that the holder of portfolio has to invest additionally into its portfolio $\mathbf{X}$ such that the portfolio is acceptable. Letting $\mathbf{X}, \mathbf{X}_1, \mathbf{X}_2 \in \mathcal{X}^N$ and $\mathbf{c} = (c_1, \ldots, c_N)$ with $c_i \in \mathbb{R}$ for $i = 1, \ldots, N$, a multivariate risk measure $\rho : \mathcal{X}^N \to \mathbb{R}$ is called coherent, if ρ satisfies the following axioms:*

(A1) *Monotonicity: If $\mathbf{X}_1 \leq \mathbf{X}_2$, then $\rho(\mathbf{X}_1) \leq \rho(\mathbf{X}_2)$.*

(A2) *Translation invariance: $\rho(\mathbf{X} + \mathbf{c}) = \rho(\mathbf{X}) + c_1 + \ldots + c_N$.*

(A3) *Convexity (Diversification of Risk): $\rho(\lambda \mathbf{X}_1 + (1 - \lambda)\mathbf{X}_2) \leq \lambda \rho(\mathbf{X}_1) + (1 - \lambda)\rho(\mathbf{X}_2)$.*

(A4) *Positive Homogeneity: $\rho(\lambda \mathbf{X}) = \lambda \rho(\mathbf{X})$ for $\lambda \geq 0$.*

Furthermore, a coherent multivariate risk measure ρ is called law-invariant if $\mathbf{X}_1, \mathbf{X}_2 \in \mathcal{X}^N$ and $\mathbb{P}(\mathbf{X}_1 \leq \boldsymbol{z}) = \mathbb{P}(\mathbf{X}_2 \leq \boldsymbol{z})$ for all $\boldsymbol{z} \in \mathbb{R}^N$ implies $\rho(\mathbf{X}_1) = \rho(\mathbf{X}_2)$. Namely, if $\mathbf{X}_1$ and $\mathbf{X}_2$ are equal in distribution, denoted by $\mathbf{X}_1 \stackrel{d}{=} \mathbf{X}_2$, their evaluated losses are equal to each other.

A fundamental law-invariant coherent risk measure in univariate case is Average-Value-at-Risk with a prespecified risk level $0 \leq \alpha_i < 1$ of a random loss $X_i \in \mathcal{X}_i$ denoted by $\mathsf{AVaR}_{\alpha_i}(X)$. We recall that $\mathsf{AVaR}_{\alpha_i}(X_i)$ is defined as

$$\mathsf{AVaR}_{\alpha_i}(X_i) \triangleq \mathbb{E}^{\mathbb{P}_i}[X_i | X_i \geq x_{\alpha_i}] \tag{2.2}$$

which is equal to

$$\mathsf{AVaR}_{\alpha_i}(X_i) \triangleq \sup_{Q_i \in \mathcal{D}_{\alpha_i}} \mathbb{E}^{Q_i}[X_i] \tag{2.3}$$

where

$$\mathcal{D}_{\alpha_i} \triangleq \left\{ Q_i \in \mathcal{M}_1^i(\mathcal{X}_i) : 0 \leq \frac{dQ_i}{d\mathbb{P}_i} \leq \frac{1}{1-\alpha_i} \right\}.$$

(See [5] for the equivalence of (2.2) and (2.3)).

Remark 2.1. *In actuarial and financial literature, the quantile in* (2.1) *is called Value-at-Risk of X_i with risk level $0 \leq \alpha_i < 1$ and is denoted by $\mathrm{VaR}_{\alpha_i}(X_i)$. In particular, $\mathsf{AVaR}_{\alpha_i}(X_i)$ in* (2.2) *is defined as $\mathsf{AVaR}_{\alpha_i} = \mathbb{E}[X_i | X_i \geq \mathrm{VaR}_{\alpha_i}(X_i)]$. That is $\mathsf{AVaR}_{\alpha_i}(X_i)$ is the tail-conditional expectation of X_i from $\mathrm{VaR}_{\alpha_i}(X_i)$. Moreover, AVaR_{α_i} has equivalent names as Conditional-Value-at-Risk at level α_i, denoted by CVaR_{α_i}, Expected-Value-at-Risk at level α_i, denoted by EVaR_{α_i} and Tail-Value-at-Risk at level α_i, denoted by TVaR_{α_i}.*

3 Multivariate Average-Value-at-Risk

3.1 Introduction of the Operator

In this section, first, the multivariate extension of the $\mathrm{mAVaR}_{\boldsymbol{\alpha}}$ is introduced and verified that it is indeed a law invariant coherent risk measure. Examples reveal the flexibility of the proposed operator.

Definition 3.1. *The multivariate Average-Value-at-Risk $\mathrm{mAVaR}_{\boldsymbol{\alpha}} : \mathcal{X}^N \to \mathbb{R}$ for risk level $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N)$ with $0 \leq \alpha_i < 1$ for $i = 1, \ldots, N$ and $\mathbf{X} \in \mathcal{X}^N$ is defined as*

$$\mathrm{mAVaR}_{\boldsymbol{\alpha}}(\mathbf{X}) \triangleq \mathbb{E}\left[\sum_{i=1}^{N} X_i | X_1 \geq x_{\alpha_1}, \ldots, X_N \geq x_{\alpha_N} \right]. \tag{3.4}$$

Remark 3.1. *By independence of X_i's with respect to $\mathbb{P}$ and using* (2.2), (3.4) *translates* *into*

$$\begin{aligned}\mathrm{mAVaR}_{\boldsymbol{\alpha}}(\mathbf{X}) &= \mathbb{E}\Big[\sum_{i=1}^{N} X_i | X_1 \geq x_{\alpha_1}, \ldots, X_N \geq x_{\alpha_N}\Big] \\ &= \sum_{i=1}^{N} \mathbb{E}^{\mathbb{P}_i}[X_i | X_i \geq x_{\alpha_i}] \\ &= \sum_{i=1}^{N} \mathsf{AVaR}_{\alpha_i}(X_i) \qquad (3.5)\end{aligned}$$

Lemma 3.1. $\mathrm{mAVaR}_{\boldsymbol{\alpha}} : \mathcal{X}^N \to \mathbb{R}$ *is a multivariate law-invariant coherent risk measure.*

Proof. Let $\mathbf{X}_1 = (X_1, \ldots, X_N) \in \mathcal{X}^N$ and $\mathbf{X}_2 = (Y_1, \ldots, Y_N) \in \mathcal{X}^N$ with $\mathbf{X}_1 \overset{d}{=} \mathbf{X}_2$. In particular, $X_i \overset{d}{=} Y_i$ for $i = 1, \ldots, N$. Since each $\mathsf{AVaR}_{\alpha_i}(\cdot)$ is law-invariant, law-invariance of $\mathrm{mAVaR}_{\boldsymbol{\alpha}}$ follows. The axioms in Definition 2.1 follow immediately by (3.4). □

We first state the following result that is by Theorem 3.1 and Remark 3.2 of [3].

Theorem 3.1. *Let* $\mathbf{X} = (X_1, \ldots, X_N) \in \mathcal{X}^N$*. Any multivariate law-invariant coherent risk measure* $\rho : \mathcal{X}^N \to \mathbb{R}$ *as in Definition 2.1 is of the form*

$$\rho(\mathbf{X}) = \sup_{(Q_1,\ldots,Q_N)\in\mathcal{D}} \Big(\mathbb{E}^{Q_1}[X_1] + \ldots + \mathbb{E}^{Q_N}[X_N]\Big), \qquad (3.6)$$

for some $\mathcal{D} \triangleq \mathcal{D}_1 \times \ldots \times \mathcal{D}_N$*, where each* $\mathcal{D}_i$ *is a subset of* $\mathcal{M}_1^i(\mathcal{X}_i)$*.*

Next corollary refines Theorem 3.1 and gives an explicit representation for (3.6).

Corollary 3.1. *Let* $\mathbf{X} \in \mathcal{X}^N$*. Any multivariate coherent risk measure* $\rho : \mathcal{X}^N \to \mathbb{R}$ *satisfying the conditions in Definition 2.1 is of the form*

$$\rho(X) = \sum_{i=1}^{N} \rho_i(X_i),$$

where $\rho_i : \mathcal{X}_i \to \mathbb{R}$ *is a univariate coherent risk measure.*

Proof. By (3.6), we have

$$\rho(X) = \sup_{(Q_1,\ldots,Q_N)\in\mathcal{D}} (\mathbb{E}^{Q_1}[X_1] + \ldots + \mathbb{E}^{Q_N}[X_N]).$$

Since $\mathcal{D} = \mathcal{D}_1 \otimes \ldots \otimes \mathcal{D}_N$ by Theorem 3.1, where each $\mathcal{D}_i \subset \mathcal{M}_i^1(\mathcal{X}_i)$, it holds that

$$\rho(X) = \sup_{\mathcal{D}_1 \subset \mathcal{M}_1^1(\mathcal{X}_1)} \mathbb{E}^{Q_1}[X_1] + \ldots + \sup_{\mathcal{D}_N \subset \mathcal{M}_1^N(\mathcal{X}_N)} \mathbb{E}^{Q_N}[X_N]$$

Denoting each $\rho_i(X_i) \triangleq \sup_{\mathcal{D}_i \subset \mathcal{M}_1^i(\mathcal{X}_1)} \mathbb{E}^{Q_i}[X_i]$, the proof is concluded. □

By Corollary 3.1, the following representation of $\mathrm{mAVaR}_{\boldsymbol{\alpha}}$ is retrieved.

Lemma 3.2. *Let* $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N) \in [0,1)^N$, $\mathbf{X} \in \mathcal{X}^N$ *and* $\mathrm{mAVaR}_{\boldsymbol{\alpha}}(\mathbf{X})$ *be as defined in Definition 3.4. Then,*

$$\begin{aligned}\mathrm{mAVaR}_{\boldsymbol{\alpha}}(\mathbf{X}) &= \sup_{Q \in \mathcal{D}_{\boldsymbol{\alpha}}} \mathbb{E}^Q\left[\sum_{i=1}^N X_i\right] \\ &= \sum_{i=1}^N \sup_{\mathcal{D}_{\alpha_i}} \mathbb{E}^{Q^i}[X_i],\end{aligned}$$

where $\mathcal{D}_{\boldsymbol{\alpha}} \in \mathcal{M}_1(\mathcal{X}^N)$ *of the form* $\mathcal{D}_{\alpha_1} \times \ldots \mathcal{D}_{\alpha_N} \subset \prod_{i=1}^N \mathcal{M}_1^i(\mathcal{X}_i)$ *with*

$$\mathcal{D}_{\alpha_i} \triangleq \left\{ Q_i \in \mathcal{M}_1^i : 0 \leq \frac{dQ_i}{d\mathbb{P}_i} \leq \frac{1}{1-\alpha_i} \right\}, \quad \text{for } i = 1, \ldots N.$$

Proof. By Lemma 3.1, $\mathrm{mAVaR}_{\boldsymbol{\alpha}}(\mathbf{X})$ is a multivariate coherent risk measure. Then, by Theorem 3.1,

$$\mathrm{mAVaR}_{\boldsymbol{\alpha}}(\mathbf{X}) = \sup_{Q \in \mathcal{D}_{\boldsymbol{\alpha}}} \mathbb{E}\left[\sum_{i=1}^N X_i\right], \tag{3.7}$$

for some $\mathcal{D}_{\boldsymbol{\alpha}} \subset \mathcal{M}_1(\mathcal{X}^N)$. Using (2.3) and (3.5), it is concluded that

$$\begin{aligned}\mathrm{mAVaR}_{\boldsymbol{\alpha}}(\mathbf{X}) &= \sum_{i=1}^N \mathsf{AVaR}_{\alpha_i}(X_i) \\ &= \sum_{i=1}^N \sup_{Q_i \in \mathcal{D}_{\alpha_i}} \mathbb{E}^{Q_i}[X_i].\end{aligned} \tag{3.8}$$

Hence, by (3.7) and (3.8), the result follows. □

The next example will shed light into flexibility of $\mathrm{mAVaR}_{\boldsymbol{\alpha}}$ compared to the sum of N independent random variables $\sum_{i=1}^N X_i$ applied to univariate $\mathsf{AVaR}_{\widetilde{\alpha}}$ with the corresponding risk level $\widetilde{\alpha}$.

Example 3.1. *Let* $\mathbf{X} = (X_1, \ldots, X_N) \in \mathcal{X}^N$ *be a random vector with independent marginals, and fix* $\alpha \in [0,1)$*. Define* $Z = \sum_{i=1}^N X_i$ *on* $\mathcal{X}^N$*, and denote the* α *quantile of* Z *as* z_α *as in*

(2.1). *Note that*

$$\mathsf{AVaR}_\alpha\big(Z\big) = \mathbb{E}\big[Z\big|Z \geq z_\alpha\big]$$
$$= \sup_{Q\in\mathcal{D}_\alpha} \mathbb{E}^Q\Bigg[\sum_{i=1}^N X_i\Bigg],$$

where

$$\mathcal{D}_\alpha \triangleq \Bigg\{Q \in \mathcal{M}_1(\mathcal{X}^N) : 0 \leq \frac{dQ}{d\mathbb{P}} \leq \frac{1}{1-\alpha}\Bigg\}.$$

On the other hand, letting $\widetilde{\alpha} \triangleq (1-\alpha)^{1/N}$,

$$\mathsf{AVaR}_\alpha\Bigg(\sum_{i=1}^N X_i\Bigg) = \sum_{i=1}^N \sup_{Q_i\in\mathcal{D}_{\widetilde{\alpha}}} \mathbb{E}^{Q_i}[X_i]$$

with

$$\mathcal{D}_{\widetilde{\alpha}} \triangleq \Bigg\{Q_i \in \mathcal{M}_1(\mathcal{X}_i) : 0 \leq \frac{dQ_i}{d\mathbb{P}_i} \leq \frac{1}{\widetilde{\alpha}}\Bigg\}.$$

Thus, choosing $\widetilde{\boldsymbol{\alpha}} = (\widetilde{\alpha}, \ldots, \widetilde{\alpha})$ *and by* (3.5),

$$\mathrm{mAVaR}_{\widetilde{\boldsymbol{\alpha}}}(\mathbf{X}) = \sum_{i=1}^N \mathsf{AVaR}_{\widetilde{\alpha}}(X_i)$$
$$= \mathsf{AVaR}_\alpha\Bigg(\sum_{i=1}^N X_i\Bigg)$$

Thus, only if the risk level α_i *for each* X_i *is equal to each other for each* $i \in [1, \ldots, N]$, *then* $\mathrm{mAVaR}_{\boldsymbol{\alpha}}(\mathbf{X})$ *can be represented via one-variate* AVaR, *choosing the corresponding risk level* $\widetilde{\alpha}$.

On the other hand, consider $\mathrm{mAVaR}_{\boldsymbol{\alpha}}$ *with* $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N)$, *where* $\alpha_i \neq \alpha_j$ *for some* $i, j \in [1, \ldots, N]$. *Then,* $\mathrm{mAVaR}_{\boldsymbol{\alpha}}$ *can not be represented using* $\mathsf{AVaR}_\alpha(\sum_{i=1}^N X_i)$ *with* N *independent* X_i*'s and a fixed* α *risk level.*

Indeed, take $N = 2$, *and let* $0 < \alpha_1 < \alpha_2 < 1$. *Consider two independent random variables* X_1 *and* X_2 *on* $[0,1]$ *with Lebesgue measure* $\mathbb{P}_1, \mathbb{P}_2$ *on* $[0,1]$ *with their density functions* $f_{X_1}(x) = x$ *and* $f_{X_2}(x) = x/2$ *for* $x \in [0,1]$. *Let* $\mathbf{X}_1 = (X_1, X_2)$ *and* $\mathbf{X}_2 = (X_2, X_1)$. *Then,* *for* $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ *and for some* $\alpha \in [0, 1)$,

$$\begin{aligned}
\mathrm{mAVaR}_{\boldsymbol{\alpha}}(\mathbf{X}_1) &= \mathbb{E}^{\mathbb{P}_1}[X_1 | X_1 \geq x_{\alpha_1}] + \mathbb{E}^{\mathbb{P}_2}[X_2 | X_2 \geq x_{\alpha_2}] \\
&= \mathsf{AVaR}_{\alpha}(X_1 + X_2) \\
&= \mathbb{E}[X_1 + X_2 | X_1 + X_2 \geq x_{\alpha}] \\
\mathrm{mAVaR}_{\boldsymbol{\alpha}}(\mathbf{X}_2) &= \mathbb{E}^{\mathbb{P}_1}[X_2 | X_2 \geq x_{\alpha_1}] + \mathbb{E}^{\mathbb{P}_2}[X_1 | X_1 \geq x_{\alpha_2}] \\
&= \mathsf{AVaR}_{\alpha}(X_2 + X_1) \\
&= \mathbb{E}[X_1 + X_2 | X_1 + X_2 \geq x_{\alpha}],
\end{aligned}$$

Thus,

$$\frac{1}{1-\alpha_1}\frac{1}{2}(1-\alpha_1^2) + \frac{1}{4(1-\alpha_2)}(1-\alpha_2^2) = \frac{1}{1-\alpha_2}\frac{1}{2}(1-\alpha_2^2) + \frac{1}{4(1-\alpha_1)}(1-\alpha_1^2)$$

which implies $\alpha_1 = \alpha_2 = \alpha$. *Hence, a contradiction.*

This shows that $\mathrm{mAVaR}_{\boldsymbol{\alpha}}$ *achieves a bigger flexibility than the sum of* N *independent risks required in multidimensional setting, while keeping the axioms of a multivariate coherent risk measure.*

The next example shows that the conditional expectation of a sum of dependent random variables, conditioned on its corresponding quantiles, might fail to be a coherent risk measure by violating the convexity axiom of a coherent risk measure.

Example 3.2. *Consider a random variable* $X \in \mathcal{X}_i$ *having a continuous distribution and taking positive values with positive measure, and let* $Y = \frac{X}{2}$. *Fix* $\alpha \in (0, 1)$ *such that* $x_\alpha > 0$ *is the corresponding* α *quantile of* X. *Choose* $\widetilde{\alpha}$ *such that* $\mathbb{P}_i(Y \leq x_\alpha) = \widetilde{\alpha}$, *i.e.* $\mathbb{P}_i(X \leq 2x_\alpha) = \widetilde{\alpha}$, *and denote the corresponding quantile as* $y_{\widetilde{\alpha}}$. *Then,*

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}_i}\Big[X + X/2 | X \geq x_\alpha, X \geq 2y_{\widetilde{\alpha}}\Big] &= \mathbb{E}^{\mathbb{P}_i}\Big[X + X/2 | X \geq x_\alpha, X \geq 2x_\alpha\Big] \\
&= \frac{3}{2}\mathbb{E}[X | X \geq 2x_\alpha]
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}_i}[3/2X | X \geq 2x_\alpha] &> \mathbb{E}^{\mathbb{P}_i}[X | X \geq x_\alpha] + \mathbb{E}^{\mathbb{P}_i}[1/2X | X \geq 2x_\alpha] \\
\mathbb{E}^{\mathbb{P}_i}[X | X \geq 2x_\alpha] &> \mathbb{E}^{\mathbb{P}_i}[X | X \geq x_\alpha]
\end{aligned}$$

In particular, diversification of risk axiom (A3) in Definition 2.1 can be violated in case the random variables are dependent.

On the other hand, $\mathrm{mAVaR}_{\boldsymbol{\alpha}}$ *with* $N = 1$ *reduces to* AVaR_{α}*. Indeed, for* $N = 1$*,* $\boldsymbol{\alpha} = \alpha_1$*,* $\mathbf{X}_1 = X$ *and* $\mathbf{X}_2 = Y$*. Denote* $X + Y = Z$ *such that the corresponding quantile of* Z *as* z_α *with* $\inf\{t \in \mathbb{R} : \mathbb{P}(Z \leq z_\alpha) \geq \alpha\}$*. Hence,*

$$\begin{aligned}\mathbb{E}[3/2X|Z \geq z_\alpha] &= \mathsf{AVaR}_\alpha(3/2X)\\ \mathsf{AVaR}_\alpha(3/2X) &\leq \mathsf{AVaR}_\alpha(X) + \mathsf{AVaR}_\alpha(Y)\\ &= \mathbb{E}[X|X \geq x_\alpha] + \mathbb{E}[Y|Y \geq y_\alpha]\end{aligned}$$

In particular, $\mathrm{mAVaR}_{\boldsymbol{\alpha}}$ *for* $N = 1$ *preserves convexity axiom of a univariate coherent risk measure. This also indicates that* $\mathrm{mAVaR}_{\boldsymbol{\alpha}}$ *is the natural extension of* AVaR_α *for* $N > 1$*.*

3.2 Continuity Properties of $\mathrm{mAVaR}_{\boldsymbol{\alpha}}$

We next introduce the relevant continuity notions in the following definition, which will be verified to hold for $\mathrm{mAVaR}_{\boldsymbol{\alpha}}$.

Definition 3.2. *Let* $\rho : \mathcal{X}^N \to \mathbb{R}$ *be a multivariate coherent risk measure,* $(\mathbf{X}_n \triangleq (X_1^n, \ldots, X_N^n))_{n\geq 1}$ *be a sequence in* $\mathcal{X}^N$ *and* $\mathbf{X} \triangleq (X_1, \ldots X_N) \in \mathcal{X}^N$*. Then,*

- ρ *is called continuous from above if* $\mathbf{X}_n \downarrow \mathbf{X}$ *implies that* $\lim_{n\to\infty} \rho(\mathbf{X}_n) = \lim \rho(\mathbf{X})$*.*
- ρ *is called continuous from below if* $\mathbf{X}_n \uparrow \mathbf{X}$ *implies that* $\lim_{n\to\infty} \rho(\mathbf{X}_n) = \lim \rho(\mathbf{X})$*.*
- ρ *is said to have Fatou property if, for every* $i = 1, \ldots N$*,* $|X_i^n| \leq Y_i$ $\mathbb{P}_i$ *a.s. for some* $Y_i \in \mathcal{X}_i$*, and* $X_i^n \to X_i$ $\mathbb{P}_i$ *a.s. imply that* $\rho(\mathbf{X}) \leq \liminf_{n\to\infty} \rho(\mathbf{X}_n)$*.*
- ρ *is called Lebesgue-continuous if* $\mathbf{X}_n \to \mathbf{X}$*,* $\mathbb{P}-$*a.s. and if, for every* $i = 1, \ldots N$*,* $|X_i^n| \leq Y_i$ $\mathbb{P}_i$ *a.s. for some* $Y_i \in \mathcal{X}_i$ *and* $X_i^n \to X_i$ $\mathbb{P}_i$ *a.s., then* $\lim_{n\to\infty} \rho(\mathbf{X}_n) = \lim \rho(\mathbf{X})$*.*

We next recall a result for Average-Value-at-Risk in univariate case, i.e. $N = 1$.

Theorem 3.2. *[17, 2] Let* $\alpha \in [0, 1)$ *be fixed. Then,* $\mathsf{AVaR}_\alpha : \mathcal{X}_i \to \mathbb{R}$ *is*

(i) continuous from below,

(ii) continuous from above,

(iii) has Fatou property,

(iv) is Lebesgue continuous.

Corollary 3.2. $\mathrm{mAVaR}_{\boldsymbol{\alpha}}$ *has the properties stated in Definition 3.2.*

Proof. The result follows by the representation (3.8) and Equation (3.5). □

3.3 A Tail Variance for $\mathrm{mAVaR}_{\boldsymbol{\alpha}}$

In this section, we assume that, for $\mathbf{X} = (X_1, \dots, X_N) \in \mathcal{X}^N$, each $X_i \in \mathcal{X}_i$ is further an element of $L^2(\Omega_i, \mathcal{F}_i, \mathbb{P}_i)$, i.e. $\mathbb{E}^{\mathbb{P}_i}[(X_i)^2] < \infty$ for all $i = 1, \dots, N$.

Definition 3.3.

(i) *Let $X_i \in \mathcal{X}_i$ and $\alpha_i \in [0, 1)$ be the risk level. The tail variance of X_i with risk level α_i is defined as*

$$\mathrm{tVar}_{\alpha_i}(X_i) \triangleq \mathbb{E}^{\mathbb{P}_i}\left[\left(X_i - \mathsf{AVaR}_{\alpha_i}(X_i)\right)^2 | X_i \geq x_{\alpha_i}\right]$$

(ii) *Let $\mathbf{X} \in \mathcal{X}^N$ and $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_N) \in [0, 1)^N$ be the risk ratio. The tail variance of total sum of risk $\mathbf{X}$ with risk ratio $\boldsymbol{\alpha}$ is defined as*

$$\mathrm{mtVar}_{\boldsymbol{\alpha}}(\mathbf{X}) \triangleq \mathbb{E}\Big[\big(\sum_{i=1}^{N} X_i - \mathrm{mAVaR}_{\boldsymbol{\alpha}}(\mathbf{X})\big)^2 | X_1 \geq x_{\alpha_1}, \dots, X_N \geq x_{\alpha_N}\Big] \tag{3.9}$$

We next list the properties that reveal the analogy between variance and multivariate tail conditional variance.

Lemma 3.3. *Let $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_N) \in [0, 1)^N$, $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_N) \in \mathbb{R}^N$ and $\boldsymbol{c} = (c_1, \dots, c_N) \in \mathbb{R}^N$, and let $\mathbf{X} = (X_1, \dots, X_N) \in \mathcal{X}^N$. Then, $\mathrm{mtVar}_{\boldsymbol{\alpha}}(\mathbf{X})$ satisfies the following properties.*

(i) $\mathrm{mtVar}_{\boldsymbol{\alpha}}(\mathbf{X}) = \sum_{i=1}^{N} \mathrm{tVar}_{\alpha_i}(X_i)$

(ii) $\mathrm{mtVar}_{\boldsymbol{\alpha}}(\boldsymbol{\lambda}\mathbf{X}) = \sum_{i=1}^{N} \lambda_i^2 \mathrm{tVar}_{\alpha_i}(X_i)$

(iii) $\mathrm{mtVar}_{\boldsymbol{\alpha}}(\mathbf{X} + \boldsymbol{c}) = \mathrm{mtVar}_{\boldsymbol{\alpha}}(\mathbf{X})$

Proof. (i) We have

$$\begin{aligned}
&\mathbb{E}\Big[\big(\sum_{i=1}^{N} X_i - \sum_{i=1}^{N} \mathsf{AVaR}_{\boldsymbol{\alpha_i}}(X_i)\big)^2 | X_1 \geq x_{\alpha_1}, \dots, X_N \geq x_{\alpha_N}\Big] \\
&= \sum_{i=1}^{N} \mathbb{E}\big[(X_i - \mathsf{AVaR}_{\alpha_i}(X_i))^2 | X_1 \geq x_{\alpha_1}, \dots, X_N \geq x_{\alpha_N}\big] \\
&\quad + 2 \sum_{1 \leq i < j \leq N} \mathbb{E}\big[(X_i - \mathsf{AVaR}_{\alpha_i}(X_i))(X_j - \mathsf{AVaR}_{\alpha_j}(X_j) | X_i \geq x_{\alpha_i}, X_j \geq x_{\alpha_j})\big] \\
&= \sum_{i=1}^{N} \mathbb{E}\big[(X_i - \mathsf{AVaR}_{\alpha_i}(X_i))^2 | X_1 \geq x_{\alpha_1}, \dots, X_N \geq x_{\alpha_N}\big] \\
&= \sum_{i=1}^{N} \mathrm{tVar}_{\alpha_i}(X_i),
\end{aligned}$$

where the last two equalities are implied by joint independence of $X_1, \dots, X_N$.

(ii) The result follows by

$$\begin{aligned}\mathrm{mtVar}_{\boldsymbol{\alpha}}(\boldsymbol{\lambda}\mathbf{X}) &= \sum_{i=1}^N \mathbb{E}\Big[\big(\sum_{i=1}^N \lambda_i X_i - \mathsf{AVaR}_{\alpha_i}(\lambda_i X_i)\big)^2 | X_1 \geq x_{\alpha_1}, \dots, X_N \geq x_{\alpha_N}\Big] \\ &= \sum_{i=1}^N \lambda_i^2 \mathbb{E}\Big[\big(\sum_{i=1}^N X_i - \mathsf{AVaR}_{\alpha_i}(X_i)\big)^2 | X_1 \geq x_{\alpha_1}, \dots, X_N \geq x_{\alpha_N}\Big] \\ &= \sum_{i=1}^N \lambda_i^2 \mathbb{E}\Big[\big(\sum_{i=1}^N X_i - \mathsf{AVaR}_{\alpha_i}(X_i)\big)^2 | X_i \geq x_{\alpha_i}\Big]\end{aligned}$$

(iii) The result follows by

$$\begin{aligned}\mathrm{mtVar}_{\boldsymbol{\alpha}}(\mathbf{X} + \boldsymbol{c}) &= \sum_{i=1}^N \mathbb{E}[(X_i + c_i - \mathsf{AVaR}_{\alpha_i}(X_i + c_i))^2 | X_i \geq x_{\alpha_i}] \\ &= \sum_{i=1}^N \mathbb{E}[(X_i - \mathsf{AVaR}_{\alpha_i}(X_i))^2 | X_i \geq x_{\alpha_i}]\end{aligned}$$

□

The following tail inequality of Chebyshev type gives the connection between $\mathrm{mtVar}_{\boldsymbol{\alpha}}(\cdot)$ and $\mathrm{mAVaR}_{\boldsymbol{\alpha}}(\cdot)$. In particular, as in the classical case, scaling by $\sqrt{\mathrm{mtVar}_{\boldsymbol{\alpha}}}$ and shifting by $\mathrm{mAVaR}_{\boldsymbol{\alpha}}$, one can get the confidence interval of how much $\mathbf{X} \in \mathcal{X}^N$ deviates from $\mathrm{mAVaR}_{\boldsymbol{\alpha}}(\mathbf{X})$ in the tail event $\{X_1 \geq x_{\alpha_1}, \dots, X_N \geq x_{\alpha_N}\}$ and how reliable $\mathrm{mAVaR}_{\boldsymbol{\alpha}}$ is.

Lemma 3.4. *Let $k > 0$ with $\mathbf{X} = (X_1, \dots, X_N) \in \mathcal{X}^N$, and each $X_i \in \mathcal{X}_i$ is in $L^2(\Omega_i, \mathcal{F}_i, \mathbb{P}_i)$, i.e. $\mathbb{E}^{\mathbb{P}_i}[(X_i)^2] < \infty$ for all $i = 1, \dots, N$. Then, we have*

$$\mathbb{P}\bigg(\frac{|\sum_{i=1}^N X_i - \mathrm{mAVaR}_{\boldsymbol{\alpha}}(\mathbf{X})|}{\sqrt{\mathrm{mtVar}_{\boldsymbol{\alpha}}(\mathbf{X})}} \geq k \big| X_1 \geq x_{\alpha_1}, \dots, X_N \geq x_{\alpha_N}\bigg) \leq \frac{1}{k^2}$$

Proof. The proof follows the same lines as in classical Chebyshev inequality. We have

$$\begin{aligned}k^2 \mathbb{P}&\left(\left(\sum_{i=1}^N X_i - \mathsf{AVaR}_{\alpha_i}(X_i)\right)^2 \geq k^2 | X_1 \geq x_{\alpha_1}, \dots, X_N \geq x_{\alpha_N}\right) \\ &\leq \mathbb{E}\left[\left(\sum_{i=1}^N X_i - \mathsf{AVaR}_{\alpha_i}(X_i)\right)^2 | X_1 \geq x_{\alpha_1}, \dots, X_N \geq x_{\alpha_N}\right]\end{aligned}$$

Thus,

$$
\begin{aligned}
&\mathbb{E}\Big[\sum_{i=1}^{N}(X_i - \mathsf{AVaR}_{\alpha_i}(X_i))^2 | X_1 \geq x_{\alpha_1}, \ldots, X_N \geq x_{\alpha_N}\Big] \\
&\qquad + 2 \sum_{1\leq i<j\leq N} \mathbb{E}\Big[(X_i - \mathsf{AVaR}_{\alpha_i}(X_i))(X_j - \mathsf{AVaR}_{\alpha_j}(X_j)) \\
&\qquad\qquad | X_1 \geq x_{\alpha_1}, \ldots, X_N \geq x_{\alpha_N}\Big] \\
&= \sum_{i=1}^{N} \mathbb{E}\big[(X_i - \mathsf{AVaR}_{\alpha_i}(X_i))^2 | X_i \geq x_{\alpha_i}\big] \qquad (3.10) \\
&= \sum_{i=1}^{N} \mathrm{tVar}_{\alpha_i}(X_i) \\
&= \mathrm{mtVar}_{\boldsymbol{\alpha}}(\mathbf{X}),
\end{aligned}
$$

where (3.10) follows by joint independence of $(X_1, \ldots, X_n)$. Hence, we conclude the proof. □

4 A Tail Conditional Expectation for Normal Mixture Models

In this section, we will work with multivariate Gaussian mixture random vectors, where the marginals can be *dependent* normal random variables. Since $\mathrm{mAVaR}_{\boldsymbol{\alpha}}$, defined as in (3.4), requires that the marginals are independent, Definition 2.1 of a coherent risk measure can not be used for an operator on a random vector with dependent marginals. However, any multivariate Gaussian random vector with nondegenerate, i.e. positive definite, covariance matrix, can be represented as a Gaussian random vector of linear combinations of independent Gaussian marginals. In particular, this gives a way to evaluate tail conditional expectation *defined* in (4.11) by applying to the sum of the linear combinations of an N-dimensional Gaussian random vector $\boldsymbol{Y} \in \mathcal{X}^N$m, whose marginals are independent standard normal random variables. This is of fundamental importance, where the cost is modeled as a Normal mixture random vector and is evaluated by keeping the axiomatic features of a coherent risk measure. We refer the reader to [20] for the class of Normal mixture multivariate distributions and its prevalent use in financial and actuarial modelling.

4.1 Multivariate Gaussian Model

We *define* first an analogous average-value-at-risk with risk level $\boldsymbol{\alpha}$ that is applied on the multivariate Gaussian random vector.

Definition 4.1. *Let* $\mathbf{X} = (X_1, \dots, X_N)$ *be a multivariate nondegenerate Gaussian random vector with positive definite covariance matrix*

$$\Sigma = \begin{bmatrix} \sigma_{11}^2 & \dots & \sigma_{1N} \\ \sigma_{21} & \dots & \sigma_{2N} \\ & \dots & \\ \sigma_{N1} & \dots & \sigma_{NN}^2 \end{bmatrix}$$

and mean vector $\boldsymbol{\mu} = (\mu_1, \dots, \mu_N)$*, denoted by* $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$*. The multivariate Average-Value-at-Risk for Gaussian random vectors for the risk level* $\boldsymbol{\alpha} \in [0,1)^N$ *is defined as*

$$\text{mAVaR}^G_{\boldsymbol{\alpha}}(\mathbf{X}) \triangleq \mathbb{E}\left[\sum_{i=1}^N X_i | X_1 \geq x_{\alpha_1}, \dots, X_N \geq x_{\alpha_N}\right]. \tag{4.11}$$

The following theorem shows that $\text{mAVaR}^G_{\boldsymbol{\alpha}}(\mathbf{X})$ is equal to a linear combination of univariate Average-Value-at-Risk operators at the risk level $\widetilde{\boldsymbol{\alpha}} = \widetilde{\alpha}_1, \dots, \widetilde{\alpha}_N \in [0,1)^N$ applied to N independent standard normal random variables.

Theorem 4.1. *Let* $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ *be a multivariate nondegenerate Gaussian random vector with positive definite covariance matrix* $\boldsymbol{\Sigma}$ *and mean vector* $\boldsymbol{\mu}$*. Suppose one wants to evaluate the tail risk of* $\mathbf{X}$ *for* $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_N) \in [0,1)^N$ *via* (4.11)*, Then,*

$$\begin{aligned} \text{mAVaR}^G_{\boldsymbol{\alpha}}(\mathbf{X}) &= \sum_{i=1}^N \mu_i + \sum_{i=1}^N \sum_{j=1}^N \lambda_{ij} \mathbb{E}^{\mathbb{P}_j}\left[Y_j | Y_j \geq y_{\widetilde{\alpha}_j}\right] \\ &= \sum_{i=1}^N \mu_i + \sum_{i=1}^N \sum_{j=1}^N \lambda_{ij} \mathsf{AVaR}_{\widetilde{\alpha}_j}(Y_j) \\ &= \sum_{i=1}^N \mu_i + \sum_{i=1}^N \sum_{j=1}^N \lambda_{ij} \frac{\phi(y_{\widetilde{\alpha}_j})}{1 - \widetilde{\alpha}_j}, \end{aligned}$$

where $\boldsymbol{Y} \triangleq (Y_1, \dots, Y_N)$ *is an* N*-dimensional standard Gaussian random vector that are mutually independent with* $\phi(\cdot)$ *being the probability density function of a standard normal random variable with* $\boldsymbol{y}_{\boldsymbol{\alpha}} \triangleq (y_{\widetilde{\alpha}_1}, \dots, y_{\widetilde{\alpha}_N}) \in \mathbb{R}^N$ *and* $\widetilde{\boldsymbol{\alpha}} = (\widetilde{\alpha}_1, \widetilde{\alpha}_2, \dots, \widetilde{\alpha}_N) \in [0,1)^N$*. Here* $\boldsymbol{y}_{\widetilde{\boldsymbol{\alpha}}}$ *and* $\widetilde{\boldsymbol{\alpha}}$ *are determined by* $\boldsymbol{x}_{\boldsymbol{\alpha}}$ *and* $\boldsymbol{\alpha}$*, respectively.*

Proof. Since $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with $\boldsymbol{\Sigma}$ being positive definite, there exists a lower triangular matrix $\boldsymbol{A}$ with positive diagonal entries of the the form $\boldsymbol{A}\boldsymbol{A}^\intercal = \boldsymbol{\Sigma}$ by Cholesky decomposition, where each X_i can be written of the form

$$X_i \stackrel{d}{=} \mu_i + \sum_{j=1}^{N} \lambda_{ij} Y_j \text{ for } i = 1, \ldots, N.$$

Here, $\boldsymbol{Y} = (Y_1, , \ldots, Y_N) \in \mathcal{X}^N$ is a Gaussian standard random vector with independent marginals for some scalars λ_{ij} for $i, j = 1, \ldots, N$ with $\lambda_{ij} = 0$ for $i < j$ and $\lambda_{ii} > 0$ for $i = \{1, 2, \ldots, N\}$ (see e.g. [4]). Namely, $\mathbf{X} = \boldsymbol{\mu} + \boldsymbol{A}\boldsymbol{Y}$, where

$$\boldsymbol{A} = \begin{bmatrix} \lambda_{11} & 0 & \ldots & 0 \\ \lambda_{21} & \lambda_{22} & \ldots & 0 \\ & \ldots & \ddots & \\ \lambda_{N1} & \lambda_{N2} & \ldots & \lambda_{NN} \end{bmatrix} \tag{4.12}$$

is an $\mathbb{R}^{N \times N}$ matrix with $\boldsymbol{A}\boldsymbol{A}^\intercal = \boldsymbol{\Sigma}$ and $\boldsymbol{\mu} \in \mathbb{R}^N$ is the N-dimensional mean vector of $\boldsymbol{X}$. Denoting $\boldsymbol{x_\alpha} \triangleq (x_{\alpha_1}, \ldots, x_{\alpha_N})$, to transform the condition $\{\boldsymbol{X} \geq \boldsymbol{x_\alpha}\}$ for $\boldsymbol{X}$ to a corresponding conditioning for $\boldsymbol{Y}$, one solves the linear system $\boldsymbol{\mu} + \boldsymbol{A}\boldsymbol{y}_\alpha = \boldsymbol{x_\alpha}$ with $\boldsymbol{x_\alpha} = (x_{\alpha_1}, \ldots, x_{\alpha_N})$. This gives

$$\begin{aligned} &\mu_1 + \lambda_{11} y_{\widetilde{\alpha}_1} = x_{\alpha_1} \\ &\mu_2 + \lambda_{21} y_{\widetilde{\alpha}_1} + \lambda_{22} y_{\widetilde{\alpha}_2} = x_{\alpha_2} \\ &\ldots \quad \ldots \quad \ldots \quad \ldots \\ &\mu_N + \lambda_{N1} y_{\widetilde{\alpha}_1} + \lambda_{N2} y_{\widetilde{\alpha}_2} + \ldots + \lambda_{NN} y_{\widetilde{\alpha}_N} = x_{\alpha_N}, \end{aligned} \tag{4.13}$$

which reveals $\boldsymbol{y_{\widetilde{\alpha}}} \triangleq (y_{\widetilde{\alpha}_1}, y_{\widetilde{\alpha}_2}, \ldots, y_{\widetilde{\alpha}_N})$ for given quantiles $\boldsymbol{x_\alpha} = (x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_N})$. Note that the system (4.13) has a unique solution since $\boldsymbol{A}\boldsymbol{A}^\intercal = \boldsymbol{\Sigma}$ is positive definite, in particular, matrix $\boldsymbol{A}$ in (4.12) is invertible. Thus, the linear system (4.13) has a unique solution with N equations and N unknowns. Furthermore, by noting that $\boldsymbol{A}$ has positive diagonal elements, we have $\{\boldsymbol{X} \geq \boldsymbol{x_\alpha}\} = \{\boldsymbol{Y} \geq \boldsymbol{y_{\widetilde{\alpha}}}\}$. Identically,

$$\{X_1 \geq x_{\alpha_1}, X_2 \geq x_{\alpha_2}, \ldots, X_N \geq x_{\alpha_N}\} = \{Y_1 \geq y_{\widetilde{\alpha}_1}, Y_2 \geq y_{\widetilde{\alpha}_2}, \ldots, Y_N \geq y_{\widetilde{\alpha}_N}\}.$$

Next, we denote

$$\begin{aligned} \widetilde{\alpha}_j &\triangleq \mathbb{P}_j(Y_j \geq y_{\widetilde{\alpha}_j}) \\ &= 1 - \Phi(y_{\widetilde{\alpha}}) \text{ for } j = 1, \ldots, N, \end{aligned}$$

where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal random variable. Hence,

$$
\begin{aligned}
\mathrm{mAVaR}^G_{\boldsymbol{\alpha}}(\mathbf{X}) &= \sum_{i=1}^N \mu_i + \sum_{i=1}^N \lambda_{ij}\mathbb{E}\Big[\sum_{j=1}^N Y_j | Y_1 \geq y_{\widetilde{\alpha}_1}, \ldots, Y_N \geq y_{\widetilde{\alpha}_N}\Big] \\
&= \sum_{i=1}^N \mu_i + \sum_{i=1}^N \sum_{j=1}^N \lambda_{ij}\mathbb{E}^{\mathbb{P}_j}\Big[Y_j | Y_j \geq y_{\widetilde{\alpha}_j}\Big] \\
&= \sum_{i=1}^N \mu_i + \sum_{i=1}^N \sum_{j=1}^N \lambda_{ij}\mathsf{AVaR}_{\widetilde{\alpha}}(Y_j) \qquad (4.14) \\
&= \sum_{i=1}^N \mu_i + \sum_{i=1}^N \sum_{j=1}^N \lambda_{ij}\frac{\phi(y_{\widetilde{\alpha}_j})}{1-\widetilde{\alpha}_j},
\end{aligned}
$$

where $\phi(\cdot)$ is the probability density function of a standard normal random variable. In (4.14), we used the closed form formula for the standard normal random variable Y_i with with risk level $\widetilde{\alpha}_i$ and $y_{\widetilde{\alpha}_i}$ as the corresponding quantile. The derivation of (4.14) is

$$
\begin{aligned}
\mathsf{AVaR}_{\widetilde{\alpha}_j}(Y_j) &= \mathbb{E}^{\mathbb{P}_j}[Y_j | Y_j \geq y_{\widetilde{\alpha}_j}] \\
&= \frac{1}{\mathbb{P}^j(Y_j \geq \widetilde{\alpha}_j)} \frac{1}{\sqrt{2\pi}} \int_{y_{\widetilde{\alpha}_j}}^\infty y e^{\frac{-y^2}{2}} dy \\
&= \frac{1}{1-\widetilde{\alpha}_j} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2_{\widetilde{\alpha}_j}}{2}} \\
&= \frac{\phi(y_{\widetilde{\alpha}_j})}{1-\widetilde{\alpha}_j}.
\end{aligned}
$$

Hence, we conclude the proof. □

Theorem 4.1 shows that the tail cost of a multivariate nondegenerate Gaussian random vector with *dependent* marginals can be represented as a linear combination of univariate $\mathsf{AVaR}_{\widetilde{\alpha}}$'s applied to *independent* marginals of $\boldsymbol{Y} = (Y_1, \ldots, Y_N) \in \mathcal{X}^N$. The following subsections reveal that this procedure can also be generalized to Normal mixture models.

4.2 Gaussian Variance Mixture Model

We will next evaluate the tail risk for losses, denoted by $\mathbf{X}$, with dependent marginals, where $\mathbf{X}$ is assumed to have a Gaussian variance mixture model using $\mathrm{mAVaR}^G_{\boldsymbol{\alpha}}$. Gaussian variance mixtures are generalizations of multivariate Gaussian random vectors generated by incorporating randomness into the covariance matrix of a Gaussian random vector. To be more specific, a random vector $\mathbf{X}$ is said to have a Gaussian variance mixture if

$$\mathbf{X} \stackrel{d}{=} \boldsymbol{\mu} + \sqrt{W}\boldsymbol{A}\boldsymbol{Z}, \tag{4.15}$$

where

(i) $\boldsymbol{Z}$ is multivariate Gaussian random vector in $\mathcal{X}^N$,

(ii) $\boldsymbol{A}$ is an $\mathbb{R}^{n\times n}$ lower triangular matrix with positive diagonals as in (4.12),

(iii) $W > 0$ is a positive real valued univariate random variable independent of $\boldsymbol{Z}$.

Note that, conditioned on W, $\mathbf{X}$ is a multivariate Gaussian random vector. This observation is the key to evaluate $\mathrm{mAVaR}^G_{\boldsymbol{\alpha}}$ on any Gaussian variance mixture models. In the next example, the N-dimensional portfolio $\boldsymbol{X}$ having a multivariate t-distribution is modelled based on that.

Example 4.1. *Let $\mathbf{X} = (X_1, \ldots, X_N)$ be a random vector having multivariate t-distribution. Namely, $\mathbf{X}$ is as in (4.15) where W has inverse Chi-square distribution for a prespecified degree of freedom $\nu > 2$, i.e.*

$$f(w;\nu) = \frac{(\nu/2)^{\nu/2}}{\Gamma(\nu/2)} w^{-\nu/2} e^{-\nu/(2w)} \quad \textit{for } w > 0, \tag{4.16}$$

where $\Gamma(\cdot)$ is the Gamma function (see e.g. [19]). We proceed as follows. For a given $\boldsymbol{\alpha} \in [0,1)^N$, the corresponding quantiles $\boldsymbol{x}_{\boldsymbol{\alpha}} = (x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_N})$ are found via

$$\boldsymbol{x}_{\boldsymbol{\alpha}} = \{(t_1, \ldots, t_N) \in \mathbb{R} : 1 - \alpha_1 = \mathbb{P}_1(X_1 \geq t_1), \ldots, 1 - \alpha_N = \mathbb{P}_N(X_N \geq t_N)\}$$

Then, conditioned on W, for a given risk level $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N) \in [0,1)^N$ and the corresponding quantiles $(x_{\alpha_1}, \ldots, x_{\alpha_N}) \in \mathbb{R}^N$, $W\boldsymbol{X}|_{W=w} \sim \mathcal{N}(\boldsymbol{\mu}, w\boldsymbol{\Sigma})$. Thus, as in (4.13),

$$\begin{aligned} \boldsymbol{y}_{\widetilde{\boldsymbol{\alpha}}} &= \frac{1}{\sqrt{w}} \boldsymbol{A}^{-1}(\boldsymbol{x}_{\boldsymbol{\alpha}} - \boldsymbol{\mu}), \\ \widetilde{\alpha}_j &= \mathbb{P}_j(Y_j \geq y_{\widetilde{\alpha}_j}) \\ &= 1 - \Phi(y_{\widetilde{\alpha}}), \quad \textit{for } j = 1, \ldots, N. \end{aligned} \tag{4.17}$$

Hence, by Theorem 4.1, and Equation (4.13), (4.14) *and* (4.15),

$$\mathbb{E}\left[\sum_{i=1}^{N}\mu_i+\sum_{i=1}^{N}\sum_{j=1}^{N}\sqrt{w}\lambda_{ij}Y_j|X_1\geq x_{\alpha_1},\ldots,X_N\geq x_{\alpha_N},W=w\right]$$
$$=\sum_{i=1}^{N}\mu_i+\sum_{i=1}^{N}\sum_{j=1}^{N}\sqrt{w}\lambda_{ij}\mathbb{E}^{\mathbb{P}_j}\left[Y_j|Y_j\geq y_{\widetilde{\alpha}_j}\right]$$
$$=\sum_{i=1}^{N}\mu_i+\sum_{i=1}^{N}\sum_{j=1}^{N}\sqrt{w}\lambda_{ij}\frac{\phi(y_{\widetilde{\alpha}_j})}{1-\widetilde{\alpha}_j}.$$

Here, note that $\boldsymbol{y}_{\widetilde{\boldsymbol{\alpha}}}$ *and* $\widetilde{\boldsymbol{\alpha}}$ *depend on* w *in the above integral via* (4.15) *and* (4.17). *Thus, the explicit expression analogous to* (4.14) *is given by taking expectation with respect to Chi-square distribution in* (4.16). *That is*

$$\mathrm{mAVaR}^G_{\boldsymbol{\alpha}}(\mathbf{X})=\mathbb{E}\left[\sum_{i=1}^{N}X_i|X_1\geq x_{\alpha_1},\ldots,X_N\geq x_{\alpha_N}\right]$$
$$=\sum_{i=1}^{N}\mu_i+\left(\int_0^\infty\sqrt{w}\frac{(\nu/2)^{\nu/2}}{\Gamma(\nu/2)}w^{-\nu/2}e^{-\nu/(2w)}\sum_{i=1}^{N}\sum_{j=1}^{N}\lambda_i^j\frac{\phi(y_{\widetilde{\alpha}_j})}{1-\widetilde{\alpha}_j}dw\right).$$

4.3 Generalised Hyperbolic Model

$\mathrm{mAVaR}^G_{\boldsymbol{\alpha}}$ can also handle Gaussian mean variance mixture models as well, which are prevalent in financial and actuarial modelling. The motivation behind the application of Gaussian mean variance mixture models, in finance particularly, is that it is observed in stock returns; negative returns (losses) have heavier tails than positive returns (gains). In particular, asymmetry is introduced by mixing normal distributions with different means as well as different variances giving the class of multivariate normal mean-variance mixtures so that a skewed distribution is retrieved.

Example 4.2. *Let* $\mathbf{X}$ *be a random vector of the form*

$$\mathbf{X}\stackrel{d}{=}\boldsymbol{\mu}+W\boldsymbol{\gamma}+\sqrt{W}\boldsymbol{A}\boldsymbol{Z},\tag{4.18}$$

where $\boldsymbol{\mu},\boldsymbol{A},\boldsymbol{Z}$ *are as in* (4.15). $\boldsymbol{\gamma}\in\mathbb{R}^N$ *is an* N *dimensional vector giving the mixture in mean term* $\boldsymbol{\mu}$. W *is assumed to be Gamma distributed, i.e. its density function is of the form*

$$f_W(w)=\frac{b^a}{\Gamma(a)}w^{a-1}e^{-bw},\quad \textit{for } w>0,$$

where $b>0$ and $a\in\mathbb{R}$ are scalars. As in multivariate t-case, we use the observation $\boldsymbol{X}$ conditioned on W as multivariate Gaussian $W\boldsymbol{X}|_{W=w}\sim\mathcal{N}(\boldsymbol{\mu}+w\boldsymbol{\gamma},\sqrt{w}\boldsymbol{\Sigma})$. Thus,

$$\begin{aligned}\boldsymbol{y}_{\widetilde{\boldsymbol{\alpha}}} &= \frac{1}{\sqrt{w}}\boldsymbol{A}^{-1}(\boldsymbol{x}_{\boldsymbol{\alpha}}-w\boldsymbol{\gamma}-\boldsymbol{\mu}),\\ \widetilde{\alpha}_j &= \mathbb{P}_j(Y_j\geq y_{\widetilde{\alpha}_j})\\ &= 1-\Phi(y_{\widetilde{\alpha}_j}),\quad \textit{for } j=1,\dots,N.\end{aligned}$$

Hence, by (4.13), (4.14) *and* (4.18)

$$\begin{aligned}\mathrm{mAVaR}^G_{\boldsymbol{\alpha}}(\mathbf{X}) &= \sum_{i=1}^N\mu_i+\mathbb{E}\Big[W\sum_{i=1}^N\gamma_i|W=w\Big]\\ &\quad+\sqrt{w}\sum_{i=1}^N\sum_{j=1}^N\lambda_{ij}\mathbb{E}\Big[Y_j|Y_1\geq y_{\widetilde{\alpha}_1},\dots,Y_N\geq y_{\widetilde{\alpha}_N},W=w\Big]\\ &=\sum_{i=1}^N\mu_i+\mathbb{E}\left[w\sum_{i=1}^N\gamma_i|W=w\right]+\mathbb{E}\left[\sqrt{w}\sum_{i=1}^N\sum_{j=1}^N\lambda_{ij}Y_j|Y_1\geq y_{\widetilde{\alpha}_1},\dots,Y_N\geq y_{\widetilde{\alpha}_N},W=w\right]\\ &=\sum_{i=1}^N\mu_i+\mathbb{E}\left[w\sum_{i=1}^N\gamma_i|W=w\right]+\sum_{i=1}^N\sum_{j=1}^N\lambda_{ij}\sqrt{w}\mathbb{E}\Big[Y_j|Y_j\geq y_{\alpha_j},W=w\Big]\\ &=\sum_{i=1}^N\mu_i+w\sum_{i=1}^N\gamma_i+\sum_{i=1}^N\sum_{j=1}^N\sqrt{w}\lambda_{ij}\frac{\phi(y_{\widetilde{\alpha}_j})}{1-\widetilde{\alpha}_j}.\end{aligned}$$

The mean of $W\sim\mathrm{Gamma}(a,b)$ distribution is ab. Thus, following the steps as in Theorem 4.1 and Example 4.1, the explicit representation

$$\mathrm{mAVaR}^G_{\boldsymbol{\alpha}}(\mathbf{X})=\sum_{i=1}^N\mu_i+ab\sum_{i=1}^N\gamma_i+\int_0^\infty\sqrt{w}\frac{b^a}{\Gamma(a)}w^{a-1}e^{-bw}\sum_{i=1}^N\sum_{j=1}^N\lambda_{ij}\frac{\phi(y_{\widetilde{\alpha}_j})}{1-\widetilde{\alpha}_j}dw$$

is retrieved.

4.4 A Tail Variance for $\mathrm{mAVaR}^G_{\boldsymbol{\alpha}}$

The tail variance of multivariate Gaussian random vector $\mathbf{X}\sim\mathcal{N}(\boldsymbol{\mu},\boldsymbol{\Sigma})$ with dependent marginals can be evaluated by adapting the definition of tail variance in (3.9) analogous to the tail conditional expectation case. The operator is defined as

$$\mathrm{mtVar}^{\mathrm{G}}_{\boldsymbol{\alpha}}(\mathbf{X})\triangleq\mathbb{E}\left[\left(\sum_{i=1}^N X_i-\mathrm{mAVaR}^G_{\boldsymbol{\alpha}}(\mathbf{X})\right)^2|X_1\geq x_{\alpha_1},\dots,X_N\geq x_{\alpha_N}\right].\qquad(4.19)$$

The following lemma gives the analogous Chebyshev type inequality of Lemma 3.4 for Gaussian random vectors.

Lemma 4.1. *Let* $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ *be a multivariate Gaussian vector and the multivariate tail variance be defined as in* (4.19). *Then,*

$$\begin{aligned}
&\mathrm{mtVar}^{\mathrm{G}}_{\boldsymbol{\alpha}}(\mathbf{X}) \\
&\quad = \left(\prod_{i=1}^{N} \frac{1}{\sqrt{2\pi}(1-\widetilde{\alpha}_i)}\right) \int_{y_{\widetilde{\alpha}_N}}^{\infty} \int_{y_{\widetilde{\alpha}_{N-1}}}^{\infty} \cdots \int_{y_{\widetilde{\alpha}_1}}^{\infty} \Big(\sum_{i=1}^{N}\sum_{j=1}^{N} \lambda_{ij} y_j\Big)^2 \exp\Big(-\frac{\sum_{i=1}^{N} y_i^2}{2}\Big) dy_1 dy_2 \dots dy_N \\
&\qquad - \left(\sum_{i=1}^{N}\sum_{j=1}^{N} \lambda_{ij} \frac{\phi(y_{\widetilde{\alpha}_j})}{1-\widetilde{\alpha}_j}\right)^2.
\end{aligned}$$

Here, $\widetilde{\boldsymbol{\alpha}} = (\widetilde{\alpha}_1, \dots, \widetilde{\alpha}_N)$, $\phi(\cdot)$ *and* $\boldsymbol{y}_{\widetilde{\boldsymbol{\alpha}}} = (y_{\widetilde{\alpha}_1}, \dots, y_{\widetilde{\alpha}_N})$ *are as in Theorem* (4.1).

Proof. We have

$$\begin{aligned}
\mathrm{mtVar}^{\mathrm{G}}_{\boldsymbol{\alpha}}(\mathbf{X}) &= \mathbb{E}\left[\left(\sum_{i=1}^{N} X_i - \mathrm{mAVaR}_{\boldsymbol{\alpha}}(\mathbf{X})\right)^2 |X_1 \geq x_{\alpha_1}, \dots, X_N \geq x_{\alpha_N}\right] \\
&= \mathbb{E}\left[\left(\sum_{i=1}^{N} X_i\right)^2 |X_1 \geq x_{\alpha_1}, \dots, X_N \geq x_{\alpha_N}\right] - \mathrm{mAVaR}^2_{\boldsymbol{\alpha}}(\mathbf{X})
\end{aligned}$$

Following the notation in Theorem 4.1, we represent each X_i

$$X_i = \mu_i + \sum_{j=1}^{N} \lambda_{ij} Y_j, \quad \text{for } i = 1, \dots, N.$$

Thus,

$$\mathrm{mAVaR}^{G}_{\boldsymbol{\alpha}}(\mathbf{X}) = \sum_{i=1}^{N} \mu_i + \sum_{i=1}^{N}\sum_{j=1}^{N} \lambda_{ij} \mathbb{E}\Big[Y_j | Y_j \geq y_{\widetilde{\alpha}_j}\Big].$$

Then,

$$\begin{aligned}
&\mathbb{E}\left[\left(\sum_{i=1}^{N}\sum_{j=1}^{N} \lambda_{ij} Y_j\right)^2 |Y_1 \geq y_{\widetilde{\alpha}_1}, \dots, Y_N \geq y_{\widetilde{\alpha}_N}\right] \\
&\quad = \left(\prod_{i=1}^{N} \frac{1}{\sqrt{2\pi}(1-\widetilde{\alpha}_i)}\right) \int_{y_{\widetilde{\alpha}_N}}^{\infty} \int_{y_{\widetilde{\alpha}_{N-1}}}^{\infty} \cdots \int_{y_{\widetilde{\alpha}_1}}^{\infty} \Big(\sum_{i=1}^{N}\sum_{j=1}^{N} \lambda_{ij} y_j\Big)^2 \exp\Big(-\frac{\sum_{i=1}^{N} y_i^2}{2}\Big) dy_1 dy_2 \dots dy_N
\end{aligned}$$

and

$$\mathbb{E}\left[\sum_{i=1}^{N}\sum_{j=1}^{N}\lambda_{ij}Y_j|Y_1\geq y_{\widetilde{\alpha}_1},\ldots,Y_N\geq y_{\widetilde{\alpha}_N}\right]$$
$$=\sum_{i=1}^{N}\sum_{j=1}^{N}\lambda_{ij}\mathbb{E}[Y_j|Y_j\geq y_{\widetilde{\alpha}_j}]$$
$$=\sum_{i=1}^{N}\sum_{j=1}^{N}\lambda_{ij}\frac{\phi(y_{\widetilde{\alpha}_j})}{1-\widetilde{\alpha}_j}$$

Thus, subtracting the latter term from the first one,

$$\mathrm{mtVar}_{\boldsymbol{\alpha}}(\mathbf{X})$$
$$=\left(\prod_{i=1}^{N}\frac{1}{\sqrt{2\pi}(1-\widetilde{\alpha}_i)}\right)\int_{y_{\widetilde{\alpha}_N}}^{\infty}\int_{y_{\widetilde{\alpha}_{N-1}}}^{\infty}\cdots\int_{y_{\widetilde{\alpha}_1}}^{\infty}\Big(\sum_{i=1}^{N}\sum_{j=1}^{N}\lambda_{ij}y_j\Big)^2\exp\Big(-\frac{\sum_{i=1}^{N}y_i^2}{2}\Big)dy_1dy_2\ldots dy_N$$
$$-\left(\sum_{i=1}^{N}\sum_{j=1}^{N}\lambda_{ij}\frac{\phi(y_{\widetilde{\alpha}_j})}{1-\widetilde{\alpha}_j}\right)^2.$$

Hence, we conclude the proof. □

For multivariate Gaussian random vectors, one can also give the analogous Chebyshev type inequality.

Lemma 4.2. *Let* $\mathbf{X}\sim\mathcal{N}(\boldsymbol{\mu},\boldsymbol{\Sigma})$ *be a multivariate Gaussian vector. Let* $\mathrm{mtVar}^{\mathrm{G}}_{\boldsymbol{\alpha}}$ *be the multivariate tail variance and the average value at risk* $\mathrm{mAVaR}_{\boldsymbol{\alpha}}$ *be as in* (4.19) *and* (4.11), *respectively. Then, we have*

$$\mathbb{P}\left(\frac{|\sum_{i=1}^{N}X_i-\mathrm{mAVaR}^{G}_{\boldsymbol{\alpha}}(\mathbf{X})|}{\sqrt{\mathrm{mtVar}^{\mathrm{G}}_{\boldsymbol{\alpha}}(\mathbf{X})}}\geq k\Big|X_1\geq x_{\alpha_1},\ldots,X_N\geq x_{\alpha_N}\right)\leq\frac{1}{k^2}.$$

Proof. The proof is almost the same as in Lemma 3.4 using the inequality

$$k^2\mathbb{P}\left(\left(\sum_{i=1}^{N}X_i-\mathrm{mAVaR}^{G}_{\boldsymbol{\alpha}}(\mathbf{X})\right)^2\geq k^2|X_1\geq x_{\alpha_1},\ldots,X_N\geq x_{\alpha_N}\right)$$
$$\leq\mathbb{E}\left[\left(\sum_{i=1}^{N}X_i-\mathrm{mAVaR}^{G}_{\boldsymbol{\alpha}}(\mathbf{X})\right)^2|X_1\geq x_{\alpha_1},\ldots,X_N\geq x_{\alpha_N}\right]$$
$$\leq\mathrm{mtVar}^{\mathrm{G}}_{\boldsymbol{\alpha}}(\mathbf{X}).$$

Hence, we conclude the proof. □

5 Numerical Illustrations

In this section, our obtained results in the previous sections are illustrated via numerical case studies. Let the dimension be $N = 3$, and let the mean vector be

$$\boldsymbol{\mu} = [0.80, 0.90, 1.00].$$

Take the covariance matrix

$$\boldsymbol{\Sigma} = \begin{bmatrix} 0.49 & 0.07 & 0.14 \\ 0.07 & 0.65 & 0.26 \\ 0.14 & 0.26 & 0.94 \end{bmatrix}$$

such that the lower triangular matrix $\boldsymbol{A}$ with $\boldsymbol{A}\boldsymbol{A}^\intercal = \boldsymbol{\Sigma}$ is

$$\boldsymbol{A} = \begin{bmatrix} 0.7 & 0.0 & 0.0 \\ 0.1 & 0.8 & 0.0 \\ 0.2 & 0.3 & 0.9 \end{bmatrix}$$

Denote $\boldsymbol{\alpha_0}, \ldots, \boldsymbol{\alpha_5}$ as the vectors, where each row corresponds to another 3-tuple.

$$\begin{bmatrix} \boldsymbol{\alpha_0} \\ \boldsymbol{\alpha_1} \\ \boldsymbol{\alpha_2} \\ \boldsymbol{\alpha_3} \\ \boldsymbol{\alpha_4} \\ \boldsymbol{\alpha_5} \end{bmatrix} = \begin{bmatrix} 0.01 & 0.01 & 0.01 \\ 0.99 & 0.01 & 0.01 \\ 0.99 & 0.99 & 0.01 \\ 0.70 & 0.70 & 0.70 \\ 0.99 & 0.10 & 0.99 \\ 0.99 & 0.99 & 0.99 \end{bmatrix}$$

We choose the different risk averse levels as follows. $\boldsymbol{\alpha_0}$ is close to $(0.00, 0.00, 0.00)$, implying that the model is close to the risk neutral case. Analogously, $\boldsymbol{\alpha_5}$ is close to $(1.00, 1.00, 1.00)$ implying that the model is considering almost only worst case scenarios. We also consider the risk-averseness levels, where we are very risk-averse to one or two risk sources and almost risk-neutral to the rest. We also consider the case, where the sum of the risk-averse levels exceeds the other scenarios, but each α_i is not close to the limit cases 0.00 and 1.00.

We illustrate how these risk-averseness level affects the multivariate tail expectation and tail variance of the multivariate Gaussian random vector $\mathbf{X} \sim (\boldsymbol{\mu}, \boldsymbol{\Sigma})$. The results are to be seen in Figure 1 and Figure 2, respectively.

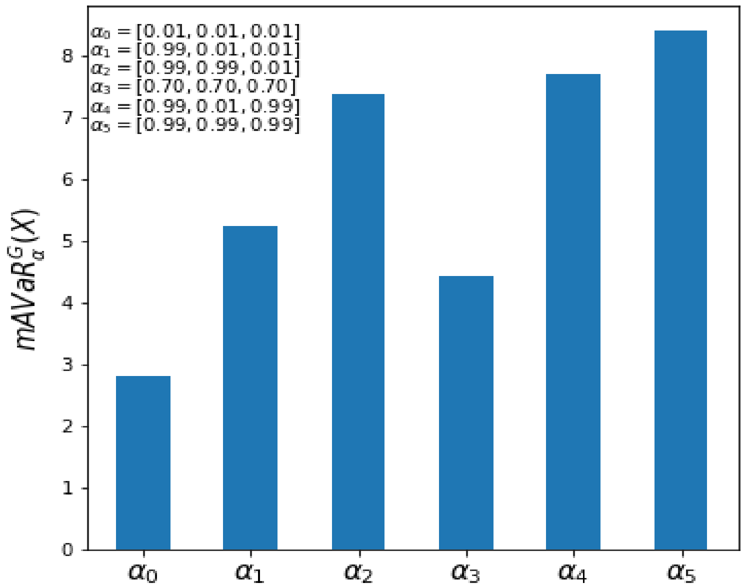

Figure 1: Multivariate Tail Expectation Results

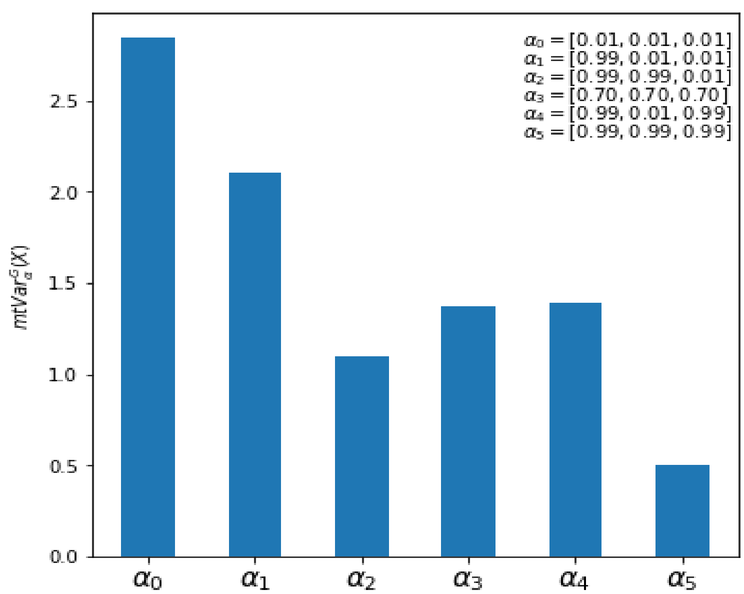

Figure 2: Multivariate Tail Variance Results

As $\boldsymbol{\alpha}$ values increase to 1, $\mathrm{mAVaR}^G_{\boldsymbol{\alpha}}(X)$ becomes higher. This is in concurrence with the monotone increase of the risk level of univariate average value-at-risk $\alpha : [0, 1) \mapsto \mathsf{AVaR}_\alpha(Y)$ for a random loss Y. On the other hand, for $\boldsymbol{\alpha_3} = (0.70, 0.70, 0.70)$, $\mathrm{AVaR}_{\boldsymbol{\alpha_3}}(\boldsymbol{X})$ is both less than $\mathrm{AVaR}_{\boldsymbol{\alpha_4}}(\boldsymbol{X})$ and $\mathrm{AVaR}_{\boldsymbol{\alpha_5}}(\boldsymbol{X})$, even though the sum of the risk-averse levels $0.7 +$

$0.7 + 0.7 = 2.1$ is larger than $0.99 + 0.99 + 0.01 = 1.99$. This shows that a larger sum of risk-averseness level does not necessarily imply a larger tail expected value.

On the other hand, for the tail variance, we see the opposite effect of the risk averseness $\boldsymbol{\alpha}$. As $\boldsymbol{\alpha}$ is closer to (0,0,0), the tail variance $\text{mtVar}^{\text{G}}_{\boldsymbol{\alpha}}(\mathbf{X})$ increases. Analogously, as $\boldsymbol{\alpha}$ is closer to (1.00,1.00,1.00), $\text{mtVar}^{\text{G}}_{\boldsymbol{\alpha}}(\mathbf{X})$ decreases. Intuitively, this is due to the fact that the variation in the tails of multivariate Gaussian is less than the variation of the whole distribution. As in tail expectation case, the larger sum of risk-averseness does not necessarily imply a larger tail variance. In particular, we see that $\text{mtVar}^{G}_{\boldsymbol{\alpha_0}}(\mathbf{X}) > \text{mtVar}^{G}_{\boldsymbol{\alpha_3}}(\mathbf{X})$. Furthermore, an obvious but important observation is that the tail variance is positive, as one would expect.

6 Conclusion and Outlook

We have explicitly characterized multi-dimensional Average-Value-at-Risk that preserves the axioms of a coherent risk measure by extending the univariate Average-Value-at-Risk to N-dimensional case delicately. In particular, we have presented a real valued multivariate extension, $\text{mAVaR}_{\boldsymbol{\alpha}}$, of the coherent risk measure Average-Value-at-Risk that takes N random sources of the risk. The framework is capable of handling N different risk-averseness levels with $\boldsymbol{\alpha} = \{\alpha_1, \cdots, \alpha_N\} \in [0,1)^N$, while keeping the axioms of a coherent risk measure. Moreover, the specific regularity properties as Fatou property, continuity from above and below and Lebesgue continuity are also preserved in our multivariate $\text{mAVaR}_{\boldsymbol{\alpha}}$. We further introduced a tail variance operator, denoted by $\text{mtVar}_{\boldsymbol{\alpha}}$, that quantifies the variation in the tail events of a random vector. We have further introduced the analogous tail mean and tail variance operators for Gaussian mixture models with N random sources with dependent marginals, denoted by $\text{mAVaR}^{G}_{\boldsymbol{\alpha}}$ and $\text{mtVar}^{\text{G}}_{\boldsymbol{\alpha}}$ respectively. The proposed operators can be used in risk management applications. In particular, considering the immense literature in modelling risks using mixture Gaussian distributions and the coherent risk measures, the current work presents a bridge between these two.

For the future outlook, we plan to extend the framework in two directions. First, we will investigate to generalize this work to the multivariate *convex* risk measure representations. Convex risk measures are generalizations of coherent ones, where positive homogeneity lacks among the axioms of a convex risk measures. In particular, we will study whether we can propose a general representation of the multidimensional law invariant real-valued convex risk measure. Second direction is to consider $\mathbb{R}^m$-valued coherent or convex risk measures instead of real-valued operators. Namely, we will investigate if there exists a unified representation, when the operator is vector-valued instead of real-valued.

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