Parametric analysis of conic linear optimization *

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Abstract

This paper focuses on the parametric analysis of a conic linear optimization problem with respect to the perturbation of the objective function along many fixed directions. We introduce the concept of the primal and dual conic linear inequality representable sets, which is very helpful for converting the correlation of the parametric conic linear optimization problems into the set-valued mapping relation. We discuss the relationships between these two sets and present the invariant region decomposition of a conic linear inequality representable set. We study the behaviour of the optimal partition and investigate the sensitivity of the optimal partition for conic linear optimization problems. All results are corroborated by examples having correlation among parameters.

Keywords: Linear programming, Semidefinite programming, Conic linear optimization, Conic linear inequality representation, Sensitivity analysis

AMS subject classifications. Primary: 90C22; Secondary: 90C51, 90C25

1 Introduction

In this paper we first discuss the relationships between the two closely related conic linear inequality representable sets and give a unified treatment for their associated parametric conic linear optimization problems. We then present a novel approach for performing sensitivity analysis of a conic linear optimization problem in which the objective function is perturbed along many fixed directions. We are motivated by recent research on parametric linear and semidefinite programs, though our results here enjoy wider applicability to conic linear programs.

We say a set $\Omega_P(d,M)$ have a conic linear inequality representation or to be a conic linear inequality representable if

$$\Omega_P(d,M)=\{v\in\mathbb{R}^r|d+M^Tv+B^Tw^1\in K\ for\ some\ w^1\in\mathbb{R}^m\},$$

*Supported by the National Natural Science Foundation of China (11871118, 11771058).

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for a vector $d \in \mathbb{R}^q$, and two matrices $B \in \mathbb{R}^{l\times q}$ and $M \in \mathbb{R}^{r\times q}$, where $K \subset \mathbb{R}^q$ is a pointed, closed, convex, solid (with non-empty interior) cone (for this formulation of a primal-dual pair and its properties, see for instance, [29]). Such a conic linear inequality representation is a closed convex set and is exactly the feasible sets of a family of the parametric conic linear optimization problems (see Section 3 below). It is a polyhedron if K is the nonnegative orthant, and a projected spectrahedron if K is the cone of symmetric positive semidefinite matrices. See, e.g., see [5, 30, 7, 2].

We define a conic linear inequality representation in the dual space of $\mathbb{R}^q$ again. A set $\Omega_D(c, M)$ is said to be a dual conic linear inequality representation or to be a dual conic linear inequality representable if

$$\Omega_D(c, M) = \{u \in \mathbb{R}^r | c + M^T u + A^T w^2 \in K^* \ for \ some \ w^2 \in \mathbb{R}^l\}$$

for a vector $c \in \mathbb{R}^q$ and a matrix $A \in \mathbb{R}^{m\times q}$ and if

$$Q = [A^T, B^T, M^T] \tag{1}$$

is of row full rank, where $M \in \mathbb{R}^{r\times q}$ is given as above, and K^* is the dual of K under the given inner-product, that is,

$$K^* = \{y \in \mathbb{R}^q | \langle y, x\rangle \geq 0, \forall x \in K\}.$$

For brevity, throughout this paper we assume that Q in the decomposition (1) is an orthogonal matrix, i.e., $Q^T Q = I_q$, a $q \times q$ identity matrix. Hence $\Omega_P(d, M)$ is said to be a primal conic linear inequality representation or to be a primal conic linear inequality representable.

Prior to the study of sensitivity analysis in recent years, the actual invariancy region plays an important role in the development of parametric linear programming (LP) and/or semidefinite programming (SDP). Adler and Monteiro [1] investigated the sensitivity analysis of LP problems first using the optimal partition approach, in which they identify the range of parameters where the optimal partition remains invariant. Other treatments of parametric analysis for LP problems based on the optimal partition approach was given by Jansen et al. [25], Greenberg [19], and Roos et al. [39], Ghaffari-Hadigheh et al. [15], Berkelaar et al.[6], Dehghan et al. [12], Hladík [24] and etc. The actual invariancy region has been studied extensively both in the setting of SDP , see, e.g., Goldfarb and Scheinberg [16], Mohammad-Nezhad and Terlaky [28]; and more generally in conic linear optimization, see Yildirim [45].

Recently there has been growing interest in finding efficient representations of convex sets by expressing them as projections of simple convex sets in higher dimensions. For instance, collections of examples of the spectrahedrons were given by Nesterov and Nemirovski [29], Ben-Tal and Nemirovski [5], and Nemirovski [31], and the semidefinite representability conditions for convex sets were discussed by Helton and Vinnikov [22], Vinnikov [46], Scheiderer [43]. An important technique for constructing spectrahedrons was introduced by Lasserre [26] and independently by Parrilo [35]. In addition, there are plenty of further results on spectrahedrons for particular kinds of sets. See, for example, [17, 18, 20, 23, 32, 33, 34, 40, 41, 42] and etc.

In this paper we define, for the first time, two set-valued mappings between the above primal and dual conic linear inequality representable sets and give a unified treatment for their associated parametric conic linear optimization problems. Such a treatment helps us to present a geometric framework that unifies and extends some of the properties of parametric LP and SDP problems to the case of a conic linear optimization. Our main goal is to develop the optimal partition approach given in [1] for parametric LP and in [15] for parametric SDP and to present theoretical results for the sensitivity of the optimal partition.

The main contribution of this paper is to present the invariant region decomposition of a conic linear inequality representable set. Theoretically, a conic linear inequality representable set can be expressed as the union of mutually disjoint invariant regions. Our result also captures and generalizes the SDP cases from Mohammad-Nezhad and Terlaky in [28]: we extend the concepts of nonlinearity region and transition point for the optimal partition to conic linear inequality representable sets, and provide sufficient conditions for the existence of a nonlinearity region and a transition point. Such concepts are very useful for the analysis of a parametric conic linear optimization problem since the nonlinearity region can be regarded as a stability region and its identification has a great influence on the post-optimal analysis of SDP problems.

The paper is organized as follows. In the next section we review some useful results from convex analysis and the duality theory of cone linear optimization. In Section 3, we introduce parametric conic linear optimization problems, and define the set-valued mappings between the primal and dual conic linear inequality representable sets, and discuss the behavior of the perturbed objective function. Section 4 extends the concept of the optimal partition to conic linear optimization by the use of the set-valued mappings and provides several examples to exhibit the invariant region decomposition of a conic linear inequality representable set. The extension of the corresponding approach to sensitivity analysis is the topic of Section 5. In particular, some properties of the nonlinearity regions and the transition points are discussed. We conclude the paper with some remarks in Section 6.

2 Preliminaries

Firstly, we review some useful facts about convex sets and convex cones. A standard reference for convex analysis is the book by Rockafellar [36].

The (topological) boundary of a set $C \in \mathbb{R}^q$ is denoted ∂C and defined as $\partial C = cl(C)\backslash int(C)$, where cl and int denote closure and interior, respectively.

Let C be a nonempty convex set in $\mathbb{R}^q$. A hyperplane

$$H = \{x \in \mathbb{R}^q | \langle f, x\rangle = \alpha\},$$

where $f \in \mathbb{R}^q$ and $\alpha \in \mathbb{R}$, is called a supporting hyperplane for C if $cl(C) \cap H_+$ is not empty and $C \subset H_+ = \{x \in \mathbb{R}^n | \langle f, x\rangle \geq \alpha\}$. If C is a cone, then $\alpha = 0$. A vector $h \in \mathbb{R}^q$ is called a recession direction of C if $c + \lambda h \in C$ for all $c \in C$ and

$\lambda > 0$. The set of all recession directions of C is called the recession cone of C and denoted by $0^+(C)$. The following results are well known:

Proposition 2.1. *If C is a nonempty closed convex set, then*
(1) $0^+(C)$ is a closed convex cone containing the origin.
(2) For any $c^0 \in C$ we have

$$0^+(C) = \{h \in \mathbb{R}^q | c^0 + \lambda h \in C,\ \lambda > 0\}.$$

The set C is called linearly bounded if $0^+(C) = \{0\}$, see [13]. Obviously, a set is linearly bounded whenever it is bounded. Furthermore, in finite-dimensional spaces, we have the following result:

Proposition 2.2. *[36, Theorem 8.4] A closed convex set in a finite-dimensional space is linearly bounded if and only if it is bounded.*

The set $F \subset C$ is an exposed face of C if there is a supporting hyperplane H such that $F = C \cap H$. An exposed point, or vertex, is a 0-dimensional exposed face, i.e., a point $x \in C$ at which there is a supporting hyperplane H of C such that $H \cap C$ reduces to $\{x\}$. This vertex x is called isolated if the supporting hyperplane of C pass through x is not unique.

A set C is called simply connected if for any two points $x, y \in C$, there is a continuous curve $\Gamma \subset C$ connecting x and y. In particular, a singleton is called a simply connected set.

Secondly, we recall the duality theory of cone linear optimization. The typical form of a conic linear optimization is a minimization problem of the form

$$\begin{array}{ll} \min & \langle c, x \rangle \\ s.t. & Ax = b, \\ & x \in K. \end{array} \tag{2}$$

The standard (Lagrangian) dual form of (2) is as follows

$$\begin{array}{ll} \min & \sum_{i=1}^{m} b_i s_i \\ s.t. & A^T s + y = c, \\ & s \in \mathbb{R}^m,\ y \in K^*. \end{array} \tag{3}$$

If $b = Ad$, then

$$\langle c, d \rangle = \langle A^T s + y, d \rangle = \langle s, Ad \rangle + \langle y, d \rangle = b^T s + \langle y, d \rangle. \tag{4}$$

Left multiplying both sides of the equality constraint in the dual program by the matrix B and M, respectively, we have $By = Bc$ and $My = Mc$. Conversely, if $y \in K^*$ satisfies $By = Bc$ and $My = Mc$, then there is $s \in \mathbb{R}^m$ such that $y = c - A^T s$ since the decomposition (1) holds. Therefore, the standard dual program (3) can be equivalently expressed as the following (nonstandard) form

$$\begin{array}{ll} \max & \langle d, c - y \rangle \\ s.t. & By = a, \\ & My = Mc, \\ & y \in K^*, \end{array} \tag{5}$$

where $a = Bc$. In the following discussions, we always give the nonstandard dual.

Corollary 2.3. *If $b = Ad$ and $a = Bc$, then for any primal feasible solution x of (2) and any dual feasible solution y of (5), the weak duality property holds, i.e.,*

$$\langle c, x\rangle \geq \langle d, c - y\rangle. \tag{6}$$

Equality holds if and only if (x, y) is a pair of optimal solutions.

Proof. It follows from (4) that

$$\langle c, x\rangle - \langle d, c - y\rangle = \langle c, x\rangle + \langle d, y\rangle - \langle c, d\rangle = \langle c, x\rangle - b^T s \geq 0,$$

which implies the desired result. □

If the primal and dual programs have optimal solutions and the duality gap is zero, i.e., the equality (6) holds, then the Karush-Kuhn-Tucker (KKT) conditions for the primal-dual pair of problems are

$$\begin{array}{lll} Ax = b, & x \in K, & \\ By = a, & My = Mc, & y \in K^*, \\ \langle x, y\rangle = 0. & & \end{array}$$

Corollary 2.4. *The optimal solutions of the primal-dual conic linear optimization pair (2) and (5) are independent of the choice of c and d only if the pair (d, c) satisfies $Ad = b$ and $Bc = a$.*

Proof. It is a direct consequence of Corollary 2.3. □

Conic linear optimization problems (2) and (5) generalize linear programs and share some of the duality theory of LP, such as Corollary 2.3 and the KKT conditions. However, in conic LP the strong duality property needs the strictly feasibility.

We say that a conic linear optimization problem is strictly feasible if there is an interior point feasible solution. For the primal program (2), a feasible solution x is called to be strictly feasible if $x \in int(K)$; and for the dual program (5), a feasible solution y is called to be strictly feasible if $y \in int(K^*)$. The following strong duality theorem is fundamental, see, e.g., [4, 8, 10, 11, 27, 44].

Theorem 2.5. *Consider the primal-dual conic linear optimization pair (2)-(5).*

(1) If the dual problem is bounded from above and if it is strictly feasible, then the primal problem attains its maximum and there is no duality gap.

(2) If the primal problem is bounded from above and if it is strictly feasible, then the dual problem attains its maximum and there is no duality gap.

Geometrically, the hyperplane

$$H_{\tau^*} = \{x \in \mathbb{R}^q | \langle c, x\rangle = \tau^*\}$$

supports the feasible region $\Omega_P = \{x \in K | Ax = b\}$ if the primal program (2) attains τ^* minimum. In particular, if x^* is an optimal solution of the primal program (2), then $x^* \in H_{\tau^*} \cap \Omega_P$. This fact will be used repeatedly in our later analysis.

3 Parametric objective function

We consider the following two closely related parametric conic linear optimizations

$$\begin{array}{ll} \min & \langle c + M^T u, x\rangle \\ s.t. & Ax = b, \\ & x \in K \end{array} \tag{7}$$

and

$$\begin{array}{ll} \min & \langle d + M^T v, y\rangle \\ s.t. & By = a, \\ & y \in K^*, \end{array} \tag{8}$$

where $c, d \in \mathbb{R}^q$, $a \in \mathbb{R}^l$, $b \in \mathbb{R}^m$ are fixed data, $u, v \in \mathbb{R}^r$ are real parameters. We study the parametric analysis of a conic linear optimization by making a connection between the above two problems.

If $b = Ad$ and $a = Bc$, then the dual of (7) can be defined as

$$\begin{array}{ll} \max & \langle d, c + M^T u - y\rangle \\ s.t. & By = a, \\ & My = Mc + u, \\ & y \in K^*; \end{array} \tag{9}$$

and the dual of (8) can be described as

$$\begin{array}{ll} \max & \langle c, d + M^T v - x\rangle \\ s.t. & Ax = b, \\ & Mx = Md + v, \\ & x \in K. \end{array} \tag{10}$$

Corollary 3.1. *Suppose that $b = Ad$ and $a = Bc$.*

(1) There is a vector $w^1 \in \mathbb{R}^l$ such that the primal slack vector

$$x = d + M^T v + B^T w^1$$

is feasible for the problem (10) if and only if $v \in \Omega_P(d, M)$;

(2) There is a vector $w^2 \in \mathbb{R}^m$ such that dual slack vector

$$y = c + M^T u + A^T w^2$$

is feasible for the problem (9) if and only if $u \in \Omega_D(c, M)$.

Proof. These results follow from the decomposition (1). □

In the following statements, we always assume that $b = Ad$ and $a = Bc$. The following corollary shows that $\Omega_P(d, M)$ (or $\Omega_D(c, M)$) denotes the set of u (or v) for which the problem (7) (or (8)) has a bounded solution.

Corollary 3.2. *(1) The optimal set of the problem (7) is either empty or unbounded for all $u \in \Omega_D(c, M)$ or nonempty and bounded for any $u \in int(\Omega_D(c, M))$;*

(2) The optimal set of the problem (8) is either empty or unbounded for all $v \in \Omega_P(d, M)$ *or nonempty and bounded for any* $v \in int(\Omega_P(d, M))$.

Proof. Let us assume that for some $\bar{u} \in int(\Omega_D(c, M))$ has an empty or unbounded optimal solution. Thus, there exists a nonzero direction $z \in K$ such that $Az = 0$ and $\langle c + M^T\bar{u}, z\rangle = 0$. This fact follows trivially in the case of an unbounded optimal set, and in the case of an empty optimal set it can be easily proved by choosing am infinite sequence of direction z^k, which converges to a direction z with the desired properties. Assume now that $\langle M^T\bar{u}, z\rangle < 0$. Since $\bar{u} \in int(\Omega_D(c, M))$, there exists a nonzero vector $e \in \mathbb{R}^l$ such that $\bar{u} + e \in int(\Omega_D(c, M))$ and $\langle c + M^T(\bar{u} + e), z\rangle < 0$. Consequently, the problem (7) with the parameter $\bar{u} + e$ is unbounded, which contradicts the definition of $\Omega_D(c, M)$. Similarly we can dismiss the case $\langle M^T\bar{u}, z\rangle > 0$. Thus $\langle M^T\bar{u}, z\rangle = 0$ has to hold, which implies that $\langle c, z\rangle = 0$ and hence $\langle c + M^T\bar{u}, z\rangle = 0$ and the problem (7) has an empty or unbounded solution for any $u \in \Omega_D(c, M)$. □

The feasible regions of problems (7) and (8) do not depend on u and v, respectively. Without causing confusion, their optimal solutions are denoted by $x^*(u)$ and $x^*(v)$, respectively; the optimal solutions of their respective dual programs are denoted by $y^*(u)$ and $y^*(v)$, respectively. Indeed, if $v = Mx^*(u) - Md$, then $x^*(v) = x^*(u)$ is of course an optimal solution of the problem (10); and if $u = My^*(v) - Mc$, then $y^*(u) = y^*(v)$ is of course an optimal solution of the problem (9). These facts show that both the primal and dual conic linear inequality representation sets $\Omega_P(d, M)$ and $\Omega_D(c, M)$ are highly correlated. To characterize the relationships between the two sets, let us define two set-valued mappings as follow

$$\phi(u) = \{M(x^*(u) - d) | x^*(u) \textit{ is an optimal solution of } (7) \textit{ if it exists}\}$$

for any $u \in \Omega_D(c, M)$ and

$$\psi(v) = \{M(y^*(v) - c) | y^*(v) \textit{ is an optimal solution of } (8) \textit{ if it exists}\}$$

for any $v \in \Omega_P(d, M)$. Here the value of the mapping $\phi(u)$ (or $\psi(v)$) could be a set if the optimal solution is not unique corresponding to u (or v). We refer to [3, 38, 37] for a detailed introduction to set-valued-mappings.

Theorem 3.3. *(1) If* $(d, c) \in K \times int(K^*)$*, then for every* $u \in int\,(\Omega_D(c, M))$*,* $\phi(u)$ *is well defined;*

(2) If $(d, c) \in int(K) \times K^*$*, then for every* $v \in int\,(\Omega_P(d, M))$*,* $\psi(v)$ *is well defined.*

Proof. If $(d, c) \in K \times int(K^*)$, then for the primal-dual conic linear optimization pair (7)-(9), the dual program is strictly feasible and is bounded for every $u \in int\,(\Omega_D(c, M))$. By Theorem 2.5, the primal program (7) is solvability, which means that the first result holds. The proof is completed. □

In LP both the set-valued mappings $\phi(u)$ and $\psi(v)$ are always well defined. However, in SDP they could be undefined on the boundary of the conic linear inequality representation sets. We illustrate Theorem 3.3 with a semidefinite system, with $\mathbb{R}^{\frac{n(n+1)}{2}} \simeq \mathbb{S}^n$ the set of order n symmetric matrices and $K = K^* = \mathbb{S}^n_+$ as the set of order n symmetric positive semidefinite (psd) matrices. The inner product of $\mathbb{S}^n$ is $c \bullet d = \langle c, d\rangle = tr(cd)$. Note that we denote the elements of $\mathbb{S}^n$ by small letters.

Example 3.4. Consider the following parametric SDP problem

$$\begin{aligned} \min_{x\in\mathbb{S}^2_+} \quad & \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix} \bullet x \\ s.t. \quad & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \bullet x = 2 \end{aligned}$$

and its dual

$$\begin{aligned} \max_{y\in\mathbb{S}^2_+} \quad & \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \bullet \left(\begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix} - y \right) \\ s.t. \quad & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \bullet y = 1, \\ & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \bullet y = u. \end{aligned}$$

Then $\Omega_D(c, M) = [0, +\infty)$. If $u > 0$, then the optimal pair $(x^*(u), y^*(u))$ of the primal-dual programs is as follows

$$x^*(u) = \begin{pmatrix} \sqrt{u} & 1 \\ 1 & \frac{1}{\sqrt{u}} \end{pmatrix}, \qquad y^*(u) = \begin{pmatrix} 1 & -\sqrt{u} \\ -\sqrt{u} & u \end{pmatrix}.$$

and there is no duality gap. However, if $u = 0$, the the primal program is not solvability although the dual program has 0 maximum at

$$y^*(0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and there is no duality gap. Therefore, for any $u \in (0, +\infty)$,

$$\phi(u) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \bullet \begin{pmatrix} \sqrt{u} & 1 \\ 1 & \frac{1}{\sqrt{u}} \end{pmatrix} = \frac{1}{\sqrt{u}}$$

is well defined and for $u = 0$, $\phi(0)$ is undefined. In additional, it is easy to verify that $\psi(v) = \frac{1}{v^2}$ for any $v = (0, +\infty)$ and $\psi(0)$ is undefined.

Corollary 3.5. *Suppose that* $(d, c) \in int(K \times K^*)$.

(1) For any $u^1, u^2 \in int\,(\Omega_D(c, M))$, *and for any* $v^1 \in \phi(u^1)$ *and* $v^2 \in \phi(u^2)$,

$$\langle u^2 - u^1, v^1 - v^2\rangle \geq 0; \tag{11}$$

(2) For any $v^1, v^2 \in int\,(\Omega_P(d, M))$, *and for any* $u^1 \in \psi(v^1)$ *and* $u^2 \in \psi(v^2)$, *the inequality (11) still holds.*

Proof. From the optimality of $x^*(u^1)$ and $x^*(u^2)$

$$\langle c + M^T u^1, x^*(u^1)\rangle \leq \langle c + M^T u^1, x^*(u^2)\rangle$$

and

$$\langle c + M^T u^2, x^*(u^1)\rangle \geq \langle c + M^T u^2, x^*(u^2)\rangle.$$

Subtracting the first inequality from the second inequality to get

$$\langle M^T u^2 - M^T u^1, x^*(u^1) - x^*(u^2)\rangle \geq 0,$$

or equivalently, for any $v^1 \in \phi(u^1)$ and $v^2 \in \phi(u^2)$,

$$\langle u^2 - u^1, v^1 - v^2\rangle = \langle u^2 - u^1, Mx^*(u^1) - Mx^*(u^2)\rangle \geq 0.$$

The proof is finished. □

In SDP, Goldfarb and Scheinberg [16] gave the similar results of Corollaries 3.2 and 3.5. They discussed a parametric SDP problem in which the objective function depends linearly on a scalar parameter.

Theorem 3.6. *(1) If $(d, c) \in K \times int(K^*)$, then for every $u \in int\,(\Omega_D(c, M))$, $\phi(u) \subset \Omega_P(d, M)$;*

(2) If $(d, c) \in int(K) \times K^*$, then for every $v \in int\,(\Omega_P(d, M))$, $\psi(v) \subset \Omega_D(c, M)$.

Proof. By Theorem 2.5, for every $u \in int\,(\Omega_D(c, M))$, there is $(x^*(u), y^*(u)) \in K \times K^*$ such that $(x^*(u), y^*(u))$ is a pair of the optimal solutions of (7) and (9). Obviously, $x^*(u)$ is feasible for the problem (8). Then by Theorem 3.3, $\phi(u)$ is well defined; and by Corollary 3.1, $\phi(u) \subset \Omega_P(d, M)$ for every $u \in int\,(\Omega_D(c, M))$. □

Corollary 3.7. *Suppose that $(d, c) \in int(K \times K^*)$.*

(1) For every $u \in int\,(\Omega_D(c, M))$, $\phi(u)$ is a closed convex set;

(2) For every $v \in int\,(\Omega_D(d, M))$, $\psi(v)$ is a closed convex set.

Proof. Without loss of generality, we assume that $\phi(u)$ is not a singleton for a given parameter $u \in int\,(\Omega_D(c, M))$. By Theorem 3.6, for any $v^1, v^2 \in \phi(u)$, there are $(x^*(v^1), y^*(u))$ and $(x^*(v^2), y^*(u))$ are optimal solutions of the primal-dual conic linear optimization pair (8) and (10) at $v = v^1$ and $v = v^2$, respectively. Therefore, the KKT conditions for the primal-dual pair of problems are

$$\begin{array}{lll} Ax^*(v^1) = b, & Mx^*(v^1) = Mc + v^1, & x^*(v^1) \in K, \\ By^*(u) = a, & y^*(u) \in K^*, & \\ \langle x^*(v^1), y^*(u)\rangle = 0 & & \end{array}$$

and

$$\begin{array}{lll} Ax^*(v^2) = b, & Mx^*(v^2) = Mc + v^2, & x^*(v^2) \in K, \\ By^*(u) = a, & y^*(u) \in K^*, & \\ \langle x^*(v^2), y^*(u)\rangle = 0. & & \end{array}$$

Then for any $\alpha \in [0, 1]$, $x^*_\alpha = \alpha x^*(v^1) + (1 - \alpha)x^*(v^2)$ satisfies

$$\begin{array}{lll} Ax^*_\alpha = b, & Mx^*_\alpha = Mc + \alpha v^1 + (1 - \alpha)v^2, & x^*_\alpha \in K, \\ By^*(u) = a, & y^*(u) \in K^*, & \\ \langle x^*_\alpha, y^*(u)\rangle = 0, & & \end{array}$$

which means that $(x^*_\alpha, y^*(u))$ is a pair of optimal solutions of the primal-dual conic linear optimization pair (8) and (10) at $v = \alpha v^1 + (1 - \alpha)v^2$. Applying Theorem 3.6

again, $(x^*_\alpha, y^*(u))$ is also a pair of optimal solutions of the primal-dual conic linear optimization pair (7) and (9) for the parameter u. So $\phi(u)$ is convex.

Finally, the intersection of the supporting hyperplane

$$H = \{x \in \mathbb{R}^q | \langle c + M^T u, x\rangle = \langle c + M^T u, x^*(u)\rangle\}$$

and $\Omega_P = \{x \in K | Ax = b\}$ is closed, which means that $\phi(u)$ is closed. The proof is finished. □

Corollary 3.7 has an interesting geometric interpretation. Let us assume that for a given parameter $u \in \Omega_D(c, M)$, $\phi(u)$ is neither empty nor a singleton. It is easy to verify that for any $v^1, v^2 \in \phi(u)$, all points of line segment $[x^*(v^1), x^*(v^2)]$ are feasible solutions of the program (10) and have the same objective value, which implies that $\phi(u)$ is convex for a given parameter u. On the other hand, the optimal solution of the program (8) with the perturbed objection function remains unchange, which means that $y^*(v) = y^*(u)$ is an isolated vertex of $\{y \in K^* | By = a\}$ for any $v \in \phi(u)$.

Corollary 3.8. *Suppose that $(d, c) \in int(K \times K^*)$, and $u^0 \in \partial\Omega_D(c, M)$ and $v^0 \in \partial\Omega_P(d, M)$ are finite.*

(1) There is a neighborhood $U(u^0)$ such that for every $u \in U(u^0) \cap int(\Omega_D(c, M))$, $\phi(u) \subset int(\Omega_P(d, M))$;

(2) There is a neighborhood $U(v^0)$ such that for every $v \in U(v^0) \cap int(\Omega_P(d, M))$, $\psi(v) \subset int(\Omega_D(c, M))$.

Proof. Suppose that $u^0 \in \partial\Omega_D(c, M)$ is finite, and let $u^1 \in int(\Omega_D(c, M))$. Then for any $\alpha \in (0, 1)$, $u^\alpha = \alpha u^0 + (1 - \alpha)u^1 \in int(\Omega_D(c, M))$. Hence the intersection of both sets $\{x \in \mathbb{R}^q | Mx = u^\alpha\}$ and $\Omega_P = \{x \in K | Ax = b\}$ has an interior point, which implies the first result. □

To conclude this section, we offer a new proof of Theorem 2.5 by the use of the above result.

Proof of Theorem 2.5. Let's prove the second result by the mathematical induction for m. Without loss of generality, we assume that $(d, c) \in int(K) \times \partial K^*$ is a pair of feasible solutions of the primal-dual conic linear optimization pair (2)-(5).

Initial step: $m = 0$. Hence the primal program (2) reduces the unconstrained optimization and has a 0 minimum; while the dual program (5) has a 0 maximum at $y^* = c$. So the dual program attains its maximum and there is no duality gap at $m = 0$.

Inductive step. Assume that the dual program (5) attains its maximum and there is no duality gap for $m = k$. Consider the primal parametric conic linear optimization program (7), where $l = 1$ and $M^T \in int(K^*)$ is assumed. Hence $\Omega_D(c, M) = [0, +\infty)$. By hypothesis, the dual program (9) has an optimal solution $y^*(u)$ and there is no duality gap for every $u \in [0, +\infty)$. In particular, the slack vector $y^*(0)$ associated with $u^0 = 0$ is an optimal solution of the program (5) and there is no duality gap. By Corollary 3.8, there is a neighborhood $U(0)$ such that for every $u \in U(0) \cap int(\Omega_D(c, M))$, $\phi(u) \subset int(\Omega_P(d, M))$. Letting u tend to 0, then the program (8) attains its infimum at $y^*(0)$ and there is no duality gap. Or equivalently, the dual program (5) attains its maximum and there is no duality gap for $m = k + 1$. This concludes the second result by the induction. □

4 Invariancy regions

In this section, we investigate the behaviour of the optimal partition under perturbation. To achieve this goal, let $|\phi(u)|$ and $|\psi(v)|$ denote the numbers of elements of $\phi(u)$ and $\psi(v)$, respectively. From Corollary 3.7, either $|\phi(u)|$ for every $u \in \Omega_D(c, M)$ or $|\psi(v)|$ for every $v \in \Omega_P(d, M)$ is equal to 0,1 or $+\infty$, where either $|\phi(u)| = 0$ or $|\psi(v)| = 0$ means that either $\phi(u)$ or $\psi(v)|$ is undefined. Define the following function

$$\pi(u, v) = (|\phi(u)|, |\psi(v)|),$$

in which either $u \in \psi(v)$ for every $v \in \Omega_P(d, M)$ or $v \in \phi(u)$ for every $u \in \Omega_D(c, M)$. We introduce and characterize the sets $\Omega_P(d, M)$ and $\Omega_D(c, M)$ on which the optimal partitions of both $\phi(u)$ and $\psi(v)$ are invariant of u and v, respectively.

The concept of the optimal partition (or the invariancy interval) has been defined for parametric LP in [1] and for parametric SDP in [15]. We now define a new notion of the invariancy region for conic linear optimization.

Definition 4.1. A simply connected set $\mathcal{U} \subset \Omega_D(c, M)$ is called invariancy in $\Omega_D(c, M)$ if either for any $u \in \mathcal{U}$, $|\phi(u)| = 0$ or for any $u^1, u^2 \in \mathcal{U}$ and any $v^1 \in \phi(u^1)$, $v^2 \in \phi(u^2)$,

$$\pi(u^1, v^1) = \pi(u^2, v^2). \tag{12}$$

A simply connected set $\mathcal{V} \subset \Omega_P(d, M)$ is called invariancy in $\Omega_P(d, M)$ if either for any $v \in \mathcal{V}$, $|\psi(v)| = 0$ or for any $v^1, v^2 \in \mathcal{V}$ and any $u^1 \in \psi(v^1)$, $u^2 \in \psi(v^2)$, the relation (12) holds. In particular, we allow that either $\mathcal{U}$ or $\mathcal{V}$ is a singleton.

Now we state the main result of this paper, which results in the invariant region decomposition of a conic linear inequality representable set.

Theorem 4.2. *Any two different invariancy regions of a conic linear inequality representable set do not intersect.*

Proof. This result follows immediately from Definition 4.1. □

We mainly care about the following three values of the function $\pi(u, v)$:

$$(1, +\infty), (+\infty, 1), (1, 1).$$

Essentially, there are two main types of invariancy regions: one type is established by the first two function values and another type is established by the last function value. In LP, there is only the first type and not others. In SDP, $\pi(u, v)$ might be equal to $(1, 1)$, for example, see Example 3.4. This case was also discussed by Mohammad-Nezhad and Terlaky [28].

Definition 4.3. A singleton invariancy set $\{\bar{u}\}$ of $\Omega_D(c, M)$ is called a transition point if $\pi(\bar{u}, v) = (+\infty, 1)$ for any $v \in int(\phi(\bar{u}))$; and a singleton invariancy set $\{\bar{v}\}$ of $\Omega_P(d, M)$ is called a transition point if $\pi(u, \bar{v}) = (1, +\infty)$ for any $u \in int(\phi(\bar{v}))$.

Definition 4.4. The invariancy sets $\mathcal{U}$ of $\Omega_D(c, M)$ and $\mathcal{V}$ of $\Omega_P(d, M)$ are called nonlinearity, respectively, if for every $u \in \psi(v) \subset \mathcal{U}$ and $v \in \psi(u) \subset \mathcal{V}$, $\pi(u, v) = (1, 1)$.

In a nonlinearity set, the set-valued mappings $\psi(v)$ and $\phi(u)$ reduce to ordinary single-valued mappings. Together they form a pair of reversible mappings in the nonlinearity set.

By Theorems 3.6 and 4.2, we can obtain the invariant region decomposition of a conic linear inequality representable set. To build intuition, we present the following two examples: one of them is a LP problem and another is a SDP problem.

Example 4.5. The following single parameter LP problem

$$\begin{array}{ll} \min & -x_1 - x_2 + 0.5u(-x_1 + x_2 - x_4 + x_5) \\ s.t. & x_1 + x_2 + x_3 = 3, \\ & x_2 + x_4 = 2, \\ & x_1 + x_5 = 2.5, \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{array}$$

has a feasible solution $d = (1, 1, 1, 1, 1.5)^T$. Since $\langle d, c + M^T u\rangle = -2 + 0.25u$, its dual can be expressed as follows

$$\begin{array}{ll} \min & -y_1 - y_2 - y_3 - y_4 - 1.5y_5 - 2 + 0.25u \\ s.t. & y_1 + y_2 - 2y_3 - y_4 - y_5 = -2, \\ & -y_1 + y_2 - y_4 + y_5 = 2u, \\ & y_1, y_2, y_3, y_4, y_5 \geq 0. \end{array}$$

If $u = 0$, then the primal feasible region may reduce a convex pentagon

$$\{x \in \mathbb{R}^2 | x_1 + x_2 \leq 3, 0 \leq x_1 \leq 2.5, 0 \leq x_2 \leq 2\}$$

in the two-dimensional plane. This convex pentagon has five isolated vertices

$$(0, 0)^T, (0, 2)^T, (2.5, 0)^T, (2.5, 0.5)^T, (1, 2)^T.$$

When $0 \leq u \leq 1$, the optimal pair $(x^*(u), y^*(u))$ is as follows

$$x^*(u) = (2.5, 0.5, 0, 1.5, 0)^T, \qquad y^*(u) = (0, 0, 1 - u, 0, 2u)^T.$$

Geometrical, the trajectory of $y^*(u)$ in the interval $[0, 1]$ is an edge of the polyhedral

$$\{y \in \mathbb{R}^5 | y_1 + y_2 - 2y_3 - y_4 - y_5 = -2, y_1, y_2, y_3, y_4, y_5 \geq 0\},$$

in which the edge connects two vertices $y^1 = (0, 0, 1, 0, 0)^T$ and $y^2 = (0, 0, 0, 0, 2)^T$. Then for every $u \in (0, 1)$, $\phi(u) = -2$. If $v = Mx^*(u) - Md = -2$, then $\psi(v) = [0, 1]$ is an interval. When u is equal to either 0 or 1, $\phi(u)$ is also an interval. The following parallel table lists the values of the set-valued mappings $\phi(u)$ and $\psi(v)$.

Table 1 *The set − valued mappings* $\phi(u)$ *and* $\psi(v)$ *parallel table*

$x^*(v)$	v	u	$y^*(u)$
$(0, 2, 1, 0, 2.5)^T$	2	$(-\infty, -1]$	$(-1 - u, 0, 0, 1 - u, 0)^T$
$(2 - v, 2, v - 1, 0, 0.5 + v)^T$	$[1, 2]$	-1	$(0, 0, 0, 2, 0)^T$
$(1, 2, 0, 0, 1.5)$	1	$[-1, 0]$	$(0, 0, 1 + u, -2u, 0)^T$
$0.5(3 - v, 3 + v, 0, 1 - v, 2 + v)$	$[-2, 1]$	0	$(0, 0, 1, 0, 0)^T$
$(2.5, 0.5, 0, 1.5, 0)^T$	-2	$[0, 1]$	$(0, 0, 1 - u, 0, 2u)^T$
$(2.5, v + 2.5, -2 - v, -0.5 - v, 0)^T$	$[-2.5, -2]$	1	$(0, 0, 0, 0, 2)^T$
$(2.5, 0, 0.5, 2, 0)^T$	-2.5	$[1, +\infty)$	$(0, 0, u - 1, 0, u + 1)$
$x^*(v)$	$\phi(u)$	$\psi(v)$	$y^*(u)$

The following observations can be understand from Table 1.

1. The primal and conic linear inequality representation sets include three and four intervals, respectively. In each intervals, the trajectory of the optimal solution is an edge of the polyhedral that connects two adjacent vertices. In particular, the interior sets of these intervals are the invariancy intervals.

2. The image of the set-valued mapping at a parametric vector is an interval if and only if the slack vector corresponding to the parameter is a vertex of the feasible regions, in which the parameter is a transition point. In additional, for two adjacent transition points u^1 and u^2, $\phi(u^1) \cap \phi(u^2)$ is a transition point of the dual conic linear inequality representation set; for two adjacent transition points v^1 and v^2, $\psi(v^1) \cap \psi(v^2)$ is a transition point of the primal conic linear inequality representation set.

3. $\psi(2) = (-\infty, 2]$ and $\psi(-2.5) = [1, +\infty)$. This implies that the perturbed objective function $-x_1 + x_2 - x_4 + x_5$ takes the maxima and minima in the primal feasible region. This fact is obviously since the perturbed objective function is continuous in the bounded primal feasible region.

Example 4.6. For the parametric SDP problem

$$\begin{aligned} \min_{x \in \mathbb{S}^3_+} \quad & \begin{pmatrix} 0 & u_1 & -u_2 \\ u_1 & 0 & -1 \\ -u_2 & -1 & 0 \end{pmatrix} \bullet x \\ s.t. \quad & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \bullet x = 1, \\ & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \bullet x = 1, \\ & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \bullet x = 1 \end{aligned}$$

and its dual

$$\begin{aligned} \max_{y \in \mathbb{S}^3_+} \quad & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \bullet \left(\left(\begin{pmatrix} 0 & u_1 & -u_2 \\ u_1 & 0 & -1 \\ -u_2 & -1 & 0 \end{pmatrix} - y \right) \right. \\ s.t. \quad & \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \bullet y = 2u_1, \\ & \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \bullet y = -2u_2, \\ & \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \bullet y = -2, \end{aligned}$$

we consider the following optimal solution pair $(x^*(u), y^*(u))$ with six indeterminate entries:

$$x^*(u) = \begin{pmatrix} 1 & x_{12} & x_{13} \\ x_{12} & 1 & x_{23} \\ x_{13} & x_{23} & 1 \end{pmatrix}, \qquad y^*(u) = \begin{pmatrix} y_{11} & u_1 & -u_2 \\ u_1 & y_{22} & -1 \\ -u_2 & -1 & y_{33} \end{pmatrix}.$$

We are interested in the six indeterminate entries x_{11}, x_{12}, x_{23} and y_{11}, y_{22}, y_{33} as a function of $(u_1, u_2) \in \mathbb{R}^2$. Let us consider the following two cases:

Case I. the rank of $x^*(u)$ is equal to 1. Under this condition, the indeterminate entries x_{11}, x_{12}, x_{23} satisfy

$$\frac{1}{x_{12}} = \frac{x_{12}}{1}, \quad \frac{1}{x_{13}} = \frac{x_{13}}{1}, \quad \frac{1}{x_{23}} = \frac{x_{23}}{1},$$

which results in $x^*(u)$ is one of four rank-one matrices

$$\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}^T, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}^T, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}^T, \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}^T.$$

Such matrices are isolated singular vertices of the surface of the primal feasible region. From the KKT conditions of the first isolated singular vertex, we have

$$\begin{pmatrix} y_{11} & u_1 & -u_2 \\ u_1 & y_{22} & -1 \\ -u_2 & -1 & y_{33} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Then y_{11}, y_{22}, y_{33} can be expressed as a function of $(u_1, u_2) \in \mathbb{R}^2$ such that

$$x^*(u) = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}, \quad y^*(u) = \begin{pmatrix} -u_1 - u_2 & u_1 & -u_2 \\ u_1 & -u_1 - 1 & -1 \\ -u_2 & -1 & -u_2 - 1 \end{pmatrix}$$

are the optimal solutions of the primal and dual programs, respectively. Here $u = (u_1, u_2)^T$ belongs to the following set

$$\Omega_1 = \{(u_1, u_2)^T | u_1 < -1, u_2 < -1, u_1 + u_2 + u_1 u_2 \geq 0\}.$$

It is a two-dimensional hyperboloid in the plane and is a convex set. Hence for any $u = (u_1, u_2)^T \in \Omega_1$, we have $\phi(u) = (2, -2)^T$ and $\pi(u, v) = (1, +\infty)$.

Analogously, for the second isolated singular vertex, we have

$$\begin{pmatrix} y_{11} & u_1 & -u_2 \\ u_1 & y_{22} & -1 \\ -u_2 & -1 & y_{33} \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

such that

$$x^*(u) = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}, \quad y^*(u) = \begin{pmatrix} u_1 + u_2 & u_1 & -u_2 \\ u_1 & u_1 - 1 & -1 \\ -u_2 & -1 & u_2 - 1 \end{pmatrix},$$

in which $u=(u_1,u_2)^T$ belongs to the following two-dimensional hyperboloid

$$\Omega_2=\{(u_1,u_2)^T|u_1>1,u_2>1,-u_1-u_2+u_1u_2\geq 0\}.$$

Then for any $u=(u_1,u_2)^T\in\Omega_2$, we have $\phi(u)=(-2,2)^T$ and $\pi(u,v)=(1,+\infty)$.

For the third isolated singular vertex, we have

$$\begin{pmatrix} y_{11} & u_1 & -u_2 \\ u_1 & y_{22} & -1 \\ -u_2 & -1 & y_{33} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

then

$$x^*(u)=\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix},\quad y^*(u)=\begin{pmatrix} u_2-u_1 & u_1 & -u_2 \\ u_1 & 1-u_1 & -1 \\ -u_2 & -1 & u_2+1 \end{pmatrix},$$

in which $u=(u_1,u_2)^T$ belongs to the following two-dimensional hyperboloid

$$\Omega_3=\{(u_1,u_2)^T|u_1<1,u_2>-1,u_1\geq u_2,u_2-u_1-u_1u_2\geq 0\}.$$

Therefore, for any $u=(u_1,u_2)^T\in\Omega_3-\{(0,0)^T\}$, we have $\phi(u)=(2,2)^T$ and $\pi(u,v)=(1,+\infty)$.

For the fourth isolated singular vertex, we have

$$\begin{pmatrix} y_{11} & u_1 & -u_2 \\ u_1 & y_{22} & -1 \\ -u_2 & -1 & y_{33} \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

then

$$x^*(u)=\begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix},\quad y^*(u)=\begin{pmatrix} u_1-u_2 & u_1 & -u_2 \\ u_1 & u_1+1 & -1 \\ -u_2 & -1 & 1-u_2 \end{pmatrix},$$

in which $u=(u_1,u_2)^T$ belongs to the following two-dimensional hyperboloid

$$\Omega_4=\{(u_1,u_2)^T|u_1>-1,u_2<1,u_1\geq u_2,u_1-u_2-u_1u_2\geq 0\}.$$

Then for any $u=(u_1,u_2)^T\in\Omega_4-\{(0,0)^T\}$, we have $\phi(u)=(-2,-2)^T$ and $\pi(u,v)=(1,+\infty)$.

The sets Ω_1, Ω_2, $\Omega_3-\{(0,0)^T\}$ and $\Omega_4-\{(0,0)^T\}$ are invariancy regions of $\Omega_D(c,M)$.

It should be noted that $\Omega_3\cap\Omega_4=\{(0,0)^T\}$. The convex combination of the third and fourth primal optimal solutions

$$x^*(u^0)=\alpha\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}+(1-\alpha)\begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}=\begin{pmatrix} 1 & 2\alpha-1 & 2\alpha-1 \\ 2\alpha-1 & 1 & 1 \\ 2\alpha-1 & 1 & 1 \end{pmatrix},\alpha\in[0,1]$$

is an optimal solution of the primal program if $u = u^0 = \{(0,0)^T\}$, which means that

$$\phi(u^0) = \{(v_1, v_1)^T | -2 \le v_1 \le 2\}.$$

So u^0 is a transition point of the dual conic linear inequality representable set $\Omega_D(c, M)$.

Case II: the rank of $y^*(u)$ is equal to 1. It follows from

$$\frac{y_{11}}{u_1} = \frac{-u_2}{-1}, \quad \frac{u_1}{-u_2} = \frac{y_{22}}{-1} = \frac{-1}{y_{33}}$$

that $y^*(u)$ can be expressed in terms of the parameters u_1 and u_2 as follows:

$$y^*(u) = \begin{pmatrix} u_1u_2 & u_1 & -u_2 \\ u_1 & \frac{u_1}{u_2} & -1 \\ -u_2 & -1 & \frac{u_2}{u_1} \end{pmatrix} = \frac{1}{u_1u_2}\begin{pmatrix} u_1u_2 \\ u_1 \\ -u_2 \end{pmatrix}\begin{pmatrix} u_1u_2 \\ u_1 \\ -u_2 \end{pmatrix}^T,$$

where $u_1u_2 > 0$. From the KKT conditions, we have

$$\begin{pmatrix} 1 & x_{12} & x_{13} \\ x_{12} & 1 & x_{23} \\ x_{13} & x_{23} & 1 \end{pmatrix}\begin{pmatrix} u_1u_2 \\ u_1 \\ -u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

or equivalently,

$$\begin{pmatrix} u_1 & -u_2 & 0 \\ u_1u_2 & 0 & -u_2 \\ 0 & u_1u_2 & u_1 \end{pmatrix}\begin{pmatrix} x_{12} \\ x_{13} \\ x_{23} \end{pmatrix} = \begin{pmatrix} -u_1u_2 \\ -u_1 \\ u_2 \end{pmatrix}.$$

Solve this algebraic system to get

$$\begin{aligned} x_{12} &= \tfrac{u_2}{2u_1^2} + \tfrac{u_2}{2} - \tfrac{1}{2u_2}, \\ x_{13} &= \tfrac{1}{2u_1} - \tfrac{u_1}{2} - \tfrac{u_1}{2u_2^2}, \\ x_{23} &= \tfrac{u_2}{2u_1} + \tfrac{u_1}{2u_2} - \tfrac{u_1u_2}{2}. \end{aligned}$$

Such six indeterminate entries are established. In particular, if $u_1 = u_2$, then

$$x^*(u) = \begin{pmatrix} 1 & \frac{u_2}{2} & -\frac{u_2}{2} \\ \frac{u_2}{2} & 1 & 1-\frac{u_2^2}{2} \\ -\frac{u_2}{2} & 1-\frac{u_2^2}{2} & 1 \end{pmatrix}, \quad y^*(u) = \begin{pmatrix} u_2^2 & u_2 & -u_2 \\ u_2 & 1 & -1 \\ -u_2 & -1 & 1 \end{pmatrix}.$$

This case was discussed by Mohammad-Nezhad and Terlaky in [28].

Since $x^*(u)$ is semidefinite, then $|x_{12}| \le 1$, $|x_{13}| \le 1$, $|x_{23}| \le 1$. It follows from $|x_{23}| \le 1$ that

$$-2 \le \frac{u_2}{u_1} + \frac{u_1}{u_2} - u_1u_2 \le 2.$$

Since $u_1u_2 > 0$, then

$$-2u_1u_2 \le u_1^2 + u_2^2 - u_1^2u_2^2 \le 2u_1u_2,$$

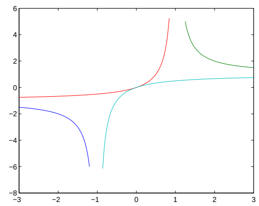

Figure 1: regions separated by four curves

or equivalently,

$$0 \leq (u_1 + u_2)^2 - (u_1 u_2)^2, \qquad \textit{and} \qquad (u_1 - u_2)^2 - (u_1 u_2)^2 \leq 0.$$

This concludes the following inequalities

$$(u_1 + u_2 + u_1 u_2)(u_1 + u_2 - u_1 u_2) \;\geq\; 0, \tag{13}$$
$$(u_1 - u_2 + u_1 u_2)(u_1 - u_2 - u_1 u_2) \;\leq\; 0. \tag{14}$$

Define a set

$$\Omega_0 = \{(u_1, u_2)^T \mathbb{R}^2 | u_1 u_2 > 0, \textit{the inequalities } (13) \textit{ and } (14) \textit{ hold}\}.$$

Then for any $(u_1, u_2) \in \Omega_0$, there is a optimal solution pair $(x^*(u), y^*(u))$ such that the rank of $y^*(u)$ is equal to one. Hence for any $u = (u_1, u_2)^T \in int(\Omega_0)$, we have

$$\phi(u) = \begin{pmatrix} \frac{u_2}{u_1^2} + u_2 - \frac{1}{u_2} \\ \frac{1}{u_1} - u_1 - \frac{u_1}{u_2^2} \end{pmatrix} \tag{15}$$

and $\pi(u, v) = (1, 1)$. So there are two different nonlinearity regions contained in Ω_0, in which the two curved triangles in the first and third quadrants are not connected.

Corollary 4.7. $int(\Omega_0) = \mathbb{R}^2 - \bigcup_{i=1}^{4} \Omega_i$.

Proof. Define four curves as follow

$$\begin{array}{lll} l_1: & u_1 + u_2 + u_1 u_2 = 0, & u_1 < -1, u_2 < -1, \\ l_2: & -u_1 - u_2 + u_1 u_2 = 0, & u_1 > 1, u_2 > 1, \\ l_3: & u_2 - u_1 - u_1 u_2 = 0, & u_1 < 1, u_2 > -1, \\ l_4: & u_1 - u_2 - u_1 u_2 = 0, & u_1 > -1, u_2 < 1. \end{array}$$

Each of them represents a unilateral branch of the hyperbola, in which l_1 and l_2 are symmetric about the line $u_1 + u_2 = 0$, and l_3 and l_4 are symmetric about the line $u_1 - u_2 = 0$. Such four curves define some areas in $\mathbb{R}^2$. Instead of a formal and tedious proof we simply plot these areas in Figure 1. The four outer two-dimensional hyperboloids are Ω_1, Ω_2, Ω_3 and Ω_4. Intuitively, $int(\Omega_0)$ lies in the two curved triangles inside, in which the two strictly inequalities (13) and (14) hold for any $(u_1, u_2) \in \{(u_1, u_2) \in \mathbb{R}^2 | u_1 u_2 > 0\}$. $\square$

Finally, the primal conic linear inequality representation $\Omega_D(c, M)$ is the whole two-dimensional space and the dual conic linear inequality representation $\Omega_P(d, M)$ is as follows

$$\{(v_1, v_2)^T | -2 \leq v_1 \leq 2,\ -2 \leq v_2 \leq 2\}.$$

For the set $\Omega_D(c, M)$, only one transition point is the origin and nonlinearity regions are two open sets contained in Ω_0. For the set $\Omega_P(d, M)$, the slack matrices corresponding to transition points are the four isolated matrices listed above and nonlinearity regions are two open triangle in the two-dimensional plane given by $\{(v_1, v_2)^T | -2 < v_1 < v_2 < 2\}$ and $\{(v_1, v_2)^T | -2 < v_2 < v_1 < 2\}$.

In LP, the actual invariancy region is convex, see Ghaffari-Hadigheh [15]. However, in SDP, the actual invariancy region could not be convex. For instance, there are six different invariancy sets of $\Omega_D(c, M)$ in Example 4.6. The two open nonlinearity regions contained in Ω_0 are not convex, although the other four invariancy regions $\Omega_1, \Omega_2, \Omega_3 - \{(0, 0)^T\}$ and $\Omega_4 - \{(0, 0)^T\}$ are convex.

5 Sensitivity analysis

In this section, we discuss how to perform sensitivity analysis based on the above set-valued mappings and the invariant region decomposition in the conic linear inequality representation sets.

Theorem 5.1. *Suppose that $(d, c) \in int(K \times K^*)$.*

(1) $h \in 0^+(\Omega_D(c, M))$ if and only if the following problem

$$\begin{array}{ll} \min & \langle M^T h, x\rangle \\ s.t. & Ax = b, \\ & x \in K \end{array} \tag{16}$$

is solvability.

(2) $h \in 0^+(\Omega_P(d, M))$ if and only if the following problem

$$\begin{array}{ll} \min & \langle M^T h, y\rangle \\ s.t. & By = a, \\ & y \in K^* \end{array} \tag{17}$$

is solvability.

Proof. Since it is trivial the case $h = 0 \in \mathbb{R}^r$, we consider the case that $h \in \mathbb{R}^r$ is not equal to zero. In the following statements, we always replace u by λh for the problem (7), where $\lambda > 0$.

Now we assume that the problem (16) is solvability. Then the problem (7) is strictly and bounded for a sufficiently large positive number λ; by Theorem 2.5, the dual of the problem (7) with $u = \lambda h$ is feasible. Since $\Omega_D(c, M)$ is a closed convex set, then for any $\lambda > 0$, the dual program (9) is feasible; by Corollary 3.1, $c + \lambda h \in \Omega_D(c, M)$, i.e., $h \in 0^+(\Omega_D(c, M))$. The sufficiency is proved.

Conversely, if $h \in 0^+(\Omega_D(c, M))$, then the dual program (9) with $u = \lambda h$ is strictly feasible for any $\lambda > 0$. Applying Theorem 2.5 again, the primal problem

(7) is solvability for any $\lambda > 0$. Therefore, we conclude that the problem (16) is solvability if λ tends to positive infinity. The proof is completed. $\square$

In LP, Theorem 5.1 provides a theoretical basis for the shadow vertex algorithm, see, e.g., [9, 14].

Corollary 5.2. *Suppose that* $(d, c) \in int(K \times K^*)$.

(1) If $0 \neq h \in 0^+(\Omega_D(c, M))$*, then there is a vector* $v^0 \in \partial(\Omega_P(d, M))$ *such that* $h \in 0^+(\psi(v^0))$*;*

(1) If $0 \neq h \in 0^+(\Omega_P(d, M))$*, then there is a vector* $u^0 \in \partial(\Omega_D(c, M))$ *such that* $h \in 0^+(\psi(u^0))$*.*

Proof. By Theorem 5.1, we assume that $x^*(h)$ is an optimal solution of the program (16) for $0 \neq h \in 0^+(\Omega_D(c, M))$. Then there is a big enough positive number Λ such that for any $\lambda > \Lambda$, the program (7) with $u = \lambda h$ attain a minimum at $x^*(h)$. Then the program (10) with $v = v^0 = Mx^*(h) - Md$ attain a minimum at $x^*(v^0) = x^*(h)$, which implies that for any $\lambda > \Lambda$, $u = \lambda h \in \psi(v^0)$. From Corollary 3.7 we have $h \in 0^+(\psi(v^0))$. Finally, from the optimality of $x^*(v^0)$ we get $v^0 \in \partial(\Omega_P(d, M))$. $\square$

In Examples 4.5 and 4.6, the dual conic linear inequality representable sets contain the recession directions.

Corollary 5.3. *Suppose that* $(d, c) \in int(K \times K^*)$.

(1) Let the slack matrix $x^*(\bar{v}) = d + M^T\bar{v} + B^T w^2$ *corresponding to* $\bar{v}$ *be an optimal solution of the problem (8). If* $x^*(\bar{v})$ *is an isolated vertex of the feasible region of the problem (8) and* $\psi(\bar{v})$ *is well defined, then* $\bar{v}$ *is a transition point of* $\Omega_P(d, M)$*;*

(2) Let the slack matrix $y^*(\bar{u}) = d + M^T\bar{u} + B^T w^2$ *corresponding to* $\bar{u}$ *be an optimal solution of the problem (9). If* $x^*(\bar{u})$ *is an isolated vertex of the feasible region of the problem (9) and* $\psi(\bar{u})$ *is well defined, then* $\bar{u}$ *is a transition point of* $\Omega_D(c, M)$*.*

Proof. Assume that $y^*(\bar{u}) = c + M^T\bar{u} + B^T w^2$ denote an optimal solution of the problem (9) corresponding to $\bar{u}$. Let $\mathcal{V}$ be a set that for every $v \in \mathcal{V}$, the hyperplane

$$H_v = \{y \in K^* | \langle d + M^T v, y\rangle = \langle d + M^T v, y^*(\bar{u})\rangle\}$$

supports the feasible region of the problem (8) at $y^*(\bar{u})$. By hypothesis, $y^*(\bar{u})$ is an isolated vertex of the feasible region of the problem (9), which implies that $\mathcal{V}$ is not empty and is not a singleton. Since $\phi(\bar{u})$ is well defined, the $\phi(\bar{u}) = \mathcal{V}$ is not a singleton, which implies that $\bar{u}$ is a transition point of $\Omega_D(c, M)$. $\square$

By Corollary 3.7, the case $\psi(v^1) \cap \psi(v^2) \neq \emptyset$ could happen. In particular, if either $\psi(v^1)$ or $\psi(v^2)$ is not a singleton and $\psi(v^1) \cap \psi(v^2) = \{\bar{u}\}$, then $\bar{u}$ is a transition point of $\Omega_D(c, M)$. For instance, in Example 4.6, if $v^1 = (2, 2)^T$ and $v^2 = (-2, -2)^T$, then $\psi(v^1) \cap \psi(v^2) = \Omega_3 \cap \Omega_4 = \{(0, 0)^T\}$ is not empty. Hence the origin is a transition point of the primal conic linear inequality representable set.

Corollary 5.3 and Example 4.6 give two existence conditions for a transition point, partially having answered the open question proposed Hauenstein et al. [21]. There are two different types of transition points: one is related to the isolated vertex, and the other is related to the intersection of the images of the set-valued mapping associated with the two isolated vertices. In Examples 3.4, 4.5 and 4.6, the numbers of transition points for $\Omega_P(d, M)$ are equal to $0, 3, 4$, respectively; and the numbers of transition points for $\Omega_D(c, M)$ are equal to $0, 4, 1$, respectively. In particular, in Example 4.6, the origin is the second type of transition points for $\Omega_P(d, M)$.

In additional, it follows from Corollary 5.3 that the number of transition points is closely related to the number of the isolated points of the feasible region. It is well-known that the number of the vertices of the polyhedron is finite. So we make the following conjecture:

Conjecture: the number of the vertices of the feasible region of a conic linear optimization problem is finite.

If this conjecture is true, then the the number of transition points is finite. Let $x^*(u^1), x^*(u^2), \cdots, x^*(u^k)$ denote all vertices of the feasible region of the problem (7) corresponding to the parameters $u^1, u^2, \cdots, u^k \in \Omega_D(c, M)$. These parameters $u^1, u^2, \cdots, u^k$ could be transition points. Now let us assume that the linear segments

$$[x^*(u^1), x^*(u^k)], [x^*(u^2), x^*(u^k)], \cdots, [x^*(u^{k-1}), x^*(u^k)]$$

do not lie in the boundary of the feasible region of the problem (7), that is, $\phi(u^k) \cap \phi(u^i) = \emptyset$, $i = 1, 2, \cdots, k-1$. By Theorem 3.6 and Corollary 3.7, the following set

$$\Omega_P(d, M) - \bigcup_{i=1}^{k} \phi(u^i)$$

is a nonempty open set. Therefore, by Theorem 4.2, there is a nonlinearity region of $\Omega_D(c, M)$. In a word, if there is not any linear segment connecting $x^*(u^k)$ and $x^*(u^j)(j = 1, 2, \cdots, k-1)$ on the boundary of the feasible region, then a nonlinearity region exists. Of course, if there is no any vertex on the boundary of the feasible region, then a nonlinearity region exists.

Corollary 5.4. *Suppose that* $(d, c) \in int(K \times K^*)$.

(1) Every nonlinearity region $\mathcal{U}$ *of* $\Omega_D(c, M)$ *is open and* $\phi(u)$ *is continuous in the region* $\mathcal{U}$*;*

(2) Every nonlinearity region $\mathcal{V}$ *of* $\Omega_P(d, M)$ *is open and* $\phi(v)$ *is continuous in the region* $\mathcal{V}$.

Proof. We first show that the nonlinearity region is open. Let us assume that $\bar{u} \in \mathcal{U}$ and $\bar{v} = \phi(\bar{u})$ such that $\pi(\bar{u}, \bar{v}) = (1, 1)$. Geometrical, the supporting hyperplane

$$H = \{x \in \mathbb{R}^q | \langle c + M^T \bar{u}, x\rangle = \langle c + M^T \bar{u}, x^*(\bar{u})\rangle\}$$

is tangent to the feasible region $\Omega_P = \{x \in K | Ax = b\}$, which implies that there is a neighbourhood $U(\bar{u})$ such that $x^*(u)$ corresponding to the parameters $u \in U(\bar{u})$ lie in the local smooth surface of Ω_P. Then for any $u \in U(\bar{u})$, the supporting hyperplane

$$H = \{x \in \mathbb{R}^q | \langle c + M^T u, x\rangle = \langle c + M^T u, x^*(u)\rangle\}$$

is tangent to the feasible region Ω_P. Furthermore, for any $u \in U(\bar{u})$ and $v \in \phi(u)$, $\pi(u, v) = (1, 1)$, which means that $\mathcal{U}$ is open.

We now show that $\phi(u)$ is continuous in the region $\mathcal{U}$. If for every $u \in \mathcal{U}$ and $v = \phi(u)$, $\pi(u, v) = (1, 1)$, then the set-valued map $\phi(u)$ on $\mathcal{U}$ degrades into a single-valued map. Then for two different vectors $u^1 \in \mathcal{U}$ and $u^2 \in \mathcal{U}$, $\phi(u^1) \neq \phi(u^2)$. By the connectivity of $\mathcal{U}$, if Γ denotes a continuous curve connecting two different points $u^1 \in \mathcal{U}$ and $u^2 \in \mathcal{U}$, then the trajectory of the optimal solution $x^*(u)$ $(u \in \Gamma)$ is a continuous curve along the boundary of the feasible region of the program (7). Therefore, $\phi(\Gamma)$ is a continuous curve connecting two points $v^1 = \phi(u^1)$ and $v^2 = \phi(u^2)$, which implies the continuity of $\phi(u)$ over the set $\mathcal{U}$. The proof is finished. □

In Example 4.6, for the primal conic linear inequality representation $\Omega_P(d, M)$, four vertices are transition points and the two open sets consisting of interior points separated by the diagonal $\{(v_1, v_2)^T | v_2 = v_2, -2 < v_1, v_2 < 2\}$ are nonlinearity regions. For the dual conic linear inequality representation $\Omega_D(c, M)$, the origin is only one transition point and the regions contained in the two curved triangles in the first and third quadrants are two different nonlinearity regions.

6 Conclusions

In this paper we established the relationships between the primal and dual conic linear inequality representable sets by the definition of the set-valued mapping and discussed the sensitivity analysis in convex conic linear optimization. We presented an efficient procedure to extend the optimal partition approach to sensitivity analysis in convex conic linear optimization and to give the invariant region decomposition of a conic linear inequality representation set. We generalized the concept of the optimal partition to arbitrary conic linear inequality representable sets and characterized a nonlinearity region of the optimal partition. Similar to the special cases of LP and SDP, it is possible to perform better parametric analysis based on the optimal partition for perturbations of the right-hand side and cost vectors.

This paper presented some existence conditions for a transition point and a nonlinear region, partially having answered the open question proposed Hauenstein et al. [21]. These conditions depend entirely on the conjecture presented in Section 5. The conjecture is helpful to understand the geometry of a conic linear inequality representable set.

Three examples in this paper confirmed our results. Currently, we are investigating more theoretical results for the conic linear optimization by the use of the parametric analysis technique presented in this paper.

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