Γ-COUNTERPARTS FOR ROBUST NONLINEAR COMBINATORIAL AND DISCRETE OPTIMIZATION

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ABSTRACT. Γ-uncertainties have been introduced for adjusting the degree of conservatism of robust counterparts of (discrete) linear optimization problems under interval uncertainty. This article's contribution is a generalization of this approach to (mixed-integer) nonlinear optimization problems. We focus on the cases in which the uncertainty is linear but also derive formulations for the general case. We present cases where the robust counterpart of a nonlinear combinatorial problem is solvable with a polynomial number of oracle calls for the underlying nominal problem and elaborate on it using a quadratic assignment problem. We show the computational efficiency with a numerical study tackling a patient transport problem and the quadratic assignment problem.

1. INTRODUCTION

Robust optimization is an established area for dealing with uncertainties in optimization problems. Several approaches have been developed in this area in the last decades, especially in the field of (mixed-integer) linear programming. However, for combinatorial optimization, those are usually not meaningful since the underlying problem's structure is changed, rendering solution algorithms for the nominal problem not applicable. To circumvent this, Bertsimas and Sim introduced Γ-uncertainties in [BS03] and [BS04] for combinatorial optimization problems with a linear objective and mixed-integer linear constraints, both under uncertainty. For $\mathcal{X} \subseteq \{0,1\}^n$, $\bar{u}$, $\Delta u \geqslant 0$, $\Delta u_0 := 0$ and a nonnegative integer Γ, they have shown that

$$\min_{x \in \mathcal{X}} \max_{u \in \mathcal{U}^\Gamma} u^\top x \tag{1}$$

where

$$\mathcal{U}^\Gamma = \{u \in [\bar{u}, \bar{u} + \Delta u] : u_i \neq \bar{u}_i \text{ for at most } \Gamma \text{ coefficients } i \in \{1,2,\dots,n\}\}$$

is equivalent to

$$\min_{k \in \{0,1,\dots,n\}} \left\{ \Gamma \Delta u_k + \min_{x \in \mathcal{X}} \left\{ \bar{u}^\top x + \sum_{j \in \{1,2,\dots,n\}} \max\{0, \Delta u_j - \Delta u_k\} x_j \right\} \right\}. \tag{2}$$

This equivalence implies oracle polynomiality of Problem (1), assuming that the objective functions of Problem (1) and the subproblems of Problem (2) underlie the same structure (cf. [BS03, Theorem 3]). Since recent research has focused on nonlinear robust programming, see, e.g., [LMM+20], [KLS22a] or [KLS22b], the main purpose of this article is extending their approach for combinatorial problems with nonlinear objective functions.

Our contribution. We propose and study a generic framework for mixed-integer nonlinear problems (MINLPs) under uncertainty that generalizes the framework of Γ-uncertainties for mixed-integer linear problems (MIPs) introduced in [BS03] and [BS04]. We focus on uncertainty in the objective. In particular, we provide reformulations for the case of linear and nonlinear uncertainties. We discuss limits of our theoretical approach. We also show that a problem being subject to so-called 'assignment structure' implies oracle polynomiality of the problem under uncertainty. We demonstrate the efficiency of our reformulations with an exemplary numerical study.

Structure. The paper is structured as follows. In Section 2, we propose the Γ-counterpart for MINLPs. We motivate our generalization with a piecewise linear objective function occurring in logistics and the quadratic assignment problem under uncertainty, which are our working examples

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throughout this article. In Section 3, we reformulate our proposed model, demonstrate our main results and show cases in which oracle-polynomiality holds. In Section 4, we conduct a numerical study for our working examples. Finally, in Section 5, we provide a conclusion and propose some interesting avenues for future research.

Notation and definitions. $\mathbb{Z}$ denotes the set of integral numbers, $\mathbb{R}$ denotes the set of real numbers. Indexing them with $\geqslant 0$ refers to the respective set of nonnegative numbers. Throughout this article, m, n denote positive integral numbers. We define $[n] := \{1, 2, \dots, n\}$ and $[n]_0 := [n] \cup \{0\}$. $\langle \cdot, \cdot \rangle$ denotes an inner product. When it is indexed, the index refers to the corresponding real vector space. For a function f_i indexed with i, we denote the uncertainty set by $\mathcal{U}_i$ and the nominal scenario by $\bar{u}^i$. When there exists a function ℓ defined on $\mathcal{X}$ and an inner product defined on a vector space containing $\mathcal{U}$ with $f(x, u) = \langle u, \ell(x) \rangle$ for every $x \in \mathcal{X}, u \in \mathcal{U}$, we call the uncertainty linear. For $m \times n$-matrices A and B, we define the matrix interval $[A, B] := \{C \in \mathbb{R}^{m \times n} : A_{i,j} \leqslant C_{i,j} \leqslant B_{i,j} \ \forall i \in [m],\ j \in [n]\}$. When the uncertainty set is an (possibly multi-dimensional) interval, we say that the uncertainty is an interval uncertainty. The matrix $A \otimes B$ denotes the Kronecker product of the matrices A and B. The vector $\operatorname{vec}(A)$ denotes the vectorization of A.

2. The setting

Our proposed generalization of Problem (1) is tailored for the nominal problem

$$\inf_{x \in \mathcal{X}} \sum_{i \in [m]} \bar{f}_i(x), \tag{3}$$

where $\bar{f}_i : \mathcal{X} \to \mathbb{R}$ is an arbitrary function with domain $\mathcal{X}$ for every $i \in [m]$. We assume that the objective function is under uncertainty. It is modeled such that the uncertainty is separable, i.e., we model it such every function $\bar{f}_i$ 'has' its own uncertainty set $\mathcal{U}_i$ that contains a nominal scenario $\bar{u}^i$ with $\bar{f}(x)_i =: f_i(x, \bar{u}^i)$ for all $x \in \mathcal{X}$ for a function $f_i : \mathcal{X} \times \mathcal{U}_i \to \mathbb{R}$.

Now, let $\Gamma \in [m]$. To adapt Bertsimas' and Sim's approach, we propose the following model as the so-called Γ-counterpart of Problem (3):

$$\inf_{x \in \mathcal{X}} \left\{ \sup_{\mathcal{S} \subseteq [m] : |\mathcal{S}| \leqslant \Gamma} \left\{ \sum_{i \in \mathcal{S}} \sup_{u^i \in \mathcal{U}_i} f_i(x, u^i) + \sum_{i \in [m] \setminus \mathcal{S}} f_i(x, \bar{u}^i) \right\} \right\}. \tag{4}$$

Showing that this is a generalization of Problem (1) is trivial: Assuming that $f_i(x, u) = u_i x_i$ and $\mathcal{U}_i$ is a one-dimensional interval, it is a matter of simple arithmetic to obtain Problem (1).

In the following, we illustrate two examples for which our introduced Γ-counterpart can be applied but the original result stated in our introduction cannot.

2.1. Working examples.

Example 1 (Vehicle Routing Problem under soft deadline uncertainty)**.** *Problems occurring in the application of logistics involving deliveries within given due times can often be modeled as combinatorial problems with (non-)linear objective functions, e.g., taxi routing, delivery of goods or patient transport. For all three of these cases, being on time is important for customer satisfaction. At the same time, it is usually not problematic when vehicle arrives too early for a pick-up.*

For tasks $i \in [m]$, we denote the due time with $b_i \in \mathbb{R}_{\geqslant 0}$. If a job is finished after b_i then a penalty cost is incurred. A program for (unweighted) penalty costs is

$$\inf_{x \in \mathcal{X}} \sum_{i \in [m]} \max\{0, x_i - b_i\}. \tag{5}$$

Problem (5) *may model, e.g., a special case of vehicle routing problems with general time windows, see* [HYI08]*. In practice, the due time can be uncertain: We assume that $b_i \in \mathcal{U}_i := [\bar{b}_i - \Delta b_i, \bar{b}_i]$ for some nominal scenario $\bar{b}_i$ and a perturbation Δb_i. To reduce conservatism, the goal is to ensure robustness against Γ deviations of the due times, resulting in the following optimization problem:*

$$\inf_{x \in \mathcal{X}} \left\{ \sup_{\mathcal{S} \subseteq [m] : |\mathcal{S}| \leqslant \Gamma} \left\{ \sum_{i \in \mathcal{S}} \max\{0, x_i - \bar{b}_i + \Delta b_i\} + \sum_{i \in [m] \setminus \mathcal{S}} \max\{0, x_i - \bar{b}_i\} \right\} \right\}. \tag{6}$$

Problem (6) *is exactly* Γ*-counterpart* (4) *by setting* $f_i(x, b) := \max\{0, x_i - b_i\}$.

Assuming that the feasible set of Problem (5) is a mixed-integer polyhedron, one could reformulate its objective function such that resulting optimization problem is a MIP. One could solve it by applying reformulations for MIPs under Γ-uncertainty. However, this would render any oracle for solving Problem (5) not applicable, contradicting the oracle-based approach.

Example 2 (Quadratic Assignment Problem under uncertainty)**.** *The Quadratic Assignment Problem (QAP) models the process of assigning n facilities to n locations such that the cost of transporting goods is minimized. With binary variables $x_{i,r}$, $i, r \in [n]$, that indicate whether facility i is assigned to location r, the feasible set can be modeled in the following fashion:*

$$\mathcal{X} = \left\{ x \in \{0,1\}^{n^2} : \sum_{i \in [n]} x_{i,r} = 1 \ \forall r \in [n], \sum_{r \in [n]} x_{i,r} = 1 \ \forall i \in [n] \right\}.$$

For each pair of facilities $(i, j) \in [n]^2$, $c_{i,j} \geqslant 0$ denotes the flow between i and j and for all pair of locations $(r, s) \in [n]^2$, $d_{r,s} \geqslant 0$ denotes the distance between r and s. Thus, the QAP can be modeled by

$$\min_{x \in \mathcal{X}} \sum_{(i,j,r,s) \in [n]^4} c_{i,j} d_{r,s} x_{i,r} x_{j,s}. \tag{7}$$

In [FF15]*, the authors have assumed that the flow is subject to linear interval uncertainty, i.e., $c_{i,j} \in \mathcal{U}_{i,j} = [\bar{c}_{i,j}, \bar{c}_{i,j} + \Delta c_{i,j}]$ for $\bar{c}_{i,j}$, $\Delta c_{i,j} \geqslant 0$ for all $(i, j) \in [n]^2$. The Γ-counterpart* (4) *is given by:*

$$\min_{x \in \mathcal{X}} \left\{ \sum_{(i,j,r,s) \in [n]^4} \bar{c}_{i,j} d_{r,s} x_{i,r} x_{j,s} + \max_{\mathcal{S} \subseteq [n]^2 : |\mathcal{S}| \leqslant \Gamma} \left\{ \sum_{(i,j) \in \mathcal{S}} \Delta \bar{c}_{i,j} \sum_{r,s \in [n]} d_{r,s} x_{i,r} x_{j,s} \right\} \right\}. \tag{8}$$

Note that it looks slightly different to the Γ-counterpart in Example 1 since the uncertainty is linear, allowing this reformulation.

It is important to point out that, while the uncertainty of Problem (7) is modeled linearly, one can not apply the original results of [BS03] since an uncertain parameter $c_{i,j}$ affects the sum $\sum_{(r,s) \in [n]^2} d_{r,s} x_{i,r} x_{j,s} \notin \{0, 1\}$ and not exactly one binary variable.

3. Reformulations of problems with uncertain objectives

In this section, we present equivalent reformulations for the Γ-counterpart introduced in Section 2. Several of the proofs are inspired by those in [BS03]. Furthermore, we discuss the applicability of oracle-based approaches with a focus on problems over matrix spaces.

3.1. **A general reformulation.** In a first step, it turns out that it is possible to obtain first reformulations of Γ-counterpart (4) without any assumptions on the functions f_i or the uncertainty sets $\mathcal{U}_i$.

Lemma 1. *Let $\Gamma \in [m]$. Then Γ-counterpart* (4) *is equivalent to*

$$\begin{aligned} &\inf_{x,p,\theta} \Gamma\theta + \sum_{i \in [m]} f_i(x, \bar{u}^i) + p_i \\ &\text{s.t. } x \in \mathcal{X}, \\ &\quad p_i + \theta \geqslant \sup_{u^i \in \mathcal{U}_i} f_i(x, u^i) - f_i(x, \bar{u}^i) \ \forall i \in [m], \\ &\quad p \in \mathbb{R}^m_{\geqslant 0}, \theta \in \mathbb{R}_{\geqslant 0}. \end{aligned} \tag{9}$$

Proof. The structure of this proof is similar to the proof of [BS03, Theorem 3]. For all $i \in [m]$, we introduce the binary variables

$$s_i := \begin{cases} 1, & \text{if } i \in \mathcal{S}, \\ 0, & \text{otherwise.} \end{cases}$$

The inner maximization problem of Γ-counterpart (4) is equivalent to

$$\begin{aligned}\sup_{s} \quad & \sum_{i\in[m]} f_i(x,\bar{u}^i) + s_i\big(\sup_{u^i\in\mathcal{U}_i} f_i(x,u^i) - f_i(x,\bar{u}^i)\big) \\ \text{s.t.} \quad & \sum_{i\in[m]} s_i \leqslant \Gamma, \\ & s \in \{0,1\}^m.\end{aligned} \tag{10}$$

Clearly, Problem (10) is equivalent to its LP relaxation because Γ is assumed to be integral. Inserting its dual into Γ-counterpart (4) proves the claim. □

In Lemma 1, to obtain a computationally tractable formulation, it is necessary to reformulate the inequality

$$p_i + \theta \geqslant \sup_{u^i\in\mathcal{U}_i} f(x,u^i) - f(x,\bar{u}^i) \tag{11}$$

for all $i \in [m]$. In [BTdHV15], the authors proposed various approaches, especially for linear (concave) uncertainties. Regarding approaches for nonconcave uncertainties, we refer to [BTdHV15] and [LMM+20]. However, their approaches are not suited for combinatorial optimization when one wishes to apply oracles.
One can reformulate Problem (9) to obtain a problem with feasible set $\mathcal{X}$ and without the new variables p and θ. This is especially important when it comes to oracle-based optimization since they usually require the feasible set not being modified (at the very last, not drastically).

Lemma 2. *If* $\Gamma \in [m]$ *then* Γ*-counterpart* (4) *is equivalent to*

$$\inf_{k\in[m]_0}\left\{\inf_{x\in\mathcal{X}}\left\{\Gamma\theta^k(x) + \sum_{i\in[m]} f_i(x,\bar{u}^i) + \sup\{0,\theta^i(x)-\theta^k(x)\}\right\}\right\}, \tag{12}$$

where $\theta^k(x) := \sup_{u^k\in\mathcal{U}_k} f_k(x,u^k) - f_k(x,\bar{u}^k)$ *and* $\theta^0(x) := 0$*, resp. for all* $x \in \mathcal{X}$*.*

Proof. Since $\Gamma \in [m]$, Γ-counterpart (4) is equivalent to Problem (9). For all $i \in [m]$, p_i occurs in exactly one inequality only (besides the nonnegativity constraint), so we obtain

$$p_i^* = \sup\{0, \sup_{u^i\in\mathcal{U}_i} f_i(x^*,u^i) - f_i(x^*,\bar{u}^i)) - \theta^*\}\ \forall i \in [m] \tag{13}$$

for an optimal solution (x^*, p^*, θ^*) of Problem (9). Inserting Equation (13) into the objective function of Problem (9) results in

$$\Gamma\theta + \sum_{i\in[m]} f_i(x,\bar{u}^i) + \sup\{0, \sup_{u^i\in\mathcal{U}_i} f_i(x,u^i) - f_i(x,\bar{u}^i)) - \theta\}. \tag{14}$$

Since (14) is convex and piecewise linear in θ, either $\theta^* = 0$ or $\theta^* = \theta^k(x)$ for one $k \in [m]$. □

In particular, Lemma 2 is already sufficient to reformulate Problem (6) given in Example 1:
Example 1 continued. Consider Problem (6):

$$\inf_{x\in\mathcal{X}}\left\{\sup_{\mathcal{S}\subseteq[m]:|\mathcal{S}|\leqslant\Gamma}\left\{\sum_{i\in\mathcal{S}}\max\{0,x_i-\bar{b}_i+\Delta b_i\} + \sum_{i\in[m]\setminus\mathcal{S}}\max\{0,x_i-\bar{b}_i\}\right\}\right\}.$$

Applying Lemma 2, an equivalent reformulation is

$$\inf_{k\in[m]_0}\left\{\inf_{x\in\mathcal{X}}\left\{\Gamma\theta^k(x) + \sum_{i\in[m]}\max\{0,x_i-\bar{b}_i,\max\{0,x_i-\bar{b}_i+\Delta b_i\}-\theta^k(x)\}\right\}\right\} \tag{15}$$

where $\theta^k(x) := \max\{0, x_k - \bar{b}_k + \Delta b_k\} - \max\{0, x_k - \bar{b}_k\}$ and $\theta^0(x) := 0$. Thus, for $k = 0$, it is necessary to solve $\inf_{x\in\mathcal{X}}\sum_{i\in[m]}\max\{0, x_i - \bar{b}_i + \Delta b_i\}$. For $k > 0$, we distinguish between three cases:

i) $x_k \geqslant \bar{b}_k$, i.e., $\theta^k(x) = \Delta b_k$ and

$$\max\{0,x_i-\bar{b}_i,\max\{0,x_i-\bar{b}_i+\Delta b_i\}-\theta^k(x)\} = \max\{0,x_i-\bar{b}_i,x_i-\bar{b}_i+\Delta b_i-\Delta b_k\}. \tag{16}$$

ii) $x_k \in [\bar{b}_k - \Delta b_k, \bar{b}_k]$, i.e., $\theta^k(x) = x_k - \bar{b}_k + \Delta b_k$ and

$$\begin{aligned} &\max\{0, x_i - \bar{b}_i, \max\{0, x_i - \bar{b}_i + \Delta b_i\} - \theta^k(x)\} = \\ &\max\{0, x_i - \bar{b}_i, x_i - \bar{b}_i + \Delta b_i - x_k + \bar{b}_k - \Delta b_k\}. \end{aligned} \tag{17}$$

iii) $x_k \leqslant \bar{b}_k - \Delta b_k$, i.e., $\max\{0, x_k - \bar{b}_k\} = \max\{0, x_k - \bar{b}_k + \Delta b_k\} = 0$. Thus, $\theta^k(x) = 0$ and we refer to the case of $k = 0$.

For each $k \in [m]$, by applying Equations (16) and (17), we solve

$$\begin{aligned} &\inf_{x \in \mathcal{X}} \sum_{i \in [m]} \max\{0, x_i - \bar{b}_i, x_i - \bar{b}_i + \Delta b_i - \Delta b_k\} + \Gamma \Delta b_k \\ &\text{s.t. } x_k \in [\bar{b}_k - \Delta b_k, \bar{b}_k] \end{aligned} \tag{18}$$

and

$$\begin{aligned} &\inf_{x \in \mathcal{X}} \sum_{i \in [m]} \max\{0, x_i - \bar{b}_i, x_i - \bar{b}_i + \Delta b_i - x_k + \bar{b}_k - \Delta b_k\} + \Gamma(x_k - \bar{b}_k + \Delta b_k) \\ &\text{s.t. } x_k \geqslant b_k. \end{aligned} \tag{19}$$

Therefore, in total, we need to solve $2m + 1$ optimization problems with a piecewise linear objective with the addition of one hard bound for exactly one variable, resp. for $k \in [m]$. Using logistics as an example, we can interpret this as follows: For $k \in [m]$ we solve two optimization problems. In particular, we distinguish whether the nominal bound $\bar{b}_k$ is satisfied or not. In both cases, we compare the violations for x_i and x_k and add Γ times the violation of the bound x_k. For $k = 0$, Γ is multiplied by 0 and vanishes, but all the bounds realize their worst-case. Summing it up, the reformulation models a trade-off between the worst-case and the effect of not satisfying the $\bar{b}_k$ bound.

We apply this reformulation to a special case of the vehicle routing problem with general time windows in Section 4.

3.2. **Linear interval uncertainties.** In recent decades, research has focused on linear interval uncertainties, and they are well studied. Therefore, in this subsection we will show how to deal with linear uncertainties in the context of MINLPs under uncertainty. In doing so, we note that many combinatorial problems model their uncertainties linearly.

3.2.1. *The special case of* $0/1$*-functions.* In this subsubsection, we discuss one more case, namely the Γ-counterpart (4) under linear one-dimensional nonnegative uncertainty with 0/1-functions, i.e., functions with co-domain $\{0, 1\}$:

Assumption 3. *For the* Γ*-counterpart* (4) *and for all* $i \in [m]$ *we assume:*

(i) *The uncertainty set* $\mathcal{U}_i$ *is a 1-dimensional positive interval, i.e.,* $\mathcal{U}_i = [\bar{u}_i, \bar{u}_i + \Delta u_i] \subseteq \mathbb{R}_{>0}$ *and* $\Delta u_i > 0$.

(ii) *There is a* $0/1$*-function* $\ell_i \colon \mathcal{X} \to \{0, 1\}$ *such that* $f_i(x, u_i) = u_i \ell_i(x)$ *for all* $u_i \in \mathcal{U}_i$.

Although Assumption 3 seems restrictive, it covers many combinatorial problems under uncertainty, e.g., the quadratic knapsack problem or the quadratic matching problem. Under this assumption, proving oracle-polynomiality is straightforward:

Theorem 4. *Let* $\Gamma \in [m]$ *and assume that Assumption 3 holds. Then* Γ*-counterpart* (4) *is equivalent to*

$$\min_{k \in [m]_0} \left\{ \Gamma \Delta u_k + \min_{x \in \mathcal{X}} \left\{ \bar{u}^\top \ell(x) + \sum_{j \in [m]} \max\{0, \Delta u_j - \Delta u_k\} \ell_j(x) \right\} \right\}. \tag{20}$$

Proof. Under Assumption 3, problem (3) is equivalent to $\min_{(x,y) \in \mathcal{X}_y} u^\top y$ with $\mathcal{X}_y := \mathcal{X} \times \ell(\mathcal{X}) \subseteq \mathbb{R}^n \times \{0, 1\}^m$. Then the result stated in the introduction implies that the modified problem's Γ-counterpart is equivalent to

$$\min_{k \in [m]_0} \left\{ \Gamma \Delta u_k + \min_{x \in \mathcal{X}_y} \left\{ \bar{u}^\top y + \sum_{j \in [m]} \max\{0, \Delta u_j - \Delta u_k\} y_j \right\} \right\}$$

where $\Delta u_0 := 0$. Since $y_j = \ell_j(x)$, the claim follows. $\square$

Theorem 4 demonstrates that one can solve the Γ-counterpart with an optimization oracle of Problem (3). This result only implies that The proofs are both heavily inspired by the resp. proofs in [BS03] and [LK14]:

Theorem 5. *Let $\Gamma \in [m]$ and assume that $\Delta u_1 \geqslant \Delta u_2 \geqslant \cdots \geqslant \Delta u_m \geqslant 0$. If Assumption 3 holds then the Γ-counterpart (4) is equivalent to*

$$\min_{k \in \mathcal{L}} \left\{ \Gamma \Delta u_k + \min_{x \in \mathcal{X}} \left\{ f(x, \bar{u}) + \sum_{i \in [k]} (\Delta u_i - \Delta u_k)\, \ell_i(x) \right\} \right\}$$

for $\mathcal{L} := \{\Gamma + 1, \ldots, \Gamma + \gamma, m + 1\}$ with γ being the largest odd integer smaller than $(m+1) - \Gamma$ and $\Delta_{m+1} := 0$. Furthermore, if the optimal value of the kth subproblem is smaller than $\Gamma \Delta u_l$ for $l \in \mathcal{L}$ then one can replace $\mathcal{L}$ with $\mathcal{L}^* := \{k \in \mathcal{L} : \ k > l\}$.

Proof. Since Assumption 3 holds, this statement is a consequence of Theorem 1 in [LK14, Theorem 1] by introducing binary variables y_i, $i \in [m]$, with $y_i = \ell_i(x)$ (as in the proof of Theorem 4). □

In particular, Theorem 5 implies that one only needs to solve at most $\lceil \frac{m-\Gamma}{2} \rceil + 1$ subproblems instead of $m + 1$, as in the case of Theorem 4. [LK14] demonstrates that this significantly reduces the number of subproblems one needs to solve. We also demonstrate that in Section 4.

Remark 6. *Applying Theorem 4, one also can determine α-approximations (for $\alpha \geqslant 1$), if Problem (3) is α-approximable*[1]*, generalizing* [BS03, Theorem 4]. *Since we do not tackle approximations in this publication and since the proof would be almost identical to the original one, we do not specifically demonstrate it.*

3.2.2. *Matrix interval uncertainties.* In this subsubsection, we consider problems in matrix spaces. This will prove beneficial when we aim to reformulate the QAP's Γ-counterpart introduced in Example 2. Before we dive into theory, we would like to recall that $\mathbb{R}^{r \times s}$ is generated by rs standard matrices $F^{p,q}$ where the entry in row p and column q equals 1 and the other entries equal 0.

Assumption 7. *Let r_i and s_i be nonnegative integers for all $i \in [k]$. Let V be a real matrix vector space and let $W^i := \mathbb{R}^{r_i \times s_i}$, $i \in [k]$, be a vector space. Furthermore, we assume that the resp. standard matrices form an orthonormalbasis w.r.t. the inner product $\langle \cdot, \cdot \rangle_{W^i}$. Γ-counterpart (4) satisfies the following:*

i) *For all $i \in [k]$, $\mathcal{U}^i = [\bar{U}^i - \Delta U^i, \bar{U}^i + \Delta U^i]$ for a nominal matrix $\bar{U}^i \in \mathbb{R}^{r \times s}$ and a matrix $\Delta U^i \in \mathbb{R}^{r \times s}_{\geqslant 0}$, and,*
ii) *the functions $f_i \colon V \times W^i \to \mathbb{R}$, $i \in [k]$, are subject to linear uncertainty w.r.t. the inner product $\langle \cdot, \cdot \rangle_{W^i}$ and a function ℓ_i.*

Note that, when the standard matrices form an orthonormalbasis of $\mathbb{R}^{r \times s}$ with respect to an inner product $\langle \cdot, \cdot \rangle$, the equation

$$\langle A, B \rangle = \sum_{i \in [r]} \sum_{j \in [s]} A_{i,j} B_{i,j}. \tag{21}$$

holds. Using Equation (21), we can provide a reformulation of the Γ-counterpart (4):

Lemma 3. *Assume that Assumption 7 holds and that additionally, for all $i \in [k]$ and $x \in \mathcal{X}$, $\ell_i(x)$ is a nonnegative matrix. Furthermore, we set $\Delta U^0 := f^0(x, \Delta U^0)$ are matrices of only zeros. Then Γ-counterpart (4) is equivalent to*

$$\inf_{l \in [k]_0} \inf_{x \in \mathcal{X}} H(l, x), \tag{22}$$

[1]The following definition is not formal and is usually applied for combinatorial problems: Assume that $f^* \in (-\infty, \infty)$ is the optimal value of Problem (3). Then Problem (3) is called α-approximable when there exists a real number $\alpha \geqslant 1$ and an algorithm `ALG` with input $(f, \mathcal{X})$, output $\tilde{x}$, the inequality $\alpha f^* \geqslant f(\tilde{x})$ holds for every instance $(f, \mathcal{X})$ and the running time of algorithm `ALG` is polynomial in the encoding length of $(f, \mathcal{X})$.

where $H\colon [k]_0 \times \mathcal{X} \to \mathbb{R}$ *is the function defined by*

$$\begin{aligned} H(l,x) :=& \Gamma \sum_{p\in[r_l]} \sum_{q\in[s_l]} \Delta U^l_{p,q} \ell^l_{p,q}(x) + \sum_{i\in[k]} \sum_{p\in[r_i]} \sum_{q\in[s_i]} \bar{U}^i_{p,q} \ell^i_{p,q}(x) + \\ & \max\left\{0, \sum_{p\in[r_i]} \sum_{q\in[s_i]} \Delta U^i_{p,q} \ell^i_{p,q}(x) - \sum_{p\in[r_i]} \sum_{q\in[s_i]} \Delta U^l_{p,q} \ell^l_{p,q}(x)\right\} \\ =& \Gamma f_l(x, \Delta U^l) + \sum_{i\in[k]} f_i(x, \bar{U}^i) + \max\{0, f_i(x, \Delta U^i) - f_l(x, \Delta U^l)\} \end{aligned}$$

Proof. We begin by applying Lemma 2. Thus, Γ-counterpart (4) is equivalent to

$$\inf_{k\in[m]_0} \left\{ \inf_{X\in\mathcal{X}} \left\{ \Gamma \theta^k(X) + \sum_{i\in[m]} f_i(X, \bar{U}^i) + \sup\{0, \theta^i(X) - \theta^k(X)\} \right\} \right\},$$

where $\theta^k(X) := \sup_{U^k\in\mathcal{U}_k} f_k(X, U^k) - f_k(X, \bar{U}^k)$ and $\theta^0(X) := 0$ for all $X \in \mathcal{X}$ and $k \in [m]_0$. Applying Equation (21), we obtain

$$\begin{aligned} \sup_{U\in[\bar{U}^i - \Delta U^i, \bar{U}^i + \Delta U^i]} f_i(x, U^i) - f_i(x, \bar{U}^i) &= \max_{U\in[\bar{U}^i - \Delta U^i, \bar{U}^i + \Delta U^i]} \langle \ell_i(x), U^i - \bar{U}^i \rangle_{W^i} \\ &= \max_{Z^i\in[-\Delta U^i, \Delta U^i]} \langle Z^i, \ell_i(x) \rangle_{W^i} \\ &= \max_{Z^i\in[-\Delta U^i, \Delta U^i]} \sum_{p\in[r_i], q\in[s_i]} Z^i_{p,q} \ell_i(x)_{p,q} \\ &= \sum_{p\in[r_i]} \sum_{q\in[s_i]} \Delta U^i_{p,q} \ell^i_{p,q}(x) \\ &\overset{\text{Eq. (21)}}{=} \langle \Delta U^i, \ell^i(x) \rangle_{W^i}. \end{aligned}$$

However, after simple arithmetic, this implies the lemma. □

While Lemma 3 provides a reformulation without including maximizing over an uncertainty set, we would still require to break the piecewise structure of H, which is, at least straight-forwardly, not possible without introducing an exponential number of subproblems or altering the feasible set, both contradicting the approach of oracle-polynomiality. However, we can solve this problem when a so-called assignment-structure is involved.

3.3. **Assignment structures.** In this subsection, we discuss a problem structure for which the Γ-counterpart is solvable in oracle-polynomial time. Throughout this subsection, we assume that, additionally to Assumption 7, that the feasible set $\mathcal{X}$ encodes the following structure:

$$\mathcal{X} \subseteq \{X \in \mathbb{R}^{m\times n} : \ \forall i \in [m] \ \exists! j \in [n] : \ X_{i,j} \neq 0\}. \tag{23}$$

When a set satisfies (23), we say that it is underlying an assignment structure. In this case, we also say that the resp. problem underlies an assignment structure.

Permutation matrices, i.e., the matrices of the set

$$\Pi_n := \{P \in \{0,1\}^{n\times n} : \sum_{i\in[n]} P_{i,j} = 1 \ \forall j \in [n], \ \sum_{j\in[n]} P_{i,j} = 1 \ \forall i \in [n]\},$$

are closely related to assignment structures. Naturally, whenever $\mathcal{X} \subseteq \Pi_n$, $\mathcal{X}$ underlies an assignment structure.

In this section, we discuss the problem

$$\min_{X\in\mathcal{X}} \langle u, Xs \rangle_{\mathbb{R}^{k_1}}$$

where $u \in \mathbb{R}^{k_1}$ is a vector under interval uncertainty, $s \in \mathbb{R}^{k_2}$ is a vector that is not under uncertainty and $X \in \mathcal{X} \subseteq \mathbb{R}^{k_1\times k_2}$ where $\mathcal{X}$ underlies an assignment structure.

In the next theorem, we show that, under these assumptions, one can 'shift' the vector s into the uncertainty set by transforming the inner product of the objective:

Lemma 4. *Let $\Gamma \in [k_1]$, $s \in \mathbb{R}^{k_2}$ and $u \in \mathbb{R}^{k_1}$ be a vector under interval uncertainty, i.e., for some $\bar{u} \in \mathbb{R}^{k_1}$ and $\Delta u \in \mathbb{R}^{k_1}_{\geqslant 0}$, $u_i \in [\bar{u}_i, \bar{u}_i + \Delta u_i]$ for all $i \in [k_1]$. Assume that the set $\mathcal{X} \subseteq \mathbb{R}^{k_1 \times k_2}$ underlies an assignment structure and that Xs is nonnegative for all $X \in \mathcal{X}$. Set*

$$\begin{aligned} \mathcal{U} &:= \bigtimes_{i \in [k_1]} [\bar{u}_i, \bar{u}_i + \Delta u_i], \\ \tilde{\mathcal{U}} &:= \bigtimes_{(i,j) \in [k_1] \times [k_2]} [s_j \bar{u}_i, s_j \bar{u}_i + s_j \Delta u_i]. \end{aligned}$$

Then the Γ-counterparts of

$$\inf_{X \in \mathcal{X}} \langle u, Xs \rangle_{\mathbb{R}^{k_1}}, \tag{24}$$

where $u \in \mathcal{U}$, and

$$\inf_{X \in \mathcal{X}} \langle Y, X \rangle_{\mathbb{R}^{k_1 \times k_2}}, \tag{25}$$

where $Y \in \tilde{\mathcal{U}}$, are equivalent.

Proof. Since Assumption 7 is satisfied, by Lemma 3, the Γ-counterpart of Problem (24) is equivalent to

$$\inf_{l \in [k_1]_0} \left\{ \inf_{X \in \mathcal{X}} \left\{ \Gamma \Delta u_l (Xs)_l + \langle \bar{u}, Xs \rangle_W + \sum_{i \in [k_1]} \max\{0, \Delta u_i (Xs)_i - \Delta u_l (Xs)_l\} \right\} \right\} \tag{26}$$

where $\Delta u_0 := (Xs)_0 := 0$. Since X underlies an assignment structure, we have that

$$\begin{aligned} (Xs)_l &= \sum_{j \in [k_2]} X_{lj} s_j \\ &= X_{l,\sigma(l)} s_{\sigma(l)} \end{aligned}$$

where, for all $l \in [k_2]$, $j =: \sigma(l) \in [n]$ denotes the column index where X_{lj} is not equal to zero. Thus, Problem (26) is equivalent to

$$\inf_{l \in [k_1]_0} \left\{ \inf_{X \in \mathcal{X}} \left\{ \Gamma \Delta u_l X_{l,\sigma(l)} s_{\sigma(l)} + \langle \bar{u}, Xs \rangle_W + \sum_{i \in [k_1]} \max\{0, \Delta u_i X_{i,\sigma(i)} s_{\sigma(i)} - \Delta u_l X_{l,\sigma(l)} s_{\sigma(l)}\} \right\} \right\} \tag{27}$$

where $\sigma(0) := X_{0,0} := 0$. Since $\sigma(l) \in [n]$, Program (27) is equivalent to

$$\inf_{(l_1,l_2) \in K} \left\{ \inf_{X \in \mathcal{X}} \left\{ \Gamma \Delta u_{l_1} X_{l_1,l_2} s_t + \langle \bar{u}, Xs \rangle_W + \sum_{i \in [k_1]} \max\{0, \Delta u_i X_{i,\sigma(i)} s_{\sigma(i)} - \Delta u_{l_1} X_{l_1,l_2} s_{l_2}\} \right\} \right\} \tag{28}$$

where $K := [k_1] \times [k_2] \cup \{(0,0)\}$, $X_{0,0} := 0$ and $s_0 := 0$. By Lemma 3 and since $\langle \bar{u}, Xs \rangle_{\mathbb{R}^{k_1}} = \langle \bar{u} s^\top, X \rangle_{\mathbb{R}^{k_1 \times k_2}}$, Problem (28) is the Γ-counterpart of Problem (25). Finally, the claim follows since $Y_{l_1,l_2} \in [s_{l_2} \bar{u}_{l_1}, s_{l_2} \bar{u}_{l_1} + s_{l_2} \Delta u_{l_1}]$ for all $l_1 \in [k_1]$ and $l_2 \in [k_2]$. □

Lemma 4 states that the Γ-counterpart of a problem subject to vector uncertainty is equivalent to the Γ-counterpart of a problem subject to matrix uncertainty by shifting a vector into the uncertainty set. This does not seem to be of importance at first. However, when it comes to problems underlying an assignment structure and $X \subseteq \{0,1\}^{k_1 \times k_2}$, we finally obtain the desired oracle-polynomiality:

Theorem 8. *Let $\Gamma \in [k_1]$, $s \in \mathbb{R}^{k_2}_{\geqslant 0}$ and $u \in \mathbb{R}^{k_1}$ be a vector under interval uncertainty, i.e., for some $\bar{u} \in \mathbb{R}^{k_1}$ and $\Delta u \in \mathbb{R}^{k_1}_{\geqslant 0}$, $u_i \in [\bar{u}_i, \bar{u}_i + \Delta u_i]$ for all $i \in [k_1]$. Define $\mathcal{U} := \bigtimes_{i \in [k_1]} [\bar{u}_i, \bar{u}_i + \Delta u_i]$ Assume that*

$$\mathcal{X} \subseteq \{\{0,1\}^{k_1 \times k_2} : \sum_{j \in [n]} X_{i,j} = 1 \ \forall i \in [k_1]\}.$$

Assume that $V := \mathbb{R}^{k_1 \times k_2}$ is equipped with an inner product such that the standard matrices form an orthonormalbasis. Then the Γ-counterpart of

$$\min_{X \in \mathcal{X}} \langle u, Xs \rangle_{\mathbb{R}^{k_1}}, \tag{29}$$

where $u \in \mathcal{U}$, is equivalent to

$$\min_{(l_1,l_2) \in [k_1]\times[k_2] \cup \{(0,0)\}} \left\{ \Gamma \Delta u_{l_1} s_{l_2} + \min_{X \in \mathcal{X}} \langle \bar{u} s^\top + Y^{l_1,l_2}, X \rangle_V \right\}$$

where $s_0 := \Delta u_0 := 0$ and, for all $(i,j,l_1,l_2) \in [k_1] \times [k_2] \times ([k_1] \times [k_2] \cup \{0,0\})$

$$Y^{l_1,l_2}_{i,j} := \max\{0, \Delta u_i s_j - \Delta u_{l_1} s_{l_2}\}.$$

Proof. Let $V := \mathbb{R}^{k_1 \times k_2}$. The conditions of Lemma 4 are satisfied, so the Γ-counterpart of Problem (29) is equivalent to the Γ-counterpart of Problem (25). Since the standard matrices of V form an orthonormalbasis, Equation (21) holds. Furthermore, X is a 0/1-matrix and $Y \in \times_{l_1 \in [k_1], l_2 \in [k_2]} [\bar{u}_{l_1} s_{l_2}, \bar{u}_{l_1} s_{l_2} + \Delta u_{l_1} s_{l_2}]$ is subject to interval uncertainty. Thus, we can apply [BS03, Theorem 3]. It implies that the Γ-counterpart of Problem (29) is equivalent to

$$\min_{(l_1,l_2) \in K} \Gamma \Delta u_{l_1} s_{l_2} + \left\{ \min_{X \in \mathcal{X}} \langle \bar{u}, Xs \rangle_W + \sum_{i \in [k_1]} \max\{0, \Delta u_i s_{\sigma(i)} - \Delta u_{l_1} s_{l_2}\} X_{i,\sigma(i)} \right\}, \tag{30}$$

since $Y \in \times_{l_1 \in [k_1], l_2 \in [k_2]} [\bar{u}_{l_1} s_{l_2}, \bar{u}_{l_1} s_{l_2} + \Delta u_{l_1} s_{l_2}]$, $K := [k_1] \times [k_2] \cup \{(0,0)\}$ and $\Delta u_0 s_0 := 0$. Furthermore, since $X \in \mathcal{X}$,

$$\begin{aligned} \sum_{i \in [k_1]} \max\{0, \Delta u_i s_{\sigma(i)} - \Delta u_{l_1} s_{l_2}\} X_{i,\sigma(i)} &= \sum_{i \in [k_1]} \sum_{j \in [k_2]} \max\{0, \Delta u_i s_j - \Delta u_{l_1} s_{l_2}\} X_{i,j} \\ &= \langle Y^{l_1,l_2}, X \rangle_V. \end{aligned}$$

Hence Problem (30) is equivalent to

$$\min_{(l_1,l_2) \in K} \Gamma \Delta u_{l_1} s_{l_2} + \left(\min_{X \in \mathcal{X}} \langle \bar{u} s^\top + Y^{l_1,l_2}, X \rangle_V \right).$$

□

Theorem 8 shows that the Γ-counterparts of problems underlying an assignment structure are oracle-polynomially solvable, when the decision variable is 0/1-valued. This allows us to reformulate the problem without needing to solve an exponential number of subproblems, incorporating piecewise linear functions or modifying the feasible set, cf. Lemma 4. We demonstrate this with the QAP that we model over matrix spaces:

Example 2 continued. We consider the QAP under interval uncertainty

$$\begin{aligned} \min_x \quad & \sum_{(i,j,r,s) \in [n]^4} c_{i,j} d_{r,s} x_{i,r} x_{j,s} \\ \text{s.t. } & x \in \mathcal{X} = \{x \in \{0,1\}^{[n]^2} : \sum_{i \in [n]} x_{i,r} = 1 \ \forall r \in [n], \ \sum_{r \in [n]} x_{i,r} = 1 \ \forall i \in [n]\} \end{aligned} \tag{31}$$

with uncertain coefficients $c_{i,j} \in \mathcal{U}_{i,j} := [\bar{c}_{i,j}, \bar{c}_{i,j} + \Delta c_{i,j}] \subseteq \mathbb{R}_{\geqslant 0}$ for all $i,j \in [n]$ and $d_{r,s} \geqslant 0$ for all $r,s \in [n]$. Let $\Gamma \in [n^2]$.

We begin by reformulating the QAP as a matrix program over the set of permutation matrices:

$$\min_{X \in \Pi_n} \ \mathrm{tr}(CXDX^\top) \tag{32}$$

where $C := (c_{i,j})_{i,j \in [n]}$ and $D := (d_{i,j})_{i,j \in [n]}$. In particular, we can reformulate Problem (32) as a problem over $\mathbb{R}^{n^2 \times n^2}$, where $\langle \cdot, \cdot \rangle$ denotes the Frobenius product:

$$\begin{aligned} \min_Z \ & \langle \mathrm{vec}(C), Z \mathrm{vec}(D) \rangle \\ \text{s.t. } & Z \in \{X \otimes X : \ X \in \Pi_n\}. \end{aligned} \tag{33}$$

Problem (33) is naturally a problem subject to an assignment structure. Thus, we can apply Theorem 8. It implies the Γ-counterpart's equivalence to the problem

$$\min_{\substack{(k_1,k_2,k_3,k_4)\in[n]^4\\ \cup\{(0,0,0,0)\}}} \left\{ \Gamma(\Delta c_{k_1,k_2}) d_{k_3,k_4} + \min_{x\in\mathcal{X}} \{F_{k_1,k_2,k_3,k_4}(x)\} \right\}, \tag{34}$$

where

$$F_{k_1,k_2,k_3,k_4}(x) = \sum_{(i,j,r,s)\in[n]^4} (\bar{c}_{i,j} d_{r,s} + \max\{0, \Delta c_{i,j} d_{r,s} - \Delta c_{k_1,k_2} d_{k_3,k_4}\}) x_{i,r} x_{j,s}$$

for $(k_1,k_2,k_3,k_4) \in [n]^4$, $F_{0,0,0,0}(x) := \sum_{(i,j,r,s)\in[n]^4} (\bar{c}_{i,j} d_{r,s} + \Delta c_{i,j} d_{r,s}) x_{i,r} x_{j,s}$, and $\Delta c_{0,0} := d_{0,0} := 0$. If we assume that the flow and the distance coefficients are symmetric, i.e., $\Delta c_{i,j} = \Delta c_{j,i}$ and $d_{r,s} = d_{s,r}$ for all $i,j,r,s \in [n]$, then we only need to solve subproblems of the set

$$\mathcal{M} := \{(k_1,k_2,k_3,k_4) \in [n]^4 : \; k_1 < k_2, k_3 < k_4\} \cup \{(0,0,0,0)\}.$$

Thus, one needs to solve $1 + (\frac{n(n-1)}{2})^2 = \frac{n^4-n^3}{2} + 1$ QAPs to solve Problem (34). By application of Theorem 5, we can reduce the number of subproblems to $\lceil \frac{n^4-n^3}{4} + \frac{1}{2} - \frac{\Gamma}{2} \rceil + 1$. In the next section, we demonstrate how the application of Theorem 5 significantly speeds up the the optimization process.

4. Numerical study

The problems were implemented in Python 3.7. To solve the optimization problems, we used Gurobi 9.0.1. [Gur20] running on machines with Xeon E3-1240 v5 CPUs (4 cores, 3.5 GHz each).

4.1. **Vehicle routing problem with soft time windows under uncertainty.** We elaborate on Example 1, based on the patient transport problem discussed in [ABLT23] and the vehicle routing program with general time windows (VRPGTW) discussed in [HYI08]. We consider a complete digraph $D = (N, A)$ with nodes $N := [n]$, a start depot 0, a copy of the start depot $n+1$, the digraph $\bar{D} = (V, \bar{A})$ with nodes $V := [n+1]_0$ and arcs

$$\bar{A} = A \cup \{(0,j) : \; j \in N\} \cup \{(i, n+1) : \; i \in N \cup \{0\}\}.$$

With $\delta^{\text{out}}(v)$ and $\delta^{\text{in}}(v)$, we denote the outgoing and the incoming arcs of $v \in V$ in $\bar{D}$. The following data are given: For every arc $a \in \bar{A}$, the travel time is modeled as $t_a \in \mathbb{R}_{\geqslant 0}$ and for every node $i \in N$, a service time $s_i \in \mathbb{R}_{\geqslant 0}$ and a soft due time $b_i \in \mathbb{R}_{\geqslant 0}$ is given. Given a homogeneous fleet of K vehicles, all nodes $i \in N$ have to be 'visited' by exactly one vehicle exactly once and with as little delay as possible. The vehicles start and end at the depot. In the following, the binary variables $x^k_{i,j} \in \{0,1\}$ for $(i,j) \in \bar{A}$ and $k \in [K]$ denote whether vehicle k 'uses' arc (i,j) and the real variables $T_i \in \mathbb{R}_{\geqslant 0}$, $i \in V$, denote the arrival time of a vehicle at node i. With this notation, we obtain the following formulation of the VRPGTW:

$$\min_{x,T} \sum_{i\in N} \max\{0, T_i - b_i\} \tag{35a}$$

$$\text{s.t.} \sum_{k\in[K]} \sum_{(i,j)\in\delta^{\text{out}}(i)} x^k_{i,j} = 1 \;\forall i \in N, \tag{35b}$$

$$\sum_{(0,j)\in\delta^{\text{out}}(0)} x^k_{0,j} = 1 \;\forall k \in [K], \tag{35c}$$

$$\sum_{(i,j)\in\delta^{\text{in}}(j)} x^k_{i,j} - \sum_{(j,i)\in\delta^{\text{out}}(j)} x^k_{j,i} = 0 \;\forall k \in [K], j \in N, \tag{35d}$$

$$\sum_{(i,n+1)\in\delta^{\text{in}}(0)} x^k_{i,0} = 1 \;\forall k \in [K], \tag{35e}$$

$$x^k_{i,j}(T_i + s_i + t_{i,j} - T_j) \leqslant 0 \;\forall k \in [K], \; (i,j) \in A, \tag{35f}$$

$$x^k_{i,j} \in \{0,1\} \;\forall k \in [K], \; (i,j) \in A, \tag{35g}$$

$$T_i \geqslant 0 \;\forall i \in V. \tag{35h}$$

Constraint (35b) ensures that each $i \in N$ is served exactly once by exactly one vehicle. Constraints (35c) and (35e) ensure that each vehicle leaves and enters the depot or stays at the depot.

In combination with constraints (35b), (35c) and (35e), constraint (35d) ensures that each node is served exactly once and by exactly one vehicle. Constraint (35f) ensures that, if vehicle k serves node j after node i, the arrival time T_j is at least as large as the arrival time T_i added to the time it requires for serving i and going from i to j. Finally, (35g) and (35h) ensure that x is binary and T is nonnegative. Note that this formulation is only one of many possibilities to formulate vehicle routing problems – for an overview, we refer the reader to [MS20]. We attempt to be robust against scenarios of the set $\times_{i\in N}[\bar{b}_i - \Delta b_i, \bar{b}_i]$. Solving the Γ-counterpart for all $\Gamma \in [m]$ would show how many shifts of the due times are possible without any (or only little) delay.
For our experiments we use the Solomon instances r101, r102, c101, c102, rc101 and rc102. If these names begin with r, the nodes are generated randomly, if they begin with c, they are clustered, and otherwise some nodes are generated randomly and some are clustered - for a detailed description of the construction, see [Sol87]. As due time b_i we chose the start time specified in the original instance for the customer, i.e., node i. The uncertainty set was constructed randomly, i.e., Δb_i is a uniformly distributed random variable in $[0, \bar{b}_i]$. Since we were ultimately aiming to find optimal solutions for the Γ-counterpart, we tested $N = [8]$ and $N = [10]$, $K \in [3]$ and all $\Gamma \in N$, and calculated the optimal solutions for the respective nominal problem. We selected the first $|N|$ customers of the list given in the resp. instance.

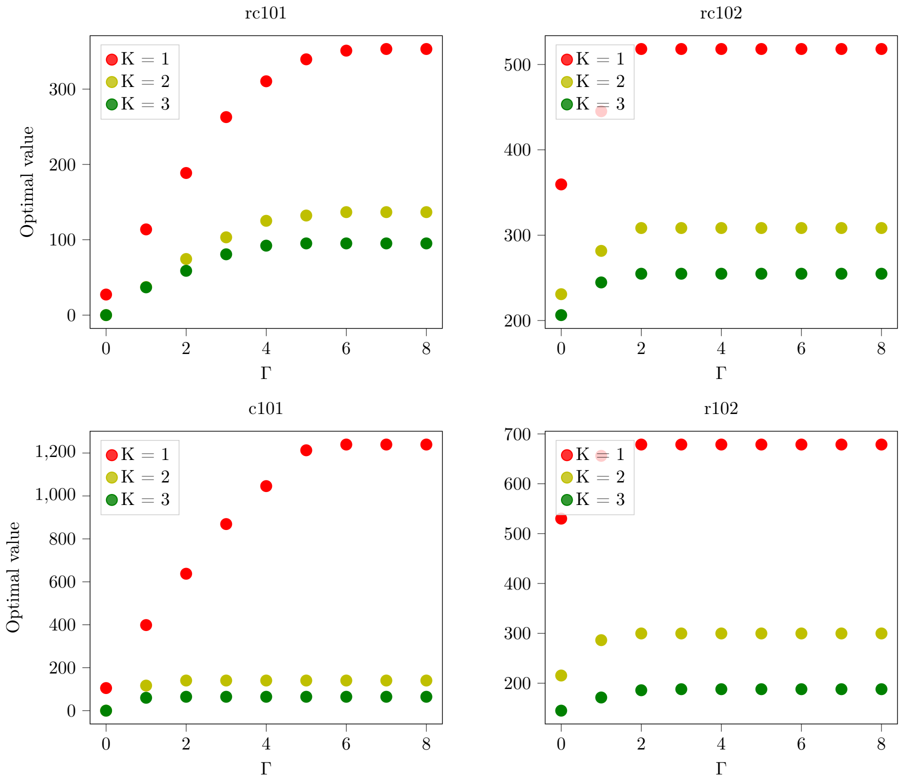


FIGURE 1. Optimal values for the respective Γ-counterparts for instances rc101 (upper left), rc102 (upper right), c101 (lower left) and r102 (lower right), with $N = [8]$ and $K \in [3]$. If a yellow point for a value of Γ is 'missing', its value coincides with the green point of the same Γ.

In Figure 1 we show that the robust optimal values for $N = [8]$, $K \in [3]$ and the instances rc101, rc102, c101 and `r102` (we have neglected the other two cases and the results for $N = [10]$ because the graphs are similar). As expected, the optimal value, i.e., the cumulated delays, increases with an increasing number of vehicles K. In addition, at $K = 1$ the optimal value for increasing Γ strongly rises, while at $K = 2, 3$ the change in the optimal value is not so marked. This is also to be expected: If there is exactly one vehicle, the changes in the due times are supposed to be met by this one vehicle, which is clearly not really possible, especially in clustered settings. However, the total delays are more robust for $K = 2, 3$ – while the robust values differ between $K = 1, 2, 3$, the

difference between $K = 1$ and $K = 2$ is much higher than in $K = 2$ and $K = 3$. So if more vehicles are available, this can lead to more robust solutions. The difference in the price of robustness is evident, e.g., in c101: For $K = 1$ the nominal optimal value is less than 200 and for $K = 2, 3$ it is 0. For $\Gamma = 1$ and $K = 1$ we obtain a delay of at least 400, while for $K = 2, 3$ we remain around the optimal nominal value for $K = 1$. For $\Gamma = 2$ the delay increases only slightly and does not change afterwards. However, for $K = 1$, the optimum value increases up to $\Gamma = 6$ and is above 1200, while for $K = 2, 3$ the optimum value is below 200. We note that for other cases, the difference between the nominal optimal values and the optimal values for $\Gamma \geqslant 1$ is not as large as can be seen in r102. In this particular case, the increase in nominal optimal values for Γ stopped at $\Gamma = 2$ for all $K = 1, 2, 3$.
Table 1 and Table 2 show the running time (in seconds) to solve the Γ-counterpart for $\Gamma \in [2]$, the nominal problem for all instances with $N = [8], [10]$ and $K \in [3]$. As the number of constraints increases with more customers and more vehicles, i.e., increasing $|N|$ and K, the running time increases in most cases. Note that when reformulating the Γ-counterpart, only the objective of the subproblems (19) will be affected, while the optimal solutions of the other subproblems can be reused. Thus, of the $2|N| + 1$ problems, only $|N|$ problems need to be solved to obtain an optimal solution of the Γ counterpart when different values of Γ are considered. This explains the fact that the running time for $\Gamma = 2$ is usually at most half as large as that for $\Gamma = 1$. We note that the value of Γ does not have any other significant influence on the running time and that the running times are relatively high, especially for $|N| = 10$.

Table 1. Running times of various instances in seconds for $\Gamma = 1, 2$, the nominal case, $N = [8]$ and $K = 1, 2, 3$.

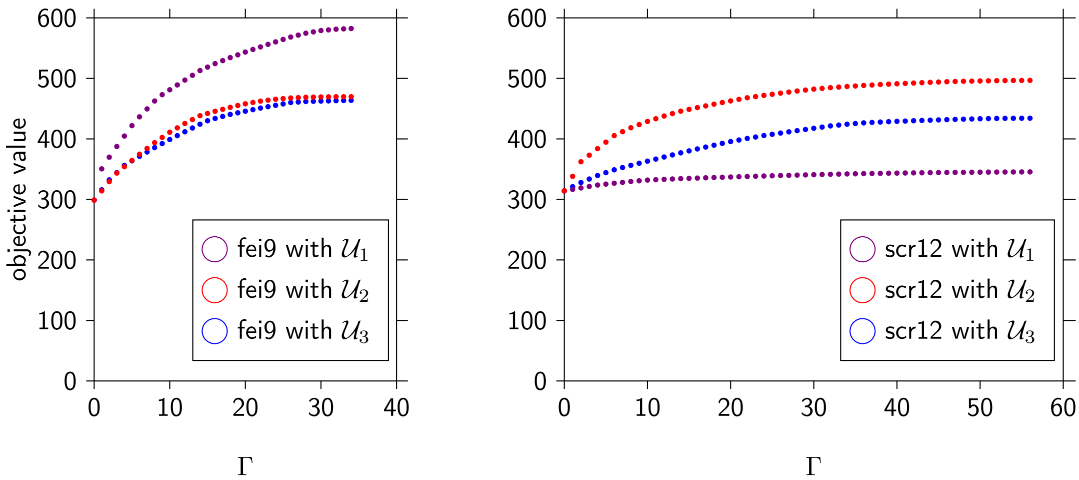
Instances:	Nominal case			$\Gamma = 1$			$\Gamma = 2$		
$N = [8]$	$K = 1$	$K = 2$	$K = 3$	$K = 1$	$K = 2$	$K = 3$	$K = 1$	$K = 2$	$K = 3$
r101	17	1	2	79	43	26	40	15	14
r102	11	22	15	105	356	246	49	136	123
c101	1	1	1	28	7	8	19	4	5
c102	10	21	24	124	394	434	62	183	298
rc101	5	1	2	107	156	224	30	19	22
rc102	7	69	291	101	823	2965	49	284	1282

Table 2. Running times of various instances in seconds for $\Gamma = 1, 2$, the nominal case, $N = [10]$ and $K = 1, 2, 3$. If no optimal solution has been obtained after 24 hours, the resp. fields are marked with –.

Instances:	Nominal case			$\Gamma = 1$			$\Gamma = 2$		
$N = [10]$	$K = 1$	$K = 2$	$K = 3$	$K = 1$	$K = 2$	$K = 3$	$K = 1$	$K = 2$	$K = 3$
r101	551	6	3	9080	5178	2316	5612	1525	376
r102	1175	1066	119	15358	54885	15637	9433	18285	5875
c101	15	2	5	8026	2444	168	4161	65	28
c102	1566	431	111	22081	32098	21335	15285	12692	12703
rc101	681	14	7	8279	11024	9364	2949	1253	350
rc102	902	2505	8425	13327	70041	–	6804	37788	–

This concludes our numerical study of the VRPGTW under uncertainty. As already mentioned, we used an optimization oracle to solve the problems given in Example 3.1 as a MINLP instead of using any VRPGTW solvers to demonstrate that our reformulation can be solved to optimality. In the future, it might be interesting to conduct experiments including instances with more customers but rather than solving them to global optimality, they could be solved only to a certain gap, i.e., to find solutions which are 'sufficiently robust' or to apply a VRPGTW oracle.

4.2. **Quadratic assignment problem under uncertainty.** Here, we solve and compare different reformulations of the Γ-counterpart of the QAP. We have chosen instances from [FF15] and from the QAPLIB [BKR97]. The goal of this section is to prototypically evaluate whether the new reformulations can be solved within a similar order of magnitude when compared to that of the nominal versions. As we do not have an efficient problem-specific QAP oracle at hand, we chose small instances where the Γ-counterpart could be solved with Gurobi within 24 hours. As expected, instances with less uncertain coefficients are computationally easier to handle. Therefore, by choosing scr12, we included an instance with $c_{i,j} = 0$ for some $(i,j) \in [12]^2$. We also chose fei9, an instance that was examined in [FF15]. For fei9, the number of facilities is $n = 9$, while for scr12, it is $n = 12$ (both taken from [FF15]). Finally, we also chose nug12 from [BKR97]. For each instance, we generated three different uncertainty sets. For fei9, the uncertainty set $\mathcal{U}_1$ is taken from [FF15]. Other uncertainty sets, denoted by $\mathcal{U}_2$ and $\mathcal{U}_3$, are generated randomly: for all $(i,j) \in [n]^2$, $\Delta c_{ij} \in [0, \bar{c}_{ij}]$ is randomly chosen. For scr12 and nug6, $\mathcal{U}_1$ is generated by setting $\Delta c_{ij} = 0.1\bar{c}_{ij}$ for all $(i,j) \in [n]^2$. Furthermore, $\mathcal{U}_2$ and $\mathcal{U}_3$ are generated randomly analogously to fei9. In Figure 2 the change in the objective value for different Γ can be observed for two of our instances are shown. As expected, the optimal objective value is increasing in Γ. As can be seen for scr12, only a mild increase in cost of robust protection can be seen for increasing values of Γ.



(A) Optimal objective value $\times 10^{-4}$ for different values of Γ for fei9.

(B) Optimal objective value $\times 10^{-2}$ for different values of Γ for scr12.

FIGURE 2. Optimal values for different instances.

Now we compare the running time of different equivalent formulation of the Γ-counterpart. In particular, we test following formulations:

- `QAP`: Formulation (34).
- $\texttt{QAP}_{\texttt{red}}$: Formulation (34) after reducing the number of subproblems with Theorem 5.
- `MIP`: A linearized QAP under uncertainty after applying Theorem 1 of [BS03].
- `BP`: A linearized QAP under uncertainty after applying Theorem 3 of [BS03].
- $\texttt{BP}_{\texttt{red}}$: A linearized QAP under uncertainty after applying the original result of Bertsimas and Sim discussed in Section 1 and Theorem 1 of [LK14].

In particular, we apply a standard linearization technique: Whenever there occurs an product of two binary variables x and y, it can be replaced by a binary variable z when adding the inequalities

$$z \leqslant x, \ z \leqslant y, \ z \geqslant x + y - 1.$$

The nominal problems can be solved within a few seconds. A comparison of running times for fei9, scr12 and nug12 and $\Gamma = 1$ can be found in Tables 3, 4 and 5. If no optimal solution could be computed after 24 hours, we stopped the process. Running times are measured in seconds.

TABLE 3. Comparison of running times for different instances for $\Gamma = 1$ and the deterministically constructed uncertainty set $\mathcal{U}_1$. – if not solvable within 24 hours.

CPU (s)	nug12	fei9	scr12
QAP	–	3420	36579
$\texttt{QAP}_{\texttt{red}}$	1054	174	221
MIP	31417	25	1718
BP	–	86243	–
$\texttt{BP}_{\texttt{red}}$	34318	4096	49126

TABLE 4. Comparison of running times for different instances for $\Gamma = 1$ and the deterministically constructed uncertainty set $\mathcal{U}_2$. – if not solvable within 24 hours.

CPU (s)	nug12	fei9	scr12
QAP	–	3542	74641
$\texttt{QAP}_{\texttt{red}}$	591	178	298
MIP	24655	25	819
BP	–	–	–
$\texttt{BP}_{\texttt{red}}$	–	4538	73847

TABLE 5. Comparison of running times for different instances for $\Gamma = 1$ and the randomly constructed uncertainty set $\mathcal{U}_3$. - if not solvable within 24 hours.

CPU (s)	nug12	fei9	scr12
QAP	–	3503	45851
$\texttt{QAP}_{\texttt{red}}$	9507	178	275
MIP	19550	30	1860
BP	–	–	–
$\texttt{BP}_{\texttt{red}}$	–	4384	56979

It is evident that the instances with $n = 12$ can be solved more efficiently than the linearizations after reducing the number of subproblems by excluding all redundant scenarios (applying Theorem 5, neglecting identical subproblems and taking symmetry of coefficients into account), for all regarded uncertainty sets. Only for the smaller instance, MIP is faster. This demonstrates the benefit of the reformulations proposed here. Without using them, the corresponding robust counterparts are algorithmically very challenging. All instances have in common that without reducing the number of problems, i.e., avoiding a repetition of scenarios or applying Theorem 5, these instances cannot be solved within the time limit, even for smaller instances.

Finally, we would like to point out two things: Firstly, if one would like to solve Γ-counterpart for different values of Γ, it is preferable to apply $\texttt{QAP}_{\texttt{red}}$ since one only has to calculate the optimal solutions of the subproblems for $\Gamma = 1, 2$, since the value of Γ does not influence the subproblems. Secondly, this computational study demonstrates that our formulations are applicable in practice. Naturally, instead of using Gurobi, one can also use algorithms that solve QAPs more efficiently. However, for our purposes, our method proved to be highly beneficial, when compared to the standard linearization approach.

5. CONCLUSION

In this article, we studied Γ-counterparts of discrete nonlinear optimization problems under uncertainty in the objective. We established reformulations of Γ-counterparts by applying reformulations techniques developed in [BTdHV15]. Similar to Γ-uncertainties in [BS03] and [BS04], our reformulations work for MINLPs in general and for combinatorial optimization problems with linear uncertainty when attempting to optimize over the original feasible set $\mathcal{X}$. Although these reformulations are not necessarily computationally tractable, we have given examples where this is indeed the case, namely for linear uncertainties with 0/1 functions and optimization problems subject to an assignment structure. In addition, we have discussed the general case with an application in logistics and have deepened it with a problem where transports are subject to a soft due date. In a prototypical numerical study, we have shown that our reformulations work in practice and are efficient. Possible further research for this topic include, naturally, more extensive numerical studies for the derived reformulations. Furthermore, one could also investigate whether the generalizations of [Pos13], [Pos14] and [Pos18] to the proposed Γ-counterpart are possible and tractable as well. Furthermore, the price of robustness, as introduced in [BS04], was not investigated in this publication as it was in [BS04] which could also be subject to future research.

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