# Sufficient condition on Schrage conjecture about the completion time variance

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**Abstract**

We consider a single machine scheduling problem to minimize the completion time variance. This problem is known to be NP-hard. We prove that if $p_{n-1} = p_{n-2}$, then there is an optimal solution of the form $(n, n-2, n-3, ..., n-4, n-1)$. A new lower bound are proposed for solving the problem. The test on more than 4000 instances shows that this lower bound is very tight and the heuristic (developed by Nessah and Chu [2010]) yields solutions very close to optimal ones since the gap between the solution given by the heuristic and the lower bound is very small.

*Keywords:* scheduling, single machine, completion time variance.

## 1 Introduction

A single machine scheduling problem consists of scheduling $n$ jobs on a machine to minimise a function of job completion times. In this paper, we have considered the scheduling of $n$ non-preemptive jobs on a single machine in order to minimise the completion time variance (CTV). The machine can process at most one job at a time. The set of jobs is denoted by $N = \{1, ..., n\}$. Each job $i$ has a positive processing time $p_i > 0$, the jobs are all available for processing at time zero. In a feasible schedule $\sigma$, the completion time of job $i$ is denoted as $C_i(\sigma)$. Without loss of generality, we have assumed that the jobs are numbered in a nondecreasing order of the processing time $p_i$. In other words, we have $p_1 \leq p_2 \leq ... \leq p_n$. The objective is to find a schedule $\sigma$ that minimises the completion time variance: $CTV(\sigma) = \frac{1}{n}\sum_{i\in N}\left[C_i(\sigma) - \overline{C}(\sigma)\right]^2$, where $\overline{C}(\sigma) = \frac{1}{n}\sum_{i\in N} C_i(\sigma)$ is the mean completion time of $\sigma$.

Kubiak [1993] proved that this problem is NP-hard. The CTV problem was introduced by Merten and Muller [1972], who motivated the variance performance measure by computer file organization problems, where it is important to provide uniform response time to users. Kanet [1981] later argued that this measure is applicable to any service and manufacturing setting where it is desirable to provide jobs or customers with approximately the same level of service. Additionally, Merten and Muller [1972] found a nice duality theorem. They proved that the sequence minimising the variance of completion time is antithetical (*For any sequence of the form* $\sigma = (i_1, i_2, i_3, ..., i_{n-1}, i_n)$*, there is an antithetical sequence of the form* $\sigma_A = (i_n, i_{n-1}, ..., i_3, i_2, i_1)$) to a sequence minimising the variance of waiting time. Eilon and Chowdhury [1977] proved that an optimal sequence must be V-shaped, which means that the jobs before the job with the shortest processing time are scheduled in the descending order of the processing time while the jobs after are scheduled in the ascending order. The bounds for the position of the smallest job in the CTV problem are established in Manna and Prasad [1999]. Kubiak [1995] formulated the CTV problem as a problem of maximising a zero-one quadratic function, which is a sub-modular function with a special cost structure. The variance of job completion time with bi-criteria extension is investigated in De *et al.* [1996] and De *et al.* [1992]. A lower bound and branch and bound algorithm to minimise CTV are given in Viswanathkumar and Srinivasan [2003], and a tabu search-based solution is developed in Al-Turki, Fediki and Andijani [2001]. Pseudo-polynomial algorithms and fast polynomial approximation schemes for CTV minimisation problems are given in Cheng and Kovalyov [1996]; Kubiak *et al.* [2002]; Manna and Prasad [1997]. A sufficient optimality condition for stochastic [1] CTV is discussed in Cai [1996] and the references therein are: Badics and Boros [1998], Bagchi *et al.* [1987], Cai [1995], Cheng and Kubiak [2005], Federgruen and Mosheiov [1996], Gupta *et al.* [1990], Li *et al.* [2007], Ye *et al.* [2007], Cheng and Kovalyov [1996], Kubiak [1995], Mittenthal *et al.* [1995] and Kellerer and Strusevich [2016].

Schrage [1975] considered the optimal sequences for problems with up to five jobs and showed that the longest job must be the first to be processed. He conjectured that there exists an optimal sequence of the form $(n, n-2, n-3, ..., n-1)$. Kanet [1981] proved by a counterexample that this sequence is not optimal. Hall and Kubiak [1991] verified later that the largest job should be scheduled first, the second largest job should be placed last and the third largest job should be scheduled in the second position. In this paper, we have proved that if $p_{n-1} = p_{n-2}$, then there is a sequence of the form $(n, n-2, n-3, ..., n-4, n-1)$. The remainder of this paper proceeds as follows. The preliminary results are presented in section 2. Section 3 proves the main result, the lower bound in Section 4 . Then, section 5 is dedicated to the computational results testing the practical effectiveness of the lower bound. A final section concludes the paper.

## 2 Notations

Below we provide our notations and present several relevant results from the literature.

*Notation.* • $N = \{1, 2, ..., n-1, n\}$, the set of jobs;

• $\Upsilon$ the set of $n!$ job sequences;

• $\sigma$ any job sequence of $\Upsilon$;

• $p_i$ the processing time of job $i$;

[1]each job $i$ requires a processing time $p_i$, which is a random variable with a mean $\mu_i > 0$ and a variance $v_i \geq 0$.

- $MS$ the makespan of any sequence which is $MS = C_{\max} = \sum_{i \in N} p_i$.

**DEFINITION** 2.1 For any sequence of the form $\sigma = (i_1, i_2, i_3, ..., i_{n-1}, i_n)$, there is a *dual sequence* of the form $\sigma_d = (i_1, i_n, i_{n-1}, ..., i_3, i_2)$.

**DEFINITION** 2.2 A sequence is $V$-shaped if jobs are arranged according to the descending order of their processing times before the shortest job and in the ascending order of processing times after the shortest job.

We have the following lemmas.

**LEMMA** 2.1 *(Schrage [1975]) For any sequence of the form $\sigma = (i_1, i_2, i_3, ..., i_{n-1}, i_n)$, then for its dual sequence $\sigma_d = (i_1, i_n, i_{n-1}, ..., i_3, i_2)$, we have*

*a)* $\overline{C}(\sigma) + \overline{C}(\sigma_d) = MS + p_{i_1}$,

*b)* $CTV(\sigma) = CTV(\sigma_d)$.

**LEMMA** 2.2 *(Eilon and Chowdhury [1977]) The optimal sequence minimising the completion time variance is $V$-shaped.*

## 3 Main Result

Schrage [1975] conjectured that there exists, for every instance of CTV, an optimal schedule of the form $(n, n-2, n-3, ..., n-1)$ or $(n, n-1, ..., n-3, n-2)$. Kanet [1981] provided a counterexample of a 8-job instance, for which no optimal solution is of these forms. Furthermore, Hall and Kubiak [1991] proved that there exists an optimal solution of the form $(n, n-2, ..., n-1)$. In the following, we will prove that if $p_{n-1} = p_{n-2}$, then there is an optimal solution of the form $(n, n-2, n-3, ..., n-4, n-1)$. Before that, let us prove the two following lemmas.

Let us consider the following job sequences:

$$\sigma = \left( \overbrace{(i_1, i_2, ..., i_r)}^{\pi_1}, \overbrace{(j_1, j_2, ..., j_s)}^{S}, \overbrace{(i_{r+1}, i_{r+2}, ..., i_k)}^{\pi_2} \right)$$

where, $k + s = n$, $\pi$ is the subsequence $(i_1, i_2, ..., i_r, i_{r+1}, i_{r+2}, ..., i_k)$, where job $i_{r+1}$ starts at time $MS - \sum_{h=r+1}^{k} p_{i_h}$ and $S$ is the subsequence $(j_1, j_2, ..., j_s)$ where job $j_1$ start at time $\sum_{h=1}^{r} p_{i_h}$. The mean completion time of the subsequences $\pi$ and $S$ is denoted by $\overline{C}(\pi)$ and $\overline{C}(S)$, respectively. The following lemma gives the relationship between $\sigma$, $\pi$ and $S$.

**LEMMA** 3.1 *We have the following decompositions*

$$CTV(\sigma) = \tfrac{k}{n} CTV(\pi) + \tfrac{s}{n} CTV(S) + \tfrac{ks}{n^2} \left( \overline{C}(S) - \overline{C}(\pi) \right)^2$$
$$CTV(\sigma) = \tfrac{k}{n} CTV(\pi) + \tfrac{s}{n} CTV(S) + \tfrac{k}{s} \left( \overline{C}(\sigma) - \overline{C}(\pi) \right)^2$$

**PROOF**. The mean completion time of $\sigma$ can be rewritten as follows:

$$\overline{C}(\sigma) = \frac{k}{n}\overline{C}(\pi) + \frac{s}{n}\overline{C}(S).$$

The quantity $\frac{1}{n}\sum_{h=1}^{n} C_h(\sigma)^2$ can be written as

$$\frac{1}{n}\sum_{h=1}^{n} C_h(\sigma)^2 = \frac{k}{n}\frac{\sum_{h=1}^{k} C_{i_h}(\pi)^2}{k} + \frac{s}{n}\sum_{h=1}^{s}\frac{C_{j_h}(S)^2}{s}.$$

We obtain then $CTV(\sigma) = \frac{k}{n}CTV(\pi) + \frac{s}{n}CTV(S) + \frac{ks}{n^2}\left(\overline{C}(\pi) - \overline{C}(S)\right)^2$. For the second equality, it suffices to replace $\overline{C}(S)$ by $\frac{n\overline{C}(\sigma) - k\overline{C}(\pi)}{s}$. ∎

The following lemma gives a new lower bound for mean completion time.

**LEMMA** 3.2 *If $p_{n-1} > p_{n-2}$, then there exists an optimal sequence $\sigma^* = (n, n-2, ..., n-1)$ for $CTV(\sigma)$ such that*

$$\overline{C}(\sigma^*) \geq \frac{1}{2}\left(MS + p_n\right) - \frac{n-2}{2n}\left(p_{n-1} - p_{n-2}\right).$$

**PROOF**. Hall and Kubiak [1991] proved that there exists an optimal sequence $\sigma^*$ of the form $\sigma^* = (n, n-2, ..., n-1)$ for $CTV(\sigma)$. Let

$$\sigma^* = \left(\overbrace{n}^{\pi_1}, \overbrace{(n-2, i_1, i_2, ..., i_{n-3})}^{S}, \overbrace{n-1}^{\pi_2}\right).$$

Let us consider the following sequence

$$\sigma = \left(\overbrace{n}^{\pi_1}, \overbrace{(n-2, i_{n-3}, i_{n-4}, ..., i_1)}^{S_d}, \overbrace{n-1}^{\pi_2}\right)$$

where, $S_d$ is the dual sequence of $S$. Then, we have $\overline{C}(\pi) = \frac{MS+p_n}{2}$ and

$$\overline{C}(S) + \overline{C}(S_d) = MS - p_{n-1} + p_n + p_{n-2}.$$

By Lemma 3.1, we have

$$\begin{aligned} CTV(\sigma^*) &= \tfrac{2}{n}CTV(\pi) + \tfrac{n-2}{n}CTV(S) + \tfrac{2(n-2)}{n^2}\left(\overline{C}(S) - \tfrac{MS+p_n}{2}\right)^2 \\ &\leq \tfrac{2}{n}CTV(\pi) + \tfrac{n-2}{n}CTV(S_d) + \tfrac{2(n-2)}{n^2}\left(\overline{C}(S_d) - \tfrac{MS+p_n}{2}\right)^2. \end{aligned}$$

By Lemma 2.1, $CTV(S_d) = CTV(S)$ then we obtain

$$\left(\overline{C}(S) - \overline{C}(S_d)\right)\left(\overline{C}(S) + \overline{C}(S_d) - MS - p_n\right) \leq 0.$$

Since $\overline{C}(S) + \overline{C}(S_d) - MS - p_n = p_{n-2} - p_{n-1} < 0$, then $\overline{C}(S) \geq \overline{C}(S_d)$ and, therefore,

$$MS - p_{n-1} + p_n + p_{n-2} = \overline{C}(S) + \overline{C}(S_d) \leq 2\overline{C}(S).$$

Hence,

$$\begin{aligned}\overline{C}(\sigma^*) &= \tfrac{1}{n}\left(MS + p_n + (n-2)\overline{C}(S)\right)\\ &\geq \tfrac{1}{n}\left(MS + p_n + (n-2)\tfrac{MS - p_{n-1} + p_n + p_{n-2}}{2}\right)\\ &\geq \tfrac{1}{2}\left(MS + p_n\right) - \tfrac{n-2}{2n}\left(p_{n-1} - p_{n-2}\right).\end{aligned}$$

■

The lower bound of the mean completion time in Lemma 3.2 dominates the lower bounds established by Ventura and Weng [1995] and Ng, Cai, Cheng [1996]. Furthermore, Ventura and Weng [1995] proved that $LB_{VW} = \frac{1}{2}\left(MS + p_n - p_{n-1}\right)$ is a lower bound for the mean completion time of $\sigma^*$. As $2p_{n-1} + (n-2)p_{n-2} > 0$, then

$$LB_{VW} < \frac{1}{2}\left(MS + p_n\right) - \frac{n-2}{2n}\left(p_{n-1} - p_{n-2}\right).$$

Ng, Cai, Cheng [1996] proved that $LB_{NCC} = \frac{MS}{2} + \frac{(n-1)(p_n - p_{n-1}) + MS}{2n}$ is a lower bound for the mean completion time of $\sigma^*$. As $MS \leq p_n + p_{n-1} + (n-2)p_{n-2}$ (Note that this inequality is strict if for some $j = 1, ..., n-3$, $p_{n-2} > p_j$), then

$$LB_{NCC} \leq \frac{1}{2}\left(MS + p_n\right) - \frac{n-2}{2n}\left(p_{n-1} - p_{n-2}\right).$$

**EXAMPLE** 3.1 This lower bound is tight in the sense that there exists a problem instance where this bound is attained. For this, let us consider a $n$-job problem described as follows

$$p_n \geq p_{n-1} > p_{n-2} \geq p_{n-3} = ... = p_2 = p_1.$$

Then, there is an optimal sequence of the form

$$\sigma^* = (n, n-2, n-3, n-4, ..., 2, 1, n-1).$$

We obtain that

$$\begin{aligned}\overline{C}(\sigma^*) &= \tfrac{1}{2n}\left(2np_n + 2p_{n-1} + 2(n-1)p_{n-2} + n(n-3)p_{n-3}\right)\\ &= \tfrac{1}{2}\left(MS + p_n\right) - \tfrac{n-2}{2n}\left(p_{n-1} - p_{n-2}\right).\end{aligned}$$

We have the following theorem.

**THEOREM** 3.1 *If $p_{n-1} = p_{n-2}$, then for any instance of CTV, there is an optimal sequence of the form*

$$\sigma^* = (n, n-2, n-3, ..., n-4, n-1).$$

**PROOF**. Where no ambiguity will arise, we may refer to a job by its processing time. Hall and Kubiak [1991] proved that there exists an optimal solution of the form $(p_n, p_{n-2}, ..., p_{n-1})$. As the optimal solution is $V$-shaped (Lemma 2.2), then the job $n-3$ can be schedule only in the third or the $n-1$ position. As $p_{n-1} = p_{n-2}$ and by the dual solution property (Lemma 2.1) remains, there exists an optimal sequence of the form $(p_n, p_{n-2}, p_{n-3}, ..., p_{n-1})$. Therefore, we need only show the assumption that job $n-4$ immediately follows job $n-3$ in $\sigma^*$ with $n \geq 6$, [2] which leads to a contradiction.

[2] If $n = 5$, Schrage [1975] proved that the sequence $(p_5, p_3, p_1, p_2, p_4)$ is optimal.

If for each $i \leq n-5$, $p_i = p_{n-4}$, then obviously $\sigma = (p_n, p_{n-2}, p_{n-3}, p_1, p_2, ..., p_{n-4}, p_{n-1})$ is optimal. Let

$$\sigma = (p_n, p_{n-2}, p_{n-3}, p_{n-4}, p_{i_1}, ..., p_{i_{n-5}}, p_{n-1})$$

be the optimal sequence of minimising CTV. Notice that $p_{n-4} > p_{i_{n-5}}$ because if $p_{n-4} = p_{i_{n-5}}$, then the sequence $\sigma' = (p_n, p_{n-2}, p_{n-3}, p_{i_{n-5}}, p_{i_1}, ..., p_{i_{n-6}}, p_{n-4}, p_{n-1})$ is optimal. To simplify the notation, denote by $\alpha_0 = p_n$, $\alpha_1 = p_{n-1} = p_{n-2}$, $\alpha_2 = p_{n-3}$, $\alpha_3 = p_{n-4}$, $\alpha_4 = p_{i_{n-5}}$ and $\beta_h = p_{i_h}$, for $h = 1, ..., n-6$. Then $\sigma$ becomes

$$\sigma = (\overbrace{(\alpha_0, \alpha_1, \alpha_2, \alpha_3)}^{\pi_1}, \overbrace{(\beta_1, ..., \beta_{n-6})}^{S}, \overbrace{(\alpha_4, \alpha_1)}^{\pi_2}) \text{ with } \alpha_3 > \alpha_4.$$

In the following, we determine the *lower and upper bounds* of the mean completion time of $\sigma$:

**An upper bound $UB$ of $\overline{C}(\sigma)$.** Construct another sequence $\omega$ by interchanging jobs $\alpha_3$ and $\alpha_4$, and keeping all the remaining jobs in their original positions. Then, we obtain

$$\omega = (\alpha_0, \alpha_1, \alpha_2, \alpha_4, \beta_1, ..., \beta_{n-6}, \alpha_3, \alpha_1).$$

The difference between the variances is:

$$CTV(\omega) - CTV(\sigma) = -10\frac{\alpha_3 - \alpha_4}{n}\overline{C}(\sigma) + \frac{\alpha_3 - \alpha_4}{n}A(\beta),$$

where, $A(\beta) = 4\beta + 10(\alpha_0 + \alpha_1) + 6\alpha_2 + \frac{9n-25}{n}\alpha_3 - \frac{n-25}{n}\alpha_4$ and $\beta = \sum_{i=1}^{n-6}\beta_i$. As $\sigma$ is optimal, then

$$\overline{C}(\sigma) \leq UB(\beta) = \frac{A(\beta)}{10} \tag{3.1}$$

**A lower bound $LB_1$ of $\overline{C}(\sigma)$.** As $\sigma = (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \overbrace{\beta_1, ..., \beta_{n-6}}^{S}, \alpha_4, \alpha_1)$, then trivially we have $\overline{C}(S) \geq \sum_{i=0}^{3}\alpha_i$. Therefore

$$\overline{C}(\sigma) \geq LB_1(\beta) = \frac{1}{n}\left(2\beta + n(\alpha_0 + \alpha_1) + (n-2)\alpha_2 + (n-3)\alpha_3 + 2\alpha_4\right). \tag{3.2}$$

Combining (3.1) and (3.2), we obtain after simplification:

$$\beta \geq (4\alpha_2 + \alpha_3 + \alpha_4))\frac{1}{4}. \tag{3.3}$$

**A lower bound $LB_2$ of $\overline{C}(\sigma)$.** Let $\epsilon > 0$ be a real number and

$$\begin{cases} \overline{\alpha}_0 = \alpha_0 + \epsilon, \\ \overline{\alpha}_1 = \alpha_1 + \epsilon, \\ \overline{\alpha}_2 = \alpha_2 - \epsilon, \\ \overline{\alpha}_3 = \alpha_3 - \epsilon. \end{cases}$$

Denote by $\sigma(\epsilon)$ the following schedule

$$\sigma(\epsilon) = (\overbrace{(\overline{\alpha}_0, \alpha_1, \overline{\alpha}_2, \overline{\alpha}_3)}^{\pi_1(\epsilon)}, S, \overbrace{(\alpha_4, \overline{\alpha}_1)}^{\pi_2(\epsilon)}).$$

Let us consider the following parametric problem:

$$P(\epsilon): \begin{cases} \underset{\pi(\epsilon)}{\text{Minimise}} & CTV(\pi(\epsilon)) \\ \text{s.t.} & 1)\ \overline{\alpha}_2 \text{ is to be scheduled on position } 3, \\ & 2)\ S \text{ are to be scheduled on positions } 5, ..., n-2. \end{cases}$$

Assume that for some $\epsilon > 0$, $\sigma(\epsilon)$ is not optimal of $P(\epsilon)$. As

$$\overline{\alpha}_0 \geq \overline{\alpha}_1 > \alpha_1 \geq \max(\overline{\alpha}_2, \overline{\alpha}_3, \alpha_4, \beta_i),$$

for each $i = 1, ..., n-6$, then according to Hall and Kubiak [1991], there exists a sequence of the form

$$\omega(\epsilon) \in \left\{ \begin{array}{l} (\overline{\alpha}_0, \alpha_1, \overline{\alpha}_2, \alpha_4, S, \overline{\alpha}_3, \overline{\alpha}_1), (\overline{\alpha}_0, \overline{\alpha}_1, \overline{\alpha}_2, \alpha_4, S, \overline{\alpha}_3, \alpha_1), \\ (\overline{\alpha}_0, \overline{\alpha}_1, \overline{\alpha}_2, \overline{\alpha}_3, S, \alpha_4, \alpha_1) \end{array} \right\}$$

such that $CTV(\omega(\epsilon)) < CTV(\sigma(\epsilon))$.

**a) If** $\omega(\epsilon) = (\overline{\alpha}_0, \alpha_1, \overline{\alpha}_2, \alpha_4, S, \overline{\alpha}_3, \overline{\alpha}_1)$. The difference between the two variances is:

$$CTV(\omega(\epsilon)) - CTV(\sigma(\epsilon)) = \frac{\alpha_4 - \overline{\alpha}_3}{n} \left( 10\overline{C}(\sigma(\epsilon)) - A(\beta) + \frac{3n-25}{n}\epsilon \right).$$

Since $CTV(\omega(\epsilon)) < CTV(\sigma(\epsilon))$, then necessarily $\alpha_4 \neq \overline{\alpha}_3$.

**a-1) If** $\alpha_4 < \overline{\alpha}_3$. Since $CTV(\omega(\epsilon)) < CTV(\sigma(\epsilon))$, then

$$\overline{C}(\sigma(\epsilon)) > \frac{A(\beta)}{10} - \frac{3n-25}{10n}\epsilon.$$

We have $\overline{C}(\sigma(\epsilon)) = \overline{C}(\sigma) - \frac{n-6}{n}\epsilon$ and then by (3.1), we obtain that $\frac{7(n-5)}{10n}\epsilon < 0$ and hence $\epsilon < 0$ or $n < 5$ which is in contradiction with $\epsilon > 0$ and $n \geq 6$.

**a-2) If** $\alpha_4 > \overline{\alpha}_3$. Since $\overline{\alpha}_1 > \alpha_1$, then by Lemma 3.2, we have:

$$\overline{C}(\omega(\epsilon)) \geq \frac{1}{2}(MS + \overline{\alpha}_0) - \frac{n-2}{2n}\epsilon.$$

Which implies that

$$\overline{C}(\sigma) \geq \alpha_0 + \alpha_1 + \frac{\alpha_2}{2} + \frac{3n-10}{2n}\alpha_3 - \frac{n-10}{2n}\alpha_4 + \frac{\beta}{2}. \tag{3.4}$$

Since $\beta \geq \frac{4\alpha_2+\alpha_3+\alpha_4}{4}$ and $\alpha_3 > \alpha_4$, by (3.1) and (3.4), we obtain that $\alpha_4 < 0$ which is impossible.

**b) If** $\omega(\epsilon) = (\overline{\alpha}_0, \overline{\alpha}_1, \overline{\alpha}_2, \alpha_4, S, \overline{\alpha}_3, \alpha_1)$. Let $\omega = (\alpha_0, \alpha_1, \alpha_2, \alpha_4, S, \alpha_3, \alpha_1)$ and $\overline{\epsilon} > 0$ defined by

$$\overline{\epsilon} = \sup\{\epsilon > 0 \text{ such that } CTV(\omega(\epsilon)) < CTV(\sigma(\epsilon))\}.$$

Let $x > 0$ be very small. Then by definition of $\overline{\epsilon}$, we deduce that

$$\begin{cases} CTV(\omega(\overline{\epsilon})) \leq CTV(\sigma(\overline{\epsilon})), \text{ and} \\ CTV(\omega(\overline{\epsilon}+x)) \geq CTV(\sigma(\overline{\epsilon}+x)). \end{cases} \tag{3.5}$$

Note that $\overline{\epsilon} < \infty$. Indeed, if $\overline{\epsilon} = \infty$, then for each $\epsilon > 0$, $CTV(\omega(\epsilon)) < CTV(\sigma(\epsilon))$. Hence, $\lim\limits_{\epsilon \to 0} CTV(\omega(\epsilon)) = \lim\limits_{\epsilon \to 0} CTV(\sigma(\epsilon)) = CVT(\omega) = CTV(\sigma)$ which contradicts the non optimality of schedule where job $n-4$ is scheduled on position $n-1$.

We have the following relations

$$\begin{cases} \overline{C}(\omega(\epsilon)) = \overline{C}(\sigma(\epsilon)) + \frac{(n-5)(\alpha_4-\alpha_3)+(2n-7)\epsilon}{n} \\ CTV(\sigma(\epsilon)) = CTV(\sigma) - \frac{12\epsilon}{n}\overline{C}(\sigma) + \frac{2\epsilon}{n} \overbrace{\left(6\alpha_0 + 5\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 + \beta + \frac{5n-18}{n}\epsilon\right)}^{f(\epsilon)} \\ CTV(\omega(\epsilon)) = CTV(\sigma) + \frac{2}{n}\left[\epsilon - 5(\alpha_3 - \alpha_4)\right]\overline{C}(\sigma) + \frac{\alpha_3-\alpha_4}{n}A(\beta) - \\ \qquad \frac{\epsilon}{n} \overbrace{\left(2\alpha_0 + 4\alpha_1 + 4\alpha_2 + 2\frac{3n-5}{n}\alpha_3 + 2\frac{n+5}{n}\alpha_4 + 4\beta - \frac{3n-1}{n}\epsilon\right)}^{g(\epsilon)} \end{cases}$$

By (3.5), we obtain that:

$$2\left(7\overline{\epsilon} - 5(\alpha_3 - \alpha_4)\right)\overline{C}(\sigma) \leq -(\alpha_3 - \alpha_4)A(\beta) + \overline{\epsilon}\left(2f(\overline{\epsilon}) + g(\overline{\epsilon})\right), \tag{3.6}$$

$$2\left(7(\overline{\epsilon}+x) - 5(\alpha_3 - \alpha_4)\right)\overline{C}(\sigma) > -(\alpha_3 - \alpha_4)A(\beta) + (\overline{\epsilon}+x)\left(2f(\overline{\epsilon}+x) + g(\overline{\epsilon}+x)\right). \tag{3.7}$$

**b-1) If** $7\overline{\epsilon} - 5(\alpha_3 - \alpha_4) \geq 0$. Then by (3.7), we obtain that

$$7A(\beta) > 5((2f(\overline{\epsilon}+x) + g(\overline{\epsilon}+x)).$$

Since $7\overline{\epsilon} \geq 5(\alpha_3 - \alpha_4)$ and $\beta \geq \frac{4\alpha_2+\alpha_3+\alpha_4}{4}$, then the last inequality implies that $(\alpha_3 + \alpha_4)\frac{5}{2} + 35\frac{n-5}{n}x < 0$ which is impossible.

**b-2) If** $7\overline{\epsilon} - 5(\alpha_3 - \alpha_4) < 0$. Then by (3.6) and (3.1), we obtain

$$\frac{7}{5}A(\beta) \leq 2f(\overline{\epsilon}) + g(\overline{\epsilon}). \tag{3.8}$$

Inequalities (3.6) and (3.7) implies that

$$\frac{7}{5}A(\beta) > 2f(\overline{\epsilon}) + g(\overline{\epsilon}) + 7\frac{n-5}{n}(\overline{\epsilon}+x). \tag{3.9}$$

Therefore by (3.8) and (3.9), we deduce that $(n-5)(\overline{\epsilon}+x) < 0$ which is in contradiction with $n \geq 6$ and $\overline{\epsilon}+x > 0$.

**c) If** $\omega(\epsilon) = (\overline{\alpha}_0, \overline{\alpha}_1, \overline{\alpha}_2, \overline{\alpha}_3, S, \alpha_4, \alpha_1)$. Let $\omega = \sigma = (\alpha_0, \alpha_1, \alpha_2, \alpha_3, S, \alpha_4, \alpha_1)$ and $\overline{\epsilon} > 0$ defined by

$$\overline{\epsilon} = \sup\{\epsilon > 0 \text{ such that } CTV(\omega(\epsilon)) < CTV(\sigma(\epsilon))\}.$$

Let $x > 0$ be very small (i.e., $\overline{\epsilon} \gg x$). Then by definition of $\overline{\epsilon}$, we deduce that

$$\begin{cases} CTV(\omega(\overline{\epsilon} - x)) < CTV(\sigma(\overline{\epsilon} - x)), \text{ and} \\ CTV(\omega(\overline{\epsilon} + x)) \geq CTV(\sigma(\overline{\epsilon} + x)). \end{cases} \tag{3.10}$$

Note that $\overline{\epsilon} < \infty$. Indeed, if $\overline{\epsilon} = \infty$, then for each $\epsilon > 0$, we have $CTV(\omega(\epsilon)) < CTV(\sigma(\epsilon))$. As $\omega = \sigma$ and $\sigma$ is optimal, with the same argument of the subsection *An upper bound* $UB$, and by the construction of a new schedule by interchanging jobs $\overline{\alpha}_3$ and $\alpha_4$ (with $\epsilon > \alpha_3 - \alpha_4$) and keeping all the remaining jobs in their original positions, we obtain $10\overline{C}(\omega) \leq 4\beta + 10(\alpha_0 + \alpha_1) + 6\alpha_2 + \frac{9n-25}{n}\alpha_3 - \frac{n-25}{n}\alpha_4$ and $10\overline{C}(\omega) \geq 4\beta + 10(\alpha_0 + \alpha_1) + 6\alpha_2 + \frac{9n-25}{n}\alpha_3 - \frac{n-25}{n}\alpha_4 + \frac{3n-15}{n}\epsilon$, which implies that $\epsilon \leq 0$ or $n \leq 5$.

For each $\epsilon$, we have

$$CTV(\sigma(\epsilon)) = CTV(\sigma) - \tfrac{12\epsilon}{n}\overline{C}(\sigma) + \tfrac{2\epsilon}{n}\overbrace{\left(6\alpha_0 + 5\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 + \beta + \frac{5n-18}{n}\epsilon\right)}^{f(\epsilon)}$$

$$CTV(\omega(\epsilon)) = CTV(\sigma) - \tfrac{8\epsilon}{n}\overline{C}(\sigma) + \tfrac{2\epsilon}{n}\overbrace{\left(4\alpha_0 + 3\alpha_1 + \alpha_2 + \frac{3n-8}{n}\epsilon\right)}^{h(\epsilon)}.$$

System (3.10) implies that

$$\begin{cases} 0 < -2\overline{C}(\sigma) + f(\overline{\epsilon} - x) - h(\overline{\epsilon} - x) \\ 0 \geq -2\overline{C}(\sigma) + f(\overline{\epsilon} + x) - h(\overline{\epsilon} + x). \end{cases}$$

Therefore,

$$h(\overline{\epsilon} - x) - f(\overline{\epsilon} - x) < -2\overline{C}(\sigma) \leq h(\overline{\epsilon} + x) - f(\overline{\epsilon} + x).$$

As $h(z) - f(z) = -MS - \alpha_0 - 2\frac{n-5}{n}z$, then we deduce that

$$-\frac{n-5}{n}(\overline{\epsilon} - x) < -\frac{n-5}{n}(\overline{\epsilon} + x).$$

Thus,

$$2x\frac{n-5}{n} < 0.$$

As $n \geq 6$ and $x > 0$, then the last inequality is impossible.

Hence, necessarily $\sigma(\epsilon)$ is optimal for $P(\epsilon)$, for each $\epsilon > 0$. As $\overline{\alpha}_1 > \alpha_1$, then by Lemma 3.2, we obtain that

$$\overline{C}(\sigma(\epsilon)) \geq \frac{MS + \overline{\alpha}_0}{2} - \frac{n-2}{2n}\epsilon = \frac{MS + \alpha_0}{2} + \frac{1}{n}\epsilon.$$

On the other hand, we have $\overline{C}(\sigma(\epsilon)) = \overline{C}(\sigma) - \frac{n-6}{n}\epsilon$. Let $\epsilon = \alpha_3 - \alpha_4 > 0$, therefore,

$$\overline{C}(\sigma) \geq LB_2(\beta) = B(\beta)\frac{1}{2n}, \tag{3.11}$$

where, $B(\beta) = n\beta + 2n(\alpha_0 + \alpha_1) + n\alpha_2 + (3n-10)\alpha_3 - (n-10)\alpha_4$.

Or by (3.1), we have

$$\overline{C}(\sigma) \leq \frac{1}{10}\left(10(\alpha_0+\alpha_1)+6\alpha_2+(9n-25)\frac{\alpha_3}{n}-(n-25)\frac{\alpha_4}{n}+4\beta\right). \tag{3.12}$$

Combining (3.11) and (3.12), we obtain after simplification

$$n\alpha_2 \geq (6n-25)\alpha_3-(4n-25)\alpha_4+n\beta.$$

By (3.3), we have $\beta \geq (4\alpha_2+\alpha_3+\alpha_4)\frac{1}{4}$, then

$$(25n-100)\alpha_3-(15n-100)\alpha_4 \leq 0. \tag{3.13}$$

As $\alpha_3 > \alpha_4 > 0$, thus, (3.13) implies that $n < 0$, which is in contradiction with $n \geq 6$. ■

By Theorem 3.1 and its proof, we deduce the following corollary.

**COROLLARY** 3.1 If $p_{n-2j-1} = p_{n-2j-2}$, for each $j = 0, 1, ..., \lfloor\frac{n-3}{2}\rfloor$, then the sequence

$$\sigma = (p_n, p_{n-2}, p_{n-4}, p_{n-6}, p_{n-8}, ..., p_1, ..., p_{n-7}, p_{n-5}, p_{n-3}, p_{n-1})$$

is optimal.

**REMARK** 3.1 The proof of Theorem 3.1 (Corollary 3.1) can be obtained by using the half-product formulation of the CTV given in Kubiak [1995].

# 4 Lower bound of CTV in case $p_{n-1} = p_{n-2}$

Assume that $n \geq 7$. By Theorem 3.1, there is an optimal sequence for the minimizing CTV of the form

$$\sigma^* = \left(\overbrace{(n, n-2, n-3}^{\pi_1}, \overbrace{(j_1, j_2, ...., j_{n-6})}^{S}, \overbrace{(j_{n-5}, n-4, n-1)}^{\pi_2}\right).$$

Then by Lemma 3.1, we have $CTV(\sigma^*) = \frac{6}{n}CTV(\pi) + \frac{n-6}{n}CTV(S) + \frac{6(n-6)}{n^2}\left(\overline{C}(S) - \overline{C}(\pi)\right)^2$. Therefore,

$$CTV(\sigma^*) \geq \frac{6}{n}CTV(\pi) + \frac{n-6}{n}CTV(S). \tag{4.1}$$

Let $m = \lfloor\frac{n-6}{2}\rfloor$ and $C_{[i]}(S)$ be the completion time of job in position $i = 1, 2, ..., n-6$ in the schedule $S$. By definition of the CTV, we have $(n-6)CTV(S) = \sum\limits_{i=1}^{n-6}\left(C_{[i]}(S) - \overline{C}(S)\right)^2$.

1) If $n-6$ is even,

$$(n-6)CTV(S) = \sum_{i=1}^{m}\overbrace{\left(\left(C_{[m+i]}(S) - \overline{C}(S)\right)^2 + \left(C_{[m+1-i]}(S) - \overline{C}(S)\right)^2\right)}^{A_i}.$$

Let $X_i = \frac{1}{2}\left(C_{[m+i]}(S) + C_{[m+1-i]}(S)\right)$. We have

$$A_i = \frac{\left(C_{[m+i]}(S) - C_{[m+1-i]}(S)\right)^2}{2} + 2\left(X_i - \overline{C}(S)\right)^2.$$

Therefore

$$A_i \geq \frac{1}{2}(p_1 + ... + p_{2i-1})^2.$$

2) If $n-6$ is odd,

$$(n-6)CTV(S) = \left(C_{[m+1]}(S) - \overline{C}(S)\right)^2 + \sum_{i=1}^{m} \overbrace{\left(\left(C_{[m+1+i]}(S) - \overline{C}(S)\right)^2 + \left(C_{[m+1-i]}(S) - \overline{C}(S)\right)^2\right)}^{B_i}.$$

Let $Y_i = \frac{1}{2}\left(C_{[m+1+i]}(S) + C_{[m+1-i]}(S)\right)$. We have

$$B_i = \frac{\left(C_{[m+1+i]}(S) - C_{[m+1-i]}(S)\right)^2}{2} + 2\left(Y_i - \overline{C}(S)\right)^2.$$

Therefore

$$B_i \geq \frac{1}{2}(p_1 + ... + p_{2i})^2.$$

We deduce the following lemma.

**LEMMA** 4.1 *Let*

$$LB_0 = \frac{1}{2}\begin{cases} \sum_{i=1}^{m}(p_1 + ... + p_{2i})^2 & \textit{if } n \textit{ is odd}, \\ \sum_{i=1}^{m}(p_1 + ... + p_{2i-1})^2 & \textit{if } n \textit{ is even}. \end{cases}$$

*Then* $(n-6)CTV(S) \geq LB_0$

Let $L_0 = 36CTV(\pi)$. After some simplification, we obtain

$$\begin{aligned} L_0 \quad = \quad & 9p_n^2 + 24p_{n-1}^2 + 5p_{n-3}^2 + 5p_{n-4}^2 + 9MS^2 + 24p_np_{n-1} + 6p_np_{n-3} \\ & + 6p_np_{n-4} + 12p_{n-1}p_{n-3} + 12p_{n-1}p_{n-4} + 2p_{n-3}p_{n-4} \\ & - 6(3p_n + 4p_{n-1} + p_{n-3} + p_{n-4})MS. \end{aligned}$$

Therefore, we deduce the following theorem.

**THEOREM** 4.1 The quantity

$$LB_N = \frac{1}{6n}L_0 + \frac{1}{n}LB_0$$

is a lower bound for the total completion time variance.

**PROOF**. Directly by inequality (4.1) and Lemma 4.1. ■

From this theorem, all the elements necessary for the computation of $LB_N$ can be obtained in $O(n)$ time as soon as the jobs are ordered in nondecreasing order of the processing times. As a consequence, the following Corollary holds.

**COROLLARY** 4.1 *The complexity of the $LB_N$ is $O(n \log n)$.*

**EXAMPLE** 4.1 Consider a 16-job problem described in following table.

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| 11 | 37 | 44 | 96 | 109 | 124 | 128 | 131 | 151 | 167 | 230 | 233 | 280 | 318 | 318 | 343 |

For this example, we obtain $LB_0 = 544378$ and $L_0 = 30572000$ and consequently $LB_N = 352481.95$. The optimal solution is given in the following: $\overline{C}(\sigma^*) = 1534$ and $CTV(\sigma^*) = 352510.12$.

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 16 | 14 | 13 | 10 | 9 | 7 | 4 | 3 | 1 | 2 | 5 | 6 | 8 | 11 | 12 | 15 |
| 343 | 318 | 280 | 167 | 151 | 128 | 96 | 44 | 11 | 37 | 109 | 124 | 131 | 230 | 233 | 318 |

Federgruen and Mosheiov [1996] and Viswanathkumar and Srinivasan [2003] proved that the following quantities

$$LB_{FM} = \frac{1}{2n} \sum_{i \leq \lfloor \frac{n}{2} \rfloor} s^2_{n-2i+1}, \; LB_{VS} = \frac{2L + VS}{2n}$$

are a lower bound for the total completion time variance, respectively, [3]. We have the following theorem.

**THEOREM** 4.2 For any instance of CTV, we have $LB_N \geq \max(LB_{FM}, LB_{VS})$.

**PROOF**. Notice that $LB_0 = \frac{1}{2n} \sum\limits_{i \leq m} s^2_{n-2i+1}$ and $\frac{L_0}{3} \geq \sum\limits_{i=m+1}^{\lfloor \frac{n}{2} \rfloor} s^2_{n-2i+1}$ where $m = \lfloor \frac{n-6}{2} \rfloor$, then $LB_N \geq LB_{FM}$. ■

**REMARK** 4.1 For Example 4.1, we obtain $LB_{FM} = LB_{VS} = 352435.9375$.

Let us consider the following definition.

**DEFINITION** 4.1 *Alternating Schedule $\sigma^{AS}$.* Assign the first (the longest) to be first in the sequence, the second to the last in the sequence, the third to be second, the fourth to be before last position, and so on until all the jobs in $n$ are assigned. Denote the obtained sequence by $\sigma^{AS}$ (Note that $\sigma^{AS}$ is a $V$-shaped sequence).

Let

$$U(\sigma^{AS}) = \left( Z(\sigma^{AS}) \frac{1}{n} + \frac{6(n-6)}{n^2} \left( \overline{C}(S^{AS}) - \overline{C}(\pi) \right)^2 \right).$$

where $m = \lfloor \frac{n-6}{2} \rfloor$,

$$\begin{cases} X_i = \frac{1}{2} \left( C_{[m+i]}(S^{AS}) + C_{[m+1-i]}(S^{AS}) \right) & \text{if } n \text{ is even} \\ Y_i = \frac{1}{2} \left( C_{[m+1+i]}(S^{AS}) + C_{[m+1-i]}(S^{AS}) \right) & \text{if } n \text{ is odd,} \end{cases}$$

[3] where $s_i = \sum\limits_{j \leq i} p_i$, $L = \{[p_n - p_a]^2 + [p_n + p_{n-2} - p_a]^2 + [MS - p_m - p_a]^2 + [MS - p_a]^2\}$, $VS = (p_1 + p_2)^2 + (p_1 + p_2 + p_3 + p_4)^2 + \ldots + (p_1 + p_2 + \ldots + p_{n-6} + p_{n-5})^2$, when $n$ is odd; $VS = p_1^2 + (p_1 + p_2 + p_3)^2 + \ldots + (p_1 + p_2 + \ldots + p_{n-6} + p_{n-5})^2$, when $n$ is even, $n \geq 5$, $p_a = (2p_n + p_{n-2} + 2MS - p_{n-1})/4$ and $MS = \sum\limits_{i=1}^{n} p_i$.

and

$$Z(\sigma^{AS}) = \begin{cases} 2\sum_{i\leq m}(X_i - \overline{C}(S^{AS}))^2 & \text{if } n \text{ is even} \\ 2\sum_{i\leq m}(Y_i - \overline{C}(S^{AS}))^2 + \left(C_{[m+1]}(S^{AS}) - \overline{C}(S^{AS})\right)^2 & \text{if } n \text{ is odd.} \end{cases}$$

By the proof of the lower bound and the definition of Alternating Schedule, we deduce the following lemma.

**LEMMA** 4.2 *Let $\sigma^*$ be the optimal sequence of minimizing the completion time variance (CTV). Then,*

$$\begin{cases} \frac{CTV(\sigma^*)-LB_N}{LB_N} \leq U(\sigma^{AS})\frac{1}{LB_N} \\ \frac{CTV(\sigma^{AS})-\sigma^*}{\sigma^*} \leq U(\sigma^{AS})\frac{1}{LB_N}. \end{cases}$$

By Theorem 6 of Federgruen and Mosheiov [1996], we deduce that if $p_i$ , $i = 1, ..., n$ are uniformly bounded in $n$, then

$$\lim_{n\to\infty} U(\sigma^{AS})\frac{1}{LB_N} = 0.$$

Therefore, $\sigma^{AS}$ is asymptotically optimal and the lower bound $LB_N$ is asymptotically accurate when $p_i$ , $i = 1, ..., n$ are uniformly bounded in $n$.

# 5 Computational results

This section describes the computational results on randomly generated instances. The lower bound algorithm was programmed and tested with C compiler on 2.50 Ghz Pentium D processor with 8 Giga RAM. For each job $i$, an integer processing time $p_i$ from the uniform distribution $[1, 100]$ and $[1, 1000]$ (in Tables 1-2, respectively) was generated. Problems with 20, 40, 60, 80, 100, 150, 250, 300 and 500 jobs were generated and 100 instances are considered for each problem size.

Table 1: Average of computational results on $p_i \in [1, 100]$

| $n$ | Time | $CTV(H_{NC})$ | $\Delta_{av}$ | $\Delta_{\min}$ | $\Delta_{\max}$ | $U(\sigma^{AS})$ |
|---|---|---|---|---|---|---|
| 20 | 1804 | 54162 | 4.62 | 0.19 | 32.79 | 50.41 |
| 40 | 2660 | 206567 | 1.66 | 0.53 | 4.50 | 49.77 |
| 60 | 4412 | 468388 | 1.04 | 0.33 | 2.64 | 52.13 |
| 80 | 3136 | 794112 | 0.80 | 0.25 | 2.37 | 52.50 |
| 100 | 4164 | 1249435 | 0.57 | 0.18 | 1.49 | 53.83 |
| 150 | 8141 | 2833312 | 0.34 | 0.13 | 0.83 | 49.65 |
| 200 | 20510 | 5110160 | 0.22 | 0.10 | 0.8 | 53.04 |
| 250 | 32308 | 8072385 | 0.14 | 0.08 | 0.36 | 48.88 |
| 300 | 52887 | 11232139 | 0.13 | 0.07 | 0.21 | 52.72 |
| 500 | 219986 | 31538851 | 0.11 | 0.06 | 0.16 | 51.39 |

The computational results for the lower bound and the upper bound are given in Tables 1-2, where Time, $CTV(H_{NC})$, $\Delta_{av}$, $\Delta_{\min}$ and $\Delta_{\max}$ represent, respectively, the average computational time in CPU milliseconds, the average completion time variance of the upper bound and finally the average (minimum and maximum) difference completion time variance and the lower bound, respectively, $\Delta = CTV(H_{NC}) - LB_N$. The difference between the average of the upper bound $H_{NC}$ compared to the average of the lower bound $LB_N$ is 33 for 20 jobs, for 100 jobs and more, the difference is about 1. In this case, lower bound is very tight. In the

Table 2: Average of computational results on $p_i \in [1, 1000]$

| $n$ | Time | $CTV(H_{NC})$ | $\Delta_{av}$ | $\Delta_{\min}$ | $\Delta_{\max}$ | $U(\sigma^{AS})$ |
|---|---|---|---|---|---|---|
| 20 | 2033 | 5159987 | 431.44 | 34.32 | 2289.09 | 4677.09 |
| 40 | 3001 | 21027409 | 184.74 | 40.95 | 530.56 | 5112.47 |
| 60 | 3639 | 46613791 | 105.67 | 35.19 | 486.01 | 5054.07 |
| 80 | 4042 | 79041533 | 72.49 | 17.64 | 217.03 | 5396.85 |
| 100 | 4458 | 122450171 | 51.66 | 17.83 | 113.27 | 4997.87 |
| 150 | 9947 | 282015081 | 29.89 | 13.04 | 70.20 | 5257.38 |
| 200 | 23026 | 497241900 | 20.41 | 10.24 | 49.68 | 5045.67 |
| 250 | 39928 | 766145254 | 14.84 | 7.35 | 33.22 | 5234.11 |
| 300 | 70379 | 1114440225 | 11.62 | 4.75 | 21.00 | 5239.28 |
| 500 | 409119 | 3061188033 | 5.53 | 2.64 | 9.84 | 5157.65 |

case where the $p_i$ is relatively large, The difference between the average of the upper bound $H_{NC}$ compared to the average of the lower bound $LB_N$ is relatively important compared to the first case. This difference is decreasing functions of $n$. For 100 jobs and more, the difference is about 100. We could reach a number of jobs equal to 500 in a reasonable amount of computational time (409119 ms CPU of average).

By results presented in Tables 1-2, the upper bound and the lower bound are very sensitive to the distribution of processing times. In Table 3, Since the range of integer processing times is likely to influence the effectiveness of the optimal solution and the lower bound, $p_j$ were generated from the uniform distribution $[1, 10 \times n \times \alpha]$ was generated. Ten $\alpha$ values $\{0.6, 0.7, 0.8, 0.9, 1, 1.25, 1.5, 1.75, 2, 3\}$ were considered. Problems with 20, 40, 40, 60, 80, 100, 120, 140, 160, 180, 200, 220, 240, 260, 280, 300, 350, 400, 500, 600 and 750 jobs were generated.

The difference between the average of the upper bound $H_{NC}$ compared to the average of the lower bound $LB_N$ is 410 for 40 jobs, 700 for 100 jobs, 900 for 200 jobs, 1663 for 300 jobs and 1700 for 500 jobs. We could reach a number of jobs equal to 500 in a reasonable amount of computational time (461614 ms CPU of average). We notice also the difference of $CTV(H_{NC}) - LB_N$ is a increasing functions of $n$ (See Table 3). These results demonstrate the efficiency of the lower bound $LB_N$.

Table 3: Average of computational results on $p_i \in [1, 10 \times n \times \alpha]$

| $n$ | Time | $CTV(H_{NC})$ | $\Delta_{av}$ | $\Delta_{\min}$ | $\Delta_{\max}$ | $U(\sigma^{AS})$ |
|---|---|---|---|---|---|---|
| 20 | 3381 | 484946 | 36.41 | 0.30 | 240.22 | 481.46 |
| 40 | 3179 | 7781173 | 66.51 | 5.35 | 409.71 | 1880.20 |
| 60 | 2911 | 35826404 | 79.43 | 6.06 | 464.54 | 4194.45 |
| 80 | 3307 | 118062782 | 97.04 | 8.09 | 505.57 | 7562.03 |
| 100 | 4175 | 282377289 | 123.67 | 9.72 | 688.28 | 12157.63 |
| 120 | 6079 | 584274769 | 143.15 | 7.74 | 797.68 | 16046.80 |
| 140 | 8734 | 1085171793 | 149.43 | 14.77 | 891.60 | 22588.93 |
| 160 | 13314 | 1823817843 | 161.14 | 17.47 | 889.77 | 30974.99 |
| 180 | 19196 | 2891719051 | 179.13 | 18.52 | 857.11 | 39547.42 |
| 200 | 25173 | 4223979205 | 178.20 | 15.44 | 889.32 | 48613.98 |
| 220 | 28176 | 6491888411 | 204.42 | 15.83 | 1479.39 | 56156.64 |
| 240 | 35018 | 8823999399 | 221.43 | 17.43 | 1437.25 | 69183.81 |
| 260 | 48568 | 12236141614 | 227.11 | 23.09 | 1402.56 | 84151.55 |
| 280 | 57731 | 16596790905 | 240.98 | 27.32 | 1567.28 | 89741.92 |
| 300 | 76711 | 20977652848 | 223.72 | 26.71 | 1663.30 | 107576.47 |
| 350 | 133231 | 39728179001 | 269.36 | 22.12 | 1652.12 | 149704.08 |
| 400 | 210054 | 65300712341 | 296.61 | 21.58 | 1781.92 | 190192.97 |
| 500 | 461614 | 152427383243 | 335.31 | 34.61 | 1696.49 | 316069.79 |
| 600 | 851436 | 323910679754 | 357.95 | 38.96 | 2627.92 | 406935.63 |
| 750 | 2063285 | 685076320132 | 466.55 | 42.82 | 2557.34 | 665871.35 |

# 6 Conclusions

In this paper, we have proved that if $p_{n-1} > p_{n-2}$, a new lower bound for the mean completion time of an optimal sequence is proposed which strictly dominates the existing lower bounds in the literature and when $p_{n-1} = p_{n-2}$, then there is an optimal solution of the form $(n, n-2, n-3, ..., n-4, n-1)$. We have also proved a new lower bound which were proved to be very efficient to prune the search tree. The test on more than 4000 instances shows a very significant result. Future research should focus more on if $p_{n-1} - p_{n-2} \leq \Delta(\mathbf{p})$, then there is an optimal solution of the form $(n, n-2, n-3, ..., n-4, n-1)$ or $(n, n-2, n-4, ..., n-3, n-1)$, where $\Delta(\mathbf{p})$ is a parameter dependents only on $\mathbf{p}$ and not on sequence.

## 7 Appendix: An upper bound of CTV

In this section, we recall that Nessah and Chu heuristic (Nessah and Chu [2010]). For an arbitrary sequence $\sigma$, denote by $\Delta_{sm}$ the total change in $CTV(\sigma)$ due to interchange of jobs scheduled at positions $m$ and $s$.

**THEOREM** 7.1 *Let $CTV(\sigma)$ be the completion time variance for the sequence $\sigma : [\sigma(1), ..., \sigma(m), \sigma(m+1), ..., \sigma(s), ..., \sigma(n)]$ and $CTV(\pi)$ be the completion time variance for the sequence $\pi : [\sigma(1), ..., \sigma(s), \sigma(m+1), ..., \sigma(m), ..., \sigma(n)]$, which is obtained by interchanging the jobs in positions $m$ and $s$. Then, the total change is given by:*

$$\begin{aligned}\Delta_{sm} &= (s-m)\frac{n-s+m}{n}\left(p_{\sigma(s)} - p_{\sigma(m)}\right)^2 \\ &+2\left(p_{\sigma(s)} - p_{\sigma(m)}\right)\sum_{j=m+1}^{s}\left[C_{\sigma(j)}(\sigma) - \overline{C}(\sigma) - p_{\sigma(j)}\right].\end{aligned}$$

*Feasible Complementary Pair (FCP).* A complementary pair of positions $\{s, m\}$ for which $\Delta_{sm} < 0$, is called a feasible complementary pair.

*Exchange Principle.* If for two positions $m$ and $s$ such that $1 \leq m < s \leq n$ in $\sigma$, we have $\Delta_{sm} < 0$, then from $\sigma$, we can obtain another sequence $\pi$ in $\Pi$ such that $CTV(\pi) < CTV(\sigma)$, and the change in $CTV(\sigma)$ is equal to $\Delta_{sm}$.

**Algorithm 1** . Heuristic Procedure (Heuristic $H_{NC}$)

**Require:** Arrange the set of jobs $N$ in descending order of weighted processing time $p_i$, compute the schedule $\sigma^{AS}$ and put flag:=1

**while** flag=1 **do**

flag := 0. While there is an FCP available in the current schedule, flag:=1 and determine the schedule by interchanging the jobs in an FCP (Select the pair with most negative value).

Find the antithetical schedule of the current schedule and set it as the current schedule.

**end while**

**return** The final schedule is the best sequence found during the process.

**EXAMPLE** 7.1 Let us again consider Example 4.1 for which the date are given in the following table.

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 11 | 37 | 44 | 96 | 109 | 124 | 128 | 131 | 151 | 167 | 230 | 233 | 280 | 318 | 318 | 343 |

Our algorithm gives the following schedule which is also optimal, $\overline{C}(H_{NC}) = 1534$ and $CTV(H_{NC}) = 352510.125$.

| 16 | 14 | 13 | 10 | 9 | 7 | 4 | 3 | 1 | 2 | 5 | 6 | 8 | 11 | 12 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 343 | 318 | 280 | 167 | 151 | 128 | 96 | 44 | 11 | 37 | 109 | 124 | 131 | 230 | 233 | 318 |

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