

Equivalent second order cone programs for distributionally robust zero-sum games

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Abstract We consider a two-player zero-sum game with stochastic linear constraints. The probability distributions of the vectors associated with the constraints are partially known. The available information with respect to the distribution is based mainly on the first two moments. In this vein, we formulate the stochastic linear constraints as distributionally robust chance constraints. We consider three different types of moments based uncertainty sets. For each uncertainty set, we show that there exists a saddle point equilibrium of the game. The latter corresponds to the optimal solution of a primal-dual pair of second order cone programs. In order to illustrate our theoretical results, we consider an instance of a zero-sum game together with distributionally robust chance constraints. A comparison with expected value constraints zero-sum game is drawn to illustrate the effectiveness of our approach.

Keywords Distributionally robust chance constraints · Zero-sum game · Saddle point equilibrium · Second-order cone program.

1 Introduction

In a two-player zero-sum game, the gain of one player is the loss of the other player. We typically represent a zero-sum game with a payoff matrix where the rows and the

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columns are the actions of player 1 and player 2, respectively. Neumann [33] showed that there exists a mixed strategy saddle point equilibrium (SPE) of a zero-sum game. Dantzig [13] showed the equivalence of SPE and the solution of a primal-dual pair of linear programs. Adler [1] studied the equivalence between linear programming problems and zero-sum games. Charnes [7] generalized the zero-sum game considered in [13, 33]. He introduced linear inequality constraints for both the players, and showed the equivalence of SPE of a constrained zero-sum game and the solution of a primal-dual pair of linear programs. Nash [25] introduced the equilibrium concept for a finite number of rational players where each player has finite number of actions. He showed that there exists a mixed strategy Nash equilibrium for finite strategic games. Since then, general strategic games have been extensively studied in the literature [2, 14, 17]. The games discussed in the above-mentioned papers have deterministic strategy set and payoff function for each player. However, the decision making process is fed by input parameters which are usually subject to uncertainties [12, 24]. The model uncertainties can be accounted for the expected value approach in case of risk neutral decision makers. Ravat and Shanbhag [27] considered the stochastic Nash games where each player optimizes his expected value payoff subject to his expected value constraints. For risk averse players, the payoff criterion with the risk measure CVaR [21, 27] and the variance was considered in the literature [11]. Singh et al. [29, 30, 31] introduced chance-constrained games by considering a risk averse payoff criterion based on the chance constraint programming for finite strategic games with random payoffs. As for the elliptically distributed random payoffs, the authors showed the existence of a Nash equilibrium for a chance-constrained game [29], and proposed an equivalent mathematical program to compute the Nash equilibria of the game [31]. In [30], the authors considered the games where the probability distributions of the players' payoffs are partially known, and belong to given distributional uncertainty sets. They formulated these games via a distributionally robust chance constraint, and showed the existence of a Nash equilibrium. Moreover, they proposed an equivalent mathematical program based method to come-up with Nash equilibria. There are several zero-sum chance-constrained games studied in past literature [3, 6, 8, 10].

The chance-constrained games in the above-mentioned papers considered the case where players' payoffs are random variables and the strategy sets are deterministic in nature. To the best of our knowledge, the research on the games with stochastic strategy sets defined by chance constraints is very scarce [26, 32]. Peng et al. [26] considered an n -player general sum game with joint chance constraints for each player. They showed the existence of a Nash equilibrium when the row vectors of the stochastic constraints are independent, and follow a multivariate normal distribution. Singh and Lisser [32] considered a stochastic version of a two-player constrained zero-sum game studied in [7] where each player has individual chance constraints. They showed the equivalence of an SPE and the optimal solution of a primal-dual pair of a second order cone programs (SOCPs) when the random row vectors of the constraints follow a multivariate elliptically symmetric distribution. Unlike stochastic two-player constrained zero-sum game [32], the only information we have on the probability distribution of a random constraint vector is that it belongs to a certain uncertainty set. In this paper, we consider three different types of

well-known uncertainty sets [9, 15, 16] based on the first and second order moments of the random constraint vectors. For each type of uncertainty set, we show that there exists an SPE and it can be obtained from the optimal solutions of a primal-dual pair of SOCPs. We illustrate our theoretical results by considering an instance of a zero-sum game and compare the distributionally robust chance constraints case with the expected value constraints case.

The structure of the rest of the paper is as follows. Section 2 contains the definition of a zero-sum game with distributionally robust chance constraints. Section 3 presents the reformulation of distributionally robust chance constraints as second order cone constraints under three different uncertainty sets. Section 4 outlines a primal-dual pair of SOCPs whose optimal solutions constitute an SPE of the game. We present our numerical results in Section 5, and conclude the paper in Section 6.

2 The Model

We consider a two-player zero-sum game defined by an $n_1 \times n_2$ matrix G , where n_1 and n_2 denote the number of pure strategies of player 1 and player 2, respectively. The set of mixed strategies of player i , $i = 1, 2$, is given by $X^i = \{x^i \in \mathbb{R}^{n_i} \mid \sum_{j=1}^{n_i} x_j^i = 1, x_j^i \geq 0, \forall j = 1, 2, \dots, n_i\}$. Each component of the matrix G corresponds to the payoff for player 1 and the cost for player 2. For a given strategy pair $(x^1, x^2) \in X^1 \times X^2$, $(x^1)^T G x^2$ represents the payoff of player 1 and the cost of player 2; T denotes the transposition. For a given $x^1 \in X^1$ (resp., $x^2 \in X^2$), player 2 (resp., player 1) chooses an optimal strategy by minimizing (resp., maximizing) $(x^1)^T G x^2$ over all $x^2 \in X^2$ (resp., $x^1 \in X^1$). A strategy pair (x^1, x^2) is an SPE of the zero-sum game if x^1 (resp., x^2) is an optimal strategy of player 1 (resp., player 2) for a fixed strategy x^2 (resp., x^1) of player 2 (resp., player 1). An SPE of a zero-sum game exists in mixed strategies [33], and can be computed via a primal-dual pair of linear programs [13]. In certain cases, the players mixed strategies are further restricted by linear inequalities. For example, in a portfolio management problem, the fractions of total amount invested in different assets incur random losses and an investor wants the total loss below certain threshold [20]. In game theoretic context these problems can be represented as a zero-sum game where nature can be considered as the adversary player. Let the mixed strategies of player 1 satisfy the following linear constraints

$$A^1 x^1 \leq b^1, \quad (1)$$

whilst the mixed strategies of player 2 satisfy the linear constraints given by

$$A^2 x^2 \geq b^2, \quad (2)$$

where A^1 is a $p \times n_1$ matrix and A^2 is a $q \times n_2$ matrix. We denote $A^1 = [a_1^1, a_2^1, \dots, a_p^1]^T$ and $A^2 = [a_1^2, a_2^2, \dots, a_q^2]^T$, where a_k^i represents k th row of the matrix A^i . Let $\mathcal{J}_1 = \{1, 2, \dots, p\}$ and $\mathcal{J}_2 = \{1, 2, \dots, q\}$ be the index sets for the constraints of player 1 and player 2, respectively.

Charnes [7] considered the case where A^1 and A^2 are deterministic matrices and called it a constrained zero-sum game. He showed that an SPE of a two-player constrained zero-sum game can be obtained from optimal solutions of a primal-dual pair

of linear programs. Singh and Lisser [32] considered the case when A^1 and A^2 are random matrices where each row vector follows a multivariate elliptically symmetric distribution. They formulated each stochastic linear constraint from (1) and (2) as a chance constraint. Such games are called zero-sum chance-constrained games. They showed that an SPE of a zero-sum chance-constrained game can be obtained from an optimal solutions of a primal-dual pair of SOCPs. In this paper, we consider the case where we have no distributional knowledge of the probability distributions of the random vectors of A^1 and A^2 except their first two moments which leads to optimize over an uncertainty set. When considering the worst case scenario, we use distributionally robust framework to formulate the stochastic linear constraints (1) and (2). The distributionally robust chance constraints of player 1 are given by

$$\inf_{F_k^1 \in \mathcal{D}_k^1} \mathbb{P}(a_k^1 x \leq b_k^1) \geq \alpha_k^1, \forall k \in \mathcal{J}_1, \quad (3)$$

where F_k^1 is a probability distribution of a_k^1 , \mathcal{D}_k^1 is the uncertainty set corresponding to the probability distribution of random vector a_k^1 , and α_k^1 is the confidence level of player 1 for the k th constraint. Similarly, the distributionally robust chance constraints of player 2 are given by

$$\inf_{F_l^2 \in \mathcal{D}_l^2} \mathbb{P}(-a_l^2 y \leq -b_l^2) \geq \alpha_l^2, \forall l \in \mathcal{J}_2, \quad (4)$$

where F_l^2 , \mathcal{D}_l^2 and α_l^2 are analogously defined. Therefore, for a given $\alpha^1 = (\alpha_k^1)_{k \in \mathcal{J}_1}$ and $\alpha^2 = (\alpha_l^2)_{l \in \mathcal{J}_2}$ the feasible strategy sets of player 1 and player 2 are given by

$$S_{\alpha^1}^1 = \left\{ x^1 \in X^1 \mid \inf_{F_k^1 \in \mathcal{D}_k^1} \mathbb{P}\{a_k^1 x^1 \leq b_k^1\} \geq \alpha_k^1, \forall k \in \mathcal{J}_1 \right\}, \quad (5)$$

and

$$S_{\alpha^2}^2 = \left\{ x^2 \in X^2 \mid \inf_{F_l^2 \in \mathcal{D}_l^2} \mathbb{P}\{-a_l^2 x^2 \leq -b_l^2\} \geq \alpha_l^2, \forall l \in \mathcal{J}_2 \right\}. \quad (6)$$

We call the matrix game G with the strategy set $S_{\alpha^1}^1$ for player 1 and the strategy set $S_{\alpha^2}^2$ for player 2 as a distributionally robust zero-sum chance-constrained game. We denote this game by Z_α . A strategy pair $(x^{1*}, x^{2*}) \in S_{\alpha^1}^1 \times S_{\alpha^2}^2$ is called an SPE of the game Z_α at $\alpha = (\alpha^1, \alpha^2) \in [0, 1]^p \times [0, 1]^q$, if

$$(x^1)^T G x^{2*} \leq (x^{1*})^T G x^{2*} \leq (x^{1*})^T G x^2,$$

for all $x^1 \in S_{\alpha^1}^1, x^2 \in S_{\alpha^2}^2$.

3 Reformulation of distributionally robust chance constraints

We consider three different uncertainty sets based on the full/partial information about the mean vectors and covariance matrices of the random vectors $a_k^1, k \in \mathcal{J}_1$, and $a_l^2, l \in \mathcal{J}_2$. For each uncertainty set, chance constraints (3) and (4) are reformulated as second order cone constraints.

3.1 Moments based uncertainty sets

First, we consider the case where the first two moments are known. The uncertainty set of player i , $i = 1, 2$, defined by the mean vector μ_k^i and the covariance matrix Σ_k^i of $(a_k^i)^T$, is given by

$$\mathcal{D}_k^i(\mu_k^i, \Sigma_k^i) = \left\{ F_k^i \mid \begin{array}{l} E_{F_k^i} [(a_k^i)^T] = \mu_k^i \\ E_{F_k^i} [((a_k^i)^T - \mu_k^i) ((a_k^i)^T - \mu_k^i)^T] = \Sigma_k^i \end{array} \right\}, \quad (7)$$

for all $k \in \mathcal{J}_i$. These uncertainty sets are considered in [5, 16]. As for the second uncertainty set, we assume that the first moment is known whilst the second moment is unknown. The uncertainty set of player i , $i = 1, 2$, defined by the mean vector μ_k^i and the upper bound Σ_k^i on covariance matrix of $(a_k^i)^T$, is given by

$$\mathcal{D}_k^i(\mu_k^i, \Sigma_k^i) = \left\{ F_k^i \mid \begin{array}{l} E_{F_k^i} [(a_k^i)^T] = \mu_k^i \\ E_{F_k^i} [((a_k^i)^T - \mu_k^i) ((a_k^i)^T - \mu_k^i)^T] \preceq \Sigma_k^i \end{array} \right\}, \quad (8)$$

for all $k \in \mathcal{J}_i$. These uncertainty sets are considered in [9]. The last uncertainty set is based on unknown first two moments. The uncertainty set of player i , $i = 1, 2$, where the mean vector of $(a_k^i)^T$ lies in an ellipsoid of size $\gamma_{k1}^i \geq 0$ centered at μ_k^i and the covariance matrix of $(a_k^i)^T$ lies in a positive semidefinite cone defined with a linear matrix inequality, is given by

$$\mathcal{D}_k^i(\mu_k^i, \Sigma_k^i) = \left\{ F_k^i \mid \begin{array}{l} \left(\mathbb{E}_{F_k^i} [(a_k^i)^T] - \mu_k^i \right)^\top (\Sigma_k^i)^{-1} \left(\mathbb{E}_{F_k^i} [(a_k^i)^T] - \mu_k^i \right) \leq \gamma_{k1}^i \\ \text{COV}_{F_k^i} [(a_k^i)^T] \preceq \gamma_{k2}^i \Sigma_k^i \end{array} \right\}, \quad (9)$$

for all $k \in \mathcal{J}_i$. $\text{COV}_{F_k^i}$ is a covariance operator under probability distribution F_k^i . These uncertainty sets are considered in [15, 22].

3.2 Second order cone constraint reformulation

It follows from Theorem 3.1 of [5] that the chance constraints (3) and (4) for uncertainty set (7) can be reformulated as

$$(\mu_k^1)^T x^1 + \sqrt{\frac{\alpha_k^1}{1 - \alpha_k^1}} \left\| (\Sigma_k^1)^{\frac{1}{2}} x^1 \right\| \leq b_k^1, \quad \forall k \in \mathcal{J}_1, \quad (10)$$

and

$$-(\mu_l^2)^T x^2 + \sqrt{\frac{\alpha_l^2}{1 - \alpha_l^2}} \left\| (\Sigma_l^2)^{\frac{1}{2}} x^2 \right\| \leq -b_l^2, \quad \forall l \in \mathcal{J}_2, \quad (11)$$

where $\|\cdot\|$ is the Euclidean norm. The same reformulation based on Lagrangian duality follows from Theorem 1 of [16]. The reformulation of (3) and (4) for uncertainty

set (8) is also given by (10) and (11), respectively (see [9]). Based on the structure of the uncertainty set (9), the constraint (3) can be written as

$$\inf_{(\mu, \Sigma) \in \mathcal{U}_k^1} \inf_{F_k^1 \in \mathcal{D}(\mu, \Sigma)} \mathbb{P}\{a_k^1 x^1 \leq b_k^1\} \geq \alpha_k^1,$$

where

$$\mathcal{D}(\mu, \Sigma) = \left\{ F_k^1 \mid \mathbb{E}_{F_k^1}[(a_k^1)^T] = \mu, \text{COV}_{F_k^1}[(a_k^1)^T] = \Sigma \right\}$$

and

$$\mathcal{U}_k^1 = \left\{ (\mu, \Sigma) \mid (\mu - \mu_k^1)^\top (\Sigma_k^1)^{-1} (\mu - \mu_k^1) \leq \gamma_{k1}^1, \Sigma \preceq \gamma_{k2}^1 \Sigma_k^1 \right\}.$$

The bound of one-sided Chebyshev inequality can be achieved by a two-point distribution given by equation (2) of [28]. Therefore, we have

$$\inf_{F_k^1 \in \mathcal{D}(\mu, \Sigma)} \mathbb{P}\{a_k^1 x^1 \leq b_k^1\} = \begin{cases} 1 - \frac{1}{1 + \frac{(\mu^T x^1 - b_k^1)^2}{(x^1)^T \Sigma x^1}}, & \text{if } \mu^T x^1 \leq b_k^1, \\ 0, & \text{otherwise.} \end{cases}$$

For the case $\mu^T x^1 > b_k^1$,

$$\inf_{F_k^1 \in \mathcal{D}(\mu, \Sigma)} \mathbb{P}\{a_k^1 x^1 \leq b_k^1\} = 0,$$

which leads to constraint (3) to be infeasible for any $\alpha_k^1 > 0$. Therefore, for $x^1 \in S_{\alpha_k^1}^1$ the condition $\mu^T x^1 \leq b_k^1$ always holds and the constraint (3) is equivalent to

$$\inf_{(\mu, \Sigma) \in \mathcal{U}_k^1} 1 - \frac{1}{1 + (\mu^T x^1 - b_k^1)^2 / ((x^1)^T \Sigma x^1)} \geq \alpha_k^1,$$

which can be reformulated as

$$h_k^1(x^1) \geq \sqrt{\frac{\alpha_k^1}{1 - \alpha_k^1}}, \quad (12)$$

where

$$h_k^1(x^1) = \begin{cases} \min_{\mu, \Sigma} \frac{b_k^1 - \mu^T x^1}{\sqrt{(x^1)^T \Sigma x^1}} \\ \text{s.t. (i) } (\mu - \mu_k^1)^\top (\Sigma_k^1)^{-1} (\mu - \mu_k^1) \leq \gamma_{k1}^1, \\ \text{(ii) } \Sigma \preceq \gamma_{k2}^1 \Sigma_k^1. \end{cases} \quad (13)$$

For the sake of simplicity, we separate problem (13) into two optimization problems

$$h_k^1(x^1) = \frac{b_k^1 + v_1(x^1)}{\sqrt{v_2(x^1)}},$$

where

$$v_1(x^1) = \begin{cases} \min_{\mu} -\mu^T x^1 \\ \text{s.t. } (\mu - \mu_k^1)^\top (\Sigma_k^1)^{-1} (\mu - \mu_k^1) \leq \gamma_{k1}^1, \end{cases} \quad (14)$$

$$v_2(x^1) = \begin{cases} \max_{\Sigma} (x^1)^T \Sigma x^1 \\ \text{s.t. } \Sigma \preceq \gamma_{k2}^1 \Sigma_k^1. \end{cases}$$

Let $\beta \geq 0$ be a Lagrange multiplier associated with the constraint of optimization problem (14). By applying the KKT conditions, the optimal solution of (14)

is given by $\mu = \mu_k^1 + \frac{\sqrt{\gamma_{k1}^1 \Sigma_k^1 x^1}}{\sqrt{(x^1)^T \Sigma_k^1 x^1}}$ and associated Lagrange multiplier is given by

$\beta = \sqrt{\frac{(x^1)^T \Sigma_k^1 x^1}{4\gamma_{k1}^1}}$. Therefore, the corresponding optimal value $v_1(x^1) = -(\mu_k^1)^T x^1 - \sqrt{\gamma_{k1}^1} \sqrt{(x^1)^T \Sigma_k^1 x^1}$. Since, $u^T \Sigma u \leq u^T \gamma_{k2}^1 \Sigma_k^1 u$ for any $u \in \mathbb{R}^n$, then, $v_2(x^1) = \gamma_{k2}^1 (x^1)^T \Sigma_k^1 x^1$. Therefore, using (12) we have the following reformulation of (3)

$$(\mu_k^1)^T x^1 + \left(\sqrt{\frac{\alpha_k^1}{1-\alpha_k^1}} \sqrt{\gamma_{k2}^1} + \sqrt{\gamma_{k1}^1} \right) \left\| (\Sigma_k^1)^{\frac{1}{2}} x^1 \right\| \leq b_k^1, \quad (15)$$

for all $k \in \mathcal{J}_1$. Similarly, the reformulation of (4) is given by

$$-(\mu_l^2)^T x^2 + \left(\sqrt{\frac{\alpha_l^2}{1-\alpha_l^2}} \sqrt{\gamma_{l2}^2} + \sqrt{\gamma_{l1}^2} \right) \left\| (\Sigma_l^2)^{\frac{1}{2}} x^2 \right\| \leq -b_l^2, \quad (16)$$

for all $l \in \mathcal{J}_2$. The reformulation of feasible strategy sets (5) and (6) for uncertainty sets (7), (8), and (9) can be written as

$$S_{\alpha^1}^1 = \left\{ x^1 \in X^1 \mid (\mu_k^1)^T x^1 + \kappa_{\alpha_k^1} \left\| (\Sigma_k^1)^{\frac{1}{2}} x^1 \right\| \leq b_k^1, k \in \mathcal{J}_1 \right\}, \quad (17)$$

and

$$S_{\alpha^2}^2 = \left\{ x^2 \in X^2 \mid -(\mu_l^2)^T x^2 + \kappa_{\alpha_l^2} \left\| (\Sigma_l^2)^{\frac{1}{2}} x^2 \right\| \leq -b_l^2, l \in \mathcal{J}_2 \right\}, \quad (18)$$

where $\kappa_{\alpha_k^i} = \sqrt{\frac{\alpha_k^i}{1-\alpha_k^i}}$, $i = 1, 2$, represents the reformulation under uncertainty sets (7)

and (8), and $\kappa_{\alpha_k^i} = \left(\sqrt{\frac{\alpha_k^i}{1-\alpha_k^i}} \sqrt{\gamma_{k2}^i} + \sqrt{\gamma_{k1}^i} \right)$, $i = 1, 2$, represents the reformulation for uncertainty set (9). We assume that the strategy sets (17) and (18) satisfy the strict feasibility condition given by Assumption 1.

Assumption 1 1. There exists an $x^1 \in S_{\alpha^1}^1$ such that the inequality constraints of $S_{\alpha^1}^1$ defined by (17) are strictly satisfied.

2. There exists an $x^2 \in S_{\alpha^2}^2$ such that the inequality constraints of $S_{\alpha^2}^2$ defined by (18) are strictly satisfied.

The conditions given in Assumption 1 are Slater's condition which are sufficient for strong duality in a convex optimization problem. We use these conditions in order to derive equivalent SOCPs for the zero-sum game Z_{α} .

4 Existence and characterization of saddle point equilibrium

In this section, we show that there exists an SPE of the game Z_α if the probability distributions of the random vectors of the constraints of both the players belong to the uncertainty sets defined in Section 3.1. We further propose a primal-dual pair of SOCPs whose optimal solutions constitute an SPE of the game Z_α .

Theorem 1 *Consider the game Z_α where the probability distributions of the row vectors a_k^i , $k \in \mathcal{J}_i$, $i = 1, 2$, belong to the uncertainty sets described in Section 3.1. Then, there exists an SPE of the game for all $\alpha \in (0, 1)^p \times (0, 1)^q$.*

Proof Let $\alpha \in (0, 1)^p \times (0, 1)^q$. For uncertainty sets described in Section 3.1, the strategy sets $S_{\alpha_1}^1$ and $S_{\alpha_2}^2$ are given by (17) and (18), respectively. It is easy to see that $S_{\alpha_1}^1$ and $S_{\alpha_2}^2$ are convex and compact sets. The function $(x^1)^T G x^2$ is a bilinear and continuous function. Hence, there exists an SPE from the minimax theorem of Neumann [33].

4.1 Equivalent primal-dual pair of second order cone programs

From the minimax theorem [33], (x^{1*}, x^{2*}) is an SPE for the game Z_α if and only if

$$x^{1*} \in \arg \max_{x^1 \in S_{\alpha_1}^1} \min_{x^2 \in S_{\alpha_2}^2} (x^1)^T G x^2, \quad (19)$$

$$x^{2*} \in \arg \min_{x^2 \in S_{\alpha_2}^2} \max_{x^1 \in S_{\alpha_1}^1} (x^1)^T G x^2. \quad (20)$$

We start with problem $\min_{x^2 \in S_{\alpha_2}^2} \max_{x^1 \in S_{\alpha_1}^1} (x^1)^T G x^2$. The inner optimization problem $\max_{x^1 \in S_{\alpha_1}^1} (x^1)^T G x^2$ can be equivalently written as

$$\left. \begin{array}{l} \max_{x^1, (t_k^1)_{k \in \mathcal{J}_1}} (x^1)^T G x^2 \\ \text{s.t.} \\ (i) \quad -(x^1)^T \mu_k^1 - \kappa_{\alpha_k^1} \|t_k^1\| + b_k^1 \geq 0, \quad \forall k \in \mathcal{J}_1 \\ (ii) \quad t_k^1 - (\Sigma_k^1)^{\frac{1}{2}} x^1 = 0, \quad \forall k \in \mathcal{J}_1 \\ (iii) \quad \sum_{j=1}^{n_1} x_j^1 = 1, \\ (iv) \quad x_j^1 \geq 0, \quad \forall j = 1, 2, \dots, n_1. \end{array} \right\} \quad (21)$$

Let $\lambda^1 = (\lambda_k^1)_{k \in \mathcal{J}_1} \in \mathbb{R}^p$, $\delta_k^1 \in \mathbb{R}^{n_1}$, $k \in \mathcal{J}_1$, and $v^1 \in \mathbb{R}$ be the Lagrange multipliers of constraints (i), (ii), and (iii) of (21), respectively. Then, the Lagrangian dual problem of the SOCP (21) is an SOCP [4, 23]. Moreover, the duality gap is zero according to

Assumption 1 . Therefore, the problem $\min_{x^2 \in \mathcal{S}_{\alpha^2}^2} \max_{x^1 \in \mathcal{S}_{\alpha^1}^1} (x^1)^T G x^2$ is equivalent to the following SOCP

$$\left. \begin{array}{l} \min_{x^2, v^1, (\delta_k^1)_{k \in \mathcal{J}_1}, (\lambda_k^1)_{k \in \mathcal{J}_1}} v^1 + \sum_{k \in \mathcal{J}_1} \lambda_k^1 b_k^1 \\ \text{s.t.} \\ (i) \quad G x^2 - \sum_{k \in \mathcal{J}_1} \lambda_k^1 \mu_k^1 - \sum_{k \in \mathcal{J}_1} (\Sigma_k^1)^{\frac{1}{2}} \delta_k^1 \leq v^1 \mathbf{1}_{n_1}, \\ (ii) \quad -(\mu_l^2)^T x^2 + \kappa_{\alpha_l^2} \left\| (\Sigma_l^2)^{\frac{1}{2}} x^2 \right\| \leq -b_l^2, \quad \forall l \in \mathcal{J}_2, \\ (iii) \quad \|\delta_k^1\| \leq \lambda_k^1 \kappa_{\alpha_k^1}, \forall k \in \mathcal{J}_1, \\ (iv) \quad \sum_{j=1}^{n_2} x_j^2 = 1, \\ (v) \quad x_j^2 \geq 0, \forall j = 1, 2, \dots, n_2 \\ (vi) \quad \lambda_k^1 \geq 0, \forall k \in \mathcal{J}_1, \end{array} \right\} \quad (\text{P})$$

where $\mathbf{1}_{n_1}$ is an $n_1 \times 1$ vector of ones. Similarly, problem $\max_{x^1 \in \mathcal{S}_{\alpha^1}^1} \min_{x^2 \in \mathcal{S}_{\alpha^2}^2} (x^1)^T G x^2$ is equivalent to the following SOCP

$$\left. \begin{array}{l} \max_{x^1, v^2, (\delta_l^2)_{l \in \mathcal{J}_2}, (\lambda_l^2)_{l \in \mathcal{J}_2}} v^2 + \sum_{l \in \mathcal{J}_2} \lambda_l^2 b_l^2 \\ \text{s.t.} \\ (i) \quad G^T x^1 - \sum_{l \in \mathcal{J}_2} \lambda_l^2 \mu_l^2 - \sum_{l \in \mathcal{J}_2} (\Sigma_l^2)^{\frac{1}{2}} \delta_l^2 \geq v^2 \mathbf{1}_{n_2}, \\ (ii) \quad (\mu_k^1)^T x^1 + \kappa_{\alpha_k^1} \left\| (\Sigma_k^1)^{\frac{1}{2}} x^1 \right\| \leq b_k^1, \forall k \in \mathcal{J}_1, \\ (iii) \quad \|\delta_l^2\| \leq \lambda_l^2 \kappa_{\alpha_l^2}, \forall l \in \mathcal{J}_2, \\ (iv) \quad \sum_{j=1}^{n_1} x_j^1 = 1, \\ (v) \quad x_j^1 \geq 0, \forall j = 1, 2, \dots, n_1, \\ (vi) \quad \lambda_l^2 \geq 0, \forall l \in \mathcal{J}_2. \end{array} \right\} \quad (\text{D})$$

It follows from the duality theory of second order cone programming problem that (P) and (D) form a primal-dual pair [4, 23].

Remark 1 For $\kappa_{\alpha_k^i} = \sqrt{\frac{\alpha_k^i}{1-\alpha_k^i}}$, $i = 1, 2$, (P) and (D) represent the primal-dual pair of SOCPs for the uncertainty sets defined by (7) and (8). For $\kappa_{\alpha_k^i} = \left(\sqrt{\frac{\alpha_k^i}{1-\alpha_k^i}} \sqrt{\gamma_{k2}^i} + \sqrt{\gamma_{k1}^i} \right)$, $i = 1, 2$, (P) and (D) represent the primal-dual pair of SOCPs for the uncertainty set defined by (9).

Next, we show that the equivalence between the optimal solutions of (P)-(D) and an SPE of the game Z_α .

Theorem 2 Consider the game Z_α where the probability distributions of the row vectors a_k^i , $k \in \mathcal{J}_i$, $i = 1, 2$, belong to the uncertainty sets defined by (7), (8), (9). Let Assumption 1 holds. Then, for a given $\alpha \in (0, 1)^p \times (0, 1)^q$, (x^{1*}, x^{2*}) is an SPE of the game Z_α if and only if there exists $(v^{1*}, (\delta_k^{1*})_{k \in \mathcal{J}_1}, \lambda^{1*})$ and $(v^{2*}, (\delta_l^{2*})_{l \in \mathcal{J}_2}, \lambda^{2*})$ such that $(x^{2*}, v^{1*}, (\delta_k^{1*})_{k \in \mathcal{J}_1}, \lambda^{1*})$ and $(x^{1*}, v^{2*}, (\delta_l^{2*})_{l \in \mathcal{J}_2}, \lambda^{2*})$ are optimal solutions of (P) and (D), respectively.

Proof Let (x^{1*}, x^{2*}) be an SPE of the game Z_α . Then, x^{1*} and x^{2*} are the solutions of (19) and (20) respectively. Therefore, there exists $(v^{1*}, (\delta_k^{1*})_{k \in \mathcal{J}_1}, \lambda^{1*})$ and $(v^{2*}, (\delta_l^{2*})_{l \in \mathcal{J}_2}, \lambda^{2*})$ such that $(x^{2*}, v^{1*}, (\delta_k^{1*})_{k \in \mathcal{J}_1}, \lambda^{1*})$ and $(x^{1*}, v^{2*}, (\delta_l^{2*})_{l \in \mathcal{J}_2}, \lambda^{2*})$ are optimal solutions of (P) and (D) respectively.

Let $(x^{2*}, v^{1*}, (\delta_k^{1*})_{k \in \mathcal{J}_1}, \lambda^{1*})$ and $(x^{1*}, v^{2*}, (\delta_l^{2*})_{l \in \mathcal{J}_2}, \lambda^{2*})$ be optimal solutions of (P) and (D) respectively. Under Assumption 1, (P) and (D) are strictly feasible. Therefore, strong duality holds for primal-dual pair (P)-(D). Then, we have

$$v^{1*} + \sum_{k \in \mathcal{J}_1} \lambda_k^{1*} b_k^1 = v^{2*} + \sum_{l \in \mathcal{J}_2} \lambda_l^{2*} b_l^2. \quad (22)$$

Take $x^1 \in S_{\alpha_1}^1$ and multiply the constraint (i) of (P) by $(x^1)^T$. Then, by using Cauchy-Schwartz inequality, we have

$$(x^1)^T G x^{2*} \leq v^{1*} + \sum_{k \in \mathcal{J}_1} \lambda_k^{1*} b_k^1, \quad \forall x^1 \in S_{\alpha_1}^1. \quad (23)$$

Similarly, we have

$$(x^{1*})^T G x^2 \geq v^{2*} + \sum_{l \in \mathcal{J}_2} \lambda_l^{2*} b_l^2, \quad \forall x^2 \in S_{\alpha_2}^2. \quad (24)$$

Take $x^1 = x^{1*}$ and $x^2 = x^{2*}$ in (23) and (24), then from (22), we get

$$(x^{1*})^T G x^{2*} = v^{1*} + \sum_{k \in \mathcal{J}_1} \lambda_k^{1*} b_k^1 = v^{2*} + \sum_{l \in \mathcal{J}_2} \lambda_l^{2*} b_l^2. \quad (25)$$

It follows from (23), (24), and (25) that (x^{1*}, x^{2*}) is an SPE of the game Z_α .

5 Numerical Results

For illustration purpose, we consider an instance of a zero-sum game with random constraints. We compute the saddle point equilibria of the game by solving the SOCPs (P) and (D). We use the convex programs solver CVX on MATLAB for solving the SOCPs [18, 19]. Consider a zero-sum game described by the following 4×4 payoff matrix G

$$G = \begin{pmatrix} 1 & 4 & 4 & 2 \\ 5 & 4 & 4 & 2 \\ 3 & 5 & 4 & 3 \\ 3 & 2 & 3 & 1 \end{pmatrix}.$$

We consider the stochastic linear constraints defined by 3×4 random matrices A^1 and A^2 . The mean vectors and covariance matrices of the row vectors of A^1 and A^2 are summarized as below

$$\begin{aligned} \mu_1^1 &= \begin{pmatrix} 11 \\ 12 \\ 9 \\ 11 \end{pmatrix}, \mu_2^1 = \begin{pmatrix} 14 \\ 14 \\ 15 \\ 12 \end{pmatrix}, \mu_3^1 = \begin{pmatrix} 10 \\ 11 \\ 19 \\ 19 \end{pmatrix}, \mu_1^2 = \begin{pmatrix} 19 \\ 17 \\ 18 \\ 11 \end{pmatrix}, \\ \mu_2^2 &= \begin{pmatrix} 18 \\ 18 \\ 14 \\ 19 \end{pmatrix}, \mu_3^2 = \begin{pmatrix} 15 \\ 14 \\ 19 \\ 9 \end{pmatrix}, b^1 = \begin{pmatrix} 18 \\ 18 \\ 18 \end{pmatrix}, b^2 = \begin{pmatrix} 12 \\ 11 \\ 12 \end{pmatrix}. \end{aligned}$$

$$\begin{aligned} \Sigma_1^1 &= \begin{pmatrix} 12 & 3 & 3 & 3 \\ 3 & 10 & 2 & 4 \\ 3 & 2 & 12 & 2 \\ 3 & 4 & 2 & 10 \end{pmatrix}, \Sigma_2^1 = \begin{pmatrix} 10 & 2 & 2 & 3 \\ 2 & 10 & 4 & 4 \\ 2 & 4 & 10 & 2 \\ 3 & 4 & 2 & 10 \end{pmatrix}, \\ \Sigma_3^1 &= \begin{pmatrix} 10 & 3 & 3 & 3 \\ 3 & 12 & 2 & 3 \\ 3 & 2 & 12 & 2 \\ 3 & 3 & 2 & 10 \end{pmatrix}, \Sigma_1^2 = \begin{pmatrix} 12 & 3 & 3 & 3 \\ 3 & 12 & 3 & 3 \\ 3 & 3 & 10 & 3 \\ 3 & 3 & 3 & 10 \end{pmatrix}, \\ \Sigma_2^2 &= \begin{pmatrix} 12 & 3 & 2 & 3 \\ 3 & 10 & 4 & 2 \\ 2 & 4 & 12 & 2 \\ 3 & 2 & 2 & 12 \end{pmatrix}, \Sigma_3^2 = \begin{pmatrix} 12 & 2 & 3 & 3 \\ 2 & 10 & 2 & 3 \\ 3 & 2 & 10 & 3 \\ 3 & 3 & 3 & 10 \end{pmatrix}. \end{aligned}$$

Table 1 summarizes the saddle point equilibria of the game Z_α for various values of α for all three uncertainty sets defined in Section 3.1. For uncertainty set (9), the SOCPs might be infeasible when the value of $\kappa_{\alpha_k^i}$ is relatively high. We also study zero-sum game problem under expected value constraints for the same data set. The constraints of each player are linear in this case. An SPE for these games can be computed by solving a primal-dual pair of linear programs given in [7]. An SPE for this game is $(x_{avg}^{1*}, x_{avg}^{2*}) = ((0, 0.25, 0.75, 0), (0.2917, 0, 0.1250, 0.5833))$. We compare the saddle point equilibria of both the problems by generating 1000 scenarios. A scenario is termed as violated if any of the three constraints is violated. We generate the scenarios via multivariate normal and multivariate Laplace random generators where the mean and covariance matrix are the same as in the example. The simulation results are shown in Figures 1 and 2. For the SPE of the game Z_α at $\alpha = 0.7$ corresponding to uncertainty set (7), the constraints of player 1 and player 2 are violated for 85 and 139 scenarios, respectively. However, for the SPE $(x_{avg}^{1*}, x_{avg}^{2*})$ of zero-sum game under expected value constraints, there are 394 and 583 violated scenarios for players 1 and 2, respectively. This shows that the saddle point equilibria of the game Z_α are more reliable compared to the zero-sum game under expected value constraints.

Table 1: Saddle point equilibrium for different uncertainty sets

α		Saddle Point Equilibrium		Value of the game	Uncertainty set
α^1	α^2	x^{1*}	x^{2*}		
(0.5, 0.5, 0.5)	(0.5, 0.5, 0.5)	(0.3070, 0.3555, 0.0565, 0.2810)	(0.0216, 0, 0.5398, 0.4386)	3.36	Uncertainty set (7) or (8)
(0.6, 0.6, 0.6)	(0.6, 0.6, 0.6)	(0.0858, 0.4166, 0.4976, 0)	(0.0226, 0, 0.5992, 0.3782)	3.42	
(0.7, 0.7, 0.7)	(0.7, 0.7, 0.7)	(0.1706, 0.4884, 0.3410, 0)	(0.0967, 0, 0.6276, 0.2757)	3.51	
(0.8, 0.8, 0.8)	(0.8, 0.8, 0.8)	(0.3070, 0.3555, 0.0565, 0.2810)	(0.2941, 0.1603, 0.5151, 0.0305)	3.44	
(0.5, 0.5, 0.5)	(0.5, 0.5, 0.5)	(0.0847, 0.4138, 0.5015, 0)	(0.0226, 0, 0.5953, 0.3821)	3.42	Uncertainty set (9) with $\gamma_1=0.1, \gamma_2=0.8$
(0.6, 0.6, 0.6)	(0.6, 0.6, 0.6)	(0.1002, 0.4555, 0.4443, 0)	(0.0233, 0, 0.6519, 0.3248)	3.49	
(0.7, 0.7, 0.7)	(0.7, 0.7, 0.7)	(0.2183, 0.4867, 0.2336, 0.0614)	(0.1371, 0, 0.6435, 0.2194)	3.50	
(0.8, 0.8, 0.8)	(0.8, 0.8, 0.8)	Infeasible	Infeasible	Does not exist	

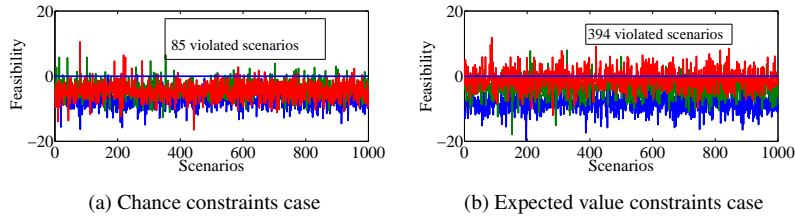


Fig. 1: Violated constraints for player 1

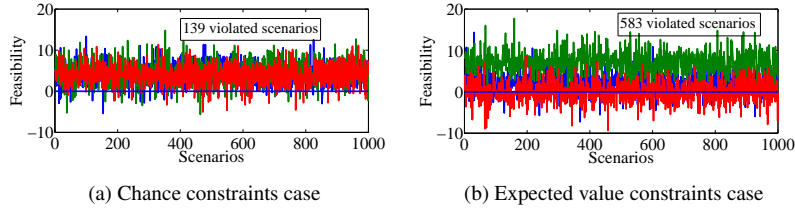


Fig. 2: Violated constraints for player 2

6 Conclusions

We show the existence of a mixed strategy SPE for a two-player distributionally robust zero-sum chance-constrained game under three different uncertainty sets. The saddle point equilibria of the game can be obtained from the optimal solutions of a primal-dual pair of SOCPs. We compute the saddle point equilibria of an instance of the game Z_α by solving SOCPs (P) and (D). For the same instance of the game, we compared the SPE of Z_α with the SPE of the zero-sum game with expected value constraints by generating 1000 scenarios. We find that the number of feasible scenarios for the game Z_α is far larger than the game with expected value constraints.

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