Deciding the Feasibility of a Booking in the European Gas Market is coNP-hard

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ABSTRACT. We show that deciding the feasibility of a booking (FB) in the European entry-exit gas market is coNP-hard if a nonlinear potential-based flow model is used. The feasibility of a booking can be characterized by polynomially many load flow scenarios with maximum potential-difference, which are computed by solving nonlinear potential-based flow models. We use this existing characterization of the literature to prove that FB is coNP-hard by reducing Partition to the infeasibility of a booking. We further prove that computing a potential-difference maximizing load flow scenario is NP-hard even if we can determine the flow direction a priori. From the literature, it is known that FB can be decided in polynomial time on trees and a single cycle. Thus, our hardness result draws the first line that separates the easy from the hard variants of FB and finally answers that FB is hard in general.

1. Introduction

The entry-exit market system has been introduced for the European gas market as the outcome of the European gas market liberalization [31–33]. One of the main goals of the entry-exit system is to decouple the trade and transport of gas. To this end, the current market system is split into different stages, which are formally described in [19] by a multilevel model of the European entry-exit gas market. In these stages, the transmission system operator (TSO) and the gas traders interact with each other via so-called bookings and nominations. A booking is a mid- to long-term capacity-right contract between the gas traders and the TSO in which the gas traders buy capacity rights for the maximum injection or withdrawal quantity of gas at entry and exit nodes of the network. On a day-ahead basis, the traders nominate the quantity of gas for an injection, respectively withdrawal, on the next day such that the nominated quantity of gas stays within the booked capacities and the total injection and withdrawal quantities are balanced. By signing the booking contracts, the TSO is obliged to guarantee the feasibility of transport of the nominated amounts of gas through the network. Since the gas traders can nominate any booking-compliant nomination, i.e., any balanced quantities of injections and withdrawals up to the booked-capacities, the TSO has to guarantee the feasibility of transport for every booking-compliant nomination. Hence, the TSO has to generally verify the feasibility of infinitely many booking-compliant nominations before the TSO signs a booking. On the one hand, this property of the entry-exit market system decouples trade and transport. On the other hand, it leads to many new

challenges, e.g., deciding the feasibility of a booking or the computation of bookable capacities, so-called technical capacities; see [9, 26, 29, 39].

In this work, we focus on deciding the feasibility of a booking in general graphs. To this end, we consider passive networks without active elements, i.e., no controllable elements such as valves or compressors are present in the network. We further focus on stationary models of gas transport with nonlinear potential-based flows; see, e.g., [20, 26, 40]. Here, nonlinear potential-based flows means that the potential function is nonlinear. We note that potential-based flows significantly differ from the frequently used linear network flows with arc capacities. The latter flows consist of flow conservation and capacities on the flow through an arc. However, linear flows with arc capacities neglect pressure levels. Thus, the corresponding flows are not pressure-driven. In the following, we refer to linear flows with arc capacities as capacitated linear flows.

One of the main differences between the physically more accurate potential-based flows and the capacitated linear flows is that in a passive potential-based flow model, no cyclic flows are possible, which is not necessarily the case for capacitated linear flows. This structural property of potential-based flows can be algorithmically exploited; see e.g., [21]. However, the link between node potentials and arc flows is usually given by nonlinear constraints. Consequently, potential-based flows generally lead to a harder class of optimization problems compared to capacitated linear flows. We note that potential-based flows are also used to model hydrogen, water, or power distribution networks; see [20, 37].

Many optimization methods for the transport of gas in pipeline networks have been researched in the recent years, e.g., see the book [25] and the survey article [35] for a comprehensive overview. In doing so, the research mainly focuses on cost-optimal transport, respectively feasible transport, of a single nomination, respectively load flow scenario. The gas flow in pipelines follows physical laws, which lead to challenging mathematical optimization models. Thus, in the literature, approximations considering different classes of optimization models are used. Based on the descriptions in [26, 27, 39], we now state some examples for these approximations. In [7], the gas physics are approximated by piecewise linear functions and the authors analyze the cost-optimal transport in the Belgian network before the European market liberalization. In [3], the cost-optimal transport in the Belgian network is again analyzed considering the different market situation due to the progressed market liberalization. A combination of piecewise linearizations together with sequential quadratic programming is applied in [8]. Furthermore, in many works piecewise linear relaxations are considered for approximating the gas transport, e.g., see [12–15]. Approaches based on continuous nonlinear optimization methods, respectively complementarity-constraint modeling, are used in, e.g., [4, 38, 41–44]. Even more sophisticated and challenging nonlinear mixed-integer models are studied, e.g., in [14, 15, 24]. For details on optimization w.r.t. partial differential equations and optimal control of gas networks, we refer to the survey [22] and the references therein.

Regarding the complexity, deciding the feasibility of a nomination for capacitated linear flows as well as for potential-based flows with linear potential functions is in P since both problems can be decided by solving linear programs; see e.g., [26, 40]. In the following, we refer to potential-based flow with linear potential functions as linear potential-based flows. If active elements such as valves are present in the network,

then deciding the feasibility of a nomination is NP-hard for the potential-based flow model; see [45].

The literature on bookings, especially on deciding the feasibility of a booking, is rather small in comparison with the research on nominations. Following the literature review in [39], we now briefly summarize the literature on bookings. We note that deciding the feasibility of a booking can be seen as a specific two-stage or adjustable robust feasibility problem, see e.g., [5, 17], since a feasible booking requires the feasibility of all, generally infinitely many, nominations within the given booking bounds. One of the first results about the mathematical properties of bookings are given within the PhD theses [23, 46]. Further structural properties such as (non-)convexity of the set of feasible nominations and bookings are shown in [40]. We note that the reservation-allocation problem considered in [10] is similar to the feasibility of a booking. Moreover, the authors of [2] develop approaches to decide robust feasibility and infeasibility of specific two-stage robust optimization problems using techniques of polynomial optimization. These approaches can also be used to decide the feasibility of a booking in pipe-only networks. In case of capacitated linear flows, deciding the feasibility of a booking is coNP-complete in general networks, but can be decided in polynomial time in tree-shaped networks; see [23]. For the more accurate potential-based flows in pipe-only networks, the feasibility of a booking can be characterized by polynomially many nominations with maximum potential-difference; see [26]. Using this characterization, the authors of [26] show that for linear potential-based flows deciding the feasibility of a booking is in P for general graphs. Additionally, they also prove that this holds for nonlinear potential-based flows in tree-shaped networks; see [26]. We note that the latter also follows from [37] under additional assumptions on the potential bounds. Moreover, the authors of [27] extend these results by proving that deciding the feasibility of a booking in a single cycle is in P. In doing so, again special structures of the nominations with maximum potential-difference are exploited. In this paper, we finally answer that deciding the feasibility of a booking is coNP-hard in general graphs. Thus, we draw the first line that separates the easy from the hard variants of deciding the feasibility of a booking.

We finally refer to the topic of computing maximal feasible bookings, which are called technical capacities. These technical capacities are introduced in [29] and first results are obtained in [23, 46]. They can also be seen as a more sophisticated application of the radius of robust feasibility, e.g., see [6, 16, 28]. Regarding the complexity exponential upper bounds for computing technical capacities w.r.t. capacitated linear flows are given in [23]. An extensive complexity analysis is provided in the recent work [39]. The authors prove that computing technical capacities, i.e., maximizing over the set of feasible bookings, is NP-hard for capacitated linear, linear-, and nonlinear-potential based flows even on trees.

Our contribution is the following. We prove that deciding the feasibility of a booking considering a nonlinear potential-based flow model is coNP-hard in general graphs. The proof is obtained by reducing Partition to the infeasibility of a booking. It is the first hardness result for deciding the feasibility of a booking w.r.t. potential-based flows since the latter is in P for linear potential-based flows in general graphs and for nonlinear potential-based flows in trees and a single-cycle network. We further prove that computing a nomination with maximum potential-difference is NP-hard even if we can determine the flow direction a priori. We summarize our
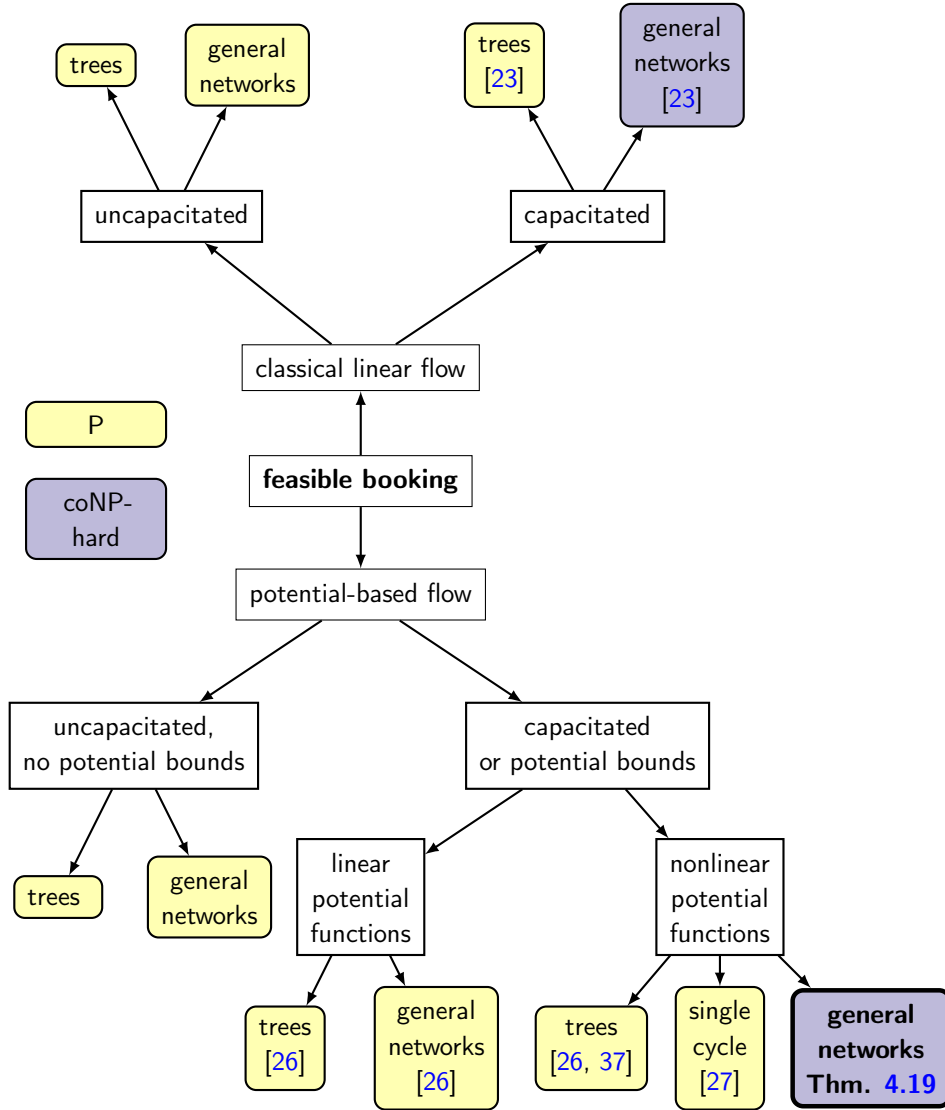
FIGURE 1. Overview of known complexity results for deciding the feasibility of a booking. The figure is an adapted and extended version of the one in Section 6 of [26]. We note that for the case of no flow bounds, respectively no potential bounds, it is not hard to observe that every booking is feasible. Thus, we do not explicitly cite this result.

contribution w.r.t. bookings together with a review of the results from the literature in Figure 1, which is an adapted and extended version of Figure 3 in Section 6 of [26].

The remainder of this paper is structured as follows. In Section 2, we formally introduce the problem of deciding the feasibility of a booking. Notations and first results, which are necessary for the hardness proof, are stated in Section 3. Afterward, in Section 4 the coNP-hardness of deciding the feasibility of a booking considering a nonlinear potential-based flow model is shown. Finally, we close with a conclusion in Section 5.

2. Problem Description

The problem description closely follows the one in [27]. We represent a gas network by a weakly connected and directed graph $G = (V, A)$ with node set V and arc set A. The set of nodes is partitioned into the set V_+ of entry nodes, at which gas is injected, the set V_- of exit nodes, where gas is withdrawn, and the set V_0 of the remaining inner nodes. These node types are equipped with a vector $\sigma \in \mathbb{R}^V$ given by

$$\sigma_u = \begin{cases} 1, & \text{if } u \in V_+, \\ -1, & \text{if } u \in V_-, \\ 0, & \text{if } u \in V_0. \end{cases}$$

In this paper, only passive networks, i.e., without active elements such as control valves or compressors, are considered. In the following, we introduce notations and definitions which are taken from [27, 40] and follow the corresponding descriptions therein.

Definition 2.1. A *load flow* is a vector $\ell \in \mathbb{R}^V_{\geq 0}$, with $\ell_u = 0$ for all $u \in V_0$. The set of load flow vectors is denoted by L.

A load flow represents an actual situation of the demand in the network at a single point in time by specifying the amount of gas ℓ_u that is injected at $u \in V_+$ or withdrawn at $u \in V_-$. Since we only consider stationary flows, we need to impose that the total injection and withdrawal quantities are balanced, which leads to the definition of a nomination.

Definition 2.2. A *nomination* is a balanced load flow ℓ, i.e., $\sigma^\top \ell = 0$. The set of nominations is given by

$$N := \left\{ \ell \in L \colon \sigma^\top \ell = 0 \right\}.$$

Definition 2.3. A *booking* is a vector $b \in \mathbb{R}^V_{\geq 0}$, with $b_u = 0$ for all $u \in V_0$.

From a mathematical perspective, load flows and bookings are the same. Further, nominations and bookings are linked as described in the following definition.

Definition 2.4. A nomination ℓ is called *booking-compliant* w.r.t. the booking b if $\ell \leq b$ holds, where "$\leq$" is meant component-wise. The set of booking-compliant (or b-compliant) nominations is given by

$$N(b) := \left\{ \ell \in N \colon \ell \leq b \right\}.$$

As in [27], we now introduce the notion of feasible nominations and bookings. To do so, we model stationary gas flows by an abstract physics model that is based on the Weymouth pressure drop equation and potential-based flows; see, e.g., [20] or [26]. It consists of arc flow variables $q \in \mathbb{R}^A$ and nodal potentials $\pi \in \mathbb{R}^V_{\geq 0}$. These arc flows couple the potentials, which leads to the nonlinear potential-based flow model (1) below. We note that in gas networks with horizontal pipes the potentials $\pi \in \mathbb{R}^V_{\geq 0}$ are closely linked to the gas pressures $p \in \mathbb{R}^V_{\geq 0}$ at the nodes of the pipeline networks by $\pi_u = p_u^2$ for $u \in V$. In case of non-horizontal pipes, we refer to [25].

Definition 2.5. A nomination $\ell \in N$ is *feasible* if a point (q, π) exists that satisfies

$$\sum_{a \in \delta^{\mathrm{out}}(u)} q_a - \sum_{a \in \delta^{\mathrm{in}}(u)} q_a = \sigma_u \ell_u, \qquad u \in V, \qquad (1a)$$

$$\pi_u - \pi_v = \Lambda_a q_a \left| q_a \right|, \qquad a = (u, v) \in A, \qquad (1b)$$

$$\pi_u \in [\pi_u^-, \pi_u^+], \qquad u \in V, \qquad (1c)$$

where $\delta^{\mathrm{out}}(u) := \{a \in A \mid a = (u, v)\}$ and $\delta^{\mathrm{in}}(u) := \{a \in A \mid a = (v, u)\}$ represent the set of arcs leaving and entering node $u \in V$, $\Lambda_a > 0$ is an arc-specific constant for any $a \in A$, and $0 < \pi_u^- \leq \pi_u^+$ are potential bounds for any $u \in V$.

Constraints (1a) ensure mass flow conservation at every node of the network. The flows q couple the potentials for every arc of the network in Constraint (1b). Since we model a gas network as a directed graph, we can interpret the direction of flow. Thus, for an arc $(u, v) \in A$, the arc flow can be negative if it flows in the opposite direction of the orientation of the arc, i.e., if it flows from v to u. Moreover, technical restrictions of the network impose that the potentials are bounded (1c). For other models of gas transport such as capacitated linear and linear potential-based flows, we refer to [26, 40].

For a given nomination $\ell \in N$ and a weakly connected pipe-only network, the corresponding flows $q = q(\ell)$, i.e., they satisfy Constraints (1a) and (1b), are unique because they are the optimal solution of a strictly convex minimization problem; see [30]. Moreover, the corresponding potentials $\pi = \pi(\ell)$ are unique up to constant shifts; see [36]. Consequently, the potential-differences $\pi_u - \pi_v$ for $u, v \in V$ are unique for a given nomination ℓ. We further note that the feasibility of a nomination can be verified by different approaches; see e.g., [18, 25]. We now turn to the feasibility of a booking, which is less studied in the literature and very challenging.

Definition 2.6. We say that a booking b is *feasible* if all booking-compliant nominations $\ell \in N(b)$ are feasible.

For deciding the feasibility of a booking, an infinite number of nominations has to be checked for feasibility, in general. As in [27], we make the following non-restrictive assumption on the potential bounds.

Assumption 2.7. The potential bound intervals have a non-empty intersection, i.e.,

$$\bigcap_{u \in V} [\pi_u^-, \pi_u^+] \neq \emptyset.$$

This assumption is a necessary condition for the feasibility of a booking, since it ensures that the zero nomination, which is always booking-compliant, with corresponding zero flows is feasible in the network. Consequently, if this assumption is not satisfied, it directly follows that all bookings are infeasible. Moreover, we can check this assumption in polynomial time a priori.

It is shown in [26], that the feasibility of a booking b can be characterized by polynomially many nominations with maximum potential-difference between all pairs of nodes. To this end, the authors introduce, for every pair of nodes $(w_1, w_2) \in V^2$,

the following problem

$$\varphi_{w_1 w_2}(b) := \max_{\ell, q, \pi} \quad \pi_{w_1} - \pi_{w_2} \tag{2a}$$

$$\text{s.t.} \quad 0 \leq \ell_u \leq b_u, \quad u \in V, \tag{2b}$$

$$(1a), (1b),$$

where $\varphi_{w_1 w_2}$ is the corresponding optimal value function that depends on the booking b. We note that the zero vector is feasible for Problem (2) due to Assumption 2.7. Consequently, an optimal solution always exists since the variables ℓ and q are bounded, and thus, we can bound the potentials π by a finite positive constant due to Theorem 7.1 of Chapter 7 in [25].

Problem (2) computes the maximum potential-difference between the given nodes w_1 and w_2 within the booking b. Thus, we do not consider any potential bounds that possibly restrict the maximum potential-difference. From Theorem 10 in [26] it then follows that these maximum potential-differences satisfy certain potential bounds if and only if booking b is feasible. More precisely, booking b is feasible if and only if

$$\varphi_{w_1 w_2}(b) \leq \pi_{w_1}^+ - \pi_{w_2}^-$$

holds for every pair of nodes $(w_1, w_2) \in V^2$. Exploiting this approach, we can decide the feasibility of a booking by solving the nonlinear and nonconvex optimization problems (2) for every pair of nodes. For trees and single-cycle networks, this can be done in polynomial time; see [26, 27, 37]. In this paper, we show that deciding the feasibility of a booking w.r.t. nonlinear potential-based flow model (1) is coNP-hard. We further prove that the decision variant of (2) is NP-hard. To this end, we formally define the problem of deciding the feasibility of a booking w.r.t. nonlinear gas flow model (1) and the decision problem corresponding to (2).

Deciding the Feasibility of a Booking (FB).

Input: Graph $G = (V, A)$ with entries V_+, exits V_-, inner nodes V_0, potential drop coefficients $\Lambda_a \in \mathbb{Q}_{>0}$ for $a \in A$, potential bounds $\pi_u^- \leq \pi_u^+, \pi_u^-, \pi_u^+ \in \mathbb{Q}_{>0}$ for all $u \in V$, and a booking $b \in \mathbb{Q}_{\geq 0}^V$.

Question: Is booking b feasible, i.e., does for every booking-compliant nomination $\ell \in N(b)$ a feasible point (q, π) for (1) exist?

Computing a Maximum Potential-Difference Nomination (MPD).

Input: Graph $G = (V, A)$ with entries V_+, exits V_-, inner nodes V_0, potential drop coefficients $\Lambda_a \in \mathbb{Q}_{>0}$ for $a \in A$, a booking $b \in \mathbb{Q}_{\geq 0}^V$, two nodes $w_1, w_2 \in V$, and a threshold $T \in \mathbb{Q}$.

Question: Does a solution of (2) with objective $\varphi_{w_1 w_2}(b) \geq T$ exist?

3. Notations and Basic Observations

We now introduce notations and basic observations, which are taken from [27]. Using these results, we prove that FB w.r.t. our nonlinear potential-based flow model (1) is coNP-hard in Section 4. In the following, we consider the directed and weakly connected graph $G = (V, A)$ that models the considered gas network. For a nomination $\ell \in N$, entry and exit nodes $v \in V_+ \cup V_-$ are called *active* if $\ell_v > 0$ holds. We denote the set of active entries by $V_+^> := \{v \in V_+ : \ell_v > 0\}$ and the set of active

exits by $V_-^> := \{v \in V_- : \ell_v > 0\}$. Furthermore, the set of non-zero nominations is given by $N^0 := N \setminus \{0\}$.

Modeling gas networks by directed graphs enables us to interpret the direction of arc flows. However, the physical flow in a potential-based network is independent from the direction of the arcs. As described in [1], we introduce a path in a directed graph $G = (V, A)$ as a sequence of nodes and arcs without any repetition of nodes. For two nodes $v_1, v_r \in V$, a path $P(v_1, v_r)$ between nodes v_1 and v_r is a sequence $v_1 - a_1 - v_2 - a_2 - \ldots - a_{r-1} - v_r$ such that for each $1 \le k \le r-1$ either $a_k = (v_k, v_{k+1}) \in A$ or $a_k = (v_{k+1}, v_k) \in A$ holds. As in [1], we partition the arcs of a path into forward and backward arcs. An arc (i, j) of the considered path is called a forward arc if the path visits node i before node j and otherwise it is called a backward arc. Further, path $P(v_1, v_r)$ is a directed path from v_1 to v_r if it does not contain any backward arcs.

To provide a more compact representation of paths, we denote a path $P(v_1, v_r)$ by the corresponding set of arcs, i.e., $P(v_1, v_r) \subseteq A$, in the following. If it is clear from the context, we may omit the start and end node and simply use $a \in P$ to express that arc a is part of the considered path. We further note that the considered paths do not contain any cycles in the graph underlying G. This is in line with the considered physics of gas transport, since in every solution (q, π) of (1) no cyclic flow exists due to (1b).

For an arc $a = (u, v) \in A$, we now introduce the following *characteristic function*

$$\chi_a(P) := \begin{cases} 1, & \text{if } a \text{ is a forward arc of path P}, \\ -1, & \text{if } a \text{ is a backward arc of path P}, \\ 0, & \text{if } a \notin P, \end{cases}$$

where P is an arbitrary path. The following lemma of [27] uses a classical result from linear flow models to construct a flow decomposition without cyclic flows.

Lemma 3.1 (Lemma 2 and Corollary 3 in [27])**.** For $\ell \in N^0$, let $\mathcal{P}_\ell := \{P(u, v) : u \in V_+^>, v \in V_-^>\}$ be the set of paths in G with an active entry as start node and an active exit as end node. Then, a decomposition of the given arc flows $q = q(\ell)$, satisfying (1a) and (1b), into path flows exists, such that

$$q_a = \sum_{P \in \mathcal{P}_\ell} \chi_a(P) \, q(P), \quad a \in A,$$

where $q(P)$ is the nonnegative flow along the path $P \in \mathcal{P}_\ell$. Furthermore, the flows q can be chosen such that if $q_a > 0$ for $a \in A$ and $\chi_a(P) = -1$ for $P \in \mathcal{P}_\ell$, then $q(P) = 0$ holds. Similarly, if $q_a < 0$ for $a \in A$ and $\chi_a(P) = 1$ for $P \in \mathcal{P}_\ell$, then $q(P) = 0$ holds. Moreover, the flows satisfy

$$\sum_{v \in V_-^>} q(P(u, v)) = \ell_u, \ u \in V_+^>, \quad \sum_{u \in V_+^>} q(P(u, v)) = \ell_v, \ v \in V_-^>.$$

Similar to [27], we express the potential-difference function along a given path.

Definition 3.2. Let a path P be given. Then, the *potential-difference function* along P is given by

$$\Pi_P : \mathbb{R}^A \to \mathbb{R}, \quad \Pi_P(q) := \sum_{a \in P} \chi_a(P) \Lambda_a q_a \, |q_a| \, .$$

As described in [27], from Constraints (1b) it follows that for any node pair $u, v \in V$ and for any path $P := P(u, v)$, the equation $\pi_u(\ell) - \pi_v(\ell) = \Pi_P(q(\ell))$ is satisfied, where $\pi(\ell)$ and $q(\ell)$ correspond to the potentials and flows of a given nomination $\ell \in N$. If P is a directed path from u to v, then the potential-difference function simplifies to

$$\Pi_P(q) = \sum_{a \in P} \Lambda_a q_a \left| q_a \right|.$$

As last basic observation, we explicitly state some properties for directed paths and the potential-difference function proven in [27].

Lemma 3.3 (Lemma 4 in [27])**.** For $u, v \in V$, let $P := P(u, v)$ be a directed path from u to v. Then, the following holds:

(a) Π_P is continuous.

(b) Π_P is strictly increasing w.r.t. every component. That means, for q fixed except for one value q_a, $a \in P$, Π_P is increasing in q_a.

(c) Π_P is unbounded w.r.t. every component, i.e., for $a \in P$,

$$\lim_{q_a \to -\infty} \Pi_P = -\infty \quad \text{and} \quad \lim_{q_a \to \infty} \Pi_P = \infty.$$

(d) Π_P is additive w.r.t. the path, i.e., for every node v' of the path P,

$$\Pi_P = \Pi_{P(u,v')} + \Pi_{P(v',v)}$$

with $P = P(u, v') \cup P(v', v)$.

(e) If (ℓ, q, π) satisfies Constraints (1a) and (1b), then $\Pi_P(q) \geq 0$ holds if and only if $\pi_u \geq \pi_v$ holds.

4. Hardness

In this section, we prove that deciding the feasibility of a booking (FB) w.r.t. nonlinear potential-based flow model (1) is coNP-hard. To this end, we reduce Partition to the infeasibility of a booking. We consider Partition as it is defined in [11].

Partition (Part).

Input: $S_1, \ldots, S_n \in \mathbb{N}$ with $n \in \mathbb{N}$, $I = \{1, \ldots, n\}$.

Question: Does $I_1 \subseteq I$ with $\sum_{i \in I_1} S_i = \sum_{I \setminus I_1} S_i$ exist?

In the following, we denote a Partition instance by $\mathcal{S}$. The value $K = \sum_{i \in I} S_i$ further represents the sum of all numbers of Partition instance $\mathcal{S}$ and we assume without loss of generality that $n \geq 3$ holds.

For a given Partition instance $\mathcal{S}$, we now construct a corresponding booking instance with its graph $G(\mathsf{Part}) = (V, A)$. To this end, we introduce for every number S_i of the Partition instance $\mathcal{S}$ an entry node z_i^+ and an exit node z_i^-, which are connected by an arc. Furthermore, we add a main entry node s and a main exit node t. Each of these two nodes is connected to every entry node z_i^+ for $i \in I$. The booking and potential bounds are chosen such that we only have to consider (2) w.r.t. (s, t) for deciding the feasibility of the considered booking. We then prove that the corresponding objective value, i.e., the potential-difference between s and t, exceeds a certain threshold value if and only if the Partition instance $\mathcal{S}$ is feasible.
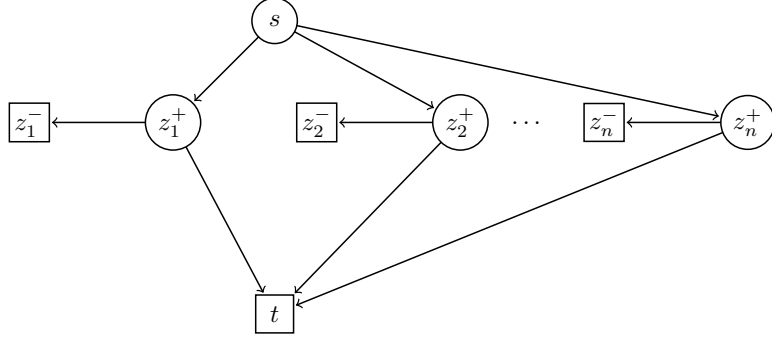
FIGURE 2. The graph $G(\mathsf{Part})$. Entry nodes are indicated by circles and exit nodes by boxes

In detail, graph $G(\mathsf{Part}) = (V, A)$ is constructed as

$$V_+ = \left\{ s, z_1^+, \ldots, z_n^+ \right\}, \quad V_- = \left\{ t, z_1^-, \ldots, z_n^- \right\}, \quad V = V_+ \cup V_-,$$
$$A = \left\{ (s, z_i^+) \colon i \in I \right\} \cup \left\{ (z_i^+, z_i^-) \colon i \in I \right\} \cup \left\{ (z_i^+, t) \colon i \in I \right\}.$$

A visualization of the graph $G(\mathsf{Part})$ is given in Figure 2. We consider for $G(\mathsf{Part})$ the booking b

$$b_s = b_t = \frac{K}{2}, \quad b_{z_i^+} = b_{z_i^-} = S_i, \quad i \in I,$$

and potential drop coefficients $\Lambda_a > 0$ for $a \in A$ given by

$$\Lambda_{(s, z_i^+)} = \Lambda_{(z_i^+, t)} = \frac{1}{S_i^2}, \ \Lambda_{(z_i^+, z_i^-)} = 1, \quad i \in I.$$

We finally set the potential bounds for this booking instance. To this end, we first introduce some parameters depending on the input K that are required throughout this section.

$$\varepsilon(K) = 1 - \left(\frac{K - \frac{1}{8K}}{K} \right)^2, \quad M(K) = \max \left\{ 1 - \varepsilon(K) + \varepsilon(K)^2, 1 - \frac{\varepsilon(K)^2}{K^2} \right\}, \quad \text{(3a)}$$

$$\tilde{\varepsilon}(K) = \frac{1}{5}(1 - M(K)), \qquad T(K) = \max \left\{ 1 - \frac{\tilde{\varepsilon}(K)^2}{K^2 n^2}, M(K) + 4\tilde{\varepsilon}(K) \right\}. \quad \text{(3b)}$$

We note that $0 < T(K) < 1$ holds due to $K \geq 1$. Further, we do not explicitly simplify these parameters by computing the maxima, because each of these parameters is separately used in the following proofs. Thus, the extensive form of (3a) and (3b) is useful for the remainder of this section.

The lower π^- and upper π^+ potential bounds for $G(\mathsf{Part})$ are given by

$$\pi_s^+ = \pi_t^+ = 9K^2 + 1, \quad \pi_u^+ = 18K^2 + 1, \ u \in V \setminus \{s, t\},$$
$$\pi_u^- = 0, \ u \in V \setminus \{t\}, \quad \pi_t^- = \pi_t^+ - T(K).$$

We note that for a given instance of $\mathsf{Partition}$, we can build the corresponding booking instance $G(\mathsf{Part})$ in polynomial time and its coding length is polynomially bounded from above by the coding length of the given $\mathsf{Partition}$ instance. We now consider $G(\mathsf{Part})$ and its booking b throughout this section. From Section 2, it follows that the booking b is feasible if and only if

$$\varphi_{w_1 w_2}(b) \leq \pi_{w_1}^+ - \pi_{w_2}^- \tag{4}$$

holds for every pair of nodes $(w_1, w_2) \in V^2$, where $\varphi_{w_1 w_2}(b)$ is the optimal value function of (2) that denotes the maximum potential-difference between w_1 and w_2 within booking b.

We first show that Inequality (4) is directly satisfied for $(w_1, w_2) \in V^2 \setminus \{(s, t)\}$, due to the choice of the lower and upper potential bounds.

Lemma 4.1. For any node pair $(w_1, w_2) \in V^2 \setminus \{(s, t)\}$ of $G(\mathsf{Part}) = (V, A)$, Inequality (4) is satisfied.

Proof. Let (q, π) be an optimal solution of $\varphi_{w_1 w_2}(b)$. We note that for the case $w_1 = w_2$, the equation $\varphi_{w_1 w_2}(b) = 0$ holds and, thus, Condition (4) is satisfied in this case due to $\pi_u^- \leq \pi_u^+$ for all $u \in V$. Consequently, we assume $w_1 \neq w_2 \in V$ in the following. From (1a), (1b), and $\sum_{u \in V_+} b_u = (3/2)K$, it follows $|q_a| \leq (3/2)K$ for every arc $a \in A$. For an arbitrary path $P(w_1, w_2)$, it holds

$$\pi_{w_1} - \pi_{w_2} = \sum_{a \in P(w_1, w_2)} \chi_a(P) \Lambda_a q_a |q_a| \leq |P(w_1, w_2)| \left(\frac{3}{2}K\right)^2.$$

Since the path $P(w_1, w_2)$ with minimal number of arcs consists of at most 4 arcs, this leads to

$$\pi_{w_1} - \pi_{w_2} \leq 4 \left(\frac{3}{2}K\right)^2 = 9K^2$$

$$< \pi_{w_1}^+ - \pi_{w_2}^- = \begin{cases} 9K^2 + 1, & \text{if } w_1 \in \{s, t\}, w_2 \neq t \\ 18K^2 + 1, & \text{if } w_1 \in V \setminus \{s, t\}, w_2 \neq t \\ 9K^2 + T(K), & \text{if } w_1 \in V \setminus \{s, t\}, w_2 = t. \end{cases}$$

$\square$

Consequently, deciding the feasibility of booking b for $G(\mathsf{Part})$ reduces to the following.

Corollary 4.2. Booking b is feasible for $G(\mathsf{Part})$ if and only if $\varphi_{st}(b) \leq T(K)$.

Proof. Booking b is feasible if and only if for every pair of nodes Inequalities (4) hold. From Lemma 4.1 it follows that for each pair of nodes except $(s, t) \in V^2$ Inequalities (4) are satisfied. Consequently, booking b is feasible if and only if

$$\varphi_{st}(b) \leq \pi_s^+ - \pi_t^- = 9K^2 + 1 - (9K^2 + 1 - T(K)) = T(K),$$

which proves the claim. $\square$

In the remainder of this section, we prove that $\varphi_{st}(b) > T(K)$ is satisfied if and only if the given $\mathsf{Partition}$ instance $\mathcal{S}$ is feasible. We first prove that if $\mathsf{Partition}$ instance $\mathcal{S}$ is feasible, then $\varphi_{st}(b)$ exceeds the threshold $T(K)$.

Lemma 4.3. If $\mathsf{Partition}$ instance $\mathcal{S}$ is feasible, then $\varphi_{st}(b) \geq 1 > T(K)$ holds.

Proof. Since $\mathsf{Partition}$ instance $\mathcal{S}$ is feasible, $M_1 \subset I$ and $M_2 \subset I$ with $M_1 \cap M_2 = \emptyset$ and $M_1 \cup M_2 = I$ exist such that $\sum_{i \in M_1} S_i = \sum_{i \in M_2} S_i = K/2$ holds. We now

construct a feasible point (ℓ, q, π) of (2) w.r.t. (s, t) that has an objective value of 1.

$$\ell_s = \ell_t = \frac{K}{2}, \quad \ell_{z_i^-} = S_i, \ \ell_{z_i^+} = 0, \ i \in M_1, \quad \ell_{z_i^+} = S_i, \ \ell_{z_i^-} = 0, \ i \in M_2,$$

$$q_{(s, z_i^+)} = q_{(z_i^+, z_i^-)} = S_i, \ q_{(z_i^+, t)} = 0, \ i \in M_1,$$

$$q_{(s, z_i^+)} = q_{(z_i^+, z_i^-)} = 0, \ q_{(z_i^+, t)} = S_i, \ i \in M_2,$$

$$\pi_s = \pi_{z_i^+} = \pi_{z_i^-} = \pi_s^+, \ i \in M_2, \quad \pi_{z_i^+} = \pi_s^+ - 1, \ \pi_{z_i^-} = \pi_{z_i^+} - S_i^2, \ i \in M_1,$$

$$\pi_t = \pi_s - 1.$$

We note that the constructed point (ℓ, q, π) satisfies Constraints (1a) as well as (1b) and ℓ respects the booking bounds. Consequently, (ℓ, q, π) is a feasible point for (2) w.r.t. (s, t). Moreover, the corresponding objective value equals 1. Thus, the optimal solution of $\varphi_{st}(b)$ has an objective value of at least $1 > T(K)$. $\qquad\square$

We next prove that if we require nonnegative flow in (2) and Partition instance $\mathcal{S}$ is infeasible, then an optimal solution does not exceed the threshold value $T(K)$, i.e., $\varphi_{st}(b) < T(K)$ holds. To this end, we introduce the nonnegative flow variant of (2) for node pair (s, t):

$$\varphi_{st}^+(b) := \max_{\ell, q, \pi} \quad \pi_s - \pi_t \tag{5a}$$

$$\text{s.t.} \quad 0 \le \ell_u \le b_u, \qquad\qquad u \in V, \tag{5b}$$

$$q_a \ge 0, \qquad\qquad a \in A, \tag{5c}$$

$$(1a), (1b). $$

With the help of Lemmas 4.5–4.13, we prove that $\varphi_{st}^+(b) < T(K)$ is satisfied if Partition instance $\mathcal{S}$ is infeasible. Afterward, we use these results to show the general case $\varphi_{st}(b) < T(K)$. The main idea is that if the considered Partition instance is infeasible, then for at least one $i \in I$ the flows satisfy $q_{(s, z_i^+)} > 0$ and $q_{(z_i^+, t)} > 0$; see Lemma 4.6. This together with the considered nonlinear potential-based flow (1b) plays an important role in the remaining section which is based on the following observation.

Observation 4.4. Let be $a > 0$ and $b > 0$ with $a^2 + b^2 = c$. Then, $a + b > \sqrt{c}$ holds.

To provide an intuition for the main idea, we give the following explanation. For every $i \in I$ and solution (ℓ, q, π) of (5), the equality

$$\pi_s - \pi_t = \Lambda_{(s, z_i^+)} q_{(s, z_i^+)}^2 + \Lambda_{(z_i^+, t)} q_{(z_i^+, t)}^2$$

holds, which we will show in Lemma 4.7. From Observation 4.4, it follows that we need strictly more flow in terms of $q_{(s, z_i^+)} + q_{(z_i^+, t)}$ to obtain a potential-difference of $\pi_s - \pi_t$ if $q_{(s, z_i^+)} > 0$ and $q_{(z_i^+, t)} > 0$ hold in comparison to the case if one of these flows is zero. We prove in Lemma 4.6 that if Partition instance $\mathcal{S}$ is infeasible, $q_{(s, z_i^+)} > 0$ and $q_{(z_i^+, t)} > 0$ for at least one $i \in I$ in an optimal solution of (5) is satisfied. The latter is not necessarily the case if $\mathcal{S}$ is feasible; see Lemma 4.3. Consequently, we need strictly more flow to obtain a certain potential-difference $\pi_s - \pi_t$ if $\mathcal{S}$ is infeasible in contrast to the case if $\mathcal{S}$ is feasible. This is one of the main reasons why the optimal value of (5) cannot exceed the threshold $T(K)$ if Partition

instance $\mathcal{S}$ is infeasible due to the monotonicity of the potential drop $\Pi_P(q)$ w.r.t. the flow q; see Lemma 3.3 (b).

We start the proof of $\varphi_{st}^+(b) < T(K)$ by characterizing optimal solutions of (5) in the next two lemmas.

Lemma 4.5. Let (ℓ, q, π) be an optimal solution of (5). Then,

$$\ell_s = \ell_t = \frac{K}{2} = b_s = b_t$$

holds.

Proof. We consider a flow decomposition of Lemma 3.1 for given (ℓ, q) and prove the claim by the distinction of three different cases.

If $\ell_s < K/2$ and $\ell_t < K/2$ hold, then from Lemma 3.3 it follows that $\varepsilon_i > 0$ for all $i \in I$ exist such that

$$\ell_s + \sum_{i \in I} \varepsilon_i \leq \frac{K}{2} = b_s, \quad \ell_t + \sum_{i \in I} \varepsilon_i \leq \frac{K}{2} = b_t$$

and we can increase the flow $q(P_i(s,t))$ by ε_i for $i \in I$, such that the potential drop $\Pi_{P_i(s,t)}$ is the same for $i \in I$. Here, $P_i(s,t)$ is the directed path from s to t via z_i^+, i.e., $P_i(s,t) = \{(s, z_i^+), (z_i^+, t)\}$. The modification increases the potential drop $\pi_s - \pi_t$ due to Lemma 3.3 (b). This is a contradiction to the optimality of (ℓ, q, π) for (5).

If $\ell_s = K/2$ and $\ell_t < K/2$ hold, then from (1a) it follows that the total flow leaving node s is $K/2$ and the total flow entering node t is strictly less than $K/2$. Moreover, no negative flow is entering s or leaving node t due to (5c). Thus, there is $i \in I$ such that $q(P(s, z_i^-)) > 0$ for $P(s, z_i^-) = \{(s, z_i^+), (z_i^+, z_i^-)\}$ holds. We additionally consider the path $P_i(s,t)$ as in the previous case. Due to Lemma 3.3 and the graph structure of $G(\mathsf{Part})$, $\varepsilon > 0$ and $0 < \tilde{\varepsilon} < \varepsilon$ exist such that decreasing $q(P(s, z_i^-))$ by ε and increasing $q(P_i(s,t))$ by $\tilde{\varepsilon}$ does not change the potential drop $\Pi_{P_i(s,t)}$ and $\ell_t < K/2$ as well as (5c) still holds. Consequently, $\ell_s < K/2$ and $\ell_t < K/2$ is satisfied and analogously to the above this is a contradiction to the optimality of the considered solution.

If $\ell_t = K/2$ and $\ell_s < K/2$ are satisfied, then, analogously to the previous case, from (1a) and (5c) it follows that $i \in I$ exist such that $q(P(z_i^+, t)) > 0$ for $P(z_i^+, t) = \{(z_i^+, t)\}$ holds. We again consider the path $P_i(s,t)$ as before. Due to Lemma 3.3 and the graph structure of $G(\mathsf{Part})$, $\varepsilon > 0$ and $0 < \tilde{\varepsilon} < \varepsilon$ exist such that decreasing $q(P(z_i^+, t))$ by ε and increasing $q(P_i(s,t))$ by $\tilde{\varepsilon}$ does not change the potential drop $\Pi_{P_i(s,t)}$ and $\ell_s < K/2$ as well as (5c) still holds. Consequently, $\ell_s < K/2$ and $\ell_t < K/2$ is satisfied and analogously to the above this is a contradiction to the optimality of the considered solution. $\square$

We now show that if $\mathsf{Partition}$ instance $\mathcal{S}$ is infeasible, then every optimal solution of (5) satisfies that at least one path $P_i(s,t) = \{(s, z_i^+), (z_i^+, t)\}$ with $i \in I$ and $q_a > 0$ for $a \in P_i(s,t)$ exists.

Lemma 4.6. If $\mathsf{Partition}$ instance $\mathcal{S}$ is infeasible, then for any optimal solution (ℓ, q, π) of (5), there exists an index $i \in I$ such that $q_{(s, z_i^+)} > 0$ and $q_{(z_i^+, t)} > 0$ hold.

Proof. We prove the contraposition. To this end, let (ℓ, q, π) be an optimal solution of (5) that satisfies $q_{(s,z_i^+)} = 0$ or $q_{(z_i^+,t)} = 0$ for each $i \in I$. Due to Lemma 4.5 and (5c), $q_{(s,z_i^+)} = q_{(z_i^+,t)} = 0$ is not satisfied for any $i \in I$ in an optimal solution of (5). Thus, the index sets

$$M_1 = \left\{ i \in I : q_{(s,z_i^+)} > 0 \right\}, \quad M_2 = \left\{ i \in I : q_{(z_i^+,t)} > 0 \right\}$$

satisfy $M_1 \cap M_2 = \emptyset$ and $M_1 \cup M_2 = I$. Consequently, if $i \in M_1$, then $q_{(s,z_i^+)} > 0$ and $q_{(z_i^+,t)} = 0$ hold. Thus, the structure of graph $G(\mathsf{Part})$ implies $q_{(s,z_i^+)} \leq S_i$ since positive flow via arc (s, z_i^+) can only be sent to exit z_i^-, which demand is bounded from above by S_i. Analogously it follows the case for $i \in M_2$ and we obtain the flow bounds

$$q_{(s,z_i^+)} \leq S_i, \ q_{(z_i^+,t)} \leq S_i, \quad i \in I. \tag{6}$$

Furthermore, from Lemma 4.5 it follows $\ell_s = \ell_t = K/2$. The latter together with flow conservation (1a) and (6), leads to

$$\frac{K}{2} = \sum_{i \in I} q_{(s,z_i^+)} = \sum_{i \in M_1} q_{(s,z_i^+)} \leq \sum_{i \in M_1} S_i,$$

$$\frac{K}{2} = \sum_{i \in I} q_{(z_i^+,t)} = \sum_{i \in M_2} q_{(z_i^+,t)} \leq \sum_{i \in M_2} S_i.$$

From this and $\sum_{i \in M_1 \cup M_2} S_i = \sum_{i \in I} S_i = K$, we obtain

$$\frac{K}{2} = \sum_{i \in M_1} S_i = \sum_{i \in M_2} S_i$$

and thus, $\mathsf{Partition}$ instance $\mathcal{S}$ is feasible. $\qquad\square$

In the next lemma, we express a relation between certain flows and the objective value of Problem (5), which we exploit multiple times in the following proofs.

Lemma 4.7. Let (ℓ, q, π) be a feasible point of Problem (5) with objective value φ. Then, for each $i \in I$ the following equality

$$q_{(s,z_i^+)}^2 + q_{(z_i^+,t)}^2 = \phi S_i^2 \tag{7}$$

holds.

Proof. For every $i \in I$, it follows from Constraints (1b) that

$$\phi = \pi_s - \pi_t = \frac{1}{S_i^2} q_{(s,z_i^+)} \left| q_{(s,z_i^+)} \right| + \frac{1}{S_i^2} q_{(z_i^+,t)} \left| q_{(z_i^+,t)} \right|.$$

In addition, the flows q are nonnegative since they satisfy Constraints (5c), which in combination with the previous equality proves the claim. $\qquad\square$

With the help of the previous three lemmas, we now prove that if $\mathsf{Partition}$ instance $\mathcal{S}$ is infeasible, then the optimal value of (5) is strictly smaller than 1.

Lemma 4.8. If $\mathsf{Partition}$ instance $\mathcal{S}$ is infeasible, then $\varphi_{st}^+(b) < 1$ holds.

Proof. Let (ℓ, q, π) be an optimal solution of (5) with objective value ϕ. Due to Lemma 4.5 and flow conservation (1a), the equalities

$$\frac{K}{2} = \sum_{i \in I} q_{(s,z_i^+)}, \quad \frac{K}{2} = \sum_{i \in I} q_{(z_i^+,t)}, \tag{8}$$

are satisfied. From Lemma 4.6, it follows that an index $l \in I$ with $q_{(s,z_l^+)} > 0$ and $q_{(z_l^+,t)} > 0$ exists. Further, Lemma 4.7 implies that for every $i \in I$ Equality (7) is satisfied. Applying Observation 4.4 then leads to

$$q_{(s,z_l^+)} + q_{(z_l^+,t)} > \sqrt{\phi} S_l, \quad q_{(s,z_i^+)} + q_{(z_i^+,t)} \geq \sqrt{\phi} S_i, \quad i \in I \setminus \{l\}.$$

From this and (8), we obtain

$$\begin{aligned}
K &= \sum_{i \in I \setminus \{l\}} \left(q_{(s,z_i^+)} + q_{(z_i^+,t)} \right) + q_{(s,z_l^+)} + q_{(z_l^+,t)} \\
&> \sqrt{\phi} \sum_{i \in I \setminus \{l\}} S_i + \sqrt{\phi} S_l = \sqrt{\phi} K,
\end{aligned}$$

which implies $\phi < 1$. $\qquad\square$

Up to now, we have shown that if Partition instance $\mathcal{S}$ is feasible, then $\varphi_{st}(b) \geq 1$ holds; see Lemma 4.3. Furthermore, if $\mathcal{S}$ is infeasible, then $\varphi_{st}^+(b) < 1$ is satisfied. In general, it now suffices to show that there is a small $\varepsilon > 0$ such that for infeasible $\mathcal{S}$ the inequality $\varphi_{st}(b) \leq 1 - \varepsilon$ holds. Then, we can set $\pi_t^- = \pi_t^+ - 1 + \varepsilon$ and analogously to Corollary 4.2 it follows that the considered booking b is feasible if and only if $\varphi_{st}(b) \leq 1 - \varepsilon$. In doing so, the coding length of $\varepsilon > 0$ has to be polynomially bounded from above by the coding length of $\mathcal{S}$ since $\varepsilon > 0$ is part of the booking instance. In the following Lemmas 4.9–4.13, we now show that if $\mathcal{S}$ is infeasible, then $\varphi_{st}^+(b) < T(K) < 1$ holds. Afterward, we use this result for nonnegative flows to prove that for infeasible $\mathcal{S}$ also the general case $\varphi_{st}(b) < T(K) < 1$ is satisfied. Consequently, booking b is feasible if and only if $\mathcal{S}$ is infeasible due to Corollary 4.2.

For a feasible point (ℓ, q, π) of (5) with positive objective value, we introduce the following partition of the indices I

$$M_1 = \left\{ i \in I : q_{(s,z_i^+)} > 0, q_{(z_i^+,t)} = 0 \right\}, \tag{9a}$$

$$M_2 = \left\{ i \in I : q_{(s,z_i^+)} = 0, q_{(z_i^+,t)} > 0 \right\}, \tag{9b}$$

$$M_3 = \left\{ i \in I : q_{(s,z_i^+)} > 0, q_{(z_i^+,t)} > 0 \right\}. \tag{9c}$$

We note that $q_{(s,z_i^+)} = q_{(z_i^+,t)} = 0$ does not hold for any $i \in I$ due to the positive objective value of the considered feasible point (ℓ, q, π). Moreover, the parameters $\varepsilon(K)$ and $M(K)$ of (3a) come now into play.

Lemma 4.9. Let Partition instance $\mathcal{S}$ be infeasible. If (ℓ, q, π) is an optimal solution of (5) such that $q_{(s,z_l^+)} > 0$ and $q_{(z_l^+,t)} > 0$ hold for exactly one index $l \in I$, then the corresponding objective value ϕ satisfies

$$\phi < \left(\frac{K - \frac{1}{8K}}{K} \right)^2 = 1 - \varepsilon(K) < 1.$$

Proof. We assume for a contradiction that an optimal solution (ℓ, q, π) of (5) exists such that the requirements are satisfied and the objective value ϕ satisfies $\phi \geq 1 - \varepsilon(K) > 0$. From Lemma 4.8, it follows $\phi < 1$. We partition the index set I according to (9).

Due to the requirements, $M_1 \cap M_2 = \emptyset$, $M_3 = \{l\}$, and $M_1 \cup M_2 \cup \{l\} = I$ hold. Consequently, Lemma 4.5 and flow conservation (1a) lead to

$$\frac{K}{2} = \sum_{i \in M_1} q_{(s,z_i^+)} + q_{(s,z_l^+)}, \quad \frac{K}{2} = \sum_{i \in M_2} q_{(z_i^+,t)} + q_{(z_l^+,t)}. \tag{10}$$

Additionally, we can apply Lemma 4.7 and thus, from Equalities (7) it follows

$$q_{(s,z_i^+)} = \sqrt{\phi} S_i, \ i \in M_1, \quad q_{(z_i^+,t)} = \sqrt{\phi} S_i, \ i \in M_2. \tag{11}$$

Combining Equations (10) and (11) leads to

$$q_{(s,z_l^+)} = \frac{K}{2} - \sqrt{\phi} \sum_{i \in M_1} S_i, \quad q_{(z_l^+,t)} = \frac{K}{2} - \sqrt{\phi} \sum_{i \in M_2} S_i. \tag{12}$$

We now distinguish different cases for the value of $\sum_{i \in M_1} S_i$, respectively $\sum_{i \in M_2} S_i$.

If $\sum_{i \in M_1} S_i = K/2$ holds, then $\sum_{i \in I \setminus M_1} S_i = K/2$ holds due to $\sum_{i \in I} S_i = K$. This is a contradiction to the infeasibility of Partition instance $\mathcal{S}$. Analogously it follows that $\sum_{i \in M_2} S_i = K/2$ cannot hold.

If $\sum_{i \in M_1} S_i > K/2$ holds, then $\sum_{i \in M_1} S_i \geq K/2 + 1/2$ holds due to the integrality of S_i for $i \in I$. This together with (12), and

$$\phi \geq 1 - \varepsilon(K) = \left(\frac{K - \frac{1}{8K}}{K} \right)^2$$

leads to

$$\begin{aligned}
q_{(s,z_l^+)} &= \frac{K}{2} - \sqrt{\phi} \sum_{i \in M_1} S_i \\
&\leq \frac{K}{2} - \sqrt{\phi} \left(\frac{K}{2} + \frac{1}{2} \right) \\
&\leq \frac{K}{2} - \left(\frac{K - \frac{1}{8K}}{K} \right) \left(\frac{K}{2} + \frac{1}{2} \right) \\
&= \frac{K^2}{2K} - \frac{K^2 - \frac{1}{8}}{2K} - \frac{K - \frac{1}{8K}}{2K} = \frac{\frac{1}{8}}{2K} - \frac{K - \frac{1}{8K}}{2K} \\
&< 0,
\end{aligned}$$

which is a contradiction to $q_{(s,z_l^+)} \geq 0$. Analogously it follows that $\sum_{i \in M_2} S_i > K/2$ cannot hold.

Consequently, $\sum_{i \in M_1} S_i < K/2$ and $\sum_{i \in M_2} S_i < K/2$ hold. Due to the integrality of S_i for $i \in I$, $\sum_{i \in M_1} S_i \leq K/2 - 1/2$ and $\sum_{i \in M_2} S_i \leq K/2 - 1/2$ are satisfied. From this, (12), and $\phi < 1$, which holds due to Lemma 4.8, it follows

$$q_{(s,z_l^+)} = \frac{K}{2} - \sqrt{\phi} \sum_{i \in M_1} S_i \geq \frac{K}{2} - \sum_{i \in M_1} S_i \geq \frac{1}{2},$$

$$q_{(z_l^+,t)} = \frac{K}{2} - \sqrt{\phi} \sum_{i \in M_2} S_i \geq \frac{K}{2} - \sum_{i \in M_2} S_i \geq \frac{1}{2}.$$

Combining the latter inequalities, Equality (7), and the strict inequalities $2 \frac{1}{8K} \sqrt{\phi} S_l < 1/4$ as well as $\left(\frac{1}{8K} \right)^2 < 1/4$, that follow from $\phi < 1$ and $S_l \leq K$,

leads to

$$
\begin{aligned}
\left(q_{(s,z_l^+)} + q_{(z_l^+,t)}\right)^2 &= q_{(s,z_l^+)}^2 + q_{(z_l^+,t)}^2 + 2q_{(s,z_l^+)}q_{(z_l^+,t)} \\
&\geq \phi S_l^2 + 2 \cdot \frac{1}{4} \\
&> \phi S_l^2 + 2\frac{1}{8K}\sqrt{\phi}S_l + \left(\frac{1}{8K}\right)^2 \\
&= \left(\sqrt{\phi}S_l + \frac{1}{8K}\right)^2,
\end{aligned}
$$

and consequently, we obtain

$$
q_{(s,z_l^+)} + q_{(z_l^+,t)} > \sqrt{\phi}S_l + \frac{1}{8K}. \tag{13}
$$

From (10), (11), (13), and $\sum_{i \in I} S_i = K$ we obtain

$$
\begin{aligned}
K &= \sum_{i \in M_1} q_{(s,z_i^+)} + \sum_{i \in M_2} q_{(z_i^+,t)} + q_{(s,z_l^+)} + q_{(z_l^+,t)} \\
&= \sqrt{\phi} \sum_{i \in I \setminus \{l\}} S_i + q_{(s,z_l^+)} + q_{(z_l^+,t)} \\
&= \sqrt{\phi}(K - S_l) + q_{(s,z_l^+)} + q_{(z_l^+,t)} \\
&> \sqrt{\phi}(K - S_l) + \sqrt{\phi}S_l + \frac{1}{8K} \\
&= \sqrt{\phi}K + \frac{1}{8K},
\end{aligned}
$$

and consequently,

$$
\frac{K - \frac{1}{8K}}{K} > \sqrt{\phi}
$$

holds. This is a contradiction to the assumption

$$
\phi \geq 1 - \varepsilon(K) = \left(\frac{K - \frac{1}{8K}}{K}\right)^2. \qquad \square
$$

We have proven that the optimal value of (5) does not exceed the threshold $1 - \varepsilon(K)$ if exactly one index $i \in I$ with $q_{(s,z_i^+)} > 0$ and $q_{(z_i^+,t)} > 0$ exists and Partition instance $\mathcal{S}$ is infeasible. We now use this result to show that if arbitrarily many indices satisfy the latter property, the optimal objective value of (5) is bounded above by a threshold value smaller than 1. To this end, we consider two cases. In Lemma 4.12 at most one index $i \in I$ satisfies the stricter property $q_{(s,z_i^+)} \geq \varepsilon(K) > 0$ and $q_{(z_i^+,t)} \geq \varepsilon(K) > 0$ where $\varepsilon(K)$, defined in (3a), is a positive lower arc flow bound. Afterward, we consider that at least two indices satisfy this lower arc flow bound; see Lemma 4.13. To this end, we first prove two technical lemmas that we use in the following.

In the next lemma, we show that a feasible point for (1a), (5b), and (5c) with a minimum potential-difference between s and t can be modified to a feasible point of (5) while preserving the potential-difference between s and t.

Lemma 4.10. Let $M \geq 0$ and (ℓ, q) be a feasible point for (1a), (5b), (5c), and for all $i \in I$

$$
\Lambda_{(s,z_i^+)}q_{(s,z_i^+)}^2 + \Lambda_{(z_i^+,t)}q_{(z_i^+,t)}^2 \geq M
$$

holds. Then, a point $(\tilde{\ell}, \tilde{q}, \pi)$ satisfying (1a), (5b), (5c), and additionally (1b) with

$$\Lambda_{(s,z_i^+)}\tilde{q}_{(s,z_i^+)}^2 + \Lambda_{(z_i^+,t)}\tilde{q}_{(z_i^+,t)}^2 = M$$

for all $i \in I$ exists. Moreover, $(\tilde{\ell}, \tilde{q}, \pi)$ is a feasible point for (5) with objective value of M.

Proof. Let (ℓ, q) be a point that satisfies the requirements. Since (ℓ, q) satisfies (1a) and (5c) for graph $G(\mathsf{Part})$, the flow q cannot contain any cycle flow. Thus, we obtain a flow decomposition as in Lemma 3.1 for (ℓ, q) by applying Theorem 3.5 of [1]. Due to $q_a \geq 0$ for $a \in A$ and the graph structure of $G(\mathsf{Part})$, this flow decomposition satisfies that a positive flow $q_{(s,z_i^+)} > 0$, respectively $q_{(z_i^+,t)} > 0$ for any $i \in I$ can only be the result of a positive flow $q(P(u,v))$ with $P(u,v)$ being one of the following paths

$$P(s, z_i^-) = \left\{(s, z_i^+), (z_i^+, z_i^-)\}\right), \tag{14a}$$

$$P(z_i^+, t) = \left\{(z_i^+, t)\right\}, \tag{14b}$$

$$P_i(s, t) = \left\{(s, z_i^+), (z_i^+, t)\right\}. \tag{14c}$$

For every $i \in I$ with

$$\Lambda_{(s,z_i^+)}q_{(s,z_i^+)}^2 + \Lambda_{(z_i^+,t)}q_{(z_i^+,t)}^2 > M$$

we decrease positive flows $q(P(u,v))$ with $P(u,v)$ of (14) until

$$\Lambda_{(s,z_i^+)}q_{(s,z_i^+)}^2 + \Lambda_{(z_i^+,t)}q_{(z_i^+,t)}^2 = M \tag{15}$$

holds. This is possible since the left-hand side of the Equality (15), respectively the previous strict inequality, is continuous and strictly increasing w.r.t. $q_{(s,z_i^+)}$ and $q_{(z_i^+,t)}$; see Lemma 3.3. Thus, we can decrease positive flows $q(P(u,v))$ with $P(u,v)$ of (14) such that Equality (15) is satisfied. Furthermore, (1a), (5b), (5c) are still satisfied.

We now can define the following potentials

$$\pi_s = \pi_s^+, \quad \pi_{z_i^+} = \pi_s - \Lambda_{(s,z_i^+)}q_{(s,z_i^+)}^2, \ i \in I, \tag{16a}$$

$$\pi_{z_i^-} = \pi_{z_i^+} - \Lambda_{(z_i^+,z_i^-)}q_{(z_i^+,z_i^-)}^2, \quad \pi_t = \pi_{z_i^+} - \Lambda_{(z_i^+,t)}q_{(z_i^+,t)}^2, \quad i \in I, \tag{16b}$$

which satisfy (1b) due to (15) and (5c). Moreover, for every $i \in I$ the potentials satisfy

$$\pi_t = \pi_{z_i^+} - \Lambda_{(z_i^+,t)}q_{(z_i^+,t)}^2 = \pi_s - \Lambda_{(s,z_i^+)}q_{(s,z_i^+)}^2 - \Lambda_{(z_i^+,t)}q_{(z_i^+,t)}^2 = \pi_s - M,$$

where the last equation follows from (15). Thus, $\pi_s - \pi_t = M$ is satisfied. $\square$

In the next lemma, we prove that for every optimal solution of (5) an optimal solution with a specific flow decomposition exists. Moreover, the flows on arcs of the paths $P_i(s,t)$ for all $i \in I$ do not differ between the two considered optimal solutions.

Lemma 4.11. Let Partition instance $\mathcal{S}$ be infeasible. Let (ℓ, q, π) be an optimal solution of (5). Then an optimal solution $(\tilde{\ell}, \tilde{q}, \tilde{\pi})$ of (5) with a flow decomposition of Lemma 3.1 for $(\tilde{\ell}, \tilde{q})$ exists such that if $q(P(u,v)) > 0$, then $P(u,v)$ is defined by either (14a) or (14b). Additionally, $q_{(s,z_i^+)} = \tilde{q}_{(s,z_i^+)}$ and $q_{(z_i^+,t)} = \tilde{q}_{(z_i^+,t)}$ for $i \in I$ is satisfied.

Proof. Let (ℓ, q, π) be an optimal solution of (5) with objective value ϕ. Consequently, for each $i \in I$ the equation $q_{(s,z_i^+)}^2 + q_{(z_i^+,t)}^2 = \phi S_i^2$ is satisfied due Lemma 4.7. Further, from Lemma 4.8, it follows $\phi < 1$ and thus, flows q satisfy $q_{(s,z_i^+)} < S_i$ as well as $q_{(z_i^+,t)} < S_i$ for $i \in I$. We now consider a flow decomposition of Lemma 3.1 corresponding to solution (ℓ, q, π). Since $q_a \geq 0$ for $a \in A$ holds and the graph structure of $G(\mathsf{Part})$, this flow decomposition can be chosen such that $q(P(u,v))$ can only be positive for a path $P(u,v)$ of (14) or $P(z_i^+, z_i^-)$ consisting of arc (z_i^+, z_i^-) for an $i \in I$, which follows from Theorem 3.5 in [1].

If $q(P(z_i^+, z_i^-)) > 0$ for an index $i \in I$ holds, then we delete this flow. We note that this does not modify flow on arcs (s, z_i^+), respectively (z_i^+, t), for $i \in I$. Thus, the modified nomination and flows $(\tilde{\ell}, \tilde{q})$ can be extended to a solution $(\tilde{\ell}, \tilde{q}, \tilde{\pi})$ of (5) with objective value ϕ, where the potentials $\tilde{\pi}$ are given by (16). Consequently, we can assume w.l.o.g. $q(P(z_i^+, z_i^-)) = 0$ for $i \in I$ in the remaining proof.

We now consider $i \in I$ with $q(P_i(s,t)) = \varepsilon > 0$. Due to $q_{(s,z_i^+)} < S_i$ and $q_{(z_i^+,t)} < S_i$ for $i \in I$, booking b, and (5c), we can set $q(P_i(s,t)) = \varepsilon > 0$ to zero and increase $q(P(s, z_i^-))$ and $q(P(z_i^+, t))$ by ε such that the corresponding solution $(\tilde{\ell}, \tilde{q})$ satisfies (5b), (5c), and (1a). We note that

$$\tilde{\ell}_{z_i^-} = \ell_{z_i^-} + \varepsilon \leq b_{z_i^-} = S_i$$

holds since before the modification $q(P(s, z_i^-)) = \ell_{z_i^-}$ and, thus,

$$q_{(s,z_i^+)} = q(P_i(s,t)) + q(P(s, z_i^-)) = \varepsilon + \ell_{z_i^-} < S_i$$

were satisfied. Analogously it follows $\tilde{\ell}_{z_i^+} = \ell_{z_i^+} + \varepsilon \leq b_{z_i^+} = S_i$. Additionally, we note that the flow on arc (s, z_i^+), respectively (z_i^+, t), for $i \in I$ is not modified. We now repeat the above procedure for every $i \in I$ with $q(P_i(s,t)) = \varepsilon > 0$. Afterward, we can extend the modified nomination and flows $(\tilde{\ell}, \tilde{q})$ to a solution $(\tilde{\ell}, \tilde{q}, \tilde{\pi})$ of (5) with objective value ϕ, where the potentials $\tilde{\pi}$ are given by (16). Due to the modification, $q_{(s,z_i^+)} = \tilde{q}_{(s,z_i^+)}$ and $q_{(z_i^+,t)} = \tilde{q}_{(z_i^+,t)}$ for $i \in I$ hold and the required flow decomposition is constructed. $\qquad\square$

For the case that $\mathsf{Partition}$ instance $\mathcal{S}$ is infeasible, we now prove that if in an optimal solution of (5) for at most one $i \in I$ the flow on the arcs (s, z_i^+) and (z_i^+, t) exceeds $\varepsilon(K)$, then the corresponding objective value is below the threshold $1 - \varepsilon(K) + \varepsilon(K)^2$.

Lemma 4.12. Let $\mathsf{Partition}$ instance $\mathcal{S}$ be infeasible. Let

$$\varepsilon(K) = 1 - \left(\frac{K - \frac{1}{8K}}{K}\right)^2 \in (0, 1)$$

and (ℓ, q, π) be an optimal solution of (5) that satisfies

$$\left|\left\{i \in I : q_{(s,z_i^+)} \geq \varepsilon(K), q_{(z_i^+,t)} \geq \varepsilon(K)\right\}\right| \leq 1. \tag{17}$$

Then, the objective value ϕ corresponding to solution (ℓ, q, π) satisfies

$$\phi < 1 - \varepsilon(K) + \varepsilon(K)^2 < 1.$$

Proof. We assume for a contradiction that (ℓ, q, π) is an optimal solution of (5) that satisfies the requirements and its objective value is $\phi \geq 1 - \varepsilon(K) + \varepsilon(K)^2$. We further assume w.l.o.g. that the corresponding flow decomposition of Lemma 3.1

for (ℓ, q) satisfies Lemma 4.11. From Lemma 4.8, it follows that $\phi < 1$. In the following, we consider the paths (14) and partition the index set I as in (9).

Due to $\phi \geq 1 - \varepsilon(K) + \varepsilon(K)^2$ and Lemmas 4.6 and 4.9, $|M_3| \geq 2$ holds. Further, the inequality $q_{(s,z_i^+)} > \varepsilon(K)$ or $q_{(z_i^+,t)} > \varepsilon(K)$ holds due to

$$q_{(s,z_i^+)}^2 + q_{(z_i^+,t)}^2 = \phi S_i^2 \geq \phi \geq 1 - \varepsilon(K) + \varepsilon(K)^2 > 2\varepsilon(K)^2,$$

where the first equality follows from Lemma 4.7 and the last inequality holds due to

$$0 < \varepsilon(K) = 1 - \left(\frac{K - \frac{1}{8K}}{K}\right)^2 = 1 - (1 - \frac{1}{8K^2})^2 \leq 1 - \frac{49}{64} = \frac{15}{64}.$$

We now set $q_a = 0$ for each $a = (u,v) \in \left\{(s,z_i^+): i \in I\right\} \cup \left\{(z_i^+,t): i \in I\right\}$ that satisfies $q_a \leq \varepsilon(K)$. This can be done in the considered flow decomposition by decreasing flows $q(P(s, z_i^-))$, respectively $q(P(z_i^+, t))$ for $i \in I$. We denote the modified flows by $\tilde{q}$ and its corresponding nomination by $\tilde{\ell}$, which satisfy (1a), (5b), and (5c). We further note that now $|M_3| \leq 1$ holds, due to (17).

For an arbitrary index $i \in I$, we assume that $q_{(z_i^+,t)}$ has been modified, i.e., $\tilde{q}_{(z_i^+,t)} = 0 < q_{(z_i^+,t)}$. Consequently, $\tilde{q}_{(s,z_i^+)} = q_{(s,z_i^+)}$ holds. This together with Lemma 4.7, $\phi \geq 1 - \varepsilon(K) + \varepsilon(K)^2$, and $S_i \geq 1$ leads to

$$\begin{aligned}
\tilde{q}_{(z_i^+,t)}^2 + \tilde{q}_{(s,z_i^+)}^2 = \tilde{q}_{(s,z_i^+)}^2 &= q_{(s,z_i^+)}^2 = \phi S_i^2 - q_{(z_i^+,t)}^2 \\
&\geq \phi S_i^2 - \varepsilon(K)^2 \geq (1 - \varepsilon(K) + \varepsilon(K)^2)S_i^2 - \varepsilon(K)^2 \\
&\geq (1 - \varepsilon(K))S_i^2.
\end{aligned}$$

Analogously we can handle the case that $q_{(s,z_i^+)}$ has been modified.

Due to this and $\Lambda_{(s,z_i^+)} = \Lambda_{(z_i^+,t)} = 1/S_i^2$, we can apply Lemma 4.10 and obtain a solution $(\tilde{\ell}, \tilde{q}, \tilde{\pi})$ for (5) with objective value $\tilde{\phi} \geq 1 - \varepsilon(K) > 0$. This solution satisfies $|M_3| \leq 1$ due to the modification, (17), and the fact that we only possibly decrease flows in Lemma 4.10 to obtain $(\tilde{\ell}, \tilde{q}, \tilde{\pi})$.

If $|M_3| = 1$ holds, then this is a direct contradiction to Lemma 4.9.

We now assume $|M_3| = 0$. Due to $|M_3| \geq 2$ before the modification and the requirements, at least one arc flow was decreased by the above. Consequently, $\tilde{\ell}_s < K/2$ or $\tilde{\ell}_t < K/2$ holds. Furthermore, $|M_1| \geq 1$ or $|M_2| \geq 1$ is satisfied after the modification. We now modify the solution such that its objective stays the same but $|M_3| = 1$ holds which is a contradiction to Lemma 4.9.

If $\tilde{\ell}_s < K/2$ and $\tilde{\ell}_t < K/2$ hold, then we assume w.l.o.g. that $q_{(s,z_i^+)} > 0$ holds for some $i \in I$. Due to the considered flow decomposition and Lemma 3.3, we now can decrease $q(P(s, z_i^-))$ by $\varepsilon > 0$ and increase $q(P(z_i^+, t))$ by $\tilde{\varepsilon} > 0$ such that both arc flows $\tilde{q}_{(s,z_i^+)}$ and $\tilde{q}_{(z_i^+,t)}$ are positive, the potential drop $\Pi_{P_i(s,t)}$ stays the same, and $\tilde{\ell} \leq b$ holds. Consequently, $|M_3| = 1$ is satisfied for the modified solution $(\tilde{\ell}, \tilde{q}, \tilde{\pi})$ and its objective value $\tilde{\phi}$ satisfies $\tilde{\phi} \geq 1 - \varepsilon(K)$, which is a contradiction to Lemma 4.9.

If $\tilde{\ell}_s < K/2$ and $\tilde{\ell}_t = K/2$ hold, then this and (1a) imply that $q_{(z_i^+,t)} > 0$ exists. Due to the considered flow decomposition and Lemma 3.3, we can decrease $q(P(z_i^+, t))$ by $\varepsilon > 0$ and increase $q(P_i(s,t))$ by $\tilde{\varepsilon}$ with $\varepsilon \geq \tilde{\varepsilon} > 0$ such that the potential drop $\Pi_{P_i(s,t)}$ stays the same, $\tilde{\ell} \leq b$, and $\tilde{q}_{(s,z_i^+)}$ and $\tilde{q}_{(z_i^-,t)}$ are positive. Consequently, $|M_3| = 1$ is satisfied for the modified solution $(\tilde{\ell}, \tilde{q}, \tilde{\pi})$ and its objective value $\tilde{\phi}$ satisfies $\tilde{\phi} \geq 1 - \varepsilon(K)$, which is a contradiction to Lemma 4.9. Analogously we handle the case $\tilde{\ell}_s = K/2$ and $\tilde{\ell}_t < K/2$. $\square$

We now consider the counterpart of Lemma 4.12, i.e.,

$$\left| \left\{ i \in I : q_{(s,z_i^+)} \geq \varepsilon(K), q_{(z_i^+,t)} \geq \varepsilon(K) \right\} \right| \geq 2. \tag{18}$$

Lemma 4.13. Let Partition instance $\mathcal{S}$ be infeasible. Let

$$\varepsilon(K) = 1 - \left(\frac{K - \frac{1}{8K}}{K} \right)^2 \in (0,1)$$

and (ℓ, q, π) be a solution of (5) that satisfies (18). Then, the corresponding objective value ϕ satisfies

$$\phi < 1 - \frac{\varepsilon(K)^2}{K^2} < 1.$$

Proof. We assume for a contradiction that a solution (ℓ, q, π) of (5) satisfying (18) with an objective value $\phi \geq 1 - \varepsilon(K)^2/K^2$ exists. From Lemma 4.8, it follows $\phi < 1$. Further, Condition (18), which is satisfied by (ℓ, q, π), implies that two indices $l \neq r \in I$ exist such that

$$q_{(s,z_l^+)} \geq \varepsilon(K), \ q_{(z_l^+,t)} \geq \varepsilon(K), \quad q_{(s,z_r^+)} \geq \varepsilon(K), \ q_{(z_r^+,t)} \geq \varepsilon(K)$$

hold. This and

$$q_{(s,z_l^+)}^2 + q_{(z_l^+,t)}^2 = \phi S_l^2, \quad q_{(s,z_r^+)}^2 + q_{(z_r^+,t)}^2 = \phi S_r^2,$$

lead, analogously to (13), to the inequalities

$$q_{(s,z_l^+)} + q_{(z_l^+,t)} > \sqrt{\phi} S_l + \frac{\varepsilon(K)^2}{2K}, \quad q_{(s,z_r^+)} + q_{(z_r^+,t)} > \sqrt{\phi} S_r + \frac{\varepsilon(K)^2}{2K}. \tag{19}$$

We now consider the partition $M_1, M_2,$ and M_3 of I according to (9). Consequently, (11) is satisfied. From this, flow conservation (1a), and booking b, it follows

$$\frac{K}{2} \geq \sum_{i \in M_1 \cup M_3} q_{(s,z_i^+)} \geq \sqrt{\phi} \sum_{i \in M_1} S_i + \sum_{i \in M_3} q_{(s,z_i^+)},$$

$$\frac{K}{2} \geq \sum_{i \in M_2 \cup M_3} q_{(z_i^+,t)} \geq \sqrt{\phi} \sum_{i \in M_2} S_i + \sum_{i \in M_3} q_{(z_i^+,t)}.$$

Combining the previous inequalities, $\sum_{i \in I} S_i = K$, Observation 4.4, (19), and $\phi \geq 1 - \varepsilon(K)^2/K^2$ lead to

$$\begin{aligned}
K &\geq \sqrt{\phi} \sum_{i \in M_1 \cup M_2} S_i + \sum_{i \in M_3} \left(q_{(s,z_i^+)} + q_{(z_i^+,t)} \right) \\
&\geq \sqrt{\phi} \sum_{i \in I \setminus \{l,r\}} S_i + q_{(s,z_l^+)} + q_{(z_l^+,t)} + q_{(s,z_r^+)} + q_{(z_r^+,t)} \\
&= \sqrt{\phi}(K - S_r - S_l) + q_{(s,z_l^+)} + q_{(z_l^+,t)} + q_{(s,z_r^+)} + q_{(z_r^+,t)} \\
&> \sqrt{\phi}(K - S_r - S_l) + \sqrt{\phi} S_l + \frac{\varepsilon(K)^2}{2K} + \sqrt{\phi} S_r + \frac{\varepsilon(K)^2}{2K} \\
&= \sqrt{\phi} K + \frac{\varepsilon(K)^2}{K}.
\end{aligned}$$

This leads to the contradiction

$$\frac{K - \frac{\varepsilon(K)^2}{K}}{K} = 1 - \frac{\varepsilon(K)^2}{K^2} > \sqrt{\phi} \geq \phi$$

because $1 > \phi \geq 1 - \varepsilon(K)^2/K^2 > 0$ holds. $\qquad\square$

Lemmas 4.12 and 4.13 prove that if Partition instance $\mathcal{S}$ is infeasible, then $\varphi_{st}^+(b)$ can be bounded above by $M(K)$ given as

$$M(K) = \max\left\{1 - \varepsilon(K) + \varepsilon(K)^2, 1 - \frac{\varepsilon(K)^2}{K^2}\right\}, \quad \varepsilon(K) = 1 - \left(\frac{K - \frac{1}{8K}}{K}\right)^2,$$

where the coding length of $M(K)$ is polynomially bounded above by the coding length of the given Partition instance. In the following two lemmas, we prove an analogue statement for the case that the flow is not necessarily nonnegative, i.e., we consider the general maximum potential-difference problem (2) w.r.t. (s,t). To this end, $M(K)$ and $\tilde{\varepsilon}(K) = (1 - M(K))/5$ come into play; see (3). Moreover, we need the following auxiliary Lemma.

Lemma 4.14. Let Partition instance $\mathcal{S}$ be infeasible. Let (ℓ, q, π) be an optimal solution of (2) w.r.t. (s,t) with objective value $\phi > 0$. Further, there is at least one arc $a \in A$ with negative arc flow $q_a < 0$. Then, for all $i \in I$, the inequalities

$$\left|q_{(s,z_i^+)}\right| \le S_i, \quad \left|q_{(z_i^+,t)}\right| \le S_i, \tag{20}$$

hold.

Proof. Let (ℓ, q, π) be an optimal solution of (2) w.r.t. (s,t) that satisfies the requirements. For all $i \in I$, the inequality $q_{(z_i^+, z_i^-)} \ge 0$ holds due to the structure of graph $G(\mathsf{Part})$. Moreover, for all $i \in I$, at least one of the arc flows $q_{(s,z_i^+)}$ and $q_{(z_i^+,t)}$ is positive due to $\phi > 0$.

For every $i \in I$ for which either $q_{(s,z_i^+)} < 0$ or $q_{(z_i^+,t)} < 0$ holds, the flow bounds (20) follow from the graph structure of $G(\mathsf{Part})$, (1a), and the considered booking b. From the requirements, it further follows that there is at least one index $l \in I$ for which either $q_{(s,z_l^+)} < 0$ or $q_{(z_l^+,t)} < 0$ holds and thus, flow bounds (20) are satisfied for this index l. Consequently,

$$\phi = \pi_s - \pi_t = \frac{1}{S_l^2} q_{(s,z_l^+)} \left|q_{(s,z_l^+)}\right| + \frac{1}{S_l^2} q_{(z_l^+,t)} \left|q_{(z_l^+,t)}\right| < 1,$$

is satisfied, where the second equality follows from (1b). Furthermore, the last inequality is satisfied since at least one of the summands is negative and the flow bounds (20) are satisfied for l. Thus, $\phi < 1$ holds.

Moreover, for every $i \in I$ with $q_{(s,z_i^+)} \ge 0$ and $q_{(z_i^+,t)} \ge 0$, Conditions (1b) and $\phi < 1$ lead to

$$1 > \phi = \frac{1}{S_l^2} q_{(s,z_l^+)}^2 + \frac{1}{S_l^2} q_{(z_l^+,t)}^2,$$

which implies that the flow bounds (20) are satisfied for the considered index i. $\quad\square$

Lemma 4.15. Let Partition instance $\mathcal{S}$ be infeasible and $n \ge 3$ denotes the number of elements of $\mathcal{S}$. Let $M(K)$ and $\tilde{\varepsilon}(K)$ be given as in (3) and (ℓ, q, π) be an optimal solution of (2) w.r.t. (s,t) with objective value ϕ. If this solution satisfies $|q_a| \le \tilde{\varepsilon}(K)/n$ for each $a \in A$ with $q_a < 0$, then

$$\phi < M(K) + 4\tilde{\varepsilon}(K) < 1$$

holds.

Proof. We assume for a contradiction that an optimal solution (ℓ, q, π) for (2) w.r.t. (s, t) with objective value ϕ that satisfies the requirements, explicitly $|q_a| \leq \tilde{\varepsilon}(K)/n$ for $a \in A$ with $q_a < 0$, exists and $\phi \geq M(K) + 4\tilde{\varepsilon}(K)$ holds.

If $q_a \geq 0$ holds for all $a \in A$, then this is a contradiction to Lemma 4.12 or 4.13 due to the objective value $\phi \geq M(K) + 4\tilde{\varepsilon}(K)$.

Now at least one arc $a \in A$ satisfies $q_a < 0$. From the graph structure of $G(\mathsf{Part})$ and booking b, it follows $q_{(z_i^+, z_i^-)} \geq 0$ for all $i \in I$. Since $\phi > 0$ and (1b), for every $i \in I$ at least one arc flow $q_{(s, z_i^+)}$ or $q_{(z_i^+, t)}$ is positive. Consequently, at most n arcs with negative flow exist. Moreover, from Lemma 4.14, it follows that the flow bounds (20) are satisfied for $i \in I$. Additionally, $q_{(s, z_i^+)} > \tilde{\varepsilon}(K)$ or $q_{(z_i^+, t)} > \tilde{\varepsilon}(K)$ holds for every $i \in I$ because of

$$q_{(s, z_i^+)} \left| q_{(s, z_i^+)} \right| + q_{(z_i^+, t)} \left| q_{(z_i^+, t)} \right| = \phi S_i^2 \geq \phi \geq M(K) + 4\tilde{\varepsilon}(K) > 2\tilde{\varepsilon}(K)^2.$$

We now consider a flow decomposition of Lemma 3.1 for (ℓ, q). For every arc $a \in A$ with $q_a < 0$, we delete every positive flow $q(P(u, v))$ where the path $P(u, v)$ satisfies $\chi_a(P) = -1$. Consequently, we obtain $\tilde{q}_a \geq 0$ for $a \in A$ with corresponding nomination $\tilde{\ell}$. Due to the chosen flow decomposition and maximally n arcs a with negative arc flow, satisfying $|q_a| \leq \tilde{\varepsilon}(K)/n$, exist, this modification decreases an arbitrary arc flow by at most $n(\tilde{\varepsilon}(K)/n) = \tilde{\varepsilon}(K)$. Further, the flow bounds (20) still hold for $i \in I$.

If either $q_{(s, z_i^+)} > \tilde{\varepsilon}(K)$ or $q_{(z_i^+, t)} > \tilde{\varepsilon}(K)$ holds, then we assume w.l.o.g. that $q_{(s, z_i^+)} > \tilde{\varepsilon}(K)$ is satisfied. This leads to

$$\begin{aligned}
\tilde{q}_{(s, z_i^+)}^2 + \tilde{q}_{(z_i^+, t)}^2 &\geq \left(\max\left\{ 0, q_{(s, z_i^+)} - \tilde{\varepsilon}(K) \right\} \right)^2 + \left(\max\left\{ 0, q_{(z_i^+, t)} - \tilde{\varepsilon}(K) \right\} \right)^2 \\
&= q_{(s, z_i^+)}^2 - 2q_{(s, z_i^+)}\tilde{\varepsilon}(K) + \tilde{\varepsilon}(K)^2 \\
&\geq q_{(s, z_i^+)}^2 - 2q_{(s, z_i^+)}\tilde{\varepsilon}(K) + q_{(z_i^+, t)}^2 \\
&\geq \phi S_i^2 - 2q_{(s, z_i^+)}\tilde{\varepsilon}(K) \\
&\geq \phi S_i^2 - 2S_i\tilde{\varepsilon}(K) \;\geq (M(K) + 4\tilde{\varepsilon}(K))S_i^2 - 2S_i\tilde{\varepsilon}(K) \\
&\geq M(K)S_i^2.
\end{aligned}$$

If $q_{(s, z_i^+)} > \tilde{\varepsilon}(K)$ and $q_{(z_i^+, t)} > \tilde{\varepsilon}(K)$ hold, then this leads to

$$\begin{aligned}
\tilde{q}_{(s, z_i^+)}^2 + \tilde{q}_{(z_i^+, t)}^2 &\geq \left(\max\left\{ 0, q_{(s, z_i^+)} - \tilde{\varepsilon}(K) \right\} \right)^2 + \left(\max\left\{ 0, q_{(z_i^+, t)} - \tilde{\varepsilon}(K) \right\} \right)^2 \\
&= q_{(s, z_i^+)}^2 - 2q_{(s, z_i^+)}\tilde{\varepsilon}(K) + \tilde{\varepsilon}(K)^2 + q_{(z_i^+, t)}^2 - 2q_{(z_i^+, t)}\tilde{\varepsilon}(K) + \tilde{\varepsilon}(K)^2 \\
&\geq \phi S_i^2 - 2q_{(s, z_i^+)}\tilde{\varepsilon}(K) - 2q_{(z_i^+, t)}\tilde{\varepsilon}(K) \\
&\geq \phi S_i^2 - 4S_i\tilde{\varepsilon}(K) \;\geq (M(K) + 4\tilde{\varepsilon}(K))S_i^2 - 4S_i\tilde{\varepsilon}(K) \\
&\geq M(K)S_i^2.
\end{aligned}$$

Thus, the modification decreases the potential drop $\Pi_{P_i(s,t)}$, but it is at least $M(K)$. Furthermore, $\tilde{q} \geq 0$ holds and we only decreased flows of q to obtain $\tilde{q}$. Consequently, $(\tilde{\ell}, \tilde{q})$ satisfy (1a), (5b), and (5c). Due to this and $\Lambda_{(s, z_i^+)} = \Lambda_{(z_i^+, t)} = 1/S_i^2$, we can apply Lemma 4.10 and obtain a feasible point of (2) w.r.t. (s, t) with $q_a \geq 0$ and objective value of at least $M(K)$. But this is a contradiction to Lemma 4.12 or 4.13. $\qquad\square$

We now prove that if Partition instance $\mathcal{S}$ is infeasible and in an optimal solution of (2) w.r.t. (s,t) at least one negative arc flow has an absolute flow of at least $\tilde{\varepsilon}(K)/n$, then we can bound the objective of (2) as follows.

Lemma 4.16. Let Partition instance $\mathcal{S}$ be infeasible. Further, let $\varepsilon(K), M(K)$, and $\tilde{\varepsilon}(K)$ be given as in (3). Moreover, let (ℓ, q, π) be a feasible point of (2) w.r.t. (s,t) such that at least one arc $a \in A$ satisfies $q_a < 0$ and $|q_a| \geq \tilde{\varepsilon}(K)/n$. Then, the corresponding objective value ϕ satisfies

$$\phi < 1 - \frac{\tilde{\varepsilon}(K)^2}{K^2 n^2}.$$

Proof. We assume for a contradiction that a solution (ℓ, q, π) of (2) w.r.t. (s,t) with objective value

$$\phi \geq 1 - \frac{\tilde{\varepsilon}(K)^2}{K^2 n^2}$$

satisfies the requirements. Consequently, an arc $a \in A$ with $q_a < 0$ and $|q_a| \geq \tilde{\varepsilon}(K)/n$ exists. We now consider the case that $a = (s, z_i^+)$ for some $i \in I$ holds. Due to Lemma 4.14 and the value of $\phi > 0$, the flow bounds $0 \leq q_{(z_i^+, t)} \leq S_i$ are satisfied. This,

$$-q_{(s, z_i^+)}^2 + q_{(z_i^+, t)}^2 = \phi S_i^2 \geq \left(1 - \frac{\tilde{\varepsilon}(K)^2}{K^2 n^2}\right) S_i^2,$$

and $S_i < K$, lead to the contradiction

$$1 - \frac{\tilde{\varepsilon}(K)^2}{K^2 n^2} \leq \frac{q_{(z_i^+, t)}^2 - q_{(s, z_i^+)}^2}{S_i^2}$$

$$\leq 1 - \frac{q_{(s, z_i^+)}^2}{S_i^2} \leq 1 - \frac{\left(\frac{\tilde{\varepsilon}(K)}{n}\right)^2}{S_i^2}$$

$$< 1 - \frac{\left(\frac{\tilde{\varepsilon}(K)}{n}\right)^2}{K^2} = 1 - \frac{\tilde{\varepsilon}(K)^2}{K^2 n^2}.$$

Analogously to the above, it follows the case of $a = (z_i^+, t)$ for some $i \in I$. $\square$

Combining now the results of Lemmas 4.12–4.16, we can bound the objective of (2) w.r.t. (s,t) by the polynomial $T(K)$ given by (3b).

Lemma 4.17. Let Partition instance $\mathcal{S}$ be infeasible. Let $M(K)$ and $\tilde{\varepsilon}(K)$ be given as in (3). Then,

$$\varphi_{st}(b) < \max\left\{1 - \frac{\tilde{\varepsilon}(K)^2}{K^2 n^2}, M(K) + 4\tilde{\varepsilon}(K)\right\} = T(K) < 1$$

holds.

Proof. Let (ℓ, q, π) be an optimal solution of (2) w.r.t. (s,t). If $q_a \geq 0$ for $a \in A$ holds, then the claim follows from Lemmas 4.12 and 4.13. If an arc $a \in A$ with $q_a < 0$ exists, then the claim follows from Lemmas 4.15 and 4.16. $\square$

Since the feasibility of a booking can be characterized by the nominations with maximum potential-difference, see (4), the previous lemma connects the feasibility of Partition instance $\mathcal{S}$ to (2) w.r.t. (s,t).

Lemma 4.18. Partition instance $\mathcal{S}$ is feasible if and only if $\varphi_{st}(b) > T(K)$ is satisfied. Further, the booking b is infeasible if and only if $\mathcal{S}$ is feasible.

Proof. If Partition instance $\mathcal{S}$ is infeasible, then from Lemma 4.17 it follows $\varphi_{st}(b) < T(K) < 1$. If $\mathcal{S}$ is feasible, then Lemma 4.3 implies $\varphi_{st}(b) \geq 1 > T(K)$, since $T(K) < 1$.

Consequently, from Corollary 4.2 it follows that booking b is infeasible if and only if $\mathcal{S}$ is feasible. $\qquad\square$

Finally, we prove our main result that deciding the feasibility of a booking (FB) is coNP-hard.

Theorem 4.19. Deciding the feasibility of a booking (FB) is coNP-hard.

Proof. Deciding the feasibility of a booking is coNP-hard due to Lemma 4.18. $\quad\square$

We further note that deciding whether there exists a nomination with maximum potential-difference (MPD), see (2), of at least $T(K)$ is NP-hard. Moreover, MPD is still NP-hard with the additional restriction of nonnegative flow; see (5).

Lemma 4.20. Computing a maximum potential-difference nomination (MPD) is NP-hard. Further, MPD with the additional restriction of nonnegative flow (5c) is also NP-hard.

Proof. The NP-hardness of MPD follows from $T(K) < 1$ and Lemmas 4.18 and 4.3.

Moreover, MPD with the additional restriction of nonnegative flow is NP-hard, which follows from the combination of Lemma 4.3, its proof, Lemmas 4.12 and 4.13, and $T(K) < 1$. $\qquad\square$

We close this section with a brief remark why the reduction of this section is not applicable to linear potential-based flows, i.e., considering q_a instead of $q_a|q_a|$ in (1b). For the latter, we know that FB is in P; see [26].

The hardness proof of FB w.r.t. nonlinear potential-based flow model (1) is based on Observation 4.4. It implies that if $q_{(s,z_i^+)} > 0$ and $q_{(z_i^+,t)} > 0$ hold, then we need strictly more flow in terms of $q_{(s,z_i^+)} + q_{(z_i^+,t)}$ to obtain a pressure drop of ϕ, i.e., $\Lambda_{(s,z_i^+)} q_{(s,z_i^+)}^2 + \Lambda_{(z_i^+,t)} q_{(z_i^+,t)}^2 = \phi$, in contrast to the case if one of the latter flows is zero. This does not apply if we consider a linear potential-based flow model. Thus, the previous reduction from Partition to the infeasibility of a booking is not applicable for a linear potential-based flow model.

5. CONCLUSION

In this paper, we prove that deciding the feasibility of a booking (FB) in the European entry-exit gas market considering a nonlinear potential-based flow model is coNP-hard in general graphs. To this end, we reduced Partition to the infeasibility of a booking. This is the first hardness result for FB w.r.t. potential-based flows, since the latter is in P for linear potential-based flows in general graphs; see [26]. It is also in P for trees and a single-cycle network considering a nonlinear potential-based flow model; see [26, 27, 37]. Thus, it is finally shown that FB is hard and a first border separating the easy from the hard variants of the problem is given. Hence, main parts of the complexity regarding the considered problem are now understood;

see Figure 1 for an overview. Moreover, we prove that computing a nomination with the maximum potential-difference is NP-hard even if we can determine the flow direction a priori. However, an open question is if we can sharpen the line that separates the easy from the hard cases of FB. The proof of our complexity results is strongly based on graphs with multiple cycles that share common arcs. Consequently, a reasonable next case could be the class of cactus graphs since for trees and a single cycle the considered problem is in P and for general graphs it is coNP-hard.

Furthermore, it remains open if FB is in coNP, i.e., it is open if there is an infeasibility certificate which coding length is polynomial in the coding length of the input and that can be checked in polynomial time. Since we consider a nonlinear potential-based flow model, this task seems to be challenging. This is stressed by Example 4.3 in the recent PhD thesis [34], which shows that even for a rational nomination the corresponding flows can be irrational in our nonlinear potential-based flow model. This result does neither show that FB is in coNP nor that it is not in coNP. However, as discussed in [34], it implies that any infeasibility certificate of polynomial size does not include possibly irrational flows corresponding to a nomination within our nonlinear potential-based flow model. Thus, we cannot directly use the potential-difference maximizing nominations and flows of (2), which are used to characterize the feasibility of a booking, as an infeasibility certificate. Consequently, it may be necessary to develop new characterizations for feasible bookings to address if FB is in coNP, which is out of scope of this paper. However, for capacitated linear flows deciding the feasibility of a booking is in coNP, see [23], which is based on the fact that for a rational nomination the corresponding flows are always rational for capacitated linear flows.

Moreover, it remains an open question if FB is strongly NP-hard since we reduced the weakly NP-hard Partition problem to the infeasibility of a booking. An answer to this question also may reveal whether there is an pseudo-polynomial algorithm that decides the feasibility of a booking.

Our complexity analysis shows that FB is indeed a challenging problem for the transmission system operator (TSO) in the European entry-exit gas market system. Moreover, FB has to be solved by the TSO whenever a booking contract is signed.

Thus, the development of effective methods for FB offers new possibilities for future research. As a consequence of our complexity analyses, it might be a smart choice to consider approximation algorithms for deciding the feasibility of a booking in general graphs. Especially, the development of approximation algorithms to compute a maximum potential-difference nomination (MPD) directly allows to decide the feasibility of a booking up to a certain tolerance.

Acknowledgments

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