

Proximity in Concave Integer Quadratic Programming ^{*}

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Abstract

A classic result by Cook, Gerards, Schrijver, and Tardos provides an upper bound of $n\Delta$ on the proximity of optimal solutions of an Integer Linear Programming problem and its standard linear relaxation. In this bound, n is the number of variables and Δ denotes the maximum of the absolute values of the subdeterminants of the constraint matrix. Hochbaum and Shanthikumar, and Werman and Magagnosc showed that the same upper bound is valid if a more general convex function is minimized, instead of a linear function. No proximity result of this type is known when the objective function is nonconvex. In fact, if we minimize a concave quadratic, no upper bound can be given as a function of n and Δ . Our key observation is that, in this setting, proximity phenomena still occur, but only if we consider also approximate solutions instead of optimal solutions only. In our main result we provide upper bounds on the distance between approximate (resp., optimal) solutions to a Concave Integer Quadratic Programming problem and optimal (resp., approximate) solutions of its continuous relaxation. Our bounds are functions of n , Δ , and a parameter ϵ that controls the quality of the approximation. Furthermore, we discuss how far from optimal are our proximity bounds.

Key words: integer quadratic programming; quadratic programming; concave minimization; proximity; sensitivity; subdeterminants

1 Introduction

The relationship between an Integer Linear Programming problem and its standard linear relaxation plays a crucial role in many theoretical and computational aspects of the field, including perfect formulations, cutting planes, and branch-and-bound. Proximity results study one of the most fundamental questions regarding this relationship: Is it possible to bound the distance between optimal solutions to an Integer Linear Programming problem and its standard linear relaxation? A classic result by Cook, Gerards, Schrijver, and Tardos [4] provides the upper bound $n\Delta$ for this distance, where n is the number of variables and Δ denotes the maximum of the absolute values of the subdeterminants of the constraint matrix. This bound has been recently extended to the mixed-integer case by Paat et al. [15] to $p\Delta$, where p is the number of integer variables. For other recent proximity results in Integer Linear Programming, we refer the reader to [11, 19, 1].

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Granot and Skorin-Kapov [12] show that the upper bound $n\Delta$ is still valid if we minimize a separable convex quadratic objective function over the integer points in a polyhedron. This result has been further extended to separable convex objective functions by Hochbaum and Shanthikumar [13] and by Werman and Magagnosc [18].

All the above results feature a convex objective function to be minimized. Therefore, a natural question is whether proximity phenomena only occur in the presence of convexity. The next example seems to indicate that this is indeed the case. In fact it shows that, with a concave objective, the distance between optimal solutions of the discrete and continuous problems cannot be bounded by any function of n, Δ .

Example 1. Consider the following optimization problem for every $t \in \mathbb{Z}$ with $t \geq 0$:

$$\begin{aligned} \min \quad & -\left(x - \frac{1}{4}\right)^2 \\ \text{s.t.} \quad & -t \leq x \leq t + \frac{3}{4} \\ & x \in \mathbb{Z}. \end{aligned} \tag{1}$$

Note that all these problems have dimension one ($n = 1$) and $\Delta = 1$. Clearly, the unique optimal solution to (1) is $x^d := -t$. If we drop the integer constraint, then the unique optimal solution is $x^c := t + \frac{3}{4}$. We have $|x^d - x^c| = 2t + \frac{3}{4}$, which goes to infinity as t approaches infinity. \diamond

Example 1 explains the lack of proximity results in the nonconvex setting. However, a key observation is that the solution $x^* = t$, while not optimal to (1), is ‘almost’ optimal. Furthermore, its distance from x^c is always $\frac{3}{4}$. This simple observation leads us to the question that is at the basis of this work: Is it possible to bound the distance between *approximate* (resp., *optimal*) solutions to a Nonconvex Integer Programming problem and *optimal* (resp., *approximate*) solutions of its continuous relaxation? This paper provides the first answers to the posed question.

The optimization problem in Example 1 belongs to perhaps the simplest class of Nonconvex Integer Programming problems, formed by Integer Quadratic Programming problems with separable concave objective functions. Therefore, in this paper we focus on this class of optimization problems. For these problems, we answer our question in the affirmative and provide explicit upper bounds. Our bounds are functions of n, Δ , and a parameter ϵ that controls the quality of the approximation. Furthermore, we discuss how far from optimal are our proximity bounds.

In the remainder of this section we formally introduce Separable Concave Integer Quadratic Programming and ϵ -approximate solutions. With the notation in place, we then formally state our results.

1.1 Separable Concave Integer Quadratic Programming

In this paper we denote by (IQP) the Separable Concave Integer Quadratic Programming problem

$$\begin{aligned} \min \quad & \sum_{i=1}^k -q_i x_i^2 + h^\top x \\ \text{s.t.} \quad & Ax \leq b \\ & x \in \mathbb{Z}^n. \end{aligned} \tag{IQP}$$

In this formulation we assume that $q_i > 0$ for every $i = 1, \dots, k$. Furthermore, we assume that the matrix A is integer, while the remaining data is real. Clearly (IQP) subsumes Integer Linear

Programming, which can be obtained by setting $k = 0$. We refer the reader to [10, 6, 9, 7, 8] for recent theoretical results on (IQP).

We denote by (QP) the Separable Concave Quadratic Programming problem obtained from (IQP) by dropping the integer constraints, i.e.,

$$\begin{aligned} \min \quad & \sum_{i=1}^k -q_i x_i^2 + h^\top x \\ \text{s.t.} \quad & Ax \leq b \\ & x \in \mathbb{R}^n. \end{aligned} \tag{QP}$$

Throughout this paper, we denote by $f(x) := \sum_{i=1}^k -q_i x_i^2 + h^\top x$ the objective function of (IQP) and (QP), which is quadratic, concave, and separable. Furthermore, we let P be the polyhedron defined by $P := \{x \in \mathbb{R}^n \mid Ax \leq b\}$, and we denote by Δ the largest absolute value of the subdeterminants of A .

1.2 ϵ -approximate solution

In order to state our proximity results, we give the definition of ϵ -approximate solution. Consider an instance of an optimization problem of the form $\min\{f(x) \mid x \in S\}$, where $S \subseteq \mathbb{R}^n$. We assume that this problem has an optimal solution, and we denote it by x^{opt} . Let f_{\max} be the maximum value of $f(x)$ on the feasible region S . For $\epsilon \in [0, 1]$, we say that a feasible point x^* is an ϵ -approximate solution if

$$f(x^*) - f(x^{\text{opt}}) \leq \epsilon \cdot (f_{\max} - f(x^{\text{opt}})).$$

An intuitive way to interpret this definition is as follows: If we let $[\alpha, \beta]$ be the smallest interval containing the image of S under f , then $f(x^*)$ should lie in the interval $[\alpha, \alpha + \epsilon(\beta - \alpha)]$. Observe that any feasible point is a 1-approximation, and only an optimal solution is a 0-approximation. If $f(x)$ has no upper bound on the feasible region, our definition loses its value because any feasible point is an ϵ -approximation for any $\epsilon > 0$. Our definition of approximation has been used in earlier works, and we refer to [14, 17, 2, 5] for more details.

In this work we consider ϵ -approximate solutions to (IQP) and to (QP). Clearly, the optimal solution x^{opt} and the quantity f_{\max} in the definition of ϵ -approximate solution differ for the two problems because the feasible regions are different. To avoid confusion, throughout the paper we denote by x^d an optimal solution to (IQP) and by x^c an optimal solution to (QP). Similarly, we denote by f_{\max}^d the value f_{\max} in the definition of ϵ -approximate solution to (IQP) and by f_{\max}^c the value f_{\max} in the definition of ϵ -approximate solution to (QP).

The definition of ϵ -approximate solution is natural for these general problems, and has several useful properties. It is well known that, for continuous optimization problems, the definition is insensitive to translations or dilations of the objective function, and that it is preserved under affine linear transformations of the problem. Similar invariance properties hold for discrete optimization problem, and are formalized in Lemma 1 in Section 2.

1.3 Our results

We are ready to state our proximity result for Separable Concave Integer Quadratic Programming.

Theorem 1. *Consider a problem (IQP), and the corresponding continuous problem (QP). Suppose that both problems have an optimal solution. Then:*

- (i) Let x^c be an optimal solution to (QP). Then, $\forall \epsilon \in (0, 1]$, there is an ϵ -approximate solution x^* to (IQP) such that $\|x^c - x^*\|_\infty \leq n\Delta \left(\frac{10\Delta}{\epsilon} + 1\right)^k$.
- (ii) Let x^d be an optimal solution to (IQP). Then, $\forall \epsilon \in (0, 1]$, there is an ϵ -approximate solution x^* to (QP) such that $\|x^* - x^d\|_\infty \leq n\Delta \left(\frac{10\Delta}{\epsilon} + 1\right)^k$.

In particular, note that the bounds in Theorem 1 do not depend on the right-hand side vector b in (IQP) and (QP). The proof of Theorem 1 is given in Section 3. Since $k \leq n$, Theorem 1 implies that, for every optimal solution to one of the two problems, and for every $\epsilon \in (0, 1]$, there is an ϵ -approximate solution to the other problem at distance bounded by a function of n, Δ, ϵ . In particular, this distance is independent on the objective function and on the vector b . Note that, for $k = 0$, problem (IQP) is an Integer Linear Programming problem, while (QP) is its standard linear relaxation. In this setting, our bounds in Theorem 1 reduce to $n\Delta$ for every $\epsilon > 0$. Therefore, the proximity bound by Cook et al. [4] can be obtained as a corollary to Theorem 1.

In Section 4, we discuss how far from optimal are our upper bounds in Theorem 1. At the heart of our tightness results lies a special polytope, denoted by \bar{P} , and which is used with several different objective functions. In particular, using the notation of Theorem 1, we show that any upper bound on $\|x^c - x^*\|_\infty$ or on $\|x^* - x^d\|_\infty$ must grow at least linearly with $\frac{1}{\epsilon}$, n , and Δ . Furthermore, we show that the neighborhood of x^c considered by Cook et al. [4], namely $\{x \in P \cap \mathbb{Z}^n \mid \|x^c - x\|_\infty \leq n\Delta\}$ might contain only arbitrarily bad solutions to (IQP), i.e., vectors that are not ϵ -approximate solution to (IQP), for any $\epsilon \in (0, 1)$. The polytope \bar{P} also allows us to show that the Integer Linear Programming bound $n\Delta$ by Cook et al. is best possible. To the best of our knowledge this tightness result was known only for $\Delta = 1$ (see page 241 in [16]).

2 Three simple lemmas

In this section we present three lemmas that will be used in the proof of Theorem 1.

Our first lemma formalizes the invariance properties of ϵ -approximate solutions to optimization problems with integer constraints. The proof is standard. This result will allow us to greatly simplify the notation in the main proof.

Lemma 1. *Consider an optimization problem of the form*

$$\begin{aligned} \min f(x) \\ \text{s.t. } x \in S \cap \mathbb{Z}^n, \end{aligned} \tag{O}$$

where $S \subseteq \mathbb{R}^n$. Let $M \in \mathbb{Z}^{n \times n}$ be a unimodular matrix, and let $t \in \mathbb{Z}^n$. For any $\alpha, \beta \in \mathbb{R}$ with $\alpha > 0$, consider the optimization problem

$$\begin{aligned} \min \alpha f(M^{-1}(y - t)) + \beta \\ \text{s.t. } y \in U(S) \cap \mathbb{Z}^n, \end{aligned} \tag{O'}$$

where $U(S) := \{y \in \mathbb{R}^n \mid y = Mx + t, x \in S\}$. Then, for every ϵ -approximate solution to (O), denoted by x^* , the vector $Mx^* + t$ is an ϵ -approximate solution to (O'). Viceversa, for every ϵ -approximate solution to (O'), denoted by y^* , the vector $M^{-1}(y^* - t)$ is an ϵ -approximate solution to (O).

Proof. We prove the first statement of the lemma, the second one being symmetric. Let x^* be an ϵ -approximate solution to (O). We show that the vector $y^* := Mx^* + t$ is an ϵ -approximate solution to (O').

Since M and t are integer, for every feasible solution x to (O), the vector $y = Mx + t$ is feasible to (O'). Viceversa, since M^{-1} and t are integer, for every feasible solution y to (O'), the vector $x = M^{-1}(y - t)$ is feasible to (O). In both cases, the relation between the cost of x and y is given by $g(y) = \alpha f(x) + \beta$, where $g(y) := \alpha f(M^{-1}(y - t)) + \beta$ denotes the objective function of (O').

Let x^d be an optimal solution to (O), and let y^d be an optimal solution to (O'). Furthermore, let f_{\max} be the maximum value of $f(x)$ on the feasible region of (O), and let g_{\max} be the maximum value of $g(y)$ on the feasible region of (O'). Since $\alpha > 0$, the above argument in particular implies $g(y^d) = \alpha f(x^d) + \beta$, and $g_{\max} = \alpha f_{\max} + \beta$. If $g_{\max} = g(y^d)$, then y^* is an optimal solution to (O') and we are done. Otherwise, we have

$$\frac{g(y^*) - g(y^d)}{g_{\max} - g(y^d)} = \frac{(\alpha f(x^*) + \beta) - (\alpha f(x^d) + \beta)}{(\alpha f_{\max} + \beta) - (\alpha f(x^d) + \beta)} = \frac{f(x^*) - f(x^d)}{f_{\max} - f(x^d)} \leq \epsilon.$$

Thus y^* is an ϵ -approximate solution to (O'). □

Next, we define a polyhedral cone which will be heavily used in the proof of Theorem 1, and we present some of its properties. We remark that this cone has been used in several papers to obtain proximity results, including [4, 12, 13, 8]. Let A be a matrix with n columns and let $x^a, x^b \in \mathbb{R}^n$. Let A_1 be the matrix that contains all rows u in A for which $ux^a \leq ux^b$. Similarly, let A_2 be the matrix that contains all rows u in A for which $ux^a \geq ux^b$. We define the polyhedral cone

$$T(A, x^a, x^b) := \{x \in \mathbb{R}^n \mid A_1 x \leq 0, A_2 x \geq 0\}.$$

From the definition of the cone, we obtain $x^a - x^b \in T(A, x^a, x^b)$.

The next lemma is well-known, see, e.g., [4]. Since we are unable to find a complete proof in the literature, we present it here.

Lemma 2. *Let A be an integer matrix with n columns, let Δ be the largest absolute value of the subdeterminants of A , and let $x^a, x^b \in \mathbb{R}^n$. Then there exists a finite subset V of \mathbb{Z}^n such that $T(A, x^a, x^b) = \text{cone } V$, and for every $v \in V$, we have $\|v\|_\infty \leq \Delta$.*

Proof. Let $T := T(A, x^a, x^b)$. We partition T into pointed polyhedral cones by intersecting it with the 2^n orthants of \mathbb{R}^n , which we denote by O_1, \dots, O_{2^n} . Namely, we let $T_i := T \cap O_i$, for $i = 1, \dots, 2^n$, and observe that $T = \bigcup_{i=1}^{2^n} T_i$. In order to prove the lemma, it suffices to show that, for every $i = 1, \dots, 2^n$, there exists a finite subset V_i of \mathbb{Z}^n such that $T_i = \text{cone } V_i$, and for every $v \in V_i$, we have $\|v\|_\infty \leq \Delta$. This is because the set $V = \bigcup_{i=1}^{2^n} V_i$ then satisfies the thesis of the lemma.

Let us now consider a single T_i , for some $i \in \{1, \dots, 2^n\}$. We assume that T_i arises from the intersection of T with the nonnegative orthant, i.e., $T_i = \{x \in \mathbb{R}^n \mid x \in T, x \geq 0\}$, the other cases being symmetric. The set T_i is a pointed polyhedral cone. Since A is integer, T_i is a rational cone. Therefore, there exists a finite set of vectors $V_i = \{r^1, \dots, r^m\} \subset \mathbb{R}^n$ such that $T_i = \text{cone } V_i$. Here we can assume that for every $j = 1, \dots, m$, the vector r^j is not a proper conic combination of other vectors in V_i , that is to say, each r^j is an extreme ray of T_i .

Let us now consider a single vector r^j , for some $j \in \{1, \dots, m\}$. We now show that we can scale r^j so that it is integer and with infinity norm at most Δ . From Theorem 3.35 in [3], we know that r^j satisfies at equality $n - 1$ linearly independent inequalities in the system $A_1 x \leq 0, A_2 x \geq 0, x \geq 0$. Let e_k be a vector of the standard basis of \mathbb{R}^n that is linearly independent from all the rows of A_1, A_2 , and the identity matrix which correspond to the $n - 1$ linearly independent inequalities. Note that we have $r_k^j \neq 0$, since otherwise we obtain $r^j = 0$, which contradicts the fact that r^j is an extreme ray of T_i . Since T_i is contained in the nonnegative orthant, we have $r_k^j > 0$. Denote by

$Dx = e_1$ the system of equations containing $x_k = 1$ and the $n - 1$ equations arising by setting to equality the $n - 1$ linearly independent inequalities discussed above, where e_1 is the first vector of the standard basis of \mathbb{R}^n . Note that the matrix D is invertible. The vector $r := D^{-1}e_1$ is a solution to the system, and is a scaling of the vector r^j . Note that each entry of r coincides with an entry of the matrix D^{-1} . By Cramer's rule, each entry of D^{-1} is a fraction with denominator $\det(D)$ and numerator with absolute value at most Δ . Thus, the vector $|\det(D)| \cdot r$ is a scaling of r^j that is integer and with $\| |\det(D)| \cdot r \|_\infty \leq \Delta$. Hence, we can assume that each vector r^j is integer and with infinity norm at most Δ . \square

The next lemma will often be used in the proof of Theorem 1 to show that a given vector is in our polyhedron P .

Lemma 3. *Let $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ be a polyhedron, let $x^a, x^b \in P$. Let x° be a vector in \mathbb{R}^n that can be written in the following two ways:*

$$x^\circ = x^1 + \sum_{i=1}^m \alpha_i v^i, \quad x^\circ = x^2 - \sum_{i=1}^m \beta_i v^i,$$

where $x^1, x^2 \in P$ and, for $i = 1, \dots, m$, α_i, β_i , are nonnegative numbers and $v^i \in T(A, x^a, x^b)$. Then $x^\circ \in P$.

Proof. Let A_1, A_2 from the definition of $T(A, x^a, x^b)$ and b_1, b_2 the corresponding sub-vectors of b . Since $x^1, x^2 \in P$, we obtain

$$\begin{aligned} A_1 x^\circ &= A_1 \left(x^1 + \sum_{i=1}^m \alpha_i v^i \right) \leq b_1 + \sum_{i=1}^m \alpha_i A_1 v^i \leq b_1 \\ A_2 x^\circ &= A_2 \left(x^2 - \sum_{i=1}^m \beta_i v^i \right) \leq b_2 - \sum_{i=1}^m \beta_i A_2 v^i \leq b_2, \end{aligned}$$

where the last inequalities follow because $A_1 v^i \leq 0$ and $A_2 v^i \geq 0$ from the definition of $T(A, x^a, x^b)$. This implies that $Ax^\circ \leq b$, hence $x^\circ \in P$. \square

We are now ready to present our proof of Theorem 1.

3 Proof of Theorem 1

Let x^c be an optimal solution to (QP), let x^d be an optimal solution to (IQP), and let $\epsilon > 0$. In this section, we present our proof of Theorem 1. To do so, we will construct an ϵ -approximate solution x^* to (IQP) and an ϵ -approximate solution x^* to (QP).

We first give a brief outline of our proof. In Section 3.1, we design a recursive algorithm which takes in input x^c, x^d, P and that outputs a point $x^\ell \in P$. In the algorithm, we use several times cones of the form $T(A, \cdot, \cdot)$ to construct a path inside P , which starts at x^c , ends at x^ℓ and contains at most $k + 1$ points. The special structure of this path enables us to upper bound $\|x^c - x^\ell\|_\infty$ by a function of n, Δ, k, ϵ . In Section 3.2, we study some properties of x^ℓ and we consider separately two cases. In the first case, $\|x^\ell - x^d\|_\infty$ can be bounded by a function of n, Δ, k, ϵ . In this case, we can then also bound $\|x^c - x^d\|_\infty$ by a function of n, Δ, k, ϵ . As a consequence, we can conclude the proof in the first case by choosing x^* to be x^d and by choosing x^* to be x^c . In the second case, $|x_i^\ell - x_i^d|$ is large for every index $i \in \{1, \dots, k\}$ such that $x_i^\ell - x_i^d \neq 0$. In this case, in Section 3.3 we use

vectors x^ℓ, x^d to construct an integer vector x^* which is close to x^ℓ , that is, $\|x^\ell - x^*\|_\infty \leq n\Delta$. This in particular implies that $\|x^c - x^*\|_\infty$ can be bounded by a function of n, Δ, k, ϵ . In Section 3.4 we further study the vector x^* . The properties obtained allow us to prove, in Section 3.5, that x^* is an ϵ -approximate solution to (IQP). This is done by providing an upper bound on $f(x^*) - f(x^d)$ and a lower bound on $f_{\max}^d - f(x^d)$. In Section 3.6 we define a vector x^\star , based on x^* , with the property that $\|x^\star - x^d\|_\infty = \|x^c - x^*\|_\infty$. Next, in Section 3.7, we prove that x^\star is an ϵ -approximate solution to (QP). This concludes the proof in the second case, and our outline of the proof of Theorem 1. We are now ready to present the full proof.

In order to simplify the notation in the remainder of the proof, in the next claim we employ Lemma 1.

Claim 1. *We can assume without loss of generality that x^d is the origin and that $f(x^d) = 0$.*

Proof. We apply Lemma 1 as follows: Problem (O) is (IQP), the matrix M is the identity, $t := -x_d$, $\alpha := 1$, and $\beta := -f(x^d)$. Problem (O') in Lemma 1 then takes the form

$$\begin{aligned} \min \quad & f(y + x^d) - f(x^d) \\ \text{s.t.} \quad & Ay \leq b - Ax^d \\ & y \in \mathbb{Z}^n. \end{aligned} \tag{2}$$

The objective function of (2) can then be explicitly written as $\sum_{i=1}^k -q_i y_i^2 + h'^T y$, where the vector h' is defined by $h'_i := h_i - 2q_i x_i^d$ for $i = 1, \dots, k$ and $h'_i := h_i$ for $i = k + 1, \dots, n$. In particular, the coefficients q_i of the quadratic monomials are identical in (IQP) and in (2). Furthermore, note that the constraint matrix of (2) is the same of (IQP) and so the two problems have the same Δ .

The transformation that maps problem (IQP) into (2) is $y = x - x_d$. Consider vectors $\bar{x}, \tilde{x}, \bar{y}, \tilde{y} \in \mathbb{R}^n$ such that $\bar{y} = \bar{x} - x_d$ and $\tilde{y} = \tilde{x} - x_d$. Then distances are maintained since $\bar{y} - \tilde{y} = \bar{x} - \tilde{x}$.

Lemma 1 implies that ϵ -approximate solutions are mapped to ϵ -approximate solutions, thus, without loss of generality we can consider problem (2) instead of (IQP). The optimal solution x^d of (IQP) is mapped to the origin, which is then an optimal solution of (2), and the optimal cost is zero. \square

3.1 Construction of the vector x^ℓ

This section of the proof is devoted to the construction of a special vector in P that we denote by x^ℓ . The vector x^ℓ is obtained via a recursive algorithm which utilizes the vectors x^c and x^d . To begin with, we introduce a claim which will play a key role in the iterative step of our algorithm.

Claim 2. *Let $x^a \in P$ and let $Z := \{i \in \{1, \dots, k\} \mid x_i^a = 0\}$. Assume $Z \neq \{1, \dots, k\}$, and let s be an index such that $|x_s^a| = \min\{|x_i^a| \mid i \in \{1, \dots, k\} \setminus Z\}$. Assume that $\|x^a\|_\infty > \Delta|x_s^a|$. Then there exists a vector $x^b \in P$ such that*

- (i) $x_i^b = 0$ for every $i \in Z \cup \{s\}$;
- (ii) $\|x^a - x^b\|_\infty \leq \Delta|x_s^a|$;
- (iii) For $i = 1, \dots, m$, there exist nonnegative scalars α_i, β_i and vectors $v^i \in T(A, x^a, x^d)$ such that x^b can be expressed in the following two ways:

$$x^b = x^d + \sum_{i=1}^m \alpha_i v^i, \quad x^b = x^a - \sum_{i=1}^m \beta_i v^i.$$

Proof. Our first task is that of defining the vector x^b . Denote by $\tilde{A}x \leq \tilde{b}$ the system obtained from $Ax \leq b$ by adding the inequalities $x_i \leq 0$, $-x_i \leq 0$, for all $i \in Z$. We remark that the largest absolute value of a subdeterminant of \tilde{A} is Δ . Let $\tilde{P} := \{x \in \mathbb{R}^n \mid \tilde{A}x \leq \tilde{b}\}$, and note that the vectors x^a and x^d are in \tilde{P} . Let $\tilde{T} := T(\tilde{A}, x^a, x^d)$. From Lemma 2, applied to \tilde{A} , x^a , and x^d , we know that there exists a finite subset \tilde{V} of \mathbb{Z}^n such that $\tilde{T} = \text{cone } \tilde{V}$, and for every $v \in \tilde{V}$, we have $\|v\|_\infty \leq \Delta$. Since $x^a - x^d \in \tilde{T}$, Caratheodory's theorem implies that there exist $m \leq n$ vectors $v^1, \dots, v^m \in \tilde{V}$ and m positive scalars $\alpha_1, \dots, \alpha_m$ such that

$$x^a - x^d = \sum_{i=1}^m \alpha_i v^i. \quad (3)$$

We pick those vectors v^i such that v_s^i has the same sign as x_s^a . Without loss of generality, we can assume that these vectors are v^1, \dots, v^r , where $r \leq m$. We now show that there exist nonnegative scalars $\lambda_1, \dots, \lambda_r$ that satisfy $\lambda_1 \leq \alpha_1, \dots, \lambda_r \leq \alpha_r$, and such that

$$x_s^a = \sum_{i=1}^r \lambda_i v_s^i. \quad (4)$$

From (3), we obtain $x_s^a = \sum_{i=1}^m \alpha_i v_s^i$. Since for $i = 1, \dots, r$, v_s^i has the same sign as x_s^a and for $i = r+1, \dots, m$, v_s^i has the opposite sign of x_s^a or $v_s^i = 0$, we have

$$0 < |x_s^a| = \left| \sum_{i=1}^r \alpha_i v_s^i \right| - \left| \sum_{i=r+1}^m \alpha_i v_s^i \right| \leq \left| \sum_{i=1}^r \alpha_i v_s^i \right|.$$

Using continuity, we know that there exist nonnegative scalars $\lambda_1 \leq \alpha_1, \dots, \lambda_r \leq \alpha_r$, such that $|x_s^a| = \left| \sum_{i=1}^r \lambda_i v_s^i \right|$. Since each v_s^i above has the same sign as x_s^a and each $\lambda_i \geq 0$, we know that (4) holds.

We are finally ready to define the vector x^b as

$$x^b := x^a - \sum_{i=1}^r \lambda_i v^i. \quad (5)$$

From (3), we can write x^b in the form

$$x^b = x^d + \sum_{i=1}^r (\alpha_i - \lambda_i) v^i + \sum_{i=r+1}^m \alpha_i v^i. \quad (6)$$

Since $x^a, x^d \in \tilde{P}$, from Lemma 3 we know that x^b is in \tilde{P} as well. Since $\tilde{P} \subseteq P$, we obtain that $x^b \in P$. Next we show that (i), (ii), (iii) hold.

(i). Note that \tilde{P} satisfies equations $x_i = 0$, for $i \in Z$, hence $x_i^b = 0$ for every $i \in Z$. Furthermore, from the definition of x^b , and using (4), we have

$$x_s^b = x_s^a - \sum_{i=1}^r \lambda_i v_s^i = x_s^a - x_s^a = 0.$$

(ii). From the definition of x^b we have $\|x^a - x^b\|_\infty = \left\| \sum_{i=1}^r \lambda_i v^i \right\|_\infty$. Denote by l the index such that $\left\| \sum_{i=1}^r \lambda_i v^i \right\|_\infty = \left| \left(\sum_{i=1}^r \lambda_i v^i \right)_l \right|$. Then we have

$$\left\| \sum_{i=1}^r \lambda_i v^i \right\|_\infty = \left| \sum_{i=1}^r \lambda_i v_l^i \right| = \left| \sum_{i=1}^r \lambda_i \frac{v_l^i}{v_s^i} v_s^i \right| \leq \sum_{i=1}^r \lambda_i \left| \frac{v_l^i}{v_s^i} \right| |v_s^i|.$$

Since, for every $i = 1, \dots, r$, the vector v^i is integer and $\|v^i\|_\infty \leq \Delta$, we know that $\left| \frac{v_l^i}{v_s^i} \right| \leq \Delta$. Thus,

$$\sum_{i=1}^r \lambda_i \left| \frac{v_l^i}{v_s^i} \right| |v_s^i| \leq \Delta \sum_{i=1}^r \lambda_i |v_s^i| = \Delta \left| \sum_{i=1}^r \lambda_i v_s^i \right| = \Delta |x_s^a|.$$

The first equality holds because all v_s^1, \dots, v_s^r have the same sign, and the last equality follows from (4). This completes the proof of (ii).

(iii). We notice that $\tilde{T} \subseteq T(A, x^a, x^d)$, which implies that $v^1, \dots, v^m \in T(A, x^a, x^d)$. So (iii) follows directly from (5) and (6). \square

We are now ready to state our algorithm that constructs the vector x^ℓ . We recursively define a sequence of vectors in P denoted by x^0, x^1, x^2, \dots . The last vector in this sequence is indeed the vector x^ℓ that we wish to obtain. To define this sequence of vectors, we first recursively define the following k scalars:

$$\begin{aligned} \chi_1 &:= \frac{8n\Delta}{\epsilon} + 2n\Delta \\ \chi_j &:= 2n\Delta + \frac{8}{\epsilon} \left(\sum_{i=1}^{j-1} \Delta \chi_i + n\Delta \right) \quad j = 2, \dots, k. \end{aligned}$$

For every vector x^j in the sequence, it will be useful to partition the set $\{1, \dots, k\}$ into the two sets

$$Z^j := \left\{ i \in \{1, \dots, k\} \mid x_i^j = 0 \right\}, \quad N^j := \left\{ i \in \{1, \dots, k\} \mid x_i^j \neq 0 \right\}.$$

We start the sequence by setting $x^0 := x^c$. Now assume that we have constructed the vectors x^0, x^1, \dots, x^j . We state the next iteration of the algorithm. In this iteration, either the algorithm sets $\ell := j$ and terminates, or it constructs the next vector x^{j+1} .

If x^j satisfies $|x_i^j| > \chi_{j+1}$ for every $i \in N^j$ then we set $\ell := j$ and terminate. Otherwise, we have $N^j \neq \emptyset$ and $|x_s^j| \leq \chi_{j+1}$, where s is an index such that $|x_s^j| = \min\{|x_i^j| \mid i \in N^j\}$.

If $\|x^j\|_\infty \leq \Delta |x_s^j|$, we set $\ell := j$ and terminate. Otherwise, we have $\|x^j\|_\infty > \Delta |x_s^j|$. Then x^j satisfies the assumptions of Claim 2. The next vector x^{j+1} in the sequence is then defined to be the vector $x^b \in P$ in the statement of Claim 2, invoked with $x^a = x^j$.

This concludes our definition of the sequence x^0, x^1, \dots, x^ℓ of vectors in P . Note that the sequence contains at most $k + 1$ points, i.e., $\ell \leq k$. In fact, according to Claim 2(i), we know that Z^{j+1} has at least one more element than Z^j , for every $j = 0, 1, \dots$.

3.2 Properties of the vector x^ℓ

For ease of notation, we define

$$\psi_j := \sum_{i=1}^j \Delta \chi_i \quad j = 1, \dots, k,$$

and obtain an upper bound on ψ_k .

Claim 3. *We have $\psi_k \leq n\Delta \left(\frac{10\Delta}{\epsilon} + 1 \right)^k - n\Delta$.*

Proof. The number $\psi_j + n\Delta$ can be upper bounded as follows:

$$\begin{aligned}\psi_j + n\Delta &= 2n\Delta^2 + \left(\frac{8\Delta}{\epsilon} + 1\right) (\psi_{j-1} + n\Delta) \\ &\leq \Delta(\psi_{j-1} + n\Delta) + \left(\frac{8\Delta}{\epsilon} + 1\right) (\psi_{j-1} + n\Delta) \\ &\leq \left(\frac{9\Delta}{\epsilon} + 1\right) (\psi_{j-1} + n\Delta).\end{aligned}$$

The equality holds by definition of ψ_j and χ_j . The first inequality follows from the fact that $n\Delta \leq \psi_i$ for every $i = 1, \dots, k$, while the second inequality is correct because $\Delta \leq \frac{\Delta}{\epsilon}$.

Then we have

$$\psi_k + n\Delta \leq \left(\frac{9\Delta}{\epsilon} + 1\right)^{k-1} (\psi_1 + n\Delta).$$

Since

$$\psi_1 + n\Delta = \frac{8n\Delta^2}{\epsilon} + 2n\Delta^2 + n\Delta = n\Delta \left(\frac{8\Delta}{\epsilon} + 2\Delta + 1\right) \leq n\Delta \left(\frac{10\Delta}{\epsilon} + 1\right),$$

we get

$$\psi_k + n\Delta \leq n\Delta \left(\frac{9\Delta}{\epsilon} + 1\right)^{k-1} \left(\frac{10\Delta}{\epsilon} + 1\right) \leq n\Delta \left(\frac{10\Delta}{\epsilon} + 1\right)^k. \quad \square$$

In the next claim we highlight some properties of x^ℓ that will be used later.

Claim 4. *The vector x^ℓ satisfies the following properties:*

- (a) $x^c - x^\ell \in P$;
- (b) $\|x^c - x^\ell\|_\infty \leq \psi_\ell$;
- (c) *At least one of the following holds:*
 - (c-1) $\|x^\ell\|_\infty \leq \Delta\chi_{\ell+1}$, with $\ell \leq k-1$;
 - (c-2) $|x_i^\ell| > \chi_{\ell+1}$ for every $i \in N^\ell$.

Proof. (a). We prove the stronger statement that $x^c - x^j \in P$ for every $j = 0, \dots, \ell$, by induction on j . The base case is $j = 0$, and it holds since $x^c - x^0 = 0 = x^d \in P$.

Next, we show the inductive step. We assume that the result is true for $j = t$, and we prove it for $j = t+1$. From our definition of the sequence x^0, x^1, \dots, x^ℓ , the vector x^{t+1} is obtained from x^t as described in Claim 2, where $x^a = x^t$ and $x^b = x^{t+1}$. Claim 2(iii) implies that for $i = 1, \dots, m$ there exist nonnegative scalars α_i, β_i and vectors $v^i \in T(A, x^t, x^d)$ such that

$$\begin{aligned}x^c - x^{t+1} &= (x^c - x^t) + (x^t - x^{t+1}) = (x^c - x^t) + \sum_{i=1}^m \beta_i v^i \\ x^c - x^{t+1} &= x^c - x^d - \sum_{i=1}^m \alpha_i v^i = x^c - \sum_{i=1}^m \alpha_i v^i.\end{aligned}$$

Clearly $x^c \in P$ and, from the induction hypothesis, $x^c - x^t \in P$ as well. Then Lemma 3 implies that $x^c - x^{t+1} \in P$. This concludes our proof that $x^c - x^j \in P$ for every $j = 0, \dots, \ell$. Therefore $x^c - x^\ell \in P$, concluding the proof of (a).

(b). According to Claim 2(ii), and from the definition of the sequence, we know that

$$\|x^{j-1} - x^j\|_\infty \leq \Delta |x_s^{j-1}| \leq \Delta \chi_j \quad j = 1, \dots, \ell.$$

Thus, we have

$$\|x^c - x^\ell\|_\infty \leq \sum_{j=1}^{\ell} \|x^{j-1} - x^j\|_\infty \leq \sum_{j=1}^{\ell} \Delta \chi_j = \psi_\ell.$$

(c). This proof follows from the definition of x^ℓ . In fact, since x^ℓ is the last point in the sequence, it must satisfy at least one of the two termination conditions. If $|x_i^\ell| > \chi_{\ell+1}$ for every $i \in N^\ell$, then (c-2) holds and we are done. Note that, if $\ell = k$, then $Z^\ell = \{1, \dots, k\}$, and this termination condition is triggered. Otherwise, we have $\|x^\ell\|_\infty \leq \Delta |x_s^\ell|$ and $\ell \leq k - 1$. Observing that $|x_s^\ell| \leq \chi_{\ell+1}$ from the construction of the sequence, we obtain (c-1). \square

From (c), the vector x^ℓ satisfies at least one of the two properties (c-1), (c-2). Next we show that, if x^ℓ satisfies (c-1), then Theorem 1 holds with $x^* = x^d$ and $x^* = x^c$. So assume that the vector x^ℓ satisfies property (c-1). We obtain

$$\|x^c - x^d\|_\infty \leq \|x^c - x^\ell\|_\infty + \|x^\ell - x^d\|_\infty \leq \psi_\ell + \Delta \chi_{\ell+1} = \psi_{\ell+1} \leq \psi_k,$$

where the second inequality follows from (b) and (c-1). Hence the distance between x^d and x^c is upper bounded by ψ_k , which is at most $n\Delta(\frac{10\Delta}{\epsilon} + 1)^k$ from Claim 3. As a consequence, in this case, we conclude the proof of Theorem 1(i) with $x^* = x^d$ and of Theorem 1(ii) with $x^* = x^c$. Therefore, in the remainder of the proof, we assume that x^ℓ satisfies (c-2).

3.3 Construction of the vector x^*

This section of the proof is devoted to the construction of the vector x^* in the statement of Theorem 1(i). In particular, x^* lies in a neighborhood of the vector x^ℓ .

Denote by $\bar{A}x \leq \bar{b}$ the system obtained from $Ax \leq b$ by adding the inequalities $x_i \leq 0$, $-x_i \leq 0$, for all $i \in Z^\ell$. Note that the largest absolute value of a subdeterminant of \bar{A} is Δ . Let

$$\bar{P} := \{x \in \mathbb{R}^n \mid \bar{A}x \leq \bar{b}\}.$$

Note that $\bar{P} \subseteq P$ and that the vectors x^ℓ and x^d are in \bar{P} . Denote by

$$\bar{T} := T(\bar{A}, x^\ell, x^d).$$

From Lemma 2, applied to \bar{A} , x^ℓ , and x^d , we know that there exists a finite subset \bar{V} of \mathbb{Z}^n such that $\bar{T} = \text{cone } \bar{V}$, and for every $v \in \bar{V}$, we have $\|v\|_\infty \leq \Delta$. Since $x^\ell - x^d \in \bar{T}$, Caratheodory's theorem implies that there exist $m \leq n$ vectors $v^1, \dots, v^m \in \bar{V}$ and m positive scalars $\gamma_1, \dots, \gamma_m$ such that

$$x^\ell - x^d = x^\ell = \sum_{i=1}^m \gamma_i v^i. \quad (7)$$

The following simple observation will be used twice in our proof.

Observation 1. For $i = 1, \dots, m$, let $\lambda_i \in \mathbb{R}$, such that $0 \leq \lambda_i \leq \gamma_i$. Then the vector $\sum_{i=1}^m \lambda_i v^i$ is in \bar{P} .

Proof. Let $x := \sum_{i=1}^m \lambda_i v^i$. Since x^d is the origin, we can write $x = x^d + \sum_{i=1}^m \lambda_i v^i$. Using (7), we can write x also in the form $x = x^\ell - \sum_{i=1}^m (\gamma_i - \lambda_i) v^i$. From Lemma 3, applied with \bar{P} and \bar{T} , we obtain $x \in \bar{P}$. \square

We are now ready to define the vector x^* as

$$x^* := \sum_{i=1}^m \lfloor \gamma_i \rfloor v^i.$$

3.4 Properties of the vector x^*

Note that $x^* \in \mathbb{Z}^n$ because $\lfloor \gamma_i \rfloor$ and v^i are all integer. From Observation 1, we have $x^* \in \bar{P}$.

The next claim introduces several properties of x^* that will be used later.

Claim 5. *The vector x^* satisfies the following properties:*

- (d) $|x_i^*| \geq \chi_{\ell+1} - n\Delta$ for every $i \in N^\ell$;
- (e) $\{i \in \{1, \dots, k\} \mid x_i^* = 0\} = Z^\ell$;
- (f) $\|x^c - x^*\|_\infty \leq \psi_\ell + n\Delta$;
- (g) $x^c - x^* \in P$.

Proof. In this proof we will be using the upper bound on $\|x^\ell - x^*\|_\infty$ given by

$$\|x^\ell - x^*\|_\infty = \left\| \sum_{i=1}^m (\gamma_i - \lfloor \gamma_i \rfloor) v^i \right\|_\infty \leq \sum_{i=1}^m \|v^i\|_\infty \leq m\Delta \leq n\Delta. \quad (8)$$

Next, we prove the properties of x^* in the statement of the claim.

(d). If $Z^\ell = \{1, \dots, k\}$ we are done, thus we assume $N^\ell \neq \emptyset$. Let $i \in N^\ell$. We have

$$|x_i^\ell| = |(x_i^\ell - x_i^*) + x_i^*| \leq |x_i^\ell - x_i^*| + |x_i^*|.$$

According to our assumption (c-2), we have $|x_i^\ell| > \chi_{\ell+1}$, thus

$$|x_i^*| \geq |x_i^\ell| - |x_i^\ell - x_i^*| > \chi_{\ell+1} - \|x^\ell - x^*\|_\infty \geq \chi_{\ell+1} - n\Delta,$$

where the last inequality holds by (8).

(e). Since the inequalities $x_i = 0$, for $i \in Z^\ell$, are valid for \bar{P} , and $x^* \in \bar{P}$, we know $\{i \in \{1, \dots, k\} \mid x_i^* = 0\} \supseteq Z^\ell$. On the other hand, given any index $i \in N^\ell$, we know from (d) that $|x_i^*| \geq \chi_{\ell+1} - n\Delta > 0$. So $\{i \in \{1, \dots, k\} \mid x_i^* = 0\} \subseteq Z^\ell$. Thus we conclude that $\{i \in \{1, \dots, k\} \mid x_i^* = 0\} = Z^\ell$.

(f). This property follows directly from (b) and (8) as follows:

$$\|x^c - x^*\|_\infty \leq \|x^c - x^\ell\|_\infty + \|x^\ell - x^*\|_\infty \leq \psi_\ell + n\Delta.$$

(g). Using the definition of x^* , the vector $x^c - x^*$ can be written as

$$\begin{aligned} x^c - x^* &= x^c - \sum_{i=1}^m \lfloor \gamma_i \rfloor v^i \\ x^c - x^* &= (x^c - x^\ell) + (x^\ell - x^*) = (x^c - x^\ell) + \sum_{i=1}^m (\gamma_i - \lfloor \gamma_i \rfloor) v^i. \end{aligned}$$

Recall that we have $\gamma_i > 0$ for every $i = 1, \dots, m$. Furthermore, we have $x^c \in P$ and, from (a), $x^c - x^\ell \in P$. Let $T := T(A, x^\ell, x^d)$, and note that $\bar{T} \subseteq T$, which implies $v^i \in T$ for every $i = 1, \dots, m$. Thus, from Lemma 3, applied with P and T , we obtain that $x^c - x^* \in P$. \square

Notice that (f) implies that $\|x^c - x^*\|_\infty$ can be upper bounded by a function of n, Δ, k, ϵ . In the next section, we will use (e) and (g) to show that the distance between x^* and x^d mainly depends on $|x_i^*|$ for $i \in N^\ell$. In particular, when $|x_i^*|$ is large enough for every $i \in N^\ell$, the vector x^* is a suitable approximation to x^d . Together with (d), this will imply that x^* is an ϵ -approximate solution to (IQP).

3.5 x^* is an ϵ -approximate solution to (IQP)

In this section we show that the vector x^* is an ϵ -approximate solution to (IQP). In Section 3.5.1 we provide an upper bound for $f(x^*) - f(x^d)$, while in Section 3.5.2 we derive a lower bound for $f_{\max}^d - f(x^d)$, where f_{\max}^d is the maximum value of f on $P \cap \mathbb{Z}^n$. In Section 3.5.3, we use the two bounds to show that x^* is an ϵ -approximate solution to (IQP).

3.5.1 Upper bound on $f(x^*) - f(x^d)$

Claim 6. *We have $f(x^*) - f(x^d) \leq 2(\psi_\ell + n\Delta) \sum_{i \in N^\ell} q_i |x_i^*|$.*

Proof. For ease of notation, let $\delta := x^c - x^*$. We have

$$\begin{aligned} f(x^c) &= f(\delta + x^*) = \left(\sum_{i=1}^k -q_i \delta_i^2 + h^\top \delta \right) + \left(\sum_{i=1}^k -q_i (x_i^*)^2 + h^\top x^* \right) - 2 \sum_{i=1}^k q_i \delta_i x_i^* \\ &= f(\delta) + f(x^*) - 2 \sum_{i \in N^\ell} q_i \delta_i x_i^* \geq f(x^c) + f(x^*) - 2 \sum_{i \in N^\ell} q_i \delta_i x_i^*. \end{aligned}$$

In the last equality we used (e). Furthermore, in the last inequality we used $f(\delta) \geq f(x^c)$ since $\delta \in P$ from (g) and x^c is an optimal solution to (QP). We obtain

$$f(x^*) \leq 2 \sum_{i \in N^\ell} q_i \delta_i x_i^* \leq 2 \sum_{i \in N^\ell} q_i |\delta_i| |x_i^*| \leq 2(\psi_\ell + n\Delta) \sum_{i \in N^\ell} q_i |x_i^*|,$$

where the last inequality holds from (f). The claim follows by recalling that $f(x^d) = 0$ by Claim 1. \square

3.5.2 Lower bound on $f_{\max}^d - f(x^d)$

In this section we give a lower bound on $f_{\max}^d - f(x^d)$. In our derivation, a fundamental role is played by the midpoint of x^d and x^* , which we denote by x^Δ , i.e.,

$$x^\Delta := \frac{x^d + x^*}{2}.$$

We first give a lower bound on $f(x^\Delta) - f(x^d)$.

Claim 7. We have $f(x^\Delta) - f(x^d) \geq \frac{1}{4} \sum_{i \in N^\ell} q_i(x_i^*)^2$.

Proof. The claim can be derived as follows:

$$\begin{aligned} f(x^\Delta) &= f\left(\frac{x^*}{2}\right) = \frac{1}{4} \sum_{i=1}^k -q_i(x_i^*)^2 + \frac{1}{2} h^\top x^* \\ &= \left(\frac{1}{2} \sum_{i=1}^k -q_i(x_i^*)^2 + \frac{1}{2} h^\top x^*\right) + \frac{1}{4} \sum_{i=1}^k q_i(x_i^*)^2 \\ &= \frac{1}{2} f(x^*) + \frac{1}{4} \sum_{i \in N^\ell} q_i(x_i^*)^2 \geq \frac{1}{4} \sum_{i \in N^\ell} q_i(x_i^*)^2. \end{aligned}$$

In the last equality we used (e). In the last inequality, we used $f(x^*) \geq f(x^d) = 0$, which holds because x^d is optimal to (IQP) and x^* is feasible to the same problem. \square

Recall that the goal of this section is to obtain a lower bound on $f_{\max}^d - f(x^d)$. Since both x^d and x^* are in P , the vector x^Δ is in P as well. Therefore, if $x^\Delta \in \mathbb{Z}^n$, then $f_{\max}^d \geq f(x^\Delta)$, and the bound of Claim 7 yields a bound on $f_{\max}^d - f(x^d)$. However, x^Δ is not always an integer vector. Thus we define two integer points x^l and x^r whose midpoint is x^Δ :

$$\begin{aligned} x^l &:= x^d + \sum_{i|\lfloor \gamma_i \rfloor \text{ odd}} \frac{\lfloor \gamma_i \rfloor - 1}{2} v^i + \sum_{i|\lfloor \gamma_i \rfloor \text{ even}} \frac{\lfloor \gamma_i \rfloor}{2} v^i \\ x^r &:= x^d + \sum_{i|\lfloor \gamma_i \rfloor \text{ odd}} \frac{\lfloor \gamma_i \rfloor + 1}{2} v^i + \sum_{i|\lfloor \gamma_i \rfloor \text{ even}} \frac{\lfloor \gamma_i \rfloor}{2} v^i. \end{aligned}$$

We now show that both x^l and x^r are in $\bar{P} \cap \mathbb{Z}^n$. Clearly, $0 \leq \frac{\lfloor \gamma_i \rfloor}{2} \leq \gamma_i$. Furthermore, if $\lfloor \gamma_i \rfloor$ is odd, we have $\lfloor \gamma_i \rfloor \geq 1$, which implies $0 \leq \frac{\lfloor \gamma_i \rfloor - 1}{2} \leq \frac{\lfloor \gamma_i \rfloor + 1}{2} \leq \gamma_i$. By Observation 1, we know that both x^l and x^r are in \bar{P} . Since all coefficients $\frac{\lfloor \gamma_i \rfloor \pm 1}{2}$ and $\frac{\lfloor \gamma_i \rfloor}{2}$ are integer, we conclude that both x^l and x^r are in $\bar{P} \cap \mathbb{Z}^n$.

Let $D \subset \mathbb{R}^n$ be the smallest box containing x^l and x^r , i.e.,

$$D := [\min\{x_1^l, x_1^r\}, \max\{x_1^l, x_1^r\}] \times \cdots \times [\min\{x_n^l, x_n^r\}, \max\{x_n^l, x_n^r\}].$$

In the reminder of the proof we denote by $q : \mathbb{R}^n \rightarrow \mathbb{R}$ the quadratic part of the objective function f , i.e.,

$$q(x) := \sum_{i=1}^k -q_i x_i^2.$$

We also define the affine function $\lambda : \mathbb{R}^n \rightarrow \mathbb{R}$ which achieves the same value as q at the vertices of the box D :

$$\lambda(x) := \sum_{i=1}^k -q_i ((x_i^l + x_i^r) x_i - x_i^l x_i^r).$$

We have the following claim.

Claim 8. For every $x \in D$ we have $\lambda(x) \leq q(x) \leq \lambda(x) + \frac{(n\Delta)^2}{4} \sum_{i \in N^\ell} q_i$.

Proof. Since λ achieves the same value as q at each vertex of D and q is a concave function, we have $\lambda(x) \leq q(x)$, for every $x \in D$.

Using the definitions of q and λ we obtain

$$\begin{aligned} q(x) - \lambda(x) &= \sum_{i=1}^k -q_i \left(x_i^2 - (x_i^l + x_i^r)x_i + x_i^l x_i^r \right) = \sum_{i=1}^k -q_i (x_i - x_i^l)(x_i - x_i^r) \\ &\leq \frac{1}{4} \sum_{i \in N^\ell} q_i (x_i^l - x_i^r)^2. \end{aligned}$$

The inequality holds because, for each $i = 1, \dots, k$, the univariate quadratic function $-q_i(x_i - x_i^l)(x_i - x_i^r)$ achieves its maximum at $\frac{x_i^l + x_i^r}{2}$. In particular, if $i \in Z^\ell$, the maximum is 0. This is because both x^l and x^r are in \bar{P} , which implies $x_i^l = x_i^r = 0$ for every $i \in Z^\ell$.

From the definition of x^l and x^r , we obtain $\|x^r - x^l\|_\infty \leq \|\sum_{i=1}^m v^i\|_\infty \leq m\Delta \leq n\Delta$. Therefore, we have $q(x) \leq \lambda(x) + \frac{(n\Delta)^2}{4} \sum_{i \in N^\ell} q_i$. \square

Claim 9. *There exists $\tilde{x} \in \{x^l, x^r\}$ such that $f(\tilde{x}) - f(x^\Delta) \geq -\frac{(n\Delta)^2}{4} \sum_{i \in N^\ell} q_i$.*

Proof. Let $g : \mathbb{R}^n \rightarrow \mathbb{R}$ be defined by $g(x) := \lambda(x) + h^\top x$. Claim 8 implies that, for every $x \in D$, we have

$$g(x) \leq f(x) \leq g(x) + \frac{(n\Delta)^2}{4} \sum_{i \in N^\ell} q_i.$$

Since g is a linear function and x^Δ is the midpoint of x^l and x^r , we know that $g(x^\Delta) \leq g(\tilde{x})$ for some $\tilde{x} \in \{x^l, x^r\}$. We derive the following relation:

$$f(x^\Delta) \leq g(x^\Delta) + \frac{(n\Delta)^2}{4} \sum_{i \in N^\ell} q_i \leq g(\tilde{x}) + \frac{(n\Delta)^2}{4} \sum_{i \in N^\ell} q_i \leq f(\tilde{x}) + \frac{(n\Delta)^2}{4} \sum_{i \in N^\ell} q_i. \quad \square$$

We are finally ready to state our lower bound on $f_{\max}^d - f(x^d)$.

Claim 10. *We have $f_{\max}^d - f(x^d) \geq \frac{1}{4} \sum_{i \in N^\ell} q_i ((x_i^*)^2 - (n\Delta)^2)$.*

Proof. Combining Claim 7 and Claim 9, we have

$$\begin{aligned} f_{\max}^d - f(x^d) &\geq f(\tilde{x}) - f(x^d) = \left(f(\tilde{x}) - f(x^\Delta) \right) + \left(f(x^\Delta) - f(x^d) \right) \\ &\geq \frac{1}{4} \sum_{i \in N^\ell} q_i ((x_i^*)^2 - (n\Delta)^2). \end{aligned} \quad \square$$

3.5.3 x^* is an ϵ -approximate solution

In order to prove that x^* is an ϵ -approximate solution, we first prove the following observation.

Observation 2. *Let $a_i, b_i > 0$, for $i = 1, \dots, k$. Then $\frac{\sum_{i=1}^k a_i}{\sum_{i=1}^k b_i} \leq \max_{i=1, \dots, k} \frac{a_i}{b_i}$.*

Proof. To prove this statement, let $j \in \{1, \dots, k\}$ such that $\frac{a_j}{b_j} = \max_{i=1, \dots, k} \frac{a_i}{b_i}$. Then,

$$\frac{\sum_{i=1}^k a_i}{\sum_{i=1}^k b_i} - \frac{a_j}{b_j} = \frac{\sum_{i=1}^k (b_j a_i - a_j b_i)}{b_j \sum_{i=1}^k b_i}.$$

We only need to show that the right-hand side of the latter equation is nonpositive. To see this, notice that, for $i = 1, \dots, k$, we have $\frac{a_i}{b_i} \leq \frac{a_j}{b_j}$, thus $b_j a_i - a_j b_i \leq 0$. \square

Claim 11. *The vector x^* is an ϵ -approximate solution to (IQP).*

Proof. Consider first the case $Z^\ell = \{1, \dots, k\}$. Then by (e), we know that $x_i^* = 0$, for $i = 1, \dots, k$. In this case, from Claim 6, we know that $f(x^*) \leq 0$. By Claim 1, this implies that x^* is an optimal solution to (IQP).

Now assume that $Z^\ell \subset \{1, \dots, k\}$, i.e., $N^\ell \neq \emptyset$. Observe that the quantity $f_{\max}^d - f(x^d)$ in the definition of ϵ -approximate solution is positive. This follows from Claim 10, since for $i \in N^\ell$, we have $q_i > 0$ by assumption and $|x_i^*| > n\Delta$ from (d). Therefore, we consider the ratio $\frac{f(x^*) - f(x^d)}{f_{\max}^d - f(x^d)}$, and our aim is to show that it is upper bounded by ϵ . Using Claim 6, Claim 10, and Observation 2, we derive the following bound:

$$\begin{aligned} \frac{f(x^*) - f(x^d)}{f_{\max}^d - f(x^d)} &\leq 8(\psi_\ell + n\Delta) \frac{\sum_{i \in N^\ell} q_i |x_i^*|}{\sum_{i \in N^\ell} q_i ((x_i^*)^2 - (n\Delta)^2)} \\ &\leq 8(\psi_\ell + n\Delta) \max_{i \in N^\ell} \frac{|x_i^*|}{(x_i^*)^2 - (n\Delta)^2}. \end{aligned}$$

In particular, the latter max can be written in the form

$$\max_{i \in N^\ell} \frac{|x_i^*|}{(x_i^*)^2 - (n\Delta)^2} = \max_{i \in N^\ell} \frac{1}{|x_i^*| - \frac{(n\Delta)^2}{|x_i^*|}}.$$

In the right-hand side, the denominator is always positive due to $|x_i^*| > n\Delta$. Let s be the index in $\{1, \dots, k\}$ that achieves $\min_{i \in N^\ell} |x_i^*|$. Note that this index exists because of our assumption $N^\ell \neq \emptyset$. Then the max is achieved by the index s . In fact, in the denominator in the right-hand side, the term $|x_i^*|$ is minimized by s , while the term $\frac{(n\Delta)^2}{|x_i^*|}$ is maximized by s .

From (d), we have $|x_s^*| \geq \chi_{\ell+1} - n\Delta = \frac{8}{\epsilon}(\psi_\ell + n\Delta) + n\Delta$, where the equality can be obtained using the definition of $\chi_{\ell+1}$ and of ψ_ℓ . We obtain

$$\begin{aligned} \frac{f(x^*) - f(x^d)}{f_{\max}^d - f(x^d)} &\leq \frac{8(\psi_\ell + n\Delta)}{|x_s^*| - \frac{(n\Delta)^2}{|x_s^*|}} \leq \frac{8(\psi_\ell + n\Delta)}{\frac{8}{\epsilon}(\psi_\ell + n\Delta) + n\Delta - \frac{(n\Delta)^2}{\frac{8}{\epsilon}(\psi_\ell + n\Delta) + n\Delta}} \\ &= \frac{8(\psi_\ell + n\Delta) \left(\frac{8}{\epsilon}(\psi_\ell + n\Delta) + n\Delta \right)}{8(\psi_\ell + n\Delta) \left(\frac{8}{\epsilon^2}(\psi_\ell + n\Delta) + \frac{2}{\epsilon}n\Delta \right)} \\ &= \epsilon \frac{8(\psi_\ell + n\Delta) + n\Delta\epsilon}{8(\psi_\ell + n\Delta) + 2n\Delta\epsilon} < \epsilon, \end{aligned}$$

where the first equality can be obtained by multiplying the numerator and the denominator by $\frac{8}{\epsilon}(\psi_\ell + n\Delta) + n\Delta$. This implies that x^* is an ϵ -approximate solution to (IQP). \square

From (f), we know that $\|x^c - x^*\|_\infty \leq \psi_\ell + n\Delta \leq \psi_k + n\Delta$. Furthermore, from Claim 3, we have $\psi_k + n\Delta \leq n\Delta \left(\frac{10\Delta}{\epsilon} + 1 \right)^k$. This completes the proof of Theorem 1(i). Therefore, in the remainder of the proof we only need to show Theorem 1(ii).

3.6 Construction of the vector x^*

In this section we introduce the vector x^* in the statement of Theorem 1(ii). The point x^* is defined by

$$x^* := x^c - x^*.$$

From the definition of x^* , we obtain

$$x^* = x^c - \sum_{i=1}^m \lfloor \gamma_i \rfloor v^i = x^c - x^\ell + \sum_{i=1}^m (\gamma_i - \lfloor \gamma_i \rfloor) v^i.$$

We know that $x^c \in P$, and, from (a), we know that $x^c - x^\ell \in P$ as well. Thus, from Lemma 3, applied with P and T , we know that $x^* \in P$.

3.7 x^* is an ϵ -approximate solution to (QP)

In this section we show that the vector x^* is an ϵ -approximate solution to (QP). To do this, we first give an upper bound on $f(x^*) - f(x^c)$, and then a lower bound on $f_{\max}^c - f(x^c)$, where f_{\max}^c is the maximum value of f on P . The two bounds are then used to show that x^* is an ϵ -approximate solution to (QP).

Claim 12. *We have $f(x^*) - f(x^c) \leq 2(\psi_\ell + n\Delta) \sum_{i \in N^\ell} q_i |x_i^*|$.*

Proof. First, we derive an upper bound on $f(x^*)$. According to the definition of x^* , we get

$$\begin{aligned} f(x^*) &= f(x^c - x^*) \\ &= \left(-\sum_{i=1}^k q_i (x_i^c)^2 + h^\top x^c \right) + \left(-\sum_{i=1}^k q_i (x_i^*)^2 - h^\top x^* \right) + 2 \sum_{i=1}^k q_i x_i^c x_i^* \\ &= f(x^c) + f(-x^*) + 2 \sum_{i \in N^\ell} q_i x_i^c x_i^* \\ &= f(x^c) + f(-x^*) + 2 \sum_{i \in N^\ell} q_i (x_i^* + x_i^c) x_i^*, \end{aligned}$$

where in the third equality we used (e).

To derive from the above formula an upper bound on $f(x^*) - f(x^c)$, we need to upper bound $f(-x^*)$. Since x^d is the optimal solution to (IQP) and $x^* \in P \cap \mathbb{Z}^n$, we know that

$$f(x^*) = -\sum_{i=1}^k q_i (x_i^*)^2 + h^\top x^* \geq f(x^d) = 0.$$

Thus, we get

$$\begin{aligned} f(-x^*) &= -\sum_{i=1}^k q_i (x_i^*)^2 - h^\top x^* \leq f(x^*) - \sum_{i=1}^k q_i (x_i^*)^2 - h^\top x^* \\ &= -2 \sum_{i=1}^k q_i (x_i^*)^2 = -2 \sum_{i \in N^\ell} q_i (x_i^*)^2. \end{aligned}$$

We obtain

$$\begin{aligned} f(x^*) - f(x^c) &\leq -2 \sum_{i \in N^\ell} q_i (x_i^*)^2 + 2 \sum_{i \in N^\ell} q_i (x_i^* + x_i^c) x_i^* = 2 \sum_{i \in N^\ell} q_i x_i^c x_i^* \\ &\leq 2 \sum_{i \in N^\ell} q_i |x_i^c| |x_i^*| \leq 2(\psi_\ell + n\Delta) \sum_{i \in N^\ell} q_i |x_i^*|. \end{aligned}$$

The last inequality holds because, from (f), we have $\|x^*\|_\infty = \|x^c - x^*\|_\infty \leq \psi_\ell + n\Delta$. \square

Claim 13. We have $f_{\max}^c - f(x^c) \geq \frac{1}{4} \sum_{i \in N^\ell} q_i (x_i^*)^2$.

Proof. Define the midpoint of x^c and x^* as

$$x^\diamond := \frac{x^c + x^*}{2}.$$

Then

$$\begin{aligned} f(x^\diamond) &= f\left(\frac{x^c + x^*}{2}\right) = \frac{1}{4} \sum_{i=1}^k -q_i ((x_i^c)^2 + 2x_i^c x_i^* + (x_i^*)^2) + \frac{1}{2} h^\top (x^c + x^*) \\ &= \left(\frac{1}{2} \sum_{i=1}^k -q_i (x_i^c)^2 + \frac{1}{2} h^\top x^c\right) + \left(\frac{1}{2} \sum_{i=1}^k -q_i (x_i^*)^2 + \frac{1}{2} h^\top x^*\right) + \frac{1}{4} \sum_{i=1}^k q_i (x_i^* - x_i^c)^2 \\ &= \frac{1}{2} f(x^c) + \frac{1}{2} f(x^*) + \frac{1}{4} \sum_{i \in N^\ell} q_i (x_i^*)^2 \geq f(x^c) + \frac{1}{4} \sum_{i \in N^\ell} q_i (x_i^*)^2. \end{aligned}$$

In the last inequality, we used $f(x^*) \geq f(x^c)$, which holds because x^c is optimal to (QP) and x^* is feasible to the same problem. Since x^\diamond is the midpoint of x^* and x^c , we know that $x^\diamond \in P$. Thus we have

$$f_{\max}^c - f(x^c) \geq f(x^\diamond) - f(x^c) \geq \frac{1}{4} \sum_{i \in N^\ell} q_i (x_i^*)^2. \quad \square$$

Claim 14. The vector x^* is an ϵ -approximate solution to (QP).

Proof. As in the proof of Claim 11, it is simple to check that the quantity $f_{\max}^c - f(x^c)$ in the definition of ϵ -approximate solution is positive. This allows us to consider the ratio $\frac{f(x^*) - f(x^c)}{f_{\max}^c - f(x^c)}$, and our aim is to show that it is upper bounded by ϵ .

Using Claim 12, Claim 13, and Observation 3, we can derive the following bound:

$$\begin{aligned} \frac{f(x^*) - f(x^c)}{f_{\max}^c - f(x^c)} &\leq 8(\psi_\ell + n\Delta) \frac{\sum_{i \in N^\ell} q_i |x_i^*|}{\sum_{i \in N^\ell} q_i (x_i^*)^2} \\ &\leq 8(\psi_\ell + n\Delta) \max_{i \in N^\ell} \frac{q_i |x_i^*|}{q_i (x_i^*)^2} = \frac{8(\psi_\ell + n\Delta)}{|x_s^*|}, \end{aligned}$$

where s is an index such that $|x_s^*| = \min\{|x_i^*| \mid i \in N^\ell\}$.

From (d), we have $|x_s^*| \geq \chi_{\ell+1} - n\Delta$, and using the definition of $\chi_{\ell+1}$ the latter quantity equals $n\Delta + \frac{8}{\epsilon}(\psi_\ell + n\Delta)$. We get

$$\frac{8(\psi_\ell + n\Delta)}{|x_s^*|} \leq \frac{8(\psi_\ell + n\Delta)}{n\Delta + \frac{8}{\epsilon}(\psi_\ell + n\Delta)} < \frac{8(\psi_\ell + n\Delta)}{\frac{8}{\epsilon}(\psi_\ell + n\Delta)} = \epsilon.$$

This implies that x^* is an ϵ -approximate solution to (QP). □

From the definition of x^* and from (f), we obtain

$$\|x^* - x^d\|_\infty = \|x^c - x^*\|_\infty \leq \psi_\ell + n\Delta \leq \psi_k + n\Delta.$$

Moreover, from Claim 3, we have $\psi_k + n\Delta \leq n\Delta \left(\frac{10\Delta}{\epsilon} + 1\right)^k$. This completes the proof of Theorem 1(ii), and of Theorem 1.

4 Lower bounds on the distance of solutions

In this section we discuss how far from optimal are the proximity bounds in Theorem 1. The main ingredient in the derivation of our lower bounds is a polyhedron \bar{P} that we introduce next.

Definition 1. For every $n, \Delta, t \in \mathbb{Z}$ with $n \geq 1$, $\Delta \geq 1$, $t \geq 0$, and $\beta \in (0, 1)$, let $\bar{P} \subset \mathbb{R}^n$ be the polyhedron defined by the following inequalities:

$$\begin{aligned} -t &\leq x_1 - \Delta \sum_{i=2}^n x_i \leq t \\ 0 &\leq x_i \leq \beta \qquad \qquad \qquad i = 2, \dots, n. \end{aligned}$$

Clearly, the polyhedron \bar{P} has dimension n if $t \geq 1$. Note that \bar{P} can be obtained from the polytope

$$\bar{P}_y := \{(y_1, x_2, \dots, x_n) \mid -t \leq y_1 \leq t, 0 \leq x_i \leq \beta, i = 2, \dots, n\}$$

by replacing variable y_1 with $x_1 := y_1 + \Delta \sum_{i=2}^n x_i$. The vertices of the polytope \bar{P}_y are all vectors with components $y_1 = \pm t$, and $x_i \in \{0, \beta\}$ for $i = 2, \dots, n$. Therefore, \bar{P} is bounded and its vertices are all vectors with components $x_i \in \{0, \beta\}$ for $i = 2, \dots, n$, and component $x_1 = \pm t + \Delta \sum_{i=2}^n x_i$. In particular, the vertex with the largest x_1 is

$$v := (t + (n-1)\beta\Delta, \beta, \dots, \beta),$$

and will play an important role in our arguments.

Next, we focus on the integer points in \bar{P} . Since $\beta < 1$, any vector in $\bar{P} \cap \mathbb{Z}^n$ satisfies $x_i = 0$, for $i = 2, \dots, n$, and the first component of such vectors are $-t, -t+1, \dots, t$. Since $t \geq 0$, the set $\bar{P} \cap \mathbb{Z}^n$ contains the origin and is therefore nonempty. In particular, the integer point in \bar{P} with the largest x_1 is

$$u := (t, 0, \dots, 0).$$

The vectors u and $-u$ will often be used in the later proofs.

Observation 3. Let A be the constraint matrix defining \bar{P} . Then each subdeterminant of A is in $\{0, \pm 1, \pm \Delta\}$.

Proof. The constraint matrix of the system defining \bar{P} is

$$A = \begin{pmatrix} 1 & -\Delta_{n-1}^\top \\ -1 & \Delta_{n-1}^\top \\ 0_{n-1} & I_{n-1} \\ 0_{n-1} & -I_{n-1} \end{pmatrix},$$

where I_{n-1} denotes the $(n-1) \times (n-1)$ identity matrix, and 0_{n-1} (resp. Δ_{n-1}) denotes the $(n-1)$ -dimensional vector with all entries equal to zero (resp. Δ). Let d be the determinant of a square submatrix M of A . If M has linearly dependent rows, then $d = 0$. Thus we now assume that M does not have linearly dependent rows. Up to multiplying rows of M by -1 , which is an operation that can only change the sign of the determinant d , the matrix M is a submatrix of

$$\begin{pmatrix} 1 & -\Delta_{n-1}^\top \\ 0_{n-1} & I_{n-1} \end{pmatrix}.$$

It is well known that adding unit rows to a matrix can only add 0, 1 and change the sign to its possible subdeterminants. Therefore, either $d \in \{0, 1\}$, or d is a subdeterminant of the matrix $\begin{pmatrix} 1 & -\Delta_{n-1}^\top \end{pmatrix}$. The latter matrix has only one row and its subdeterminants are 1, $-\Delta$. \square

For brevity, in this section, we say that a (IQP) or (QP) has *subdeterminant* Δ if the maximum of the absolute values of the subdeterminants of the constraint matrix A is Δ .

4.1 Tightness in Integer Linear Programming

In this section we consider our problems (IQP) and (QP) under the additional assumption $k = 0$. In this special case, (IQP) is a general Integer Linear Programming (ILP) problem, while (QP) is the corresponding Linear Programming (LP) problem, also known as the standard linear relaxation of (ILP).

We remark that, for $k = 0$, Theorem 1 reduces to the proximity bound by Cook et al. [4] for Integer Linear Programming. In particular, this result yields the upper bound

$$\min \left\{ \|x^c - x^d\|_\infty \mid x^d \text{ opt. to (ILP), } x^c \text{ opt. to (LP)} \right\} \leq n\Delta.$$

The impact of the polytope \bar{P} is immediately apparent, as it allows us to prove that the above upper bound $n\Delta$ is asymptotically best possible. To the best of our knowledge this tightness result was previously known only for $\Delta = 1$ [16, 15].

Proposition 1. *For every $n, \Delta \in \mathbb{Z}$ with $n \geq 1$, $\Delta \geq 1$, and $\beta \in (0, 1)$, there exists an instance of (ILP) with subdeterminant Δ for which*

$$\min \left\{ \|x^c - x^d\|_\infty \mid x^d \text{ opt. to (ILP), } x^c \text{ opt. to (LP)} \right\} = (n - 1)\beta\Delta \in \Omega(n\Delta).$$

Proof. Let n, Δ, β be as in the statement. Consider the (ILP) problem

$$\begin{aligned} \max x_1 \\ \text{s.t. } x \in \bar{P} \cap \mathbb{Z}^n, \end{aligned} \tag{9}$$

where the parameter t in the definition of \bar{P} can be chosen to be any integer greater than or equal to zero. From Observation 3, problem (9) has subdeterminant Δ . The unique optimal solution of (9) is the vector u , while the unique optimal solution of the corresponding (LP) is the vertex v of \bar{P} . We obtain $\|v - u\|_\infty = (n - 1)\beta\Delta$. \square

4.2 Lower bounds in Integer Quadratic Programming

Let us now get back to the general case of (IQP) where k can be positive. In this setting, even for $k = 1$, problem (1) in Example 1 shows that it is not possible to upper bound the distance

$$\min \left\{ \|x^c - x^d\|_\infty \mid x^d \text{ opt. to (IQP), } x^c \text{ opt. to (QP)} \right\}$$

with a function that depends only on n and Δ . Therefore, we focus instead on the two quantities

$$\begin{aligned} \delta_\epsilon^* &:= \min \left\{ \|x^c - x^*\|_\infty \mid x^* \text{ } \epsilon\text{-approx. to (IQP), } x^c \text{ opt. to (QP)} \right\}, \\ \delta_\epsilon^* &:= \min \left\{ \|x^* - x^d\|_\infty \mid x^d \text{ opt. to (IQP), } x^* \text{ } \epsilon\text{-approx. to (QP)} \right\}. \end{aligned}$$

Our Theorem 1 implies that both δ_ϵ^* and δ_ϵ^* are upper bounded by

$$n\Delta \left(\frac{10\Delta}{\epsilon} + 1 \right)^k \in O \left(\frac{n\Delta^{k+1}}{\epsilon^k} \right).$$

In the next two sections we gain insight on how far from optimal are our proximity results. This is done by providing lower bounds on both δ_ϵ^* , in Section 4.2.1, and on δ_ϵ^* , in Section 4.2.2. Note that our bounds can be further improved, as we are only interested here in the asymptotic behaviour of δ_ϵ^* and δ_ϵ^* .

We remark that the problems that we present in the following results are of the form (IQP) and (QP) with an additional constant in the objective function. We decided to keep these constants to simplify the presentation, and we observe that the presence of these constants does not affect optimal or ϵ -approximate solutions thanks to Lemma 1.

4.2.1 Lower bounds on δ_ϵ^*

To begin with, we present a special (IQP) problem, which will be useful in the subsequent discussion. For every $n, \Delta, t \in \mathbb{Z}$ with $n \geq 1$, $\Delta \geq 1$, $t \geq 0$, and $a \in \mathbb{R}$, $\beta \in (0, 1)$, consider the (IQP)

$$\begin{aligned} \min f(x) &= -(x_1 - a)^2 - \frac{(t + n\Delta)^2}{\beta^2} \sum_{i=2}^n x_i^2 \\ \text{s.t. } x &\in \bar{P} \cap \mathbb{Z}^n, \end{aligned} \tag{10}$$

where the polytope \bar{P} is given in Definition 1. Note that problem (10) has $k = n$ and, from Observation 3, subdeterminant Δ .

The next lemma provides some information about (10) and its corresponding (QP). We remind the reader that the vectors u, v are defined right after Definition 1.

Lemma 4. *If $0 < a < (n - 1)\beta\Delta$, then the vector $-u$ is the unique optimal solution to (10), and the vector v is the unique optimal solution to the corresponding (QP).*

Proof. Consider problem (10) and assume $0 < a < (n - 1)\beta\Delta$.

We first show that the vector $-u$ is the unique optimal solution to (10). We have seen that any vector in $\bar{P} \cap \mathbb{Z}^n$ satisfies $x_i = 0$, for $i = 2, \dots, n$, and the first component ranges in $-t, -t+1, \dots, t$. Our assumption $a > 0$ then implies that the vector $-u = (-t, 0, \dots, 0)$ is the unique optimal solution to (10).

Next, we show that the vertex v of \bar{P} is the unique optimal solution to the corresponding (QP). Note that

$$f(v) = -(t + (n - 1)\beta\Delta - a)^2 - (n - 1)(t + n\Delta)^2.$$

Since \bar{P} is a polytope and the objective is concave, we only need to show that any other vertex v' of \bar{P} has cost strictly larger than v .

First, assume that $v'_i = \beta$ for every $i = 2, \dots, n$. Then $v'_1 = -t + (n - 1)\beta\Delta$. Notice that, if $t = 0$, then v' and v are the same point. Therefore, we assume that $t > 0$. We have

$$f(v') = -(-t + (n - 1)\beta\Delta - a)^2 - (n - 1)(t + n\Delta)^2.$$

Since $(n - 1)\beta\Delta - a > 0$, due to the fact that $t > 0$, we obtain $|t + (n - 1)\beta\Delta - a| > |-t + (n - 1)\beta\Delta - a|$. We have therefore shown $f(v) < f(v')$.

We can now assume that $v'_i = 0$ for some $i = 2, \dots, n$. If we denote by m the number of components among v'_2, \dots, v'_n that are equal to β , then we have $m \leq n - 2$. Hence

$$\begin{aligned} f(v') &= -(v'_1 - a)^2 - m(t + n\Delta)^2 \geq -(v'_1 - a)^2 - (n - 2)(t + n\Delta)^2 \\ &> -(t + n\Delta)^2 - (n - 2)(t + n\Delta)^2 = -(n - 1)(t + n\Delta)^2 \geq f(v). \end{aligned}$$

Now we explain how we obtain the second inequality. From the definition of \bar{P} , we know that $-t \leq v'_1 \leq t + (n-1)\beta\Delta$. Since $0 \leq a \leq (n-1)\beta\Delta$, we get

$$-t - (n-1)\beta\Delta \leq v'_1 - a \leq t + (n-1)\beta\Delta.$$

We obtain

$$(v'_1 - a)^2 \leq (t + (n-1)\beta\Delta)^2 < (t + n\Delta)^2,$$

which implies the second inequality. In this second case we have shown that $f(v) < f(v')$ holds for every $t \geq 0$. This concludes the proof that v is the unique optimal solution to the (QP) corresponding to (10). \square

In the next proposition we highlight a key difference between Separable Concave Integer Quadratic Programming and Integer Linear Programming. More in detail, we discuss an important difference between problem (IQP) with $k \geq 1$ and the same problem with $k = 0$. Consider a feasible instance of (IQP), and let x^c be an optimal solution to the corresponding (QP). According to Cook et al. [4], we can always find integer points x in P with $\|x^c - x\|_\infty \leq n\Delta$. Furthermore, if $k = 0$, one of these vectors is optimal to (IQP). However, this is not true for the case $k \geq 1$. In fact, when $k \geq 1$, the set $\{x \in P \cap \mathbb{Z}^n \mid \|x^c - x\|_\infty \leq n\Delta\}$ not only might contain no optimal solution to (IQP), but it also might contain only arbitrarily bad solutions, i.e., vectors that are not ϵ -approximate solution to (IQP), for any $\epsilon \in (0, 1)$.

Proposition 2. *For every $\Delta \in \mathbb{Z}$ with $\Delta \geq 1$, and $\epsilon \in (0, 1)$, there is an instance of (IQP) with subdeterminant Δ and $k = n$ for which $\delta_\epsilon^* > n\Delta$.*

Proof. Let Δ, ϵ be as in the statement, and consider problem (10) with $n := \left\lceil \frac{4-3\sqrt{\epsilon}}{1-\sqrt{\epsilon}} \right\rceil \geq 5$, $\beta := \frac{n-4}{n-3} \in [\frac{1}{2}, 1)$, $a := (n-3)\beta\Delta$, and $t := \frac{n+(n-3)\beta}{2}\Delta \geq 3$. Note that, both a and t are integer. From Lemma 4, we know that $x^d = -u$ is the unique optimal solution to (10) and $x^c = v$ is the unique optimal solution to the corresponding (QP).

Let $S := \{x \in \bar{P} \cap \mathbb{Z}^n \mid \|x^c - x\|_\infty \leq n\Delta\}$. It suffices to show that there is no ϵ -approximate solution to (10) in S . Using the definition of v , we derive

$$S = \{x \in \mathbb{R}^n \mid t - (n - (n-1)\beta)\Delta \leq x_1 \leq t, x_1 \in \mathbb{Z}, x_i = 0, i = 2, \dots, n\},$$

and it can be checked that the quantity $t - (n - (n-1)\beta)\Delta$ is in $(-t, t)$. Using $n \geq 5$, it can be checked that a is smaller than the midpoint between the two points $t - (n - (n-1)\beta)\Delta$ and t . Due to the concavity of the objective, and the fact that $t \in \mathbb{Z}$, this implies that the vector u is a minimizer of the objective function $f(x)$ over the set S . Therefore, it suffices to show that the vector u is not an ϵ -approximate solution to (10).

Let $u' := (a, 0, \dots, 0)$. Since $-t < a < t$, we have $u' \in \bar{P}$. Moreover, since a is integer, it is simple to check that $f_{\max}^d = 0$ and it is achieved at u' . Furthermore, we have

$$\begin{aligned} f(u) &= -(t-a)^2 = -(t - (n-3)\beta\Delta)^2 = -\left(\frac{n - (n-3)\beta}{2}\Delta\right)^2 = -4\Delta^2, \\ f(x^d) &= -(t+a)^2 = -(t + (n-3)\beta\Delta)^2 = -\left(\frac{n + 3(n-3)\beta}{2}\Delta\right)^2 = -(2n-6)^2\Delta^2. \end{aligned}$$

We obtain

$$\frac{f(u) - f(x^d)}{f_{\max}^d - f(x^d)} = \frac{4(n-4)(n-2)\Delta^2}{4(n-3)^2\Delta^2} > \frac{(n-4)^2}{(n-3)^2} = \left(1 - \frac{1}{n-3}\right)^2 \geq \epsilon,$$

where the last inequality can be checked by plugging in n . Therefore, the vector u is not an ϵ -approximate solution to (10). \square

In particular, Proposition 2 shows that if $k = n$, then δ_ϵ^* can grow at least linearly with respect to both n and Δ . In the next proposition, we use Lemma 4 to derive our main lower bound on δ_ϵ^* .

Proposition 3. *For every $n, \Delta \in \mathbb{Z}$ with $n \geq 2$, $\Delta \geq 1$, and $\epsilon \in (0, 1]$, there exists an instance of (IQP) with subdeterminant Δ and $k = n$ for which*

$$\delta_\epsilon^* \geq 4 \left(\frac{1}{\epsilon} - 1 \right) + \frac{2}{3}(n-1)\Delta \in \Omega \left(\frac{1}{\epsilon} + n\Delta \right).$$

Proof. Let n, Δ, ϵ be as in the statement, and consider problem (10) with $a := \frac{1}{2}$, $\beta := \frac{2}{3}$, and $t := \lceil \frac{2}{\epsilon} - 1 \rceil - 1 \geq 0$. Our assumptions imply $0 < a < (n-1)\beta\Delta$. Therefore, Lemma 4 implies that $x^d = -u$ is the unique optimal solution to (10) and that $x^c = v$ is the unique optimal solution to the corresponding (QP).

To prove the proposition, it suffices to show that the only ϵ -approximate solution to (10) is the optimal solution $-u$. In fact, this implies $\delta_\epsilon^* = \|x^c - x^d\|_\infty = \|v + u\|_\infty$, and the latter norm can be bounded as follows:

$$\begin{aligned} \|v + u\|_\infty &= 2t + (n-1)\beta\Delta = 2 \left(\left\lceil \frac{2}{\epsilon} - 1 \right\rceil - 1 \right) + (n-1)\beta\Delta \\ &\geq 2 \left(\frac{2}{\epsilon} - 2 \right) + (n-1)\beta\Delta = 4 \left(\frac{1}{\epsilon} - 1 \right) + (n-1)\beta\Delta. \end{aligned}$$

Therefore, in the remainder of the proof we show that the only ϵ -approximate solution to (10) is the optimal solution $-u$. If $\epsilon = 1$, this is easy to see. In fact, our definition of t implies $t = 0$. Therefore, the unique feasible point for (10) is the origin. Therefore, in the remainder of the proof we assume $\epsilon \in (0, 1)$.

Let $u' := (-t + 1, 0, \dots, 0)$. Note that u' is in \bar{P} since $\epsilon < 1$ implies $t \geq 1$. We have $f(u') = -(t - \frac{1}{2})^2 = f(u)$. Furthermore, it is simple to see that any feasible vector for (10) different from $\pm u, u'$ has cost strictly larger than $f(u)$.

It is simple to check that $f_{\max}^d = -\frac{1}{4}$, since the maximum is achieved at the origin. The vector u is an ϵ -approximate solution to (10) if and only if

$$\frac{f(u) - f(x^d)}{f_{\max}^d - f(x^d)} = \frac{f(u) - f(-u)}{f_{\max}^d - f(-u)} = \frac{-(t - \frac{1}{2})^2 + (t + \frac{1}{2})^2}{-\frac{1}{4} + (t + \frac{1}{2})^2} = \frac{2}{t + 1} \leq \epsilon.$$

Note that our definition of t implies $t < \frac{2}{\epsilon} - 1$, thus $\frac{2}{t+1} > \epsilon$. This shows that the vector u is not an ϵ -approximate solution to (10). Since $-u$ is the only vector in $\bar{P} \cap \mathbb{Z}^n$ with cost strictly smaller than $f(u)$, the only ϵ -approximate solution to (10) is the optimal solution $-u$. \square

Similarly to Proposition 2, also Proposition 3 implies that if $k = n$, then δ_ϵ^* can grow at least linearly with respect to both n and Δ . However, the bound in Proposition 3 is a function of ϵ as well. It implies that δ_ϵ^* can grow at least linearly with respect to $\frac{1}{\epsilon}$.

4.2.2 Lower bounds on δ_ϵ^*

In this section we study the tightness of Theorem 1(ii) by providing a lower bound on δ_ϵ^* that is a function of n , Δ , and $\frac{1}{\epsilon}$.

Proposition 4. For every $n, \Delta \in \mathbb{Z}$ with $n \geq 2$, $\Delta \geq 2$ and $\epsilon \in (0, \frac{1}{2})$, there exists an instance of (IQP) with subdeterminant Δ and $k = 1$ for which

$$\delta_\epsilon^* \geq \frac{(n-1)\Delta - 1}{\epsilon} - 2 \in \Omega\left(\frac{n\Delta}{\epsilon}\right).$$

Proof. Let n, Δ, ϵ be as in the statement, and consider the (IQP)

$$\begin{aligned} \min f(x) &= -x_1^2 \\ \text{s.t. } x &\in \tilde{P} \cap \mathbb{Z}^n, \end{aligned} \tag{11}$$

where $\tilde{P} := \bar{P} \cap \{x \in \mathbb{R}^n \mid x_1 - \Delta \sum_{i=2}^n x_i \leq \beta - 1 + t\}$. Here, the polytope \bar{P} is given in Definition 1 with $\beta := \frac{1}{2}$ and $t := \left\lfloor \frac{(n-1)\beta\Delta + \beta - 1}{\epsilon} \right\rfloor \geq 1$. Notice that the constraint matrix defining \tilde{P} coincides with the one defining \bar{P} . Therefore, Observation 3 implies that problem (11) has subdeterminant Δ .

Since $\tilde{P} \subseteq \bar{P}$, we have $\tilde{P} \cap \mathbb{Z}^n \subseteq \bar{P} \cap \mathbb{Z}^n$. It can be easily checked that $-u \in \tilde{P}$, while $u \notin \tilde{P}$, therefore $x^d = -u$ is the unique optimal solution to (11).

Let $w := ((n-1)\beta\Delta + \beta - 1 + t, \beta, \dots, \beta)$. Observe that w is a vector in \tilde{P} with the largest x_1 . In fact, we know that every $x \in \bar{P}$ satisfies $x_i \leq \beta$, for $i = 2, \dots, n$, thus for every $x \in \tilde{P}$ we have $x_1 \leq \Delta \sum_{i=2}^n x_i + \beta - 1 + t \leq (n-1)\beta\Delta + \beta - 1 + t$. Furthermore, the vector $-u$ is a vector in \tilde{P} with the smallest x_1 . In fact, we know that $-u$ is the vertex of \bar{P} with the smallest x_1 , and $-u \in \tilde{P}$. It can be checked that

$$f(w) = -((n-1)\beta\Delta + \beta - 1 + t)^2 < -t^2 = f(-u),$$

because $(n-1)\beta\Delta + \beta - 1 > 2\beta - 1 = 0$. In particular, we conclude that $x^c = w$ is an optimal solution to the (QP) corresponding to (11).

Next, we show that $-u$ is not an ϵ -approximate solution to (QP). Notice that $f_{\max}^c = 0$ as it is achieved at the origin. We have

$$\begin{aligned} \frac{f(-u) - f(x^c)}{f_{\max}^c - f(x^c)} &= 1 - \frac{t^2}{((n-1)\beta\Delta + \beta - 1 + t)^2} \\ &= 1 - \frac{1}{1 + \frac{((n-1)\beta\Delta + \beta - 1)^2}{t^2} + 2\frac{(n-1)\beta\Delta + \beta - 1}{t}} \\ &\geq 1 - \frac{1}{1 + \epsilon^2 + 2\epsilon} = \epsilon \frac{2 + \epsilon}{(1 + \epsilon)^2} > \epsilon, \end{aligned}$$

where the first inequality holds because $t \leq \frac{(n-1)\beta\Delta + \beta - 1}{\epsilon}$, while the second inequality is correct because $\frac{2 + \epsilon}{(1 + \epsilon)^2} > 1$, when $\epsilon \in (0, \frac{1}{2})$. Thus, $-u$ is not an ϵ -approximate solution to (QP).

Since $-u$ is a vector in \tilde{P} with the smallest x_1 , and due to the form of the objective function, every ϵ -approximate solution x^* to (QP) must satisfy $x_1^* > u_1 = t$, which implies

$$\|x^* - x^d\|_\infty > 2t \geq 2 \frac{(n-1)\beta\Delta + \beta - 1}{\epsilon} - 2 = \frac{(n-1)\Delta - 1}{\epsilon} - 2.$$

This implies $\delta_\epsilon^* \geq \frac{(n-1)\Delta - 1}{\epsilon} - 2$. □

In particular, Proposition 4 shows that, in Theorem 1(ii), the dependence of δ_ϵ^* on n is tight. Furthermore, it shows that δ_ϵ^* can grow at least linearly with respect to both Δ and $\frac{1}{\epsilon}$. Therefore, in Theorem 1(ii), the dependence on ϵ is tight for $k = 1$.

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