

# Refinements of Kusuoka Representations on $L^\infty$

Kerem Uğurlu \*

## Abstract

We study Kusuoka representations of law invariant coherent risk measures on the space of bounded random variables, which says that any law invariant coherent risk measure is the supremum of integrals of Average-Value-at-Risk measures. We refine this representation by showing that the supremum in Kusuoka representation is attained for some probability measure in the unit interval. Namely, we prove that any law invariant coherent risk measure on the space of bounded random variables can be written as an integral of the Average-Value-at-Risk measures on the unit interval with respect to some probability measure. This representation gives a numerically constructive way to bound any law invariant coherent risk measure on the space of essentially bounded random variables from above and below. The results are illustrated on specific law invariant coherent risk measures along with numerical simulations.

*Keywords: coherent risk measures; law invariance; average value-at-risk; comonotonic risk measures; robust performance measures; disutility minimization*

## 1 Introduction

Coherent risk measures have been introduced in seminal paper [1] to put risk averse performance evaluation into an axiomatic framework in utility maximization or disutility minimization in financial and actuarial mathematics. It has seen huge development both in theory and practice since then. We refer the reader to the monographs [5] and [9] and the references therein for the basics of these operators. In the seminal paper, [6], Kusuoka has shown that any law invariant coherent risk measure on the space of bounded random variables  $L^\infty$  can

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\*e-mail:kerem.ugurlu@nu.edu.kz Department of Mathematics, Nazarbayev University.

be represented in a unifying way. In particular, this representation states that any law invariant coherent risk measure is a supremum of integrals of Average-Value-at-Risk (AVaR) operators taken with respect to different measures on  $[0, 1]$ . Namely, it is shown that any law invariant coherent risk measure  $\rho$  is a limit of the combination of AVaR $_{\alpha}$ 's with different  $\alpha \in [0, 1]$ , that determines the risk awareness level of  $\rho$ . In [8], this representation has been extended to higher orders as suprema of integrals of Average-Value-at-Risk with respect to probability measures on  $[0, 1]$ , where the random variables are in  $L^p$  for  $p \geq 1$ . In [3], the existence of Kusuoka representation in  $L^p$  spaces, in particular in the case of unbounded random variables, has been positively answered. It is left as an open question, whether this representation is unique in a specialised sense. This question has been positively answered in a subsequent work in [4]. In [13] and in [12], coherent risk measures have been generalized to convex risk measures by removing the positive homogeneity axiom of coherent risk measures. In [14], the authors have introduced a new notion of qualitative robustness for risk measures applied to the coherent risk measures that concern the sensitivity of a risk measure to the uncertainty of dependence in risk aggregation. In a recent work, [11] shows that the expectation functional is the only law-invariant convex functional that is linear along the direction of a nonconstant random variable with nonzero expectation up to an affine transformation under some assumptions. In a broader context robust optimization has been a very active research area in various operations research fields. In [16], robust optimization techniques have been merged with mixed-integer linear programming models and applied to supply chain planning problem. In [17], a scheduling problem has been studied in the robust setting. In [18], a routing problem has been studied in a robust setting.

An important aspect of Kusuoka representation is that  $\rho$  is given only as a supremum of those integrals. A natural question is when/if the supremum is attained in this representation, and whether the attaining measure is the same for all bounded random variables. We give a positive answer to the first question of attainment of supremum, whereas a counterexample has been shown for the second one. Namely, we work on  $L^{\infty}$  space, and indeed obtain that supremum in Kusuoka representation of law invariant coherent risk measures is always attained. This refinement of Kusuoka representation allows for a numerically straightforward upper and lower bounds of any law invariant coherent risk measure using finite convex combinations of AVaR $_{\alpha}$ 's with different  $\alpha \in [0, 1]$  values. Moreover, we show that the attaining measure for a fixed law invariant coherent risk measure for two random variables with different distributions can be different by giving an example. We summarise the results regarding the Kusuoka representation in Table 1 below for the convenience of the reader. The second column checks whether the corresponding works on bounded or unbounded random variables. The third column checks whether the supremum is attained in the Kusuoka

Table 1: Literature Survey

References	Bounded/ Unbounded	Attainment of Supremum	Lower Bound	Upper Bound
[6]	Bounded	No	Yes	No
[3]	Unbounded	No	Yes	No
[4]	Unbounded	No	Yes	No
[8]	Unbounded	No	Yes	No
This Research	Bounded	Yes	Yes	Yes

representation. The fourth and fifth column check whether a lower and upper bound for the Kusuoka representation are given. We emphasize that our work is the only one giving both lower and upper bounds of the Kusuoka representation of a law invariant coherent risk measure.

The rest of the paper is as follows. In Section 2, we introduce the notation and theoretical background. In Section 3, we present our main theoretical results along with the approximation schemes of Kusuoka representations. In Section 4, we introduce specific law invariant coherent risk measures and the numerical illustrations of them. In Section 5, we briefly summarize our results and finalize the paper.

## 2 Notation and Preliminaries

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be an atomless probability space. That is, for every  $A \in \mathcal{F}$  with  $\mathbb{P}(A) > 0$ , there exists  $B \subset A$  and  $B \in \mathcal{F}$  that satisfies  $0 < \mathbb{P}(B) < \mathbb{P}(A)$ . We further denote  $\mathcal{Z} \triangleq L^\infty(\Omega, \mathcal{F}, \mathbb{P})$  as the set of essentially bounded random variables. The Banach space  $\mathcal{Z}$  is equipped with  $\text{ess sup}(Z) = \|Z\|_\infty$  norm.

**Definition 2.1.** *A real valued functional  $\rho : \mathcal{Z} \rightarrow \mathbb{R}$  is called a coherent risk measure, if it satisfies the following axioms:*

- (A1) *If  $X, Y \in \mathcal{Z}$  and  $X \leq Y$ ,  $\mathbb{P}$ -a.s., then  $\rho(X) \leq \rho(Y)$ .*
- (A2) *For all  $X, Y \in \mathcal{Z}$  and all  $\alpha \in [0, 1]$ ,*

$$\rho(\alpha X + (1 - \alpha)Y) \leq \alpha\rho(X) + (1 - \alpha)\rho(Y). \quad (2.1)$$

- (A3) *If  $c \in \mathbb{R}$  and  $Z \in \mathcal{Z}$ , then  $\rho(Z + c) = \rho(Z) + c$ .*
- (A4) *If  $\lambda \geq 0$ , and  $Z \in \mathcal{Z}$ , then  $\rho(\lambda Z) = \lambda\rho(Z)$ .*

**Definition 2.2.** Let  $X, Y \in \mathcal{Z}$  and let  $F_X(\cdot), F_Y(\cdot)$  be their corresponding cumulative distribution functions. A coherent risk measure  $\rho$  is called law invariant, if  $F_X(x) = F_Y(x)$  for all  $x \in \mathbb{R}$ , denoted by  $X \stackrel{d}{=} Y$ , implies  $\rho(X) = \rho(Y)$ .

**Definition 2.3.** A fundamental law invariant coherent risk measure is the Average-Value-at-Risk of  $Z \in \mathcal{Z}$  that is defined by

$$\text{AVaR}_\alpha(Z) \triangleq \frac{1}{1-\alpha} \int_\alpha^1 F_Z^{-1}(t) dt, \quad Z \in \mathcal{Z}, \quad (2.2)$$

where  $\alpha \in [0, 1]$  and  $\text{AVaR}_0(Z) = \mathbb{E}[Z]$  and  $\lim_{t \rightarrow 1} \text{AVaR}_t(Z) = \text{ess sup}(Z)$ . Here,

$$F_Z^{-1}(\tau) \triangleq \inf\{t \in [0, 1] : F_Z(t) \geq \tau\} \quad (2.3)$$

is the left quantile of  $Z$ . We further denote by  $\mathfrak{P}$  the set of all probability measures on  $[0, 1]$ .

**Remark 2.1.** In actuarial and financial literature, the quantile in (2.3) is called Value-at-Risk of  $Z$  with risk level  $0 \leq \alpha < 1$  and is denoted by  $\text{VaR}_\alpha(Z)$ . In particular,  $\text{AVaR}_\alpha(Z)$  in (2.2) is defined as  $\text{AVaR}_\alpha = \mathbb{E}[Z | Z \geq \text{VaR}_\alpha(Z)]$ . That is  $\text{AVaR}_\alpha(Z)$  is the tail-conditional expectation of  $Z$  from  $\text{VaR}_\alpha(Z)$ . Moreover,  $\text{AVaR}_\alpha$  has equivalent names as Conditional-Value-at-Risk at level  $\alpha$ , denoted by  $\text{CVaR}_\alpha$ , Expected-Value-at-Risk at level  $\alpha$ , denoted by  $\text{EVaR}_\alpha$  and Tail-Value-at-Risk at level  $\alpha$ , denoted by  $\text{TVaR}_{\alpha_i}$ .

Since  $(\Omega, \mathcal{F}, \mathbb{P})$  is atomless and the coherent risk measures that we will work on are law invariant, without loss of generality, we can take  $\Omega$  as  $[0, 1]$ ,  $\mathbb{P}_U$  as uniform measure on  $[0, 1]$  and any  $Z$  in  $\mathcal{Z}$  as  $F_Z^{-1}$  by  $Z \stackrel{d}{=} F_Z^{-1}$ . We denote  $\mathcal{Z}_{\text{cont}}$  as  $([0, 1], \mathcal{G}, \mathbb{P}_U)$ , where  $\mathcal{G}$  is the Borel  $\sigma$ -algebra on  $[0, 1]$ . We call this  $\mathcal{Z}_{\text{cont}}$  as the standard probability space. The analogue of  $\mathcal{Z}_{\text{cont}}$  in discrete probability space, denoted by  $\mathcal{Z}_{\text{disc}}$ , is of the form  $\Omega = (\omega_1, \omega_2, \dots, \omega_N)$  with  $\mathbb{P}(\{\omega_i\}) = 1/N$  for some positive integer  $N$ , and  $\mathcal{G}$  is the power set of  $\Omega$ .

### 3 Main Results

We first recall the following Kusuoka representation of law invariant coherent risk measures.

**Theorem 3.1.** [6] Let  $Z \in \mathcal{Z}$ . Every law invariant, coherent risk measure  $\rho : \mathcal{Z} \rightarrow \mathbb{R}$  in an atomless probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  can be represented by the form

$$\rho(Z) = \sup_{\mu \in \mathfrak{M}} \int_{[0,1]} \text{AVaR}_t(Z) d\mu(t), \quad (3.4)$$

where  $\mathfrak{M}$  is a set of probability measures on  $[0, 1]$ , and the integral is taken in the Lebesgue sense. Here  $\mathfrak{M}$  does not depend on  $Z \in \mathcal{Z}$ .

**Definition 3.1.** We say that a set  $\mathfrak{M}$  of probability measures on  $[0, 1]$  is a Kusuoka set, if the representation (3.4) holds for all  $Z \in \mathcal{Z}$ .

**Definition 3.2.** We say that a sequence of probability measures  $(\mu_n)_{n \geq 1}$  on  $[0, 1]$  converges in weak\*-topology to a probability measure  $\mu^*$  on  $[0, 1]$ , if

$$\int_{[0,1]} g(t) d\mu_n(t) \rightarrow \int_{[0,1]} g(t) d\mu^*(t), \quad (3.5)$$

as  $n \rightarrow \infty$  for all continuous bounded  $g$  on  $[0, 1]$ .

### 3.1 Attainment of Kusuoka Sets in $\mathcal{Z}$

The following lemma refines Theorem 3.1 by showing that supremum in (3.4) is attained for a probability measure on  $[0, 1]$  for  $Z \in \mathcal{Z}$ .

**Lemma 3.1.** Let  $\rho : \mathcal{Z} \rightarrow \mathbb{R}$  be a law invariant coherent risk measure and  $\mathfrak{M}$  be a Kusuoka set as in (3.4). Then, the closure of  $\mathfrak{M}$ , denoted by  $\bar{\mathfrak{M}}$ , taken with respect to weak\* topology in the sense of Definition 3.2, is also a Kusuoka set and the supremum in (3.4) is attained for some  $\mu^* \in \bar{\mathfrak{M}}$  on  $[0, 1]$ .

*Proof.* Let  $Z \in \mathcal{Z}$  be fixed.  $t : [0, 1] \rightarrow \text{AVaR}_t(Z)$  is a continuous and monotone function with  $\text{AVaR}_0(Z) = \mathbb{E}[Z]$  and  $\lim_{t \rightarrow 1} \text{AVaR}_t(Z) = \|Z\|_\infty$ . Since  $[0, 1]$  is compact and  $t \rightarrow \text{AVaR}_t(Z)$  is a continuous function, by Prokhorov theorem  $\mathfrak{M}$  is weak\* relatively compact (see e.g. [7]). Thus, by (3.4), there exists a sequence of  $(\mu_n)_{n \geq 1} \subset \mathfrak{P}$  such that

$$\rho(Z) = \sup_{\mu \in \mathfrak{M}} \int_{[0,1]} \text{AVaR}_t(Z) d\mu(t) \quad (3.6)$$

$$= \lim_{n \rightarrow \infty} \int_{[0,1]} \text{AVaR}_t(Z) d\mu_n(t) \quad (3.7)$$

$$= \int_{[0,1]} \text{AVaR}_t(Z) d\mu^*(t). \quad (3.8)$$

Here, the limit  $\mu^*$  is a finite signed measure that is positive with  $\mu^*([0, 1]) = 1$ . Namely,  $\mu^* \in \mathfrak{P}$ , as well with  $\mu^*$  being in the closure of  $\bar{\mathfrak{M}}$ , i.e.  $\mu^* \in \bar{\mathfrak{M}}$ . Hence, we conclude the proof.  $\square$

**Definition 3.3.** Two random variables  $X, Y \in \mathcal{Z}$  are called comonotone, if they satisfy for almost all  $(\omega, \omega') \in \Omega \times \Omega$

$$(X(\omega) - Y(\omega))(X(\omega') - Y(\omega')) \geq 0. \quad (3.9)$$

A coherent risk measure  $\rho : \mathcal{Z} \rightarrow \mathbb{R}$  is called comonotonic, if it satisfies

$$\rho(X + Y) = \rho(X) + \rho(Y), \quad (3.10)$$

for all comonotone  $X, Y \in \mathcal{Z}$ .

It is shown in [11] recently, that the expectation operator is the only law-invariant convex functional that is linear along the direction of a nonconstant random variable with nonzero expectation up to an affine transformation. It is also known that  $\text{AVaR}_\alpha : \mathcal{Z} \rightarrow \mathbb{R}$  for  $\alpha \in [0, 1]$  is law invariant and comonotonic (see e.g. [2]). Hence, it is tempting to conclude that any law invariant coherent risk measure  $\rho : \mathcal{Z} \rightarrow \mathbb{R}$  is comonotonic. However, this is not the case, as the following example shows.

**Example 3.1.** Consider the random variables  $X(u) = \max(0, 2u - 1)$  for  $u \in [0, 1]$  and  $Y(u) = 0$  for  $u \in [0, 1/2)$  and  $Y(u) = 1$  for  $u \in [1/2, 1]$ . Since,  $X$  and  $Y$  are nondecreasing, they are comonotonic. Using (2.2), we have  $\text{AVaR}_0(X) = 0.25$ ,  $\text{AVaR}_{1/2}(X) = 0.5$  and  $\text{AVaR}_1(X) = 1$ , whereas  $\text{AVaR}_0(Y) = 0.5$ ,  $\text{AVaR}_{1/2}(Y) = 1$  and  $\text{AVaR}_1(Y) = 1$ . It is known that  $\text{AVaR}_\alpha : \mathcal{Z} \rightarrow \mathbb{R}$  for  $\alpha \in [0, 1]$  is law invariant and comonotonic (see e.g. [2]). Thus,  $\text{AVaR}_0(X + Y) = 0.75$ ,  $\text{AVaR}_{1/2}(X + Y) = 1.5$  and  $\text{AVaR}_1(X + Y) = 2$ . However, consider the law invariant coherent risk measure  $\rho : \mathcal{Z} \rightarrow \mathbb{R}$  with  $\rho(X) \triangleq \max(\text{AVaR}_{1/2}(X), 1/2\text{AVaR}_0(X) + 1/2\text{AVaR}_1(X))$ . Note that  $\rho$  is law invariant, as well. Then, by the calculations above, we have  $\rho(X) = 0.625$ ,  $\rho(Y) = 1$  and  $\rho(X + Y) = 1.5$ . Hence,  $\rho$  is not comonotonic.

**Remark 3.1.** The Kusuoka representation in Theorem 3.1 says that any law invariant coherent risk measure  $\rho : \mathcal{Z} \rightarrow \mathbb{R}$  can be represented by a set of probability measures  $\mathfrak{M}$  on  $[0, 1]$ , where  $\mathfrak{M}$  does not depend on  $\mathcal{Z}$ . We show in Lemma 3.1 that supremum is attained for each  $Z \in \mathcal{Z}$  for some  $\mu^* \in \bar{\mathfrak{M}}$ , but  $\mu^*$  depends on  $Z \in \mathcal{Z}$ , otherwise any law invariant, coherent risk measure would have been necessarily comonotonic, which is not the case as shown in Example 3.1. On the other hand, another question is the minimality of Kusuoka set  $\mathfrak{M}$  in (3.4) as in [3]. Here, minimality is in the following sense. Suppose a fixed law invariant coherent risk measure  $\rho$  has the representation for two sets of probability measures  $\mathfrak{M}_1$  and another set  $\mathfrak{M}$  in (3.4). Then,  $\mathfrak{M}_1 \subset \mathfrak{M}$ . In that sense,  $\mathfrak{M}_1$  is also unique. The question of existence of minimal Kusuoka set has been positively answered, when we are working in  $L^p(\Omega, \mathcal{F}, \mathbb{P})$  space. Since  $\mathcal{Z} \subset L^p(\Omega, \mathcal{F}, \mathbb{P})$ , we necessarily have a unique minimal set of probabilities in  $\mathcal{Z}$ , as well. Hence, there exists a minimal unique Kusuoka set  $\mathfrak{M}$  for any law invariant coherent risk measure. But by taking the weak closure, denoted by  $\bar{\mathfrak{M}}$ , the representing measure for (3.8) can be different for different random variables among the set  $\bar{\mathfrak{M}}$ .

### 3.2 Approximations of Kusuoka Representations in $\mathcal{Z}$

In this section, we give two recipes to approximate any law invariant coherent risk measure using convex combinations of  $\text{AVaR}_\alpha$ 's with different  $\alpha \in [0, 1]$  values and convex combination of expectation and essential supremum.

**Lemma 3.2.** *Let  $N \geq 1$  be a fixed integer. Any law invariant coherent risk measure  $\rho : \mathcal{Z} \rightarrow \mathbb{R}$  can be bounded from below by*

$$\sum_{i=0}^{N-1} \lambda_{i/N} \text{AVaR}_{i/N}(\cdot) \quad (3.11)$$

and from above by

$$\sum_{i=1}^N \lambda_{i/N} \text{AVaR}_{i/N}(\cdot) \quad (3.12)$$

for some specific sequence of  $(\lambda_{i/N})_{i=0}^N$  with  $\lambda_{i/N} \geq 0$  and  $\sum_{i=0}^N \lambda_{i/N} = 1$ .

*Proof.* Let  $Z \in \mathcal{Z}$  be fixed. Since  $t \rightarrow \text{AVaR}_t(Z)$  is a continuous and monotone increasing function, it is Riemann integrable. Hence, the Lebesgue integral in (3.8) can be approximated in the Riemann sense. Namely, (3.8) can be approximated from below by step functions in the Riemann integral sense as  $N \rightarrow \infty$  with

$$\sum_{i=0}^{N-2} \text{AVaR}_{i/N}(Z) \mu^*\left(\left[\frac{i}{N}, \frac{i+1}{N}\right)\right) + \text{AVaR}_{N-1/N}(Z) \mu^*([N-1/N, 1]) \nearrow \int_{[0,1]} \text{AVaR}_t(Z) d\mu^*(t). \quad (3.13)$$

Similarly, (3.8) can be approximated from above as  $N \rightarrow \infty$  with

$$\sum_{i=0}^{N-2} \text{AVaR}_{i+1/N}(Z) \mu^*\left(\left[\frac{i}{N}, \frac{i+1}{N}\right)\right) + \|Z\|_\infty \mu^*\left(\left[\frac{N-1}{N}, 1\right]\right) \searrow \int_{[0,1]} \text{AVaR}_t(Z) d\mu^*(t). \quad (3.14)$$

Hence, letting  $\mu^*\left[\frac{i}{N}, \frac{i+1}{N}\right) \triangleq \lambda_{i/N}$  for  $i = 0, 1, \dots, N-2$  and  $\mu^*\left[\frac{N-1}{N}, 1\right] \triangleq \lambda_1$ , we conclude the proof.  $\square$

**Lemma 3.3.** *Let  $\epsilon > 0$  be given arbitrarily small and  $\rho : \mathcal{Z} \rightarrow \mathbb{R}$  be a given law invariant coherent risk measure. Then, there exists  $\lambda \in [0, 1]$  depending on  $Z \in \mathcal{Z}$  and on  $\epsilon$  such that*

$$|\rho(Z) - (\lambda \mathbb{E}[Z] + (1 - \lambda) \|Z\|_\infty)| \leq \epsilon. \quad (3.15)$$

*Proof.* The proof follows by the observation that any coherent risk measure  $\rho : \mathcal{Z} \rightarrow \mathbb{R}$  satisfies

$$\mathbb{E}[Z] \leq \rho(Z) \leq \|Z\|_\infty. \quad (3.16)$$

Hence, adjusting  $\lambda \in [0, 1]$ , we conclude (3.15).  $\square$

## 4 Examples

In this section, we give examples of unique Kusuoka sets for specific law invariant coherent risk measures motivated by financial and insurance mathematics. The first two have been adapted from [4], where it was worked on  $L^p(\Omega, \mathcal{F}, \mathbb{P})$  for  $1 \leq p < \infty$ . Hence, we derive their correspondent representations in  $\mathcal{Z}$ .

**Example 4.1.** (*Higher Order Premiums*) Let  $Z \in \mathcal{Z}$  and  $c \geq 1$ . Higher order premium on  $\mathcal{Z}$  is then defined as

$$\rho_{hop}(Z) \triangleq \inf_{t \in \mathbb{R}} \{t + c\|(Z - t)_+\|_\infty\}. \quad (4.17)$$

Without loss of generality, we can take  $Z$  to be monotone with taking  $F_Z^{-1}$  instead of  $Z$ . We consider two cases for infimum above. For  $t^* = \|Z\|_\infty$  in the above infimum, we have  $(Z - \|Z\|_\infty)_+ = 0$ . For  $t^* < \|Z\|_\infty$ , we have by definition of  $\|\cdot\|_\infty$  norm,  $\|(Z - t^*)_+\|_\infty = \|Z\|_\infty - t^*$ . Since  $c \geq 1$ , we have  $t^* + c(\|Z\|_\infty - t^*)$ , and since we want to minimize with respect to  $t^*$  subject to  $t^* < \|Z\|_\infty$ , we choose  $t^* = \|Z\|_\infty$ . Hence, (4.17) equals to  $\rho_{hop}(Z) = \|Z\|_\infty$ . In particular, unique attaining Kusuoka set  $\mu^*$  in (3.8) is  $\bar{\mathfrak{M}}_{hop} = \{\delta_1\}$ , where  $\delta_1$  is the measure that puts whole mass 1 on  $\{1\}$ .

**Example 4.2.** (*The Dutch Premium Principle*) Let  $Z \in \mathcal{Z}$  and  $\lambda \in [0, 1]$ . Consider

$$\rho_{dp}(Z) = \mathbb{E}[Z] + \lambda\|(Z - \mathbb{E}[Z])_+\|_\infty. \quad (4.18)$$

By definition of  $\|\cdot\|_\infty$ , we have

$$\|(Z - \mathbb{E}[Z])_+\|_\infty = \|Z\|_\infty - \mathbb{E}[Z] \quad (4.19)$$

Thus, (4.18) reads as

$$\rho_{dp}(Z) = (1 - \lambda)\mathbb{E}[Z] + \lambda\|Z\|_\infty, \quad (4.20)$$

for  $\lambda \in [0, 1]$ . In particular, attaining Kusuoka set  $\bar{\mathfrak{M}}_{dp} = \{(1 - \lambda)\delta_0 + \lambda\delta_1\}$ .

The next example investigates the lower and upper bounds from Lemma 3.2 in a discrete uniform probability space.

**Example 4.3.** (*Coherent Risk Measure on Discrete Space*) The Kusuoka representation for  $Z \in \mathcal{Z}_{disc}$  holds as in (3.8). Let  $N \geq 1$  be a fixed integer. Consider, then  $\mu^*$  being the uniform measure on  $[0, 1]$ . Define the law invariant coherent risk measure as

$$\rho_{disc}(Z) \triangleq \int_{[0,1]} \text{AVaR}_t(Z) d\mu^*(t) \quad (4.21)$$

$$= \int_0^1 \text{AVaR}_t(Z) dt. \quad (4.22)$$

Consider, then the upper bound for  $\rho_{disc}$  as in (4.21)

$$\frac{1}{N-1} \sum_{i=2}^N \frac{1}{N-i+1} \sum_{j=i}^N Z_{(j)} \quad (4.23)$$

and the lower bound

$$\frac{1}{N-1} \sum_{i=1}^{N-1} \frac{1}{N-i+1} \sum_{j=i}^N Z_{(j)} \quad (4.24)$$

with  $Z_{(j)}$  for  $j = 1, \dots, N$  being the order statistics of  $Z$ . Here, note that

$$\text{AVaR}_{i/N}(Z) = \frac{1}{N-i+1} \sum_{j=i}^N Z_{(j)}, \quad (4.25)$$

for  $i = 1, \dots, N$ .

## 4.1 Numerical Illustrations

**Example 4.4.** (*Higher Order Premium*) For illustrative verifying, let  $c = 3$  and  $Z \in \mathcal{Z}$  take values between 0 and 3 such that  $\|Z\|_\infty = 3$ . Define

$$\rho_t(Z) \triangleq t + 3\|(Z-t)_+\|_\infty \quad (4.26)$$

In particular,

$$\rho_{hop}(Z) = \min_{t \in \mathbb{R}} \rho_t(Z) = 3. \quad (4.27)$$

The graph of  $\rho_t(Z)$  with respect to  $t$  is to be seen in Figure 1.

**Example 4.5.** (*The Dutch Premium*) Consider  $Z \in \mathcal{Z}$  such that  $\mathbb{E}[Z] = 3.5$  and  $\|Z\|_\infty = 6$ . Then using (4.20), the graph of

$$\lambda : [0, 1] \rightarrow (1-\lambda)\mathbb{E}[Z] + \lambda\|Z\|_\infty \quad (4.28)$$

is to be seen in Figure 2.

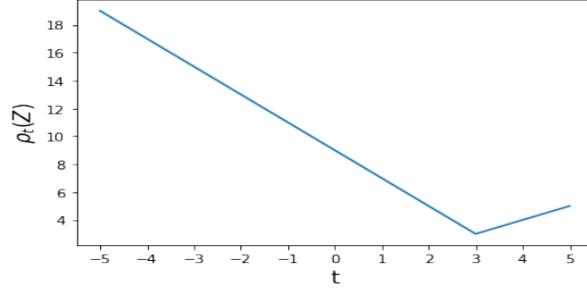


Figure 1: Higher Order Premium

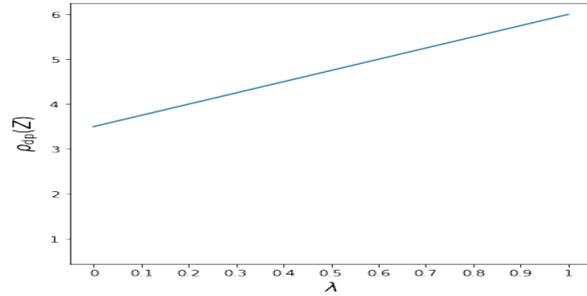


Figure 2: Dutch Premium

**Example 4.6.** Let  $Z \in \mathcal{Z}_{disc}$  with  $N = 6$  and  $Z_{(i)} = i$  for  $i \in \{1, 2, \dots, 6\}$ . Then, we let

$$\rho_{disc}(Z) \triangleq \sum_{i=1}^6 \frac{\lambda_i}{6-i+1} \sum_{j=i}^6 Z_{(j)} \quad (4.29)$$

$$\lambda_1 \triangleq [0.00, 0.10, 0.15, 0.20, 0.25, 0.30] \quad (4.30)$$

$$\lambda_2 \triangleq [0.15, 0.17, 0.23, 0.25, 0.07, 0.13] \quad (4.31)$$

$$\lambda_3 \triangleq [0.05, 0.08, 0.42, 0.11, 0.09, 0.25] \quad (4.32)$$

$$\lambda_4 \triangleq [0.00, 0.00, 0.00, 0.20, 0.30, 0.50] \quad (4.33)$$

Hence, the corresponding values of  $\rho_{disc}(Z)$  with respect to  $\lambda_1, \dots, \lambda_4$  are to be seen as in Figure 3.

## 5 Conclusion

In this paper, we have studied attainment of supremum in Kusuoka representations of coherent risk measures in the space of essentially bounded random variables. We have shown the supremum representation in [6] can be given as a maximal representation with the attaining corresponding probability measure  $\mu^*$ . Furthermore, we show that law invariant coherent

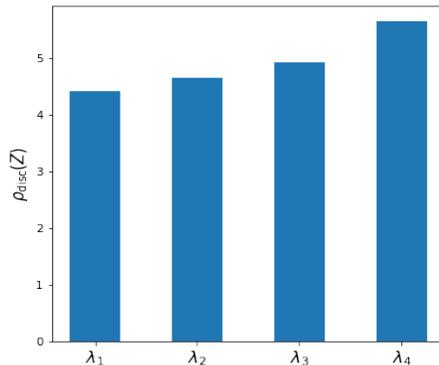


Figure 3: Coherent Risk Measure on Discrete Space

risk measures on  $\mathcal{Z}$  are not necessarily comonotonic. In fact, it is shown in [11] the only law invariant comonotonic operators that is linear along the direction of a nonconstant random variable with nonzero expectation are expectation functionals. Our work deals with the essentially bounded random variables on  $\mathcal{Z}$ . This is in contrast with [3] and [4], where the Kusuoka representation is studied in  $L^p(\Omega, \mathcal{F}, \mathbb{P})$  for  $1 \leq p < \infty$ . We show that the attaining measure for random variables with different distributions can be different among the set of alternative probability measures defining Kusuoka set of probability measures. We have also given a numerical scheme to approximate any law invariant coherent risk measure from below and above.

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