

Dual bounds for periodical stochastic programs

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Abstract. In this paper we discuss construction of the dual of a periodical formulation of infinite horizon linear stochastic programs with a discount factor. The dual problem is used for computing a deterministic upper bound for the optimal value of the considered multistage stochastic program. Numerical experiments demonstrate behavior of that upper bound especially when the discount factor is close to one.

Key Words: multistage programs, dynamic programming, Bellman equations, linear programming duality, SDDP algorithm, decision rules

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1 Introduction

To solve, even linear, multistage stochastic programs is difficult and in a generic sense could be computationally intractable [12]. In some settings the considered multistage problems have a periodical behavior. In such cases it was suggested in [11] to use a periodical variant of the Wald-Bellman (WB) equations for the corresponding infinite horizon problem with a discount factor $\gamma \in (0, 1)$. Moreover a cutting plane - Stochastic Dual Dynamic Programming (SDDP) type - algorithm can be applied to approximate the value functions defined by these WB equations. The statistical upper bound of that algorithm is constructed by employing T -stage approximation of the considered infinite horizon problem. The error of that approximation is of the order $O(\gamma^T/(1-\gamma))$, which can be large when the discount factor is close to one even for reasonably large T . This motivates to consider a different approach to construction of a valid upper bound based on a dual formulation of the considered linear multistage program. An idea of such upper bounds was coined in [7]. In this paper we follow an approach to construction of the dual upper bound developed in [6].

2 Periodical multistage stochastic programs

Consider the multistage linear stochastic program

$$\begin{aligned} \min_{x_t \geq 0} \quad & \mathbb{E} \left[\sum_{t=1}^T \gamma^{t-1} c_t^\top x_t \right] \\ \text{s.t.} \quad & A_1 x_1 = b_1, \\ & B_t x_{t-1} + A_t x_t = b_t, \quad t = 2, \dots, T. \end{aligned} \tag{2.1}$$

Here vectors $c_t = c_t(\xi_t) \in \mathbb{R}^{n_t}$, $b_t = b_t(\xi_t) \in \mathbb{R}^{m_t}$ and matrices $B_t = B_t(\xi_t)$, $A_t = A_t(\xi_t)$ are functions of random process $\xi_t \in \mathbb{R}^{d_t}$, $t = 1, \dots, T$, and $\gamma \in (0, 1)$ is the discount factor. We denote by $\xi_{[t]} = (\xi_1, \dots, \xi_t)$ the history of the data process up to time t and by $\mathbb{E}_{|\xi_{[t]}}$ the corresponding conditional expectation. The optimization in (2.1) is performed over functions (policies) $x_t = x_t(\xi_{[t]}) \in \mathbb{R}^{n_t}$, $t = 1, \dots, T$, of the data process satisfying the feasibility constraints. Vector ξ_1 and the first stage solution x_1 are deterministic, i.e., the first stage decision is made before knowing (observing) realizations of the data process ξ_2, \dots, ξ_T .

- We assume that the data process is *stagewise independent*, i.e., random vector ξ_{t+1} is independent of $\xi_{[t]}$, and that each ξ_t has a discrete distribution with finite support $\Xi_t = \{\xi_t^1, \dots, \xi_t^{N_t}\}$, with respective probabilities p_{tj} , $j = 1, \dots, N_t$, $t = 2, \dots, T$.

When the number T of stages is finite, it is possible to write the following dynamic programming equations for problem (2.1) (cf., [10]). At stages $t = T, \dots, 2$, the value function $Q_t(x_{t-1}, \xi_t)$ is given by the optimal value of the problem

$$\begin{aligned} \min_{x_t \geq 0} \quad & c_t^\top x_t + \gamma Q_{t+1}(x_t) \\ \text{s.t.} \quad & B_t x_{t-1} + A_t x_t = b_t, \end{aligned} \tag{2.2}$$

with

$$Q_{t+1}(x_t) = \mathbb{E}[Q_{t+1}(x_t, \xi_{t+1})] \tag{2.3}$$

and $\mathcal{Q}_{T+1}(\cdot) \equiv 0$. At the first stage the following problem should be solved

$$\begin{aligned} \min_{x_1 \geq 0} \quad & c_1^\top x_1 + \gamma \mathcal{Q}_2(x_1) \\ \text{s.t.} \quad & A_1 x_1 = b_1. \end{aligned} \tag{2.4}$$

Consider now the infinite horizon setting of $T = \infty$. Following [11] we make the following assumptions about periodical behavior, with period $\mathbf{m} \in \mathbb{N}$, of problem (2.1).

- (A1) The random vectors ξ_t and $\xi_{t+\mathbf{m}}$ have the same distribution for $t \geq 2$ (recall that ξ_1 is deterministic).
- (A2) The sequence of functions $b_t(\cdot)$, $B_t(\cdot)$, $A_t(\cdot)$ and $c_t(\cdot)$ has period \mathbf{m} , i.e., these functions are the same for $t = \tau$ and $t = \tau + \mathbf{m}$, $t \geq 2$.

We also make the following assumptions ensuring the relatively complete recourse and boundedness properties of problem (2.1).

- (A3) For every $x_{t-1} \geq 0$ the set $\{x_t : B_t(\xi_t)x_{t-1} + A_t(\xi_t)x_t = b_t(\xi_t), x_t \geq 0\}$ is nonempty for all $\xi_t \in \Xi_t$ and $t \geq 2$.
- (A4) There exist bounded sets $\mathcal{X}_t \subset \mathbb{R}^{n_t}$ such that adding the constraints $x_t \in \mathcal{X}_t$, $t = 1, \dots, T$, to the problem (2.1) does not change its optimal value.

In applications the sets \mathcal{X}_t typically are sufficiently large boxes containing the considered decision variables. Under these assumptions the value functions $Q_t(\cdot, \cdot)$ and $Q_{t+\mathbf{m}}(\cdot, \cdot)$, and the expected value functions $\mathcal{Q}_t(\cdot)$ and $\mathcal{Q}_{t+\mathbf{m}}(\cdot)$, are the same for all $t \geq 2$, in particular $\mathcal{Q}_{\mathbf{m}+2}(\cdot) \equiv \mathcal{Q}_2(\cdot)$. This leads to the following periodical variant of Wald-Bellman (WB) equations for the value functions (cf., [1]):

$$\mathcal{Q}_\tau(x_{\tau-1}) = \mathbb{E}[Q_\tau(x_{\tau-1}, \xi_\tau)], \tag{2.5}$$

with

$$Q_\tau(x_{\tau-1}, \xi_\tau) = \inf_{x_\tau \geq 0} \{c_\tau^\top x_\tau + \gamma \mathcal{Q}_{\tau+1}(x_\tau) : B_\tau x_{\tau-1} + A_\tau x_\tau = b_\tau\}, \tag{2.6}$$

for $\tau = 2, \dots, \mathbf{m} + 1$, and $\mathcal{Q}_{\mathbf{m}+2}$ replaced by \mathcal{Q}_2 for $\tau = \mathbf{m} + 1$. It is possible to show that there exists a unique set of value functions $Q_\tau(x_{\tau-1}, \xi_\tau)$, $\mathcal{Q}_\tau(x_{\tau-1})$, $\tau = 2, \dots, \mathbf{m} + 1$, satisfying these WB equations and that these value functions are convex in $x_{\tau-1}$ (cf., [11]). The first stage solution is obtained by solving problem (2.4) with $\mathcal{Q}_2(\cdot)$ being the solution of these WB equations.

In order to solve the WB equations (2.5) - (2.6) a cutting plane algorithm was suggested in [11]. That algorithm can be viewed as a variant of the Stochastic Dual Dynamic Programming (SDDP) method introduced in Pereira and Pinto [8] based on the nested cutting plane method of Birge [2]. An upper bound for the optimal value in that algorithm is based on a statistical estimate of the value of the current iterate approximation of the optimal policy. When the discount factor γ is close to one, the convergence is slow and the computational effort to reduce the optimality gap becomes prohibitive. This motivates development of dual upper bounds which we discuss in the next section.

3 Dual bounds

The Lagrangian of problem (2.1) is

$$L(x, \pi) = \mathbb{E} \left[\sum_{t=1}^T \gamma^{t-1} c_t^\top x_t + \pi_t^\top (b_t - B_t x_{t-1} - A_t x_t) \right] \quad (3.1)$$

in variables $x = (x_1(\xi_{[1]}), \dots, x_T(\xi_{[T]}))$ and $\pi = (\pi_1(\xi_{[1]}), \dots, \pi_T(\xi_{[T]}))$ with the convention that $x_0 = 0$. Dualization of the feasibility constraints leads to the following dual of problem (2.1) (cf., [10, Section 3.2.3]):

$$\begin{aligned} \max_{\pi} \quad & \mathbb{E} \left[\sum_{t=1}^T b_t^\top \pi_t \right] \\ \text{s.t.} \quad & A_T^\top \pi_T \leq \gamma^{T-1} c_T, \\ & A_{t-1}^\top \pi_{t-1} + \mathbb{E}_{|\xi_{[t-1]}} [B_t^\top \pi_t] \leq \gamma^{t-2} c_{t-1}, \quad t = 2, \dots, T. \end{aligned} \quad (3.2)$$

The optimization in (3.2) is over policies $\pi_t = \pi_t(\xi_{[t]})$, $t = 1, \dots, T$. Note that in the considered framework of finite number of scenarios, problem (2.1) can be viewed as a large linear program and problem (3.2) as its dual. By the theory of linear programming we have that the optimal values of primal problem (2.1) and its dual (3.2) are equal to each other.

It is convenient for the subsequent analysis to make change of variables $\lambda_t = \gamma^{-(t-1)} \pi_t$, $t = 1, \dots, T$, in order to remove the powers of γ in the right hand sides of the feasibility constraints. In terms of variables λ_t problem (3.2) can be written as

$$\begin{aligned} \max_{\lambda} \quad & \left\{ \mathbb{E} \left[\sum_{t=1}^T \gamma^{t-1} b_t^\top \lambda_t \right] = b_1^\top \lambda_1 + \gamma \mathbb{E}_{|\xi_1} [b_2^\top \lambda_2 + \dots + \gamma \mathbb{E}_{|\xi_{[T-1]}} [b_T^\top \lambda_T]] \right\} \\ \text{s.t.} \quad & A_T^\top \lambda_T \leq c_T, \\ & A_{t-1}^\top \lambda_{t-1} + \gamma \mathbb{E}_{|\xi_{[t-1]}} [B_t^\top \lambda_t] \leq c_{t-1}, \quad t = 2, \dots, T. \end{aligned} \quad (3.3)$$

Recall that the process ξ_1, \dots, ξ_T is assumed to be stagewise independent, and distribution of ξ_t has a finite support, $\Xi_t = \{\xi_t^1, \dots, \xi_t^{N_t}\}$, with respective probabilities p_{tj} , $j = 1, \dots, N_t$, $t = 2, \dots, T$. We denote by $A_t^j, B_t^j, c_t^j, b_t^j$ the respective scenarios corresponding to ξ_t^j .

We can write the following dynamic programming equations for the dual problem (3.3) (cf., [6]). At the last stage $t = T$, given λ_{T-1} and $\xi_{[T-1]}$, we need to solve the following problem with respect to λ_T :

$$\begin{aligned} \max_{\lambda_T} \quad & \mathbb{E}[b_T^\top \lambda_T] \\ \text{s.t.} \quad & A_T^\top \lambda_T \leq c_T, \\ & A_{T-1}^\top \lambda_{T-1} + \gamma \mathbb{E} [B_T^\top \lambda_T] \leq c_{T-1}. \end{aligned} \quad (3.4)$$

In terms of scenarios the above problem can be written as

$$\begin{aligned} \max_{\lambda_{T1}, \dots, \lambda_{TN_T}} \quad & \sum_{j=1}^{N_T} p_{Tj} (b_T^j)^\top \lambda_{Tj} \\ \text{s.t.} \quad & A_{Tj}^\top \lambda_{Tj} \leq c_{Tj}, \quad j = 1, \dots, N_T, \\ & A_{T-1}^\top \lambda_{T-1} + \gamma \sum_{j=1}^{N_T} p_{Tj} (B_T^j)^\top \lambda_{Tj} \leq c_{T-1}. \end{aligned} \quad (3.5)$$

The optimal value $V_T(\lambda_{T-1}, \xi_{T-1})$ and an optimal solution $(\bar{\lambda}_{T1}, \dots, \bar{\lambda}_{TN_T})$ of problem (3.5) are functions of vectors λ_{T-1} and c_{T-1} and matrix A_{T-1} . And so on going backward in time, using the stagewise independence assumption, we can write the respective dynamic programming equations for $t = T - 1, \dots, 2$, as

$$\begin{aligned} \max_{\lambda_{t1}, \dots, \lambda_{tN_t}} \quad & \sum_{j=1}^{N_t} p_{tj} [(b_t^j)^\top \lambda_{tj} + \gamma V_{t+1}(\lambda_{tj}, \xi_{tj})] \\ \text{s.t.} \quad & A_{t-1}^\top \lambda_{t-1} + \gamma \sum_{j=1}^{N_t} p_{tj} (B_t^j)^\top \lambda_{tj} \leq c_{t-1}, \end{aligned} \quad (3.6)$$

with $V_t(\lambda_{t-1}, \xi_{t-1})$ being the optimal value of problem (3.6). Note that unlike the primal problem, the dual dynamic equations do not decompose into individual scenarios - the optimization problem (3.6) is formulated jointly with respect to the dual variables $\lambda_{t1}, \dots, \lambda_{tN_t}$.

Finally at the first stage the following problem should be solved

$$\max_{\lambda_1} b_1^\top \lambda_1 + \gamma V_2(\lambda_1). \quad (3.7)$$

Note that if A_t and c_t are deterministic, then $V_{t+1}(\lambda_t)$ does not depend on ξ_t .

Suppose now that the costs of the primal and dual problems are bounded and consider the infinite horizon case $T = \infty$. Suppose further that the problem is periodical with period m , i.e., the assumptions (A1) - (A3) hold. Consider first the case of $m = 1$. In that case the random process ξ_t is iid with the corresponding scenarios $\xi^j = (A^j, B^j, c^j, b^j)$, $j = 1, \dots, N$, which do not depend on $t \geq 2$. The WB equations for the value function $V(\lambda_j, \xi^j)$ of the dual problem then become

$$V(\lambda, \xi^j) = \sup_{\lambda_1, \dots, \lambda_N} \left\{ \sum_{k=1}^N p_k [(b^k)^\top \lambda_k + \gamma V(\lambda_k, \xi^k)] : (A^j)^\top \lambda + \gamma \sum_{k=1}^N p_k (B^k)^\top \lambda_k \leq c^j \right\}, \quad (3.8)$$

$j = 1, \dots, N$. Note that solution $V(\lambda, \xi^j)$ of this equation is concave in λ , and if $A^j \equiv A$ and $c^j \equiv c$ are deterministic, then $V(\lambda)$ does not depend on ξ^j and is given by the equation

$$V(\lambda) = \sup_{\lambda_1, \dots, \lambda_N} \left\{ \sum_{k=1}^N p_k [(b^k)^\top \lambda_k + \gamma V(\lambda_k)] : A^\top \lambda + \gamma \sum_{k=1}^N p_k (B^k)^\top \lambda_k \leq c \right\}. \quad (3.9)$$

Consider the general case of $m \geq 1$. Then the WB equations for the value functions of the dual problem are

$$\begin{aligned} V_\tau(\lambda_{\tau-1}, \xi_{\tau-1}^j) = \sup_{\lambda_{\tau1}, \dots, \lambda_{\tau N_\tau}} \left\{ \sum_{k=1}^{N_\tau} p_{\tau k} [(b_\tau^k)^\top \lambda_{\tau k} + \gamma V_{\tau+1}(\lambda_{\tau k}, \xi_\tau^k)] : \right. \\ \left. (A_{\tau-1}^j)^\top \lambda_{\tau-1} + \gamma \sum_{k=1}^{N_\tau} p_{\tau k} (B_\tau^k)^\top \lambda_{\tau k} \leq c_{\tau-1}^j \right\}, \end{aligned} \quad (3.10)$$

for $\tau = 2, \dots, m + 1$, and V_{m+2} replaced by V_2 . If $A_\tau^j \equiv A_\tau$ and $c_\tau^j \equiv c_\tau$, $\tau = 2, \dots, m + 1$, are deterministic, then value functions do not depend on A_τ and c_τ and are given by equations

$$V_\tau(\lambda_{\tau-1}) = \sup_{\lambda_{\tau1}, \dots, \lambda_{\tau N_\tau}} \left\{ \sum_{k=1}^{N_\tau} p_{\tau k} [(b_\tau^k)^\top \lambda_{\tau k} + \gamma V_{\tau+1}(\lambda_{\tau k})] : A_{\tau-1}^\top \lambda_{\tau-1} + \gamma \sum_{k=1}^{N_\tau} p_{\tau k} (B_\tau^k)^\top \lambda_{\tau k} \leq c_{\tau-1} \right\}, \quad (3.11)$$

for $\tau = 2, \dots, m + 1$, and V_{m+2} replaced by V_2 .

Under the specified assumptions, there is no duality gap between the primal and dual problems also in the case of the infinite number of stages. Indeed, the difference between the optimal values of problem (2.1) for the infinite and finite number T of stages can be bounded by $\kappa\gamma^T/(1 - \gamma)$, where $\kappa > 0$ is a constant ensured by assumption (A4). As T tends to ∞ this difference tends to zero, and also the optimal value of the dual problem (3.3) tends to the optimal value of its counterpart for the infinite number of stages. By invoking the no duality gap property of (finite dimensional) linear programs and going to the limit as $T \rightarrow \infty$ completes the arguments.

4 Numerical experiments

In this section we report empirical results of applying variants of the periodical SDDP algorithm to the inventory model and the Brazilian Inter-connected Power System problem, with different discount factors. Details of implementation of the SDDP type algorithm to the primal problem are discussed in [11]. In a similar way, an SDDP type cutting plane algorithm is applied to the dual problem, details are given in the Appendix. It could be mentioned that although the Relatively Complete Recourse (RCR) property for the primal problem is guaranteed by assumption (A3), RCR does not necessarily hold for the dual problem. As suggested in [6], this is dealt with by introducing a certain penalty term in formulation of the dual optimization problems, we discuss this further in section 5.1 of the Appendix.

In both models, convergence is measured by relative gap computed by deterministic upper bound of the dual problem and deterministic lower bound of the primal problem (see (4.4) below). Both implementations were written in Python 3 using Dual SDDP solver `dualsddp`, <https://github.com/starrycheng/dualsddp>, which is developed from MSPPy described in [3].

4.1 Inventory model

Consider the classical inventory model (cf., [14])

$$\begin{aligned} \min_{y_t \geq x_{t-1}} \quad & \mathbb{E} \left[\sum_{t=1}^T \gamma^{t-1} (c_t(y_t - x_{t-1}) + b_t[D_t - y_t]_+ + h_t[y_t - D_t]_+) \right] \\ \text{s.t.} \quad & x_t = y_t - D_t, \quad t = 1, \dots, T, \end{aligned} \quad (4.1)$$

where c_t, b_t, h_t are ordering cost, backorder penalty cost and holding cost per unit, respectively and $[\cdot]_+ := \max\{\cdot, 0\}$. Here x_t denotes the current inventory level, in particular x_0 denotes the initial level, $y_t - x_{t-1}$ represents the order quantity at stage t , and D_1, \dots, D_T is the demand process. We assume that the demand process is stagewise independent and has periodical behavior specified in the respective assumptions (A1) - (A2).

In order to formulate model (4.1) as a linear programming problem and hence to construct its dual, we proceed as follows. An equivalent formulation of (4.1) is to replace $[D_t - y_t]_+$ and $[y_t - D_t]_+$ with $w_t \geq 0$ and $v_t \geq 0$, respectively, and simultaneously to add feasibility

constraints $y_t + w_t \geq D_t$ and $y_t - v_t \leq D_t$. The Lagrangian of this problem then becomes

$$L(p, d) = \mathbb{E} \left[\sum_{t=1}^T \gamma^{t-1} (c_t(y_t - x_{t-1}) + b_t w_t + h_t v_t + \Psi_t(p, d)) \right], \quad (4.2)$$

where

$$\Psi_t(p, d) := \pi_t(D_t + x_t - y_t) + \mu_t(y_t - x_{t-1}) + u_t(y_t + w_t - D_t) + s_t(D_t + v_t - y_t),$$

$p := (p_1(\xi_1), \dots, p_T(\xi_T))$, $d := (d_1(\xi_1), \dots, d_T(\xi_T))$ with $p_t(\xi_t) := (x_t(\xi_t), y_t(\xi_t), w_t(\xi_t), v_t(\xi_t))$ and $d_t(\xi_t) := (\pi_t(\xi_t), \mu_t(\xi_t), u_t(\xi_t), s_t(\xi_t))$. Dualization of feasibility constraints and change of variables $d_t \leftarrow \gamma^{-(t-1)} d_t$ result in the following periodical multistage linear stochastic inventory dual model

$$\begin{aligned} \max_{\pi, \mu, u, s} \quad & \mathbb{E} \left[\sum_{t=1}^T \gamma^{t-1} D_t (\pi_t - u_t + s_t) \right] - c_1 x_0 - \mu_1 x_0 \\ \text{s.t.} \quad & \pi_t - \mu_t - u_t + s_t = c_t, \\ & -u_t \leq b_t, \quad -s_t \leq h_t, \quad \mu_t, u_t, s_t \leq 0, \quad t = 1, \dots, T, \\ & -\gamma^{-1} \pi_{t-1} + \mathbb{E}[\mu_t] = -c_t, \quad t = 2, \dots, T. \end{aligned} \quad (4.3)$$

Consider model (4.3) with infinite horizon $T = \infty$ and period $\mathbf{m} = 12$. At the first stage, D_1 is assumed to be deterministic with $D_1 = 5.5$. At stages $\tau = 2, \dots, \mathbf{m} + 1$, we assume the following setting. The demands are discrete random variables such that $D_\tau^j = \alpha + \beta \xi_\tau^j$, where $\alpha = 9.0$, $\beta = 0.6$, and values ξ_τ^j , $j = 1, \dots, 50$, are generated by taking random samples of size 50 from the uniform distribution on the interval $[0, 1]$ independently for each $\tau = 2, \dots, \mathbf{m} + 1$. The assigned probabilities $p_{\tau j} = 0.02$ are the same for all τ and j . The backlog costs and holding costs are static for all stages with $b_\tau = 2.8$ and $h_\tau = 0.2$, $\tau = 1, 2, \dots, \mathbf{m} + 1$, and $c_\tau = \cos(\frac{\pi}{6}\tau) + 1.5$ for $\tau = 2, \dots, \mathbf{m} + 1$. For $t \geq \mathbf{m} + 2$ the above setting is repeated periodically with period $\mathbf{m} = 12$. Optimization of (4.3) is performed over the respective policies satisfying the feasibility constraints.

We conduct experiments with the following values of the discount factor: $\gamma = 0.8$, $\gamma = 0.9906$, $\gamma = 0.9990$, $\gamma = 0.9999$. These settings aim at investigating rate of convergence when discount factor approaches one. To solve the dual problem we apply the periodical Dual SDDP algorithm with penalization (see Algorithm 1 in the Appendix), equipped with penalty parameter sequence: $r_t^k = 10^4$, $t = 1, 2, \dots$, for every iteration k .

In Table 1, we use ‘Primal-PSDDP’ and ‘Dual-PSDDP’ to denote the periodical Primal SDDP and Dual SDDP algorithms, respectively. Deterministic (upper) bounds of the dual and deterministic (lower) bound of the primal problem are represented by (D.-UB.) and (D.-LB), respectively. For example, Dual-PSDDP(D.-UB.) refers to the deterministic (upper) bound output from periodical Dual SDDP. Gap(%) is computed by

$$\frac{\text{Dual-PSDDP(D.-UB.)} - \text{Primal-PSDDP(D.-LB)}}{\text{Dual-PSDDP(D.-UB.)}} \times 100\%. \quad (4.4)$$

Different rows of the table are associated with different discount factors. At each row, we display deterministic bounds of the primal and dual problems when the algorithm stabilizes. The results in the table suggest that as the discount factor approaches one, the convergence

slows down. This is not surprising and such effect is well known. On the other hand, results in Table 1 show that when the algorithm stabilizes, the optimality gap does not differ much in scale even when the discount factor is close to one. It can also be seen that the optimal value of the problem is almost proportional to $(1 - \gamma)^{-1}$. This of course is in accordance with the geometric series view of the considered problem (4.1).

γ	Dual-PSDDP(D.-UB.)	Primal-PSDDP(D.-LB.)	Gap(%)
0.8	43.783186	43.782698	1.115×10^{-3}
0.9906	1173.345945	1173.204425	1.206×10^{-2}
0.9990	11059.03217	11051.86157	6.485×10^{-2}
0.9999	110590.2919	110514.1413	6.886×10^{-2}

Table 1: Inventory problem: evolution of bounds of primal and dual periodical programs.

When the discount factor γ approaches one (e.g., $\gamma = 0.999$, $\gamma = 0.9999$), our experiments indicate that the convergence is much slower than for smaller discount factors. In order to deal with this we apply the trust-bound strategy to the periodical Dual SDDP algorithm, as it drastically saves CPU time and yields a faster convergence (see Remark 4.1 below and section 5.1 of the Appendix).

Remark 4.1 (trust-bound strategy) The dual SDDP algorithm starts by setting an initial (constant) upper bound for the value functions, and proceeds by adding cuts during the iterations. In order to make sure that this initial upper bound is bigger than the respective optimal values, the corresponding constant is taken to be sufficiently large. After a significant number of cuts are generated, large linear programs should be solved at consecutive iterations, and this slows down the progress of the numerical procedure. An idea is to restart the algorithm after a certain number of iterations by removing all generated cuts and setting the current upper bounds of the value functions at each stage of the optimization problem. This strategy worked quite well especially when the discount factor was close to one (see section 5.1 of the Appendix for a further discussion).

It could be mentioned that other cut selection strategies have been proposed in the literature, e.g., [4],[5],[9].

Remark 4.2 When the discount factor is very close to one, it becomes very challenging to compute the classical statistical upper bound for the optimal value especially of large-scale problems. To illustrate this, consider for instance the inventory model (4.3) with $\gamma = 0.999$ and period $m = 12$, and compute its statistical upper bound (with 95% confidence level). When the algorithm for solving the primal model stabilizes, we evaluate value of the constructed policy on the discretized model using Monte Carlo simulation with number of simulations equal to 3000.

Note that when $\gamma = 0.999$, the error of a finite horizon approximation is of order $O(\gamma^T/(1 - \gamma))$ (cf., [11]), which is small enough (≈ 0.045) only when $T \geq 10000$. The CPU time to compute the statistical upper bound using $T = 10000$ exceeds 24 hours. If we decrease T to 5000, the CPU time to compute the statistical bound is around 18.7 hours.

However, the obtained statistical bound turns out to be smaller than the primal deterministic bound, which indicates that such T is too small to provide a valid upper bound. On the other hand, a valid (deterministic) upper bound obtained by solving the dual problem employing the periodical Dual SDDP method with trust bound strategy, only consumes CPU time less than 1 hour with the corresponding gap less than 0.1%.

4.2 Hydro-thermal generation problem

In this section we consider the Brazilian Inter-connected Power System operation planning problem discussed in [13]. This problem is much larger than the inventory problem considered in the previous section. The original problem has $T = 120$ stages corresponding to 10 years of monthly planning with the discount factor $\gamma = 0.9906$ (this discount factor corresponds to the annual discount rate of 12%), and 4 state variables representing energy equivalent reservoirs of four interconnected main regions. The random data process is represented by the respective 4-dimensional vectors of monthly inflows. We assume that the monthly inflows are stagewise independent and are sampled from 4-dimensional log-normal distributions calibrated by the historical data. Here we follow the periodical variant of this problem discussed in [11] with period $m = 12$ corresponding to the monthly cycle of one year. The explicit formulations of the primal and dual model are presented in section 5.2 of the Appendix.

We apply the periodical Dual SDDP (Algorithm 1 in the Appendix) to solve the SAA of the dual problem, with 50 samples per stage. In order to approximate the infinite horizon setting, we run $T = 120$ stages in the forward pass. The error of that finite horizon approximation is of order $O(\gamma^T/(1-\gamma))$ (cf., [11]). By exploring the periodical behavior, we only need to perform m stages in the backward pass to approximate the value functions. Objective coefficients of the penalty terms in the algorithm are chosen as $\{r_\tau^k\} = 1 \times 10^9$ throughout all stages and all iterations. The initial upper bounds of value functions approximation is set as 1×10^9 for all stages.

Empirical results are reported for two cases: $\gamma = 0.8$ and $\gamma = 0.9906$. We solve the first model without applying trust-bound strategy. It can be observed that the periodical Dual SDDP method, with the trust-bound strategy, signifies fast convergence in the dual problem, especially when the discount factor is close to one. As it was discussed in the previous section, for $\gamma = 0.9906$ in order to employ the classical statistical upper bound procedure, the corresponding time horizon T should be so large that makes it computationally infeasible.

Table 2 reports deterministic bounds and relative gaps of primal and dual problems with $\gamma = 0.8$ for iterations 100, 200, 300, 400, 500, 800, 1000 and an extra iteration 3000 for the primal problem. We use same notations here as in Table 1. In the last row of the table, gap is computed by the dual bound at the 1500-th iteration and the primal lower bound at 3000-th iteration, as the dual bound stabilizes around iteration 1500.

Iter.	Dual-PSDDP (D.-UB.)($\times 10^6$)	Primal-PSDDP (D.-LB.)($\times 10^6$)	Gap(%)
100	20.454	6.261	69.39
200	11.959	6.589	44.90
300	9.499	6.739	29.06
400	8.602	6.824	20.67
500	8.182	6.851	16.26
800	7.616	6.897	9.43
1000	7.477	6.915	7.51
1500	7.328	6.941	5.28
3000	-	6.964	<4.96

Table 2: Hydro-thermal problem with $\gamma = 0.8$: deterministic bounds of primal and dual periodical programs.

In Figure 1 we demonstrate evolution of deterministic primal and dual bounds produced by the algorithm in solving the hydro-thermal problem with discount factor $\gamma = 0.9906$. To solve the dual problem, we utilize the trust-bound strategy by restarting the algorithm every 100 iterations and run the algorithm for 1900 iterations in total when it stabilizes.

In view of the evolution of the dual bounds displayed by the figure, we add a few remarks. Firstly, it can be observed that the dual bounds are monotonically decreasing in each epoch (between two consecutive restarts). Such property is not maintained by consecutive restarts, that is, at the beginning of current restart, the dual bound may be larger than the one at the end of the last restart. One reason for this is that initialized bounds of the value functions are still larger than the potential tightest upper bounds of the problem. It should be noticed that, the displayed dual bound is the optimal value of the value function at the first stage while at each restart, only value functions from stage 2 and onwards are initialized using the information from the last iteration. Therefore, at the beginning of each restart, the multistage problem is re-optimized and a new optimal value of the first stage problem is computed.

Secondly, it can be seen from the figure that the algorithm converges faster in the first few restarts and becomes slower afterwards. In the last few restarts, the algorithm stabilizes and precludes the dual bound from descending below the primal lower bound.

In Table 3, we present results of the values of both deterministic bounds and the relative gaps at iterations 100, 500, 1000, 1200, 1500, 1700, 1900 and an extra value of the primal bound at iteration 3000. The gap corresponding to the last row with iteration 3000 is estimated by the dual bound at iteration 1900 and the primal bound at iteration 3000. It could be observed that the computed gaps are significantly better than the ones in Table 2, this is due to the employed trust-bound strategy.

Evolution of deterministic bounds(samples sizes per stage 50, $\gamma = 0.9906$)

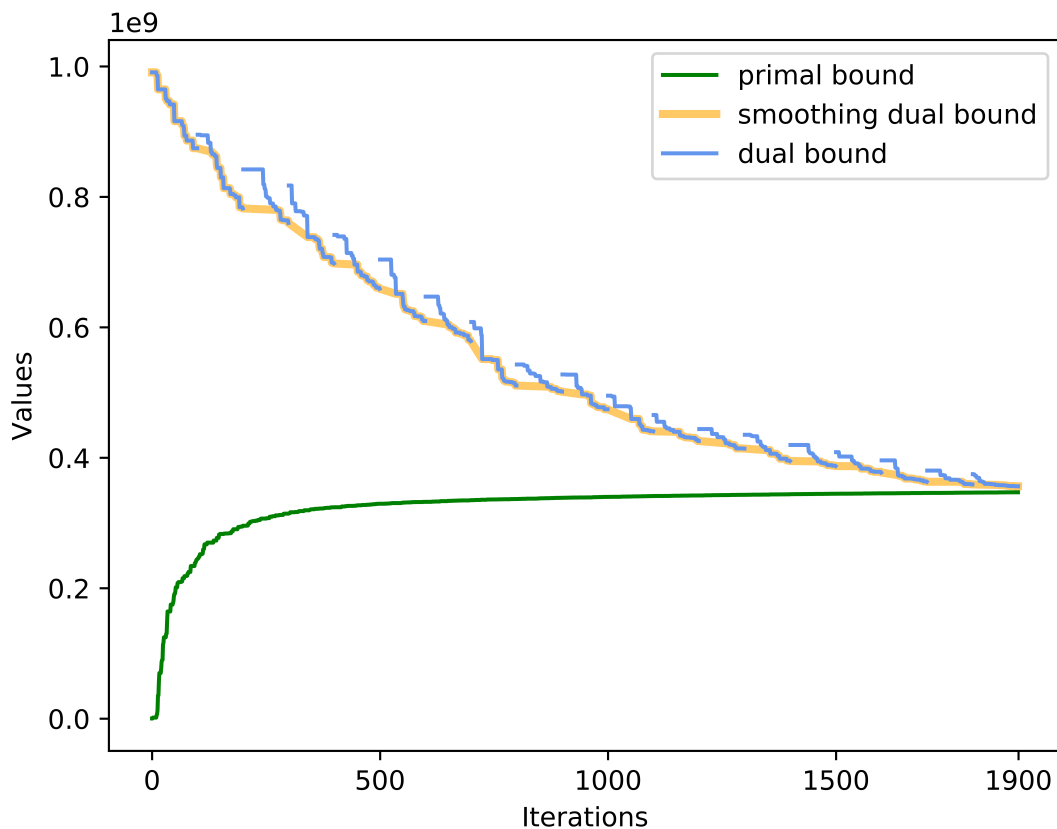


Figure 1: Hydro-thermal problem with $\gamma = 0.9906$: evolution of deterministic bounds of primal and dual periodical multistage stochastic programs. The orange line is obtained by smoothing the dual bounds (in blue) to exhibit the descending trend.

Iter.	Dual-PSDDP (D.-UB.)($\times 10^8$)	Primal-PSDDP (D.-LB.)($\times 10^8$)	Gap(%)
100	8.7443	2.4405	72.09
500	6.5912	3.2941	52.61
1000	4.7428	3.3995	28.32
1200	4.2559	3.4213	19.61
1500	3.8719	3.448	10.95
1700	3.6315	3.4601	4.72
1900	3.5621	3.4698	2.59
3000	-	3.5052	<1.6

Table 3: Hydro-thermal problem with $\gamma = 0.9906$: deterministic bounds of primal and dual periodical programs.

5 Appendix

5.1 Periodical Dual SDDP

In this section we present a detail discussion of the periodical Dual SDDP method with period $m \geq 1$. In particular, we elaborate on the trial point selection in the forward step and cutting plane method in the backward step of the algorithm.

5.1.1 Trial points selection

In the forward step, we fix a finite value for the time horizon T , and solve the corresponding T -stage problems. The obtained feasible solutions of the dual problem are state variables $\{\lambda_{tj}\}$. Notice that each optimization problem at stage $t = 2, \dots, T$, is not separable with respect to solution corresponding to each (discretized) sample. For each stage $t \geq 2$, by selecting $\lambda'_t := \lambda_{t\hat{j}}$ with probability $p_{\tau\hat{j}}$, $\hat{j} \in \{1, \dots, N_\tau\}$, where $\tau = t \pmod{m}$, we construct a set of solutions $\{\lambda'_t : t = 2, \dots, T\}$. Next, consider T -stages divided into consecutive groups with periods m , that is, $\{(\ell, \ell + 1, \dots, \ell + m - 1) : \ell = 1, m + 1, 2m + 1, 3m + 1, \dots, \ell + m - 1 \leq T\}$. Randomly select ℓ from $\{1, m + 1, 2m + 1, \dots, \ell + m - 1 \leq T\}$, then we construct a group of trial points $\{\bar{\lambda}_{\ell+\tau-1} : \tau = 1, \dots, m\}$.

5.1.2 Cutting Plane Algorithm

To deal with the issue of possible violation of the Relatively Complete Recourse (RCR), a penalty term ν is introduced for the dual problem with objective coefficient r . With the penalty term, the value function at each stage τ is finite valued on a compact set formed by linear constraints, thus maximum is attainable. The WB equations can be written as, for $\tau = m + 1, \dots, 2$,

$$\begin{aligned} V_\tau(\lambda_{\tau-1}) &= \max_{\lambda_\tau, \nu_\tau \geq 0} \sum_{j=1}^{N_\tau} p_{\tau j} \left(b_\tau^{j\top} \lambda_{\tau j} + \gamma V_{\tau+1}(\lambda_{\tau j}) \right) - r_\tau^\top \nu_\tau \\ \text{s.t.} \quad & A_{\tau-1}^\top \lambda_{\tau-1} + \gamma \sum_{j=1}^{N_\tau} p_{\tau j} B_\tau^{j\top} \lambda_{\tau j} - \nu_\tau \leq c_{\tau-1}, \end{aligned} \quad (5.1)$$

where $V_{m+2}(\cdot)$ is replaced by $V_2(\cdot)$. For $\tau = 1$, the problem is deterministic and can be written as

$$V_1 = \max b_1^\top \lambda_1 + \gamma V_2(\lambda_1). \quad (5.2)$$

For each stage $\tau \in \{1, 2, \dots, m + 1\}$, given a current upper approximation $\mathfrak{V}_{\tau+1}(\cdot)$ of the value function $V_{\tau+1}(\cdot)$ and a trial point $\bar{\lambda}_{\tau-1}$, new cuts $\{\psi_{\tau j}(\cdot)\}$ are constructed by computing the (sub)gradient g_τ at $\bar{\lambda}_{\tau-1}$ of the current estimate of the value function. That is,

$$\psi_{\tau j}(\lambda_{\tau-1j}) := g_\tau^\top (\lambda_{\tau-1j} - \bar{\lambda}_{\tau-1}) + \bar{V}_\tau(\bar{\lambda}_{\tau-1}), j = 1, \dots, N_{\tau-1}, \quad (5.3)$$

and a new supporting plane for $V_\tau(\cdot)$ at $\bar{\lambda}_{\tau-1}$ is generated by $\{\psi_{\tau j}\}$ as

$$l_\tau(\lambda_{\tau-1}) := \sum_{j=1}^{N_{\tau-1}} p_{\tau-1j} \psi_{\tau j}(\lambda_{\tau-1j}), \quad (5.4)$$

where

$$\begin{aligned} \bar{V}_\tau(\bar{\lambda}_{\tau-1}) = & \max_{\lambda_\tau, \nu_\tau \geq 0} \sum_{j=1}^{N_\tau} p_{\tau j} b_\tau^j \top \lambda_{\tau j} + \gamma \mathfrak{V}_{\tau+1}(\lambda_\tau) - r_\tau \top \nu_\tau \\ \text{s.t.} \quad & A_{\tau-1} \top \bar{\lambda}_{\tau-1} + \gamma \sum_{j=1}^{N_\tau} p_{\tau j} B_\tau^j \top \lambda_{\tau j} - \nu_\tau \leq c_{\tau-1}, \end{aligned} \quad (5.5)$$

with $\lambda_\tau = [\lambda_{\tau 1}, \dots, \lambda_{\tau j}, \dots, \lambda_{\tau N_\tau}]$. Then collection of supporting planes of $\mathfrak{V}_\tau(\cdot)$ is updated by $\mathfrak{V}_\tau(\cdot) \leftarrow \min\{\mathfrak{V}_\tau(\cdot), l_\tau(\cdot)\}$. Specifically, cutting plane approximation for value function at stage $m+2$ is equal to the approximation at stage 2 by periodic property.

5.1.3 Trust-bound strategy

Note that for each $\tau \in \{2, \dots, m+1\}$, $\mathfrak{V}_\tau(\cdot)$ is formed by the minimum of piecewise linear functions, and hence problem (5.5) can be formulated as a linear programming problem. Suppose at iteration k in the backward step, we are solving the following problem at stage τ :

$$\begin{aligned} \max \quad & \sum_{j=1}^{N_\tau} p_{\tau j} b_\tau^j \top \lambda_{\tau j} + \gamma \alpha_{\tau+1} - r_\tau \top \nu_\tau \\ \text{s.t.} \quad & A_{\tau-1} \top \bar{\lambda}_{\tau-1} + \gamma \sum_{j=1}^{N_\tau} p_{\tau j} B_\tau^j \top \lambda_{\tau j} - \nu_\tau \leq c_{\tau-1}, \\ & \alpha_{\tau+1} \leq l_{\tau+1}^s(\lambda_\tau), s = 1, \dots, k, \end{aligned} \quad (5.6)$$

where $l_{\tau+1}^s(\cdot)$ denotes the supporting plane generated at iteration $s \leq k$. Consequently, we obtain optimal value of $\alpha_{\tau+1}$, denoted by $\tilde{\alpha}_{\tau+1}^k$, as the best upper approximation for value function at stage $\tau+1$ for current iteration.

If the algorithm restarts after the k -th iteration, then all the generated cuts are eliminated and for each stage $\tau, \tau = 2, \dots, m+1$, value function \bar{V}_τ is set to $\tilde{\alpha}_\tau^k$. The algorithmic scheme is the same after each restart.

Finally, we refer to Algorithm 1 for details of the Periodical Dual Stochastic Dynamic Dual Programming.

Algorithm 1 Periodical Dual SDDP with penalization

- 1: Given sample size N_τ and discretizations $\{u_\tau^j, v_\tau^j, b_\tau^j, A_\tau^j, B_\tau^j, C_\tau^j\}_{j=1}^{N_\tau}$, for $\tau = 2, \dots, m+1$.
 - 2: Initialization of cutting planes: $\mathfrak{W}_\tau^0 = \text{LargeBound}$, $\tau = 2, \dots, m+1$, $\mathfrak{W}_{m+2}^0 = \mathfrak{W}_2^0$.
 - 3: **for** $k = 1, 2, \dots$ **do**
 - 4: **for** $t = 1, \dots, T$ **do** ▷ Forward Pass
 - 5: **if** $t = 1$ **then** $\tau = 1$
 - 6: **else** $\tau \equiv (t \bmod m)$
 - 7: **end if**
 - 8: $\{\lambda_{tj}\}_{j=1}^{N_\tau} = \arg \max \left\{ \sum_{j=1}^{N_\tau} p_{\tau j} b_\tau^j \lambda_{tj} + \gamma \mathfrak{W}_{\tau+1}^{k-1}(\lambda_t) - r_\tau^{k \top} \nu_\tau : \right.$

$$A_{\tau-1}^\top \bar{\lambda}_{t-1} + \gamma \sum_{j=1}^{N_\tau} p_{\tau j} B_\tau^j \lambda_{tj} - \nu_\tau \leq c_{\tau-1},$$

$$\left. \nu_\tau \geq 0 \right\}$$
 - 9: Select forward solutions $\lambda'_t \leftarrow \lambda_{t\hat{j}}$
 - 10: **end for**
 - 11: Trial points selection: $(\bar{\lambda}_1, \dots, \bar{\lambda}_m) \leftarrow (\lambda'_\ell, \lambda'_{\ell+1}, \dots, \lambda'_{\ell+m-1})$
 - 12: **for** $\tau = m+1, \dots, 2$ **do** ▷ Backward Pass
 - 13: $(\bar{V}_\tau(\bar{\lambda}_{\tau-1}), g_\tau) = \max \left\{ \sum_{j=1}^{N_\tau} p_{\tau j} b_\tau^j \lambda_{\tau j} + \gamma \mathfrak{W}_{\tau+1}^k(\lambda_\tau) - r_\tau^{k \top} \nu_\tau : \right.$

$$A_{\tau-1}^\top \bar{\lambda}_{\tau-1} + \gamma \sum_{j=1}^{N_\tau} p_{\tau j} B_\tau^j \lambda_{\tau j} - \nu_\tau \leq c_{\tau-1},$$

$$\left. \nu_\tau \geq 0 \right\}$$
 - 14: Update cutting planes: $\mathfrak{W}_\tau^k \leftarrow \left\{ \alpha \in \mathfrak{W}_\tau^{k-1} : \alpha \leq \sum_{j=1}^{N_{\tau-1}} p_{\tau-1j} \psi_{\tau j}(\lambda_{\tau-1j}), \right.$

$$\left. \psi_{\tau j}(\lambda_{\tau-1j}) \leq g_\tau(\lambda_{\tau-1j} - \bar{\lambda}_{\tau-1}) + \bar{V}_\tau(\bar{\lambda}_{\tau-1}) \right\}$$
 - 15: **if** $\tau = 2$ **then** update cutting planes: $\mathfrak{W}_{m+2}^k \leftarrow \left\{ \alpha \in \mathfrak{W}_{m+2}^{k-1} : \alpha \leq \sum_{j=1}^{N_{m+1}} p_{m+1j} \psi_{2j}(\lambda_{m+1j}), \right.$

$$\left. \psi_{2j}(\lambda_{m+1j}) \leq g_2(\lambda_{m+1j} - \bar{\lambda}_1) + \bar{V}_2(\bar{\lambda}_1) \right\}$$
 - 16: **end if**
 - 17: **end for**
 - 18: Deterministic bound $V_1^k = \max\{b_1^\top \lambda_1 + \gamma \mathfrak{W}_2^k(\lambda_1)\}$
 - 19: **end for**
-

5.2 Hydro-thermal planning problem

The explicit primal model of the infinite-horizon problem with discount factor $\gamma = 0.9906$ is the following:

$$\begin{aligned}
\min \quad & \sum_{t=1}^T \gamma^{t-1} \left[\sum_{i=1}^4 b_i s_{i,t} + \sum_{i=1}^4 \sum_{j=1}^4 e_j df_{i,j,t} + \sum_{i=1}^4 u_i \sum_{k \in \Omega_i} g_{i,k,t} + \sum_{i=1}^5 \sum_{j=1}^5 c_{j,i} \text{ex}_{j \rightarrow i,t} \right] \\
\text{s.t. for } & t = 1, 2, \dots, T, \\
& \sum_{k \in \Omega_i} g_{i,k,t} + q_{i,t} + \sum_{j=1}^4 df_{i,j,t} - \sum_{j=1}^5 \text{ex}_{i \rightarrow j,t} + \sum_{j=1}^5 \text{ex}_{j \rightarrow i,t} = d_{i,t}, \quad i = 1, \dots, 4, \\
& \sum_{j=1}^5 \text{ex}_{j \rightarrow 5,t} - \sum_{j=1}^5 \text{ex}_{5 \rightarrow j,t} = 0, \\
& q_{i,t} + s_{i,t} + v_{i,t} - v_{i,t-1} = a_{i,t} \quad i = 1, \dots, 4, \\
& s_{i,t} \geq 0, \quad i = 1, \dots, 4, \\
& 0 \leq v_{i,t} \leq \bar{v}_i, \quad i = 1, \dots, 4, \\
& 0 \leq q_{i,t} \leq \bar{q}_i, \quad i = 1, \dots, 4, \\
& 0 \leq df_{i,j,t} \leq \bar{df}_{i,j}, \quad i, j = 1, \dots, 4, \\
& 0 \leq \text{ex}_{i \rightarrow j,t} \leq \bar{\text{ex}}_{i,j}, \quad i, j = 1, \dots, 4, \\
& \underline{g}_i \leq g_{i,k,t} \leq \bar{g}_i, \quad i = 1, \dots, 4, \quad k \in \Omega_i.
\end{aligned} \tag{5.7}$$

We refer to [13] and [11] for the details of the primal model and variables/parameters notations correspondingly.

By writing the Lagrangian of (5.7) and dualization of the feasibility constraints, the dual

model can be written as

$$\begin{aligned}
\max \quad & \sum_{t=1}^T \gamma^{t-1} \left[\sum_{i=1}^4 \left(d_{i,t} \lambda_{i,t} + \bar{v}_i x_{i,t} + \bar{q}_i o_{i,t} + \sum_{j=1}^4 \bar{d}f_j h_{i,j,t} + \sum_{k \in \Omega_i} (\bar{g}_i z_{i,k,t} + \underline{g}_i \omega_{i,k,t}) + a_{i,t} \mu_{i,t} \right) \right. \\
& \left. + \sum_{i=1}^5 \sum_{j=1}^5 \bar{e}x_{i,j} f_{i,j,t} \right] + \sum_{i=1}^4 a_{i,1} \mu_{i,1} + \sum_{i=1}^4 v_{i,0} y_{i,1} \\
\text{s.t.} \quad & \text{for } t = 1, \dots, T, \\
& \mu_{t,i} \leq b_i, \quad i = 1, \dots, 4, \\
& \lambda_{i,t} + \mu_{i,t} + o_{i,t} \leq 0, \quad i = 1, \dots, 4, \\
& \lambda_{i,t} + h_{i,j,t} \leq e_j, \quad i, j = 1, \dots, 4, \\
& \lambda_{i,t} + z_{i,k,t} + \omega_{i,k,t} = u_i, \quad i = 1, \dots, 4, \quad k \in \Omega_i, \\
& \text{for } i \in \{1, \dots, 5\}, j \in \{1, \dots, 5\}, \\
& \text{if } i = j : f_{i,j,t} \leq c_{i,j}, \\
& \text{if } i \neq j, (i, j) \leq 4 : -\lambda_{i,t} + \lambda_{j,t} + f_{i,j,t} \leq c_{i,j}, \\
& \text{if } i \neq j, i = 5, j < 5 : \lambda_{j,t} - \eta_t + f_{i,j,t} \leq c_{i,j}, \\
& \text{if } i \neq j, i < 5, j = 5 : \lambda_{i,t} + \eta_t + f_{i,j,t} \leq c_{i,j}, \\
& x_{i,t} \leq 0, \quad i = 1, \dots, 4, \\
& o_{i,t} \leq 0, \quad i = 1, \dots, 4, \\
& h_{i,j,t} \leq 0 \quad i, j = 1, \dots, 4, \\
& z_{i,k,t} \leq 0, \quad i = 1, \dots, 4, \quad k \in \Omega_i, \\
& \omega_{i,k,t} \geq 0, \quad i = 1, \dots, 4, \quad k \in \Omega_i, \\
& f_{i,j,t} \leq 0, \quad i, j = 1, \dots, 5, \\
& \text{for } t = 2, \dots, T, \\
& \mu_{i,t-1} + x_{i,t-1} - \gamma \mathbb{E} [\mu_{i,t}] \leq 0, \quad i = 1, \dots, 4.
\end{aligned} \tag{5.8}$$

Here in (5.8), we denote the states as $\{x_{i,t}, i = 1, \dots, 4\}$ and $\{\mu_{i,t}, i = 1, \dots, 4\}$ for $t = 1, 2, \dots, T$. Control variables are denoted by $\{\lambda_{i,t}\}, \{o_{i,t}\}, \{h_{i,j,t}\}, \{z_{i,k,t}\}, \{\omega_{i,k,t}\}, \{f_{i,j,t}\}$ and $\{y_{i,1}\}$. In both models, initial stored energy $v_{i,0}$ and initial inflow $a_{i,1}$, $i = 1, \dots, 4$ are given and inflow $\mathbf{a}_t := (a_{1,t}, \dots, a_{4,t}), t = 2, \dots, T$ is periodical and modeled as stagewise independent stochastic process, such that $\mathbf{a}_t \sim \text{lognormal}(\boldsymbol{\mu}_\tau, \boldsymbol{\Sigma}_\tau)$ for each t if $t \bmod m = \tau$, $\tau = 2, \dots, m+1$, where $\boldsymbol{\mu}_\tau$ and $\boldsymbol{\Sigma}_\tau$ is the mean and covariance matrix of log of the historical inflow data for each month, respectively.

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