BCOL RESEARCH REPORT 19.02

Industrial Engineering & Operations Research
University of California, Berkeley, CA 94720–1777

SUBMODULAR FUNCTION MINIMIZATION AND POLARITY

ALPER ATAMTÜRK AND VISHNU NARAYANAN

Abstract. Using polarity, we give an outer polyhedral approximation for the epigraph of set functions. For a submodular function, we prove that the corresponding polar relaxation is exact; hence, it is equivalent to the Lovász extension. The polar approach provides an alternative proof for the convex hull description of the epigraph of a submodular function. Computational experiments show that the inequalities from outer approximations can be effective as cutting planes for solving submodular as well as non-submodular set function minimization problems.

Keywords: Polarity, Lovász extension, submodular functions, polymatroids, greedy algorithm, cutting planes.

December 2019

1. Introduction

Given a finite ground set N and a rational valued set function $f : 2^N \to \mathbb{Q}$, we consider finding the minimum value of f:

$$\min_{S \subseteq N} f(S). \tag{1}$$

Throughout, by abuse of notation, we also use $f(x)$, where $x \in \{0,1\}^N$ is the indicator vector for the set of subsets of N. Introducing an auxiliary variable z for the objective value, let us restate problem (1) as

$$\min \left\{ z : (x,z) \in \mathcal{Q}_f \right\}, \tag{2}$$

where $\mathcal{Q}_f$ is the convex hull of the epigraph of f, i.e.,

$$\mathcal{Q}_f := \operatorname{conv} \left\{ (x,z) \in \{0,1\}^N \times \mathbb{R} : f(x) \le z \right\}.$$

In this paper, we give a polyhedral relaxation for $\mathcal{Q}_f$ based on polarity. Whereas the Lovász extension [17] for a set function gives an inner piecewise polyhedral approximation of the convex hull of its epigraph, the polar relaxation is a (convex) polyhedral outer approximation.

A. Atamtürk: Department of Industrial Engineering and Operations Research, University of California, Berkeley, CA 94720-1777 USA `atamturk@berkeley.edu`
V. Narayanan: Industrial Engineering and Operations Research, Indian Institute of Technology Bombay, Mumbai India 400076 `vishnu@iitb.ac.in` .

A set function $f : 2^N \to \mathbb{Q}$ is *submodular* if

$$f(S \cup T) + f(S \cap T) \le f(S) + f(T) \text{ for all } S, T \subseteq N.$$

For a submodular set function, we show that the corresponding polar relaxation is exact; hence, it is equivalent to the Lovász extension. The polar approach, thus, provides a new polyhedral proof of the convex hull description of the epigraph of a submodular function. Furthermore, through computational experiments, we show that the inequalities from outer approximations can be effective as cutting planes for solving submodular as well as non-submodular set function minimization problems.

A short literature review. Submodular set functions play an important role in many fields and have received much interest in the literature [7, 8, 26]. Combinatorial optimization problems such as the min-cut problem, entropy minimization, matroids, binary quadratic function minimization with a non-positive matrix, among many others, are special cases. Submodular functions can be minimized in polynomial time [11, 14, 23, 24]. For comprehensive reviews on this subject we refer the reader to [10, 13, 25]. While the majority of research on submodularity has been devoted to optimization on binary variables, submodularity has been useful for deriving strong inequalities for mixed 0–1 optimization as well [2, 3, 4, 27]. Unlike minimization, maximization of submodular functions is $\mathcal{NP}$-hard (the max-cut problem is a special case). Most of the research for this case has been on approximation algorithms. The greedy algorithm and its extensions [9, 16, 19, 21] provide constant factor approximation. There is so far limited work on polyhedral analysis of submodular function maximization [1, 15, 18, 28].

Assumption and notation. Without loss of generality, we assume that $f(\emptyset) = 0$ as, otherwise, one can solve the equivalent minimization problem for $f' := f - f(\emptyset)$, i.e., with $f'(S) = f(S) - f(\emptyset)$ for all $S \subseteq N$. Let χ_S be the indicator vector for a set $S \subseteq N$ and S_x be the support set of a binary vector $x \in \{0,1\}^N$. For a vector $x \in \mathbb{R}^N$ and $S \subseteq N$, define $x(S) := \sum_{i \in S} x_i$.

2. Polar outer approximation

We start this section with a simple property of the facets of $\mathcal{Q}_f$. We refer to the variable bounds $\mathbf{0} \le x \le \mathbf{1}$ as the *trivial inequalities* of $\mathcal{Q}_f$.

Proposition 1. *Any non-trivial facet-defining inequality* $\pi x \le \alpha z + \pi_0$ *for* $\mathcal{Q}_f$ *satisfies* $\pi_0 \ge 0$ *and* $\alpha = 1$ *(up to scaling).*

Proof. Note that $\pi_0 \ge 0$ is necessary for validity, since otherwise inequality $\pi x \le z + \pi_0$ is invalid for $(\mathbf{0}, 0) \in \mathcal{Q}_f$. Because $(\mathbf{0}, 1)$ is a ray of $\mathcal{Q}_f$, inequality is invalid unless $\alpha \ge 0$. However, any valid $\pi x \le \pi_0$ is implied by the trivial inequalities $\mathbf{0} \le x \le \mathbf{1}$ as $\{(x, f(x)) : x \in \{0,1\}^n\} \subseteq \mathcal{Q}_f$. Thus, $\alpha > 0$ and, by scaling, it may be assumed to be one. □

For a set function f with $f(\emptyset) = 0$, let the *associated polyhedron*[1] be

$$\mathcal{P}_f := \left\{\pi \in \mathbb{R}^N : \pi(S) \le f(S) \text{ for all } S \subseteq N\right\},$$

where $\pi(S)$ denotes $\sum_{i \in S} \pi_i,\ S \subseteq N$. Consider the polar of $\mathcal{P}_f$

$$\mathcal{P}_f^\circ := \left\{(x, z) \in \mathbb{R}^N \times \mathbb{R} : \pi x \le z,\ \text{for all } \pi \in \mathcal{P}_f\right\}\cdot$$

The next proposition shows a polarity relationship between $\mathcal{P}_f$ and the homogeneous ($\pi_0 = 0$) valid inequalities for $\mathcal{Q}_f$.

Proposition 2. *Inequality $\pi x \le z$ is valid for $\mathcal{Q}_f$ if and only if $\pi \in P_f$.*

Proof. For $\pi \in \mathcal{P}_f$, we have $\pi x = \pi(S_x) \le f(S_x) \le z$. Conversely, if $\pi \notin \mathcal{P}_f$, then $\pi(S) > f(S)$ for some $S \subseteq N$; but then for $z = f(S)$, $\pi(S) = \pi\chi_S > z$, contradicting the validity of $\pi x \le z$. □

We refer to inequalities of Proposition 2 as the *polar inequalities*. By Proposition 2, we have $\mathcal{Q}_f \subseteq \mathcal{P}_f^\circ$. Indeed, each facet of $\mathcal{P}_f^\circ$ is a facet of $\mathcal{Q}_f$ as well, as shown below.

Proposition 3. *Inequality $\pi x \le z$ is facet-defining for $\mathcal{Q}_f$ if and only if π is an extreme point of P_f.*

Proof. From Proposition 2 if $\pi \notin \mathcal{P}_f$, inequality $\pi x \le z$ is invalid for $\mathcal{Q}_f$. If $\pi \in \mathcal{P}_f$ is not an extreme point, then $\pi = \lambda\pi^1 + (1-\lambda)\pi^2$ for some $0 < \lambda < 1$ and distinct $\pi^1, \pi^2 \in \mathcal{P}_f$ and $\pi x \le z$ is implied by $\pi^1 x \le z$ and $\pi^2 x \le z$. Conversely, if π is an extreme point of P_f, it is the unique solution to a set of n linearly independent equations $\pi(S_i) = f(S_i)$ for $i = 1, \ldots, n$. Then, the corresponding linearly independent points $(\chi_{S_i}, f(S_i))$, $i = 1, \ldots, n$ of $\mathcal{Q}_f$ and $(\mathbf{0}, 0)$ are on the face $\{x \in \mathcal{Q}_f : \pi x = z\}$. Finally, as $(\mathbf{0}, 1) \in \mathcal{Q}_f$ but not on the face, the face is proper. □

The polar relaxation $\mathcal{P}_f^\circ$ gives a (convex) polyhedral outer approximation for $\mathcal{Q}_f$. It indeed gives all nontrivial homogeneous facets of $\mathcal{Q}_f$. It is interesting to contrast it with the Lovász extension [17] for a set function.

Definition 1. For a set function f, the *Lovász extension* $\hat{f} : [0,1]^N \to \mathbb{R}$ is defined as

$$\hat{f}(x) := \sum_{i=1}^{n-1} (x_i - x_{i+1}) f(S_i) + x_n f(S_n),$$

where $1 \ge x_1 \ge \ldots \ge x_n \ge 0$ and $S_i = \{1, 2, \ldots, i\},\ i \in N =: \{1, \ldots, n\}$.

Observe that $\hat{f}$ is homogeneous and $\hat{f}(\chi_S) = f(S),\ \forall S \subseteq N$. It is easy to see that for a general set function, Lovász extension is a piecewise polyhedral *inner* approximation of $\mathcal{Q}_f$. Thus, we have

$$\boxed{\textbf{epi } \hat{f} \ \subseteq\ \mathcal{Q}_f \ \subseteq\ \mathcal{P}_f^\circ \cap [0,1]^n \times \mathbb{R}.}$$

[1]For the submodular case, $\mathcal{P}_f$ is referred to as the submodular polyhedron or the extended polymatroid.

Example 1. In this example, we compare the polar outer approximation with the Lovász extension for a non-submodular set function. Consider the function f defined as $f(\emptyset) = \ \ 0$ (A), $f(\{1\}) = -1$ (B), $f(\{2\}) = -1$ (C), $f(\{1,2\}) = -1$ (D). Function f is, in fact, supermodular.

The inequalities describing $\mathbf{epi}\ \hat{f}$, $\mathcal{Q}_f$, and $\mathcal{P}_f^\circ$, other than the bounds $\mathbf{0} \leq x \leq \mathbf{1}$, are listed below and displayed in Figure 1.

$$\mathbf{epi}\ \hat{f} : \begin{cases} \text{ABD:} & -x_1 \leq z \quad \text{if } x_1 \geq x_2 \\ \text{ACD:} & -x_2 \leq z \quad \text{if } x_1 \leq x_2 \end{cases}$$

$$\mathcal{Q}_f : \begin{cases} \text{ABC:} & -x_1 - x_2 \leq z \\ \text{BCD:} & -1 \leq z \end{cases}$$

$$\mathcal{P}_f^\circ : \begin{cases} \text{ABC:} & -x_1 - x_2 \leq z \end{cases}$$

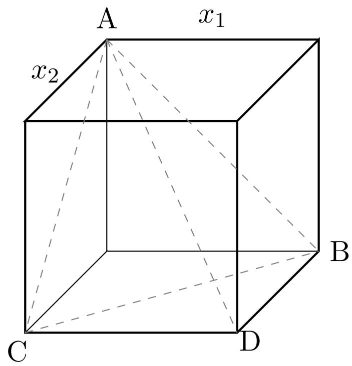

Figure 1. $\mathbf{epi}\ \hat{f}$, $\mathcal{Q}_f$, $\mathcal{P}_f^\circ$.

Observe that $\mathbf{epi}\ \hat{f}$ is non-convex and $\mathbf{epi}\ \hat{f} \subsetneq \mathcal{Q}_f \subsetneq \mathcal{P}_f^\circ \cap [0,1]^n \times \mathbb{R}$.

The submodular case. Lovász [17] has shown that $\hat{f}$ is convex if and only if f is submodular, establishing the relationship between convexity and submodular set functions. Next we show that for a submodular function f, all nontrivial facets of $\mathcal{Q}_f$ are homogeneous; consequently, the polar $\mathcal{P}_f^\circ$ gives an exact relaxation. So, for a submodular function f, it holds

$$\boxed{\mathbf{epi}\ \hat{f} \ = \ \mathcal{Q}_f \ = \ \mathcal{P}_f^\circ \cap [0,1]^n \times \mathbb{R}.}$$

Proposition 4. *For a submodular function f, any non-trivial facet-defining inequality*

$$\pi x \leq z + \pi_0 \tag{3}$$

of $\mathcal{Q}_f$ satisfies $\pi_0 \leq 0$.

Proof. For contradiction, suppose $\pi_0 > 0$. Consider $f' : 2^N \to \mathbb{R}$ defined as $f'(\emptyset) = 0$ and $f'(S) = f(S) + \pi_0$ for $\emptyset \neq S \subseteq N$. Observe that f' is submodular as well. Since (3) is valid for $\mathcal{Q}_f$, $\pi(S) \leq f(S) + \pi_0$ for all $S \subseteq N$ and hence, $\pi \in \mathcal{P}_{f'}$ and, by Proposition 2, inequality

$$\pi x \leq z \tag{4}$$

is valid for $\mathcal{Q}_{f'}$. Now, since (3) is facet-defining for $\mathcal{Q}_f$, (4) is facet-defining for $\mathcal{Q}_{f'}$. Hence, by Proposition 3, π is an extreme point of $P_{f'}$. After permuting variables, if necessary, we may assume that $\pi_1 = f'(S_1) = f(S_1) + \pi_0$, and $\pi_i = f'(S_i) - f'(S_{i-1}) = f(S_i) - f(S_{i-1})$ for $i = 2, \dots, n$. Then, $\gamma \in \mathcal{P}_f$ for $\gamma_1 = \pi_1 - \pi_0$, and $\gamma_i = \pi_i$ for all $i = 2, \dots, n$. However, by Proposition 2, $\gamma x \leq z$ is valid for $\mathcal{Q}_f$ and, together with $x_1 \leq 1$, it dominates inequality (3). □

Remark 1. Note that if $f(\emptyset) \neq 0$, in order to define the inequalities for $\mathcal{Q}_f$, we may use the polyhedron $\mathcal{P}_{f'}$ for $f' := f - f(\emptyset)$. In general, polar inequalities for $\mathcal{Q}_f$ take the form

$$\pi x \leq z - f(\emptyset), \ \pi \in \mathcal{P}_{f - f(\emptyset)}. \tag{5}$$

Remark 2. For a submodular function, $\mathcal{P}_f$ is an extended polymatroid and the polar inequalities reduce to the extended polymatroid inequalities. The separation problem for extended polymatroid inequalities is optimization of a linear objective over $\mathcal{P}_f$, which can be solved by the greedy algorithm of Edmonds [7]: Given $\bar{x} \in \mathbb{R}^n_+$ and $\bar{z} \in \mathbb{R}$, checking whether $(\bar{x}, \bar{z})$ violates an extended polymatroid inequality is equivalent to solving the problem

$$\zeta := \max\big\{\pi \bar{x} : \pi \in \mathcal{P}_f\big\}. \tag{6}$$

For a nonincreasing order $\bar{x}_{(1)} \geq \bar{x}_{(2)} \geq \cdots \geq \bar{x}_{(n)}$, let $S_{(i)} = \{(1), (2), \ldots, (i)\}$ and $\bar{\pi}_{(i)} = f(S_{(i)}) - f(S_{(i-1)})$ for $1 \leq i \leq n$. Then, $(\bar{x}, \bar{z})$ is violated by the corresponding extended polymatroid inequality if and only if $\zeta = \bar{\pi}\bar{x} > \bar{z}$.

3. Inequalities for general set functions

Unlike the submodular case, homogeneous inequalities of the polar relaxation are not sufficient to describe $\mathcal{Q}_f$ for a general set function. One can, however, generate non-homogeneous inequalities by decomposing a set function into a sum of a submodular function and a supermodular function and utilizing inequalities for each.

It is well-known that for any set function f, such a decomposition exists. For a strictly submodular h and sufficiently large $\lambda > 0$, the first term below is submodular, whereas the second term is supermodular:

$$f = \big(f + \lambda h\big) + \big(- \lambda h\big)\cdot$$

Although, in general, a submodular–supermodular decomposition may be difficult to compute, in many cases, such a decomposition is either readily available or easy to construct. Below we give a few examples that will also be used in the computations in Section 4:

- *Optimization with higher moments*: When random variables deviate significantly from the normal distribution, optimization of utility functions with higher moments, such as skewness and kurtosis, in addition to the expectation and standard deviation, are preferred for more accurate modeling:

  $$-\mu' x + \lambda_2 \left(\sum \sigma_i^2 x_i^2\right)^{1/2} - \lambda_3 \left(\sum \gamma_i^3 x_i^3\right)^{1/3} + \lambda_4 \left(\sum \kappa_i^4 x_i^4\right)^{1/4}$$

  Here, the first term is modular, the second and fourth (even) terms submodular, and the third (odd) term is supermodular.

- *Quadratic optimization on binaries*: Let Q be a square matrix and Q^- and Q^+ be the square matrices with the negative and positive elements of Q, respectively, and zeros elsewhere. Since $g(x) = x'Q^-x$

is submodular and $-h(x) = x'Q^+x$ is supermodular [20], we have the corresponding submodular–supermodular decomposition:

$$f(x) = x'Qx = x'Q^-x + x'Q^+x.$$

- *Fractional linear functions*: Although not as immediate as in the cases above, a submodular-supermodular decomposition can be constructed for a fractional linear function with positive coefficients $a, c > 0$ as follows:

$$f(x) := \frac{c'x}{1+a'x} = \left(f(x) + \lambda\frac{a'x}{1+a'x}\right) - \left(\lambda\frac{a'x}{1+a'x}\right).$$

  Observe that $h(x) = a'x/(1 + a'x)$ is submodular as it is the composite of the univariate function $x/(1 + x)$, which is concave over nonnegative values and the modular function $a'x$ [1]. Han et al. [12] show that $f(x)$ is submodular if $f(N) \leq r_{min} := \min_{i\in N} c_i/a_i$. Then letting $\lambda \geq \lambda_{min} := \big(c(N) - r_{min}(1 + a(N))\big)^+$ ensures that the first term $f(x) + \lambda h(x)$ in the decomposition is always submodular: Observe that if all ratios c_i/a_i are equal, then $f(x)$ is submodular and $\lambda_{min} = 0$ is sufficient. Otherwise, $\lambda = \lambda_{min}$ implies $f(N) + \lambda_{min}h(N) \leq r_{min} + \lambda_{min}$.

Writing a general set function as the difference of two submodular functions, $f = g - h$, an outer approximation for f can be formed by using the polar inequalities for g and the *submodular inequalities* of Nemhauser and Wolsey [20, pg 710] for h:

$$w \leq h(S) - \sum_{i\in S}\rho_i(N\setminus i)(1 - x_i) + \sum_{i\in N\setminus S}\rho_i(S)x_i \ \text{ for all } S \subseteq N, \quad (7)$$

$$w \leq h(S) - \sum_{i\in S}\rho_i(S\setminus i)(1 - x_i) + \sum_{i\in N\setminus S}\rho_i(\emptyset)x_i \ \text{ for all } S \subseteq N, \quad (8)$$

where $\rho_i(S) = h(S\cup\{i\}) - h(S)$. Submodular inequalities (7)–(8) are valid for the *hypograph* of h, $\textbf{hyp}\ h := \Big\{(x, w) \in \{0, 1\}^N \times \mathbb{R} : h(x) \geq w\Big\}\cdot$

4. Computations

In this section we report on our computational experiments with using the inequalities from submodular–supermodular decompositions for non-submodular functions. In each experiment, we control the deviation of the set functions from submodularity. All computations are done with Gurobi version 9.0 with default solver options (except heuristics and presolve are turned off and single thread is used) on a Xeon workstation.

The first set of experiments are on binary quadratic optimization of the form: $\min\left\{x'Qx + \Omega c'x : x \in \{0,1\}^n\right\}$. The data is generated following Carter [6] : Q_{ij} is drawn from Uniform$[-100\lambda, 100(1-\lambda)]$ for $i \neq j$ and $Q_{ii} = 0$; c_i is drawn from Uniform$[-100(1-\lambda)(n-1), 100\lambda(n-1)]$. The

TABLE 1. Binary quadratic optimization.

λ	0.0	0.2	0.4	0.6	0.8	1.0
gap (%)	0.1	0.8	1.4	7.1	35.2	40.6
cgap (%)	0.1	0.6	0.2	0.1	0.1	0.0
time (sec.)	0.2	2.4	2.7	2.4	5.6	8.6
ctime (sec.)	0.2	1.8	3.1	1.4	3.0	0.0
# nodes	286.2	258.0	259.0	258.0	258.2	257.0
# cnodes	286.2	360.0	348.2	210.2	207.2	0.2
# cuts	0.0	1.6	4.0	6.8	13.0	2.0

parameter $\lambda \in [0, 1]$ controls the distance from submodularity: for $\lambda = 1$, Q is nonpositive and the quadratic function is submodular, whereas for $\lambda = 0$, Q is nonnegative and the quadratic function is supermodular. Thus, the quadratic function is a convex combination of a submodular function and a supermodular function. The parameter Ω is chosen to ensure that quadratic and linear terms are well-balanced to avoid trivial solutions. Gurobi 9.0 uses McCormick inequalities to automatically build convex relaxations of nonconvex quadratic functions.

In Table 1 we report the integrality gap at the root node, solution time, and the number of nodes explored with and without adding cuts. Each row shows the average for five instances with $n = 200$. We observe in Table 1 that the integrality gap of the convex relaxations increases with λ, achieving the highest value for the submodular case ($\lambda = 1$). The inequalities from the outer relaxations are particularly effective for these cases with high integrality gap. Indeed, for the submodular case, the gap is closed completely at the root node and the problems are solved without branching, as expected. For all values of λ we observe a substantial reduction in the integrality gaps, leading to reduction in the number of nodes as well as the solution times.

The second set of experiments are done on the 0–1 knapsack problem with a mean-risk objective involving higher moments:

$$\begin{aligned}&\min -\Omega\mu' x + \lambda \left(\sum \sigma_i^2 x_i^2\right)^{1/2} - (1-\lambda)\left(\sum \gamma_i^3 x_i^3\right)^{1/3} + \lambda\left(\sum \kappa_i^4 x_i^4\right)^{1/4} \\ &\text{s.t. } a'x \le b, \;\; x \in \{0,1\}^n .\end{aligned}$$

The data is generated following Bergman and Cire [5]. The parameters μ_i and a_i are drawn from Uniform[0,100], σ_i, γ_i, κ_i are drawn from Uniform[0, μ_i]. The knapsack capacity is set to $0.5 \sum_i a_i$. As before, the parameter λ controls the distance of the objective function from submodularity. For $\lambda = 1$ the objective is submodular; for $\lambda = 0$ it is supermodular. Ω is chosen to ensure that positive and negative terms in the objective are well-balanced to avoid trivial solutions. Conic quadratic formulations of the epigraphs of (convex) standard deviation and kurtosis functions are standard [22]. We utilize the following convex formulation for the hypograph of the skewness

Table 2. Optimization with higher moments.

λ	0.0	0.2	0.4	0.6	0.8	1.0
gap (%)	0.0	7.5	50.2	48.3	37.1	29.9
cgap (%)	0.0	1.2	25.8	14.8	2.8	0.0
time (sec.)	0.1	71.9	1,800.0	1,553.0	1,279.3	388.1
ctime (sec.)	0.1	1.4	814.6	448.5	3.3	0.8
# nodes	2.6	748.6	13,554.6	20,717.0	18,044.0	10,598.8
# cnodes	2.6	57.6	3,720.2	4,107.4	87.8	2.2
# cuts	0.0	52.6	100.0	100.0	95.4	64.2

function over binary x:

$$s \leq \left(\sum \gamma_i^3 x_i^3\right)^{1/3} \iff z \leq \sum_i \gamma_i^3 x_i;\ w^2 \leq zs;\ s^2 \leq w.$$

The results for this experiment are summarized in Table 2. Each row shows the average for five instances with $n = 100$. For this problem the integrality gap is highest for $\lambda = 0.4$. Out of 30, 13 instances could not be solved to optimality within the half hour time limit without cuts. Of those, ten are solved to optimality within time limit with the cuts. Utilizing the cuts reduced the root gaps substantially and, consequently, led to smaller search trees, faster solution times and better solutions.

Finally, the last set of experiments are done on fractional linear optimization problems of the form

$$\min\left\{\frac{c'x}{1+a'x} - \Omega s'x :\ x \in \{0,1\}^n\right\}.$$

The parameters a_i and s_i are drawn from Uniform[0,10] and r_i are drawn from Uniform$[1+\lambda, 2]$. We let $c_i = r_i a_i$, $i = 1, \ldots, n$. Ω is chosen to ensure that positive and negative terms are well-balanced to avoid trivial solutions. Observe that for $\lambda = 1$, $c = 2a$ and the objective is submodular. Otherwise, we form the submodular-supermodular decomposition as discussed in Section 3. The convex relaxation used for the formulation is

$$\begin{aligned}&\min\ z + \lambda_{min}(t-1) - \Omega s'x \\ &\text{s.t. } w = 1 + a'x;\ \sum_i \tilde{c}_i x_i^2 \leq zw;\ 1 \leq tw;\ x, w, z, t \geq 0,\end{aligned}$$

where $\tilde{c}_i = c_i + \lambda_{min} a_i$, $i = 1, \ldots, n$. Here z corresponds to the convex relaxation of the submodular function $\tilde{c}'x/(1+a'x)$ and t corresponds to the convex complement $1/(1+a'x)$ of $a'x/(1+a'x)$. We utilize polymatroid inequalities for $\tilde{c}'x/(1+a'x)$ and for the rotated cone constraint above [2].

Observe in Table 3 that the percentage integrality gaps are very large. This is due to the integer optimal values being close to zero. We see large reduction in integrality gaps when cuts ar added. For all instances the

Table 3. Optimization with fractional linear functions.

λ	0.0	0.2	0.4	0.6	0.8	1.0
gap (%)	1,326.8	856.8	645.3	347.1	178.7	44.0
cgap (%)	90.8	61.5	41.3	21.5	9.6	0.0
time (sec.)	83.3	117.2	206.6	261.4	84.5	40.8
ctime (sec.)	44.1	88.9	69.5	71.1	12.3	0.0
# nodes	31,590.0	36,075.6	41,933.0	57,776.4	32,800.6	24,170.2
# cnodes	16,048.6	21,166.4	23,353.6	22,840.2	9,707.4	0.0
# cuts	27.6	27.8	30.4	23.2	23.4	22.0

number of nodes and the computation time are reduced substantially with the largest improvement for the submodular case ($\lambda = 1$) as expected.

These computational experiments demonstrate that, when used as cutting planes, the valid inequalities from a submodular-supermodular decomposition of general set functions can be effective to improve the branch-and-bound algorithms.

References

[1] Ahmed, S. and Atamtürk, A. (2011). Maximizing a class of submodular utility functions. *Mathematical Programming*, 128:149–169.

[2] Atamtürk, A. and Gómez, A. (2016). Submodularity in conic quadratic mixed 0-1 optimization. *arXiv preprint arXiv:1705.05918*. BCOL Research Report 16.02, UC Berkeley. Forthcoming in *Operations Research*.

[3] Atamtürk, A. and Gómez, A. (2018). Strong formulations for quadratic optimization with M-matrices and indicator variables. *Mathematical Programming*, 170:141–176.

[4] Atamtürk, A. and Narayanan, V. (2008). Polymatroids and risk minimization in discrete optimization. *Operations Research Letters*, 36:618–622.

[5] Bergman, D. and Cire, A. A. (2017). Discrete nonlinear optimization by state-space decompositions. *Management Science*, 64(10):4700–4720.

[6] Carter, M. W. (1984). The indefinite zero-one quadratic problem. *Discrete Applied Mathematics*, 7:23–44.

[7] Edmonds, J. (1971). Submodular functions, matroids and certain polyhedra. In Guy, R., editor, *Combinatorial structres and their applications*, volume 11, pages 69–87. Gordon and Breach, New York, NY.

[8] Edmonds, J. and Giles, R. (1977). A min-max relation for submodular functions on graphs. In Hammer, P., Johnson, E., Korte, B., and Nemhauser, G., editors, *Studies in Integer Programming*, volume 1 of *Annals of Discrete Mathematics*, pages 185 – 204. Elsevier.

[9] Feige, U., Mirrokni, V. S., and Vondrák, J. (2011). Maximizing non-monotone submodular functions. *SIAM Journal on Computing*, 40:1133–1153.

[10] Fujishige, S. (2005). *Submodular Functions and Optimization*, volume 58 of *Annals of Discrete Mathematics*. Elsevier, 2nd edition.

[11] Grötschel, M., Lovász, L., and Schrijver, A. (1981). The ellipsoid method and its consequences in combinatorial optimization. *Combinatorica*, 1:169–197.

[12] Han, S., Gómez, A., and Prokopyev, O. A. (2019). Assortment optimization and submodularity.

[13] Iwata, S. (2008). Submodular function minimization. *Mathematical Programming*, 112:45–64.

[14] Iwata, S., Fleischer, L., and Fujishige, S. (2001). A combinatorial strongly polynomial algorithm for minimizing submodular functions. *Journal of the ACM*, 48:761–777.

[15] Lee, H., Nemhauser, G. L., and Wang, Y. (1996). Maximizing a submodular function by integer programming: Polyhedral results for the quadratic case. *European Journal of Operational Research*, 94:154–166.

[16] Lee, J., Mirrokni, V. S., Nagarajan, V., and Sviridenko, M. (2010). Maximizing nonmonotone submodular functions under matroid or knapsack constraints. *SIAM Journal on Discrete Mathematics*, 23:2053–2078.

[17] Lovász, L. (1983). Submodular functions and convexity. In Bachem, A., Grötschel, M., and Korte, B., editors, *Mathematical Programming– State of the Art*, pages 235–257. Springer, Berlin.

[18] Nemhauser, G. and Wolsey, L. (1981). Maximizing submodular set functions: Formulations and analysis of algorithms. In Hansen, P., editor, *Annals of Discrete Mathematics (11)*, volume 59 of *North-Holland Mathematics Studies*, pages 279 – 301. North-Holland.

[19] Nemhauser, G. L. and Wolsey, L. A. (1978). Best algorithms for approximating the maximum of a submodular set function. *Mathematics of Operations Research*, 3:177–188.

[20] Nemhauser, G. L. and Wolsey, L. A. (1988). *Integer and Combinatorial Optimization*. John Wiley and Sons, New York.

[21] Nemhauser, G. L., Wolsey, L. A., and Fisher, M. L. (1978). An analysis of approximations for maximizing submodular set functions–I. *Mathematical Programming*, 14:265–294.

[22] Nesterov, Y. and Nemirovski, A. (1993). *Interior-point polynomial algorithms for convex programming*. SIAM, Philedelphia.

[23] Orlin, J. B. (2009). A faster strongly polynomial time algorithm for submodular function minimization. *Mathematical Programming*, 118:237–251.

[24] Schrijver, A. (2000). A combinatorial algorithm minimizing submodular functions in strongly polynomial time. *Journal of Combinatorial Theory, Series B*, 80:346–355.

[25] Schrijver, A. (2003). *Combinatorial Optimization: Polyhedra and Efficiency*. Springer Verlag, Berlin.

[26] Topkis, D. M. (1978). Minimizing a submodular function on a lattice. *Operations Research*, 26:305–321.

[27] Wolsey, L. A. (1988). Submodularity and valid inequalities in capacitated fixed charge networks. *Operations Research Letters*, 8:119–124.

[28] Yu, J. and Ahmed, S. (2017). Maximizing a class of submodular utility functions with constraints. *Mathematical Programming*, 162:145–164.