The block mutual coherence property condition for signal recovery

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Abstract. Compressed sensing shows that a sparse signal can stably be recovered from incomplete linear measurements. But, in practical applications, some signals have additional structure, where the nonzero elements arise in some blocks. We call such signals as block-sparse signals. In this paper, the $\ell_2/\ell_1-\alpha\ell_2$ minimization method for the stable recovery of block-sparse signals is investigated. Sufficient conditions based on block mutual coherence property and associating upper bound estimations of error are established to ensure that block-sparse signals can be stably recovered in the presence of noise via the $\ell_2/\ell_1-\alpha\ell_2$ minimization method. For all we know, it is the first block mutual coherence property condition of stably reconstructing block-sparse signals by the $\ell_2/\ell_1-\alpha\ell_2$ minimization method.

Key words. Compressed sensing; block-sparse recovery; block mutual coherence property; $\ell_2/\ell_1-\alpha\ell_2$ minimization method

1 Introduction

Compressed sensing (CS) is a novel genre of sampling theory, which has attracted a large number of attention in different areas including applied mathematics, machine learning, pattern recognition, image processing, and so forth. The sparsity of signal is elementary precondition of compressed sensing. In general, one thinks over the model as follows:

$$y=\Phi x+z, \tag{1.1}$$

where Φ is an $M\times N$ measurement matrix ($M\ll N$) and $z\in\mathbb{R}^M$ is a vector of measurement errors. The aim is to reconstruct the unknown signal $x\in\mathbb{R}^N$ based on y and Φ.

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Now we all understand that the ℓ_1 minimization method presents an efficient approach for recovery of the sparse signal in numerous scenarios. The ℓ_1 minimization problem in this settings is

$$\min_{\tilde{x}} \|\tilde{x}\|_1 \ \text{ subject to } \ y - \Phi\tilde{x} \in \mathcal{B}. \tag{1.2}$$

In the noisefree situation, we get $\mathcal{B} = \{0\}$. In the noisy situation, we can put $\mathcal{B}^{\ell_2} = \{z| \ \|z\|_2 \leq \epsilon\}$ [1] or $\mathcal{B}^{DS} = \{z| \ \|\Phi^\top z\|_\infty \leq \epsilon\}$, where $\Phi^\top$ stands for the conjugate transpose of the matrix Φ [2]. Now it is well known that the problem of sparse signal recovery has been well investigated in the framework of the mutual coherence property introduced in [3]. Let

$$\mu = \max_{i \neq j} \frac{|\Phi_i^\top \Phi_j|}{\|\Phi_i\|_2 \|\Phi_j\|_2}. \tag{1.3}$$

It has been shown that a sparse signal can been reconstructed by ℓ_1 minimization with a small or zero error under some appropriate conditions regarding MIP[3] [1] [4] [5] [6]. In order to further enhance the reconstruction performance, Yin et al. [7] has recently proposed the approach (i.e., ℓ_{1-2} minimization method) as follows:

$$\min_{\tilde{x}} \|\tilde{x}\|_1 - \|\tilde{x}\|_2 \ \text{ subject to } \ y - \Phi\tilde{x} \in \mathcal{B}. \tag{1.4}$$

Additionally, Yin et al. conducted simulations to show that the method (1.4) behaves better than the method (1.2) in recovering sparse signals. Based on this fact, numerous researches [8] [9] [10] on the ℓ_{1-2} minimization approach have been developed. Besides, for recovering $x \in \mathbb{R}^N$, the researchers [11] [12] proposed $\ell_1 - \alpha\ell_2$ $(0 < \alpha \leq 1)$ minimization method:

$$\min_{\tilde{x}} \|\tilde{x}\|_1 - \alpha\|\tilde{x}\|_2 \ \text{ subject to } \ y - \Phi\tilde{x} \in \mathcal{B}. \tag{1.5}$$

When $\alpha = 1$, (1.5) degenerates to the ℓ_{1-2} minimization method (1.4).

However, in practical applications, there exist signals which have special structure form, where the nonzero coefficients appear in some blocks. Such structural signal we called block spare signal in this paper. Such structural sparse signals commonly arise in all kinds of applications, e.g. foetal electrocardiogram (FECG) [13], motion segmentation[15], color image [14], and reconstruction of multi-band signals [16] [17]. Without loss of generality, suppose that there exist n blocks with block size $d = N/n$ in x. Then, x can be expressed as

$$x = [\underbrace{x_1, \cdots, x_d}_{x[1]}, \underbrace{x_{d+1}, \cdots, x_{2d}}_{x[2]}, \cdots, \underbrace{x_{N-d+1}, \cdots, x_N}_{x[n]}]^T, \tag{1.6}$$

where $x[i] \in \mathbb{R}^d$ represents the ith block of x. We call a vector $x \in \mathbb{R}^N$ block s-sparse signal if x has at most s nonzero blocks, i.e., $\|x\|_{2,0} = \sum_{i=1}^n I(\|x[i]\|_2) \leq s$. Therefore, the measurement matrix $\Phi \in \mathbb{R}^{M \times N}$ can also be described as

$$\Phi = [\underbrace{\Phi_1, \cdots, \Phi_d}_{x[1]}, \underbrace{\Phi_{d+1}, \cdots, \Phi_{2d}}_{\Phi[2]}, \cdots, \underbrace{\Phi_{N-d+1}, \cdots, \Phi_N}_{\Phi[n]}]^T, \tag{1.7}$$

where Φ_i and $\Phi[j]$ respectively stand for the ith column vector and jth sub-block matrix of Φ.

In this paper, we propose the following $\ell_2/\ell_1-\alpha\ell_2$ minimization to recover block sparse signal:

$$\min_{\tilde{x}} \|\tilde{x}\|_{2,1}-\alpha\|\tilde{x}\|_2 \ \text{ subject to } \ y-\Phi\tilde{x}\in\mathcal{B}, \tag{1.8}$$

where $\|x\|_{2,1}=\sum_{i=1}^{n}\|x[i]\|_2$. Furthermore, mixed norm $\|x\|_{2,2}=(\sum_{i=1}^{n}\|x[i]\|_2^2)^{1/2}$. Observe that $\|x\|_{2,2}=\|x\|_2$. When $\alpha=1$, (1.8) returns to ℓ_2/ℓ_{1-2} minimization [18]. And when the block size $d=1$, (1.8) reduces to the $\ell_1-\alpha\ell_2$ minimization (1.5).

In this paper, we study the block mutual coherence conditions for the stable recovery of signals with blocks structure from (1.6) via $\ell_2/\ell_1-\alpha\ell_2$ minimization in noise case. Sufficient conditions for stable signal reconstruction by $\ell_2/\ell_1-\alpha\ell_2$ minimization are established. Moreover, we also gain upper bound estimation of error concerning the recovery of block sparse signal. As far as we know, this is the first block mutual coherence based sufficient condition of stably reconstructing x via solving (1.8).

The remainder of the paper is organized as follows. In, Section 2, we present some notations and lemmas that will be used. The main theoretical results and their proofs are given in Section 3. Finally, the conclusion is summarized in Section 4.

2 Preliminaries

In this section, we primarily present several lemmas to prove our main results. Before giving these lemmas, we first of all explain some symbols in this paper.

Notations: $T\subset\{1,2,\cdots,n\}$ denotes block indices, T^c is the complement of T in $\{1,2,\cdots,n\}$. For any vector $x\in\mathbb{R}^N$, denote x_T to imply that x_T maintains the blocks indexed by T of x and displaces other blocks by zero. $E=\{i:\ \|x[i]\|_2\neq 0\}$ represents the block support of x. In addition, we often assume that $h=\hat{x}-x$, where $\hat{x}$ is the solution of (1.8) and x is the signal to be recovered.

Definition 2.1. *(block mutual coherence) Given matrix $\Phi\in\mathbb{R}^{M\times N}$, we define its block mutual coherence as*

$$\mu_\tau=\max_{1\le i<j\le n}\frac{1}{d}\frac{\|(\Phi[i])^\top\Phi[j]\|_2}{\|\Phi[i]\|_2\cdot\|\Phi[j]\|_2}. \tag{2.9}$$

Lemma 2.2. *([19], Lemma 3) For any block s-sparse vector x, we have*

$$(1-(s-1)d\mu_\tau)\|x\|_2^2\le\|\Phi x\|_2^2\le(1+(s-1)d\mu_\tau). \tag{2.10}$$

Lemma 2.3. *We have*

$$\|h_{E^c}\|_{2,1}\le\|h_E\|_{2,1}+\alpha\|h\|_2. \tag{2.11}$$

Proof. Recollect that $h=\hat{x}-x$. Since $\hat{x}$ is a minimizer of (1.8), we get

$$\begin{aligned}\|h+x\|_{2,1}-\alpha\|h+x\|_2&=\|\hat{x}\|_{2,1}-\alpha\|\hat{x}\|_2\\&\le\|x\|_{2,1}-\alpha\|x\|_2.\end{aligned}$$

By the reverse triangular inequality of $\|\cdot\|_2$, we get

$$\|h+x\|_{2,1}-\|x\|_{2,1}\le \alpha\|h+x\|_2-\alpha\|x\|_2\le \alpha\|h\|_2.$$

Note that x is block s-sparse and $E=\{i: \ \|x[i]\|_2\neq 0\}$, then

$$\begin{aligned}\alpha\|h\|_2 &\ge \|h+x\|_{2,1}-\|x\|_{2,1}\\ &=\|(h+x)_E\|_{2,1}+\|(h+x)_{E^c}\|_{2,1}-\|x\|_{2,1}\\ &=\|h_E+x_E\|_{2,1}+\|h_{E^c}+x_{E^c}\|_{2,1}-\|x\|_{2,1}\\ &\ge \|x_E\|_{2,1}-\|h_E\|_{2,1}+\|h_{E^c}\|_{2,1}-\|x\|_{2,1}\\ &=\|h_{E^c}\|_{2,1}-\|h_E\|_{2,1},\end{aligned}$$

which brings about the result. □

3 Main results

With the preparations provided in Section 2, we establish the main results in this section-block mutual coherence conditions for the stable reconstruction of block s-sparse signals. We will reveal that the measurement matrix Φ satisfies the block mutual coherence property with $\mu_\tau<1/3sd$, then every block s-sparse signal can be stably reconstructed via the $\ell_2/\ell_1-\alpha\ell_2$ minimization method in presence of noise. We first think about stable reconstruction of block s-sparse signals with ℓ_2-error.

Theorem 3.1. *Consider the model (1.1) with $\|z\|_2\le\epsilon$. Let $\hat{x}$ be the solution of (1.8) with $\mathcal{B}^{\ell_2}=\{z: \ \|z\|_2\le\eta\}$, and $\epsilon\le\eta$. Assume that x is block s-sparse with $\mu_\tau<1/3sd$. Then $\hat{x}$ fulfills*

$$\|\hat{x}-x\|_2\le\begin{cases}\frac{2(1-d\mu_\tau)(1+3\alpha d\mu_\tau)}{1-(2+\alpha^2)d\mu_\tau+(1-\alpha^2)d^2\mu_\tau^2}(\epsilon+\eta), & s=1,\\ \frac{(1-3d\mu_\tau)(25\alpha d\mu_\tau+\sqrt{30})}{2[1-(6+\alpha^2)d\mu_\tau+(9-\alpha^2)d^2\mu_\tau^2]}(\epsilon+\eta), & s=2,\\ \frac{24\sqrt{3}s\alpha d\mu_\tau+\sqrt{17[1+(1-9\alpha^2)d\mu_\tau]}}{1+(1-9\alpha^2)d\mu_\tau}(\epsilon+\eta), & s\ge 3.\end{cases} \tag{3.12}$$

We then consider stable reconstructing of block s-sparse signals with error in the bounded set $\mathcal{B}^{DS}=\{z: \ \|\Phi^\top z\|_\infty\le\epsilon\}$.

Theorem 3.2. *Let $y=\Phi x+z$ be noisy measurement of a signal x with $\|\Phi^\top z\|_\infty\le\epsilon$. If the block s-sparse signal x obeys the block mutual coherence property with $\mu_\tau<1/3sd$, then the solution of (1.8) with $\mathcal{B}^{DS}=\{z: \ \|\Phi^\top z\|_\infty\le\eta\}$ fulfills*

$$\|\hat{x}-x\|_2\le\begin{cases}\frac{\sqrt{d}(1-d\mu_\tau)(3\alpha+\sqrt{6})}{1-(2+\alpha^2)d\mu_\tau+(1-\alpha^2)d^2\mu_\tau^2}(\epsilon+\eta), & s=1,\\ \frac{\sqrt{d}(1-3d\mu_\tau)(4\alpha+\sqrt{19})}{1-(6+\alpha^2)d\mu_\tau+(9-\alpha^2)d^2\mu_\tau^2}(\epsilon+\eta), & s=2,\\ \frac{\sqrt{d}(15\alpha+3\sqrt{2s})}{1+(1-9\alpha^2)d\mu_\tau}(\epsilon+\eta), & s\ge 3.\end{cases} \tag{3.13}$$

Proof of Theorem 3.1.

Due to the feasibility of $\hat{x}$, we get

$$\|\Phi h\|_2 \leq \|\Phi x - \Phi \hat{x}\|_2 \leq \|\Phi x - y\|_2 + \|\Phi \hat{x} - y\|_2 \leq \epsilon + \eta. \tag{3.14}$$

Notice that $E = \{i : \ \|x[i]\|_2 \neq 0\}$. It follows from the facts $\|\Phi[i]\|_2 = 1$, $\|(\Phi[i])^\top \Phi[j]\|_2 \leq d\mu_\tau$ for $i \neq j, \ i, \ j = 1, \ 2, \cdots, n$, and (2.10) that

$$\begin{aligned}
|\langle \Phi h, \Phi h_E \rangle| &\geq |\langle \Phi h_E, \Phi h_E \rangle| - |\langle \Phi h_{E^c}, \Phi h_E \rangle| \\
&\geq (1-(s-1)d\mu_\tau)\|h_E\|_2^2 - |\sum_{j\in E^c}\sum_{i\in E}(h[j])^\top(\Phi[j])^\top \Phi[i]h[i]| \\
&\geq (1-(s-1)d\mu_\tau)\|h_E\|_2^2 - \sum_{j\in E^c}\sum_{i\in E}\|(\Phi[j])^\top \Phi[i]\|_2\|h[i]\|_2\|h[j]\|_2 \\
&\geq (1-(s-1)d\mu_\tau)\|h_E\|_2^2 - d\mu_\tau\|h_E\|_{2,1}\|h_{E^c}\|_{2,1} \\
&\geq (1-(s-1)d\mu_\tau)\|h_E\|_2^2 - \sqrt{s}d\mu_\tau\|h_E\|_2(\|h_E\|_{2,1} + \alpha\|h\|_2) \\
&\geq (1-(2s-1)d\mu_\tau)\|h_E\|_2^2 - \alpha\sqrt{s}d\mu_\tau\|h_E\|_2\|h\|_2.
\end{aligned} \tag{3.15}$$

On the other hand, by (2.10), we get

$$\|\Phi h_E\|_2^2 \leq (1+(s-1)d\mu_\tau)\|h_E\|_2^2. \tag{3.16}$$

It follows from the Cauchy-Schwarz inequality, (3.14) and (3.16) that

$$|\langle \Phi h, \Phi h_E \rangle| \leq \|\Phi h\|_2\|\Phi h_E\|_2 \leq (\epsilon+\eta)\sqrt{1+(s-1)d\mu_\tau}\|h_E\|_2. \tag{3.17}$$

Combining with (3.15) and $\mu_\tau < 1/3sd$, it implies

$$\begin{aligned}
\|h_E\|_2 &\leq \frac{\sqrt{1+(s-1)d\mu_\tau}}{1-(2s-1)d\mu_\tau}(\epsilon+\eta) + \frac{\alpha\sqrt{s}d\mu_\tau}{1-(2s-1)d\mu_\tau}\|h\|_2 \\
&\leq \frac{\sqrt{1+(s-1)/3s}}{1-(2s-1)/3s}(\epsilon+\eta) + \frac{\alpha\sqrt{s}d\mu_\tau}{1-(2s-1)d\mu_\tau}\|h\|_2.
\end{aligned}$$

Then, one can easily check that

$$\|h_E\|_2 \leq \begin{cases} \frac{3}{2}(\epsilon+\eta) + \frac{\alpha d\mu_\tau}{1-d\mu_\tau}\|h\|_2, & s=1, \\ \frac{\sqrt{42}}{3}(\epsilon+\eta) + \frac{\sqrt{2}\alpha d\mu_\tau}{1-3d\mu_\tau}\|h\|_2, & s=2, \\ 2\sqrt{3}(\epsilon+\eta) + \frac{\alpha}{\sqrt{s}}\|h\|_2, & s\geq 3. \end{cases} \tag{3.18}$$

Because of the fact $\|\Phi[i]\|_2 = 1$, $\|(\Phi[i])^\top \Phi[j]\|_2 \leq d\mu_\tau$ for $i \neq j, \ i, \ j = 1, \ 2, \cdots, n$, we get

$$\begin{aligned}
\|\Phi h\|_2^2 = \langle \Phi h, \Phi h \rangle &= \sum_{i,j}\langle \Phi[i]h[i], \Phi[j]h[j] \rangle \\
&= \sum_i (h[i])^\top(\Phi[i])^\top \Phi[i]h[i] + \sum_{i\neq j}(h[i])^\top(\Phi[i])^\top \Phi[j]h[j] \\
&\geq \sum_i \|(\Phi[i])^\top \Phi[i]\|_2\|h[i]\|_2^2 - \sum_{i\neq j}\|(\Phi[i])^\top \Phi[j]\|_2\|h[i]\|_2\|h[j]\|_2
\end{aligned}$$

$$
\begin{aligned}
&\geq \|h\|_{2,2}^2 - d\mu_\tau \sum_{i\neq j} \|h[i]\|_2 \|h[j]\|_2 \\
&= \|h\|_2^2 + d\mu_\tau \sum_i \|h[i]\|_2^2 - d\mu_\tau \sum_{i,j} \|h[i]\|_2 \|h[j]\|_2 \\
&= \|h\|_2^2 + d\mu_\tau \|h\|_{2,2}^2 - d\mu_\tau \|h\|_{2,1}^2 \\
&= (1+d\mu_\tau)\|h\|_2^2 - d\mu_\tau (\|h_E\|_{2,1} + \|h_{E^c}\|_{2,1})^2 \\
&\overset{(a)}{\geq} (1+d\mu_\tau)\|h\|_2^2 - d\mu_\tau (2\|h_E\|_{2,1} + \alpha\|h\|_2)^2 \\
&\overset{(b)}{\geq} (1+d\mu_\tau)\|h\|_2^2 - d\mu_\tau (2\sqrt{s}\|h_E\|_2 + \alpha\|h\|_2)^2, \qquad (3.19)
\end{aligned}
$$

where (a) follows from (2.11), and (b) is due to the Cauchy-Schwarz inequality.

Next, we estimate (3.12) by discussing three cases: $s = 1$, $s = 2$, and $s \geq 3$. We first of all discuss the situation that $s = 1$. A combination of (3.14), (3.18) and (3.19), we get

$$
(1+d\mu_\tau)\|h\|_2^2 - d\mu_\tau \left[3(\epsilon+\eta) + \frac{\alpha(1+d\mu_\tau)}{1-d\mu_\tau}\|h\|_2\right]^2 \leq (\epsilon+\eta)^2.
$$

The equation above can be adapted as

$$
[1-(2+\alpha^2)d\mu_\tau + (1-\alpha^2)d^2\mu_\tau^2]\|h\|_2^2 - 6\alpha d\mu_\tau(1-d\mu_\tau)(\epsilon+\eta)\|h\|_2 - (1+9d\mu_\tau)\frac{(1-d\mu_\tau)^2}{1+d\mu_\tau}(\epsilon+\eta)^2 \leq 0.
$$

Therefore, due to $\mu_\tau < 1/3d$, we get

$$
[1-(2+\alpha^2)d\mu_\tau + (1-\alpha^2)d^2\mu_\tau^2]\|h\|_2^2 - 6\alpha d\mu_\tau(1-d\mu_\tau)(\epsilon+\eta)\|h\|_2 - \frac{4(1-d\mu_\tau)^2}{1+d\mu_\tau}(\epsilon+\eta)^2 \leq 0.
$$

Accordingly, by Quadratic Formula, we get

$$
\begin{aligned}
\|h\|_2 \leq& \frac{1}{2[1-(2+\alpha^2)d\mu_\tau + (1-\alpha^2)d^2\mu_\tau^2]}\Bigg\{6\alpha d\mu_\tau(1-d\mu_\tau)(\epsilon+\eta) \\
&+ \left\{[6\alpha d\mu_\tau(1-d\mu_\tau)(\epsilon+\eta)]^2 + 16[1-(2+\alpha^2)d\mu_\tau + (1-\alpha^2)d^2\mu_\tau^2](\epsilon+\eta)^2\frac{(1-d\mu_\tau)^2}{1+d\mu_\tau}\right\}^{1/2}\Bigg\} \\
&\overset{(a)}{\leq} \frac{2(1-d\mu_\tau)(\epsilon+\eta)}{[1-(2+\alpha^2)d\mu_\tau + (1-\alpha^2)d^2\mu_\tau^2]}\left\{3\alpha d\mu_\tau + \sqrt{\frac{1-(2+\alpha^2)d\mu_\tau + (1-\alpha^2)d^2\mu_\tau^2}{1+d\mu_\tau}}\right\} \\
&\overset{(b)}{\leq} \frac{2(1-d\mu_\tau)(1+3\alpha d\mu_\tau)}{[1-(2+\alpha^2)d\mu_\tau + (1-\alpha^2)d^2\mu_\tau^2]}(\epsilon+\eta),
\end{aligned}
$$

where (a) is from the fact $(a+b)^{1/2} \leq a^{1/2} + b^{1/2}$ for any nonnegative constants a and b, and (b) is because both $1-(2+\alpha^2)d\mu_\tau + (1-\alpha^2)d^2\mu_\tau^2$ and $1/(1+d\mu_\tau)$ are monotonically reducing when $0 < \mu_\tau < 1/3d$.

In the case of $s = 2$, it follows from (3.14), (3.18) and (3.19) that

$$
(1+d\mu_\tau)\|h\|_2^2 - d\mu_\tau \left[\frac{4\sqrt{21}}{3}(\epsilon+\eta) + \frac{\alpha(1+d\mu_\tau)}{1-3d\mu_\tau}\|h\|_2\right]^2 \leq (\epsilon+\eta)^2.
$$

The above equation can be recast as

$$
(1+d\mu_\tau)\left[1 - \frac{\alpha^2 d\mu_\tau(1+d\mu_\tau)}{(1-3d\mu_\tau)^2}\right]\|h\|_2^2 - \frac{8\sqrt{21}\alpha d\mu_\tau(1+d\mu_\tau)(\epsilon+\eta)}{3(1-3d\mu_\tau)}\|h\|_2 - \left(\frac{112}{3}d\mu_\tau + 1\right)(\epsilon+\eta)^2 \leq 0.
$$

Owing to the condition of Theorem 3.1, $\mu_\tau < 1/6d$, thereby,

$$[1-(6+\alpha^2)d\mu_\tau+(9-\alpha^2)d^2\mu_\tau^2]\|h\|_2^2-\frac{25}{2}\alpha d\mu_\tau(1-3d\mu_\tau)(\epsilon+\eta)\|h\|_2-\frac{15}{2}(\epsilon+\eta)^2\frac{(1-3d\mu_\tau)^2}{1+d\mu_\tau}\leq 0.$$

By utilizing Quadratic Formula, we obtain

$$\begin{aligned}\|h\|_2 &\leq \frac{1}{2[1-(6+\alpha^2)d\mu_\tau+(9-\alpha^2)d^2\mu_\tau^2]}\left\{\frac{25}{2}\alpha d\mu_\tau(1-3d\mu_\tau)(\epsilon+\eta)+\left\{\left[\frac{25}{2}\alpha d\mu_\tau(1-3d\mu_\tau)(\epsilon+\eta)\right]^2\right.\right.\\ &\quad\left.\left.+30[1-(6+\alpha^2)d\mu_\tau+(9-\alpha^2)d^2\mu_\tau^2](\epsilon+\eta)^2\frac{(1-3d\mu_\tau)^2}{1+d\mu_\tau}\right\}^{1/2}\right\}\\ &\leq \frac{(1-3d\mu_\tau)(\epsilon+\eta)}{2[1-(6+\alpha^2)d\mu_\tau+(9-\alpha^2)d^2\mu_\tau^2]}\left\{25\alpha d\mu_\tau+\sqrt{\frac{30[1-(6+\alpha^2)d\mu_\tau+(9-\alpha^2)d^2\mu_\tau^2]}{1+d\mu_\tau}}\right\}\\ &\overset{(a)}{\leq} \frac{(1-3d\mu_\tau)(25\alpha d\mu_\tau+\sqrt{30})}{2[1-(6+\alpha^2)d\mu_\tau+(9-\alpha^2)d^2\mu_\tau^2]}(\epsilon+\eta),\end{aligned}$$

where (a) is from the fact that both $1-(6+\alpha^2)d\mu_\tau+(9-\alpha^2)d^2\mu_\tau^2$ and $1/(1+d\mu_\tau)$ are monotonically descending when $0<\mu_\tau<1/6d$.

When $s\geq 3$, through (3.14), (3.18) and (3.19), we gain

$$(1+d\mu_\tau)\|h\|_2^2-d\mu_\tau(4\sqrt{3s}(\epsilon+\eta)+3\alpha\|h\|_2)^2\leq(\epsilon+\eta)^2.$$

The above equation can be reworded as

$$[1+(1-9\alpha^2)d\mu_\tau]\|h\|_2^2-24\sqrt{3s}\alpha d\mu_\tau(\epsilon+\eta)\|h\|_2-(1+48sd\mu_\tau)(\epsilon+\eta)^2\leq 0.$$

From $\mu_\tau\leq 1/3sd$, $1+(1-9\alpha^2)d\mu_\tau>0$ and $48sd\mu_\tau<16$ when $s\geq 3$, hence

$$[1+(1-9\alpha^2)d\mu_\tau]\|h\|_2^2-24\sqrt{3s}\alpha d\mu_\tau(\epsilon+\eta)\|h\|_2-17(\epsilon+\eta)^2\leq 0.$$

Consequently,

$$\begin{aligned}\|h\|_2 &\leq \frac{1}{2[1+(1-9\alpha^2)d\mu_\tau]}\left\{24\sqrt{3s}\alpha d\mu_\tau(\epsilon+\eta)\right.\\ &\quad\left.+\left\{[24\sqrt{3s}\alpha d\mu_\tau(\epsilon+\eta)]^2+68[1+(1-9\alpha^2)d\mu_\tau](\epsilon+\eta)^2\right\}^{1/2}\right\}\\ &\leq \frac{24\sqrt{3s}\alpha d\mu_\tau+\sqrt{17[1+(1-9\alpha^2)d\mu_\tau]}}{1+(1-9\alpha^2)d\mu_\tau}(\epsilon+\eta).\end{aligned}$$

□

Proof of Theorem 3.2.

Notice that from the first portion of the proof of Theorem 3.1, we get

$$|\langle\Phi h,\Phi h_E\rangle|\geq(1-(2s-1)d\mu_\tau)\|h_E\|_2^2-\alpha\sqrt{s}d\mu_\tau\|h_E\|_2\|h\|_2. \tag{3.20}$$

Employing the fact $\|(\Phi_E)^\top\Phi h\|_2\leq\sqrt{sd}(\epsilon+\eta)$, where $E=\{i:\ \|x[i]\|_2\neq 0\}$, we have

$$|\langle\Phi h,\Phi_E h_E\rangle|\leq\|h_E\|_2\|(\Phi_E)^\top\Phi h\|_2\leq\|h_E\|_2\sqrt{sd}(\epsilon+\eta),$$

which combines with (3.20) and the condition $\mu_\tau \leq 1/3sd$, it leads to

$$\begin{aligned}\|h_E\|_2 &\leq \frac{\sqrt{sd}}{1-(2s-1)d\mu_\tau}(\epsilon+\eta) + \frac{\alpha\sqrt{s}d\mu_\tau}{1-(2s-1)d\mu_\tau}\|h\|_2 \\ &\leq \frac{\sqrt{sd}}{1-(2s-1)/3s}(\epsilon+\eta) + \frac{\alpha\sqrt{s}d\mu_\tau}{1-(2s-1)d\mu_\tau}\|h\|_2.\end{aligned}$$

Thus, it is easy to check that

$$\|h_E\|_2 \leq \begin{cases} \frac{3\sqrt{d}}{2}(\epsilon+\eta) + \frac{\alpha d\mu_\tau}{1-d\mu_\tau}\|h\|_2, & s=1, \\ 2\sqrt{2d}(\epsilon+\eta) + \frac{\sqrt{2}\alpha d\mu_\tau}{1-3d\mu_\tau}\|h\|_2, & s=2, \\ 3\sqrt{sd}(\epsilon+\eta) + \frac{\alpha}{\sqrt{s}}\|h\|_2, & s\geq 3. \end{cases} \tag{3.21}$$

By (3.19), we get

$$\langle \Phi h, \Phi h\rangle \geq (1+d\mu_\tau)\|h\|_2^2 - d\mu_\tau(2\sqrt{s}\|h_E\|_2 + \alpha\|h\|_2)^2. \tag{3.22}$$

By (2.11), the fact that $\|\Phi^\top \Phi h\|_\infty = \|\Phi^\top[(\Phi x - y) - (\Phi\hat{x} - y)\|_\infty \leq \epsilon+\eta$ and Cauchy-Schwarz inequality, we get

$$\begin{aligned}\langle \Phi h, \Phi h\rangle &= h^\top \Phi^\top \Phi h \leq \|h\|_1 \|\Phi^\top \Phi h\|_\infty = \sum_{i=1}^n \|h[i]\|_1(\epsilon+\eta) \\ &\leq (\epsilon+\eta)\sum_{i=1}^n \sqrt{d}\|h[i]\|_2 = \sqrt{d}(\epsilon+\eta)\|h\|_{2,1} \\ &= \sqrt{d}(\epsilon+\eta)(\|h_E\|_{2,1} + \|h_{E^c}\|_{2,1}) \leq \sqrt{d}(\epsilon+\eta)(2\|h_E\|_{2,1} + \alpha\|h\|_2) \\ &\leq \sqrt{d}(\epsilon+\eta)(2\|h_E\|_{2,1} + \alpha\|h\|_2) \leq \sqrt{d}(\epsilon+\eta)(2\sqrt{s}\|h_E\|_{2,2} + \alpha\|h\|_2) \\ &= \sqrt{d}(\epsilon+\eta)(2\sqrt{s}\|h_E\|_2 + \alpha\|h\|_2),\end{aligned}$$

which combines with (3.22), it implies that

$$\sqrt{d}(\epsilon+\eta)(2\sqrt{s}\|h_E\|_2 + \alpha\|h\|_2) \geq (1+d\mu_\tau)\|h\|_2^2 - d\mu_\tau(2\sqrt{s}\|h_E\|_2 + \alpha\|h\|_2)^2. \tag{3.23}$$

Hereafter, we give the proof of (3.13) by taking into account three situations: $s=1$, $s=2$, and $s\geq 3$. Firstly, we think over the situation that $s=1$. Due to (3.21) and (3.23), we get

$$(1+d\mu_\tau)\|h\|_2^2 - d\mu_\tau\left[3(\epsilon+\eta)\sqrt{d} + \frac{\alpha(1+d\mu_\tau)}{1-d\mu_\tau}\|h\|_2\right]^2 \leq \sqrt{d}(\epsilon+\eta)\left[3(\epsilon+\eta)\sqrt{d} + \frac{\alpha(1+d\mu_\tau)}{1-d\mu_\tau}\|h\|_2\right].$$

The above equation can be adapted as

$$(1+d\mu_\tau)\left[1 - \frac{\alpha^2 d\mu_\tau(1+d\mu_\tau)}{(1-d\mu_\tau)^2}\right]\|h\|_2^2 - \alpha\sqrt{d}(6d\mu_\tau+1)(\epsilon+\eta)\frac{1+d\mu_\tau}{1-d\mu_\tau}\|h\|_2 - (9d\mu_\tau+3)d(\epsilon+\eta)^2 \leq 0.$$

By the condition $\mu_\tau < 1/3sd$, $\mu_\tau < 1/3d$, accordingly,

$$[1-(2+\alpha^2)d\mu_\tau + (1-\alpha^2)d^2\mu_\tau^2]\|h\|_2^2 - 3\alpha\sqrt{d}(\epsilon+\eta)(1-d\mu_\tau)\|h\|_2 - 6d(\epsilon+\eta)^2\frac{(1-d\mu_\tau)^2}{1+d\mu_\tau} \leq 0.$$

Therefore, it is not hard to check that

$$
\begin{aligned}
\|h\|_h \leq & \frac{1}{2[1-(2+\alpha^2)d\mu_\tau+(1-\alpha^2)d^2\mu_\tau^2]}\Bigg\{3\alpha\sqrt{d}(\epsilon+\eta)(1-d\mu_\tau)+\bigg\{[3\alpha\sqrt{d}(\epsilon+\eta)(1-d\mu_\tau)]^2 \\
& + 24[1-(2+\alpha^2)d\mu_\tau+(1-\alpha^2)d^2\mu_\tau^2]d(\epsilon+\eta)^2\frac{(1-d\mu_\tau)^2}{1+d\mu_\tau}\bigg\}^{1/2}\Bigg\} \\
\leq & \frac{\sqrt{d}(1-d\mu_\tau)(\epsilon+\eta)}{[1-(2+\alpha^2)d\mu_\tau+(1-\alpha^2)d^2\mu_\tau^2]}\left\{3\alpha+\sqrt{\frac{6[1-(2+\alpha^2)d\mu_\tau+(1-\alpha^2)d^2\mu_\tau^2]}{1+d\mu_\tau}}\right\} \\
\leq & \frac{\sqrt{d}(1-d\mu_\tau)(3\alpha+\sqrt{6})}{[1-(2+\alpha^2)d\mu_\tau+(1-\alpha^2)d^2\mu_\tau^2]}(\epsilon+\eta).
\end{aligned}
$$

In the situation of $s=2$, a combination of (3.21) and (3.23), it leads to

$$
(1+d\mu_\tau)\|h\|_2^2-d\mu_\tau\left[8\sqrt{d}(\epsilon+\eta)+\frac{\alpha(1+d\mu_\tau)}{1-3d\mu_\tau}\|h\|_2\right]^2\leq\sqrt{d}(\epsilon+\eta)\left[8\sqrt{d}(\epsilon+\eta)+\frac{\alpha(1+d\mu_\tau)}{1-3d\mu_\tau}\|h\|_2\right].
$$

We can rewrite the above equation as

$$
\begin{aligned}
& [1-(6+\alpha^2)d\mu_\tau+(9-\alpha^2)d^2\mu_\tau^2]\|h\|_2^2-\alpha\sqrt{d}(\epsilon+\eta)(1+16d\mu_\tau)(1-3d\mu_\tau)\|h\|_2 \\
& -8d(\epsilon+\eta)^2(1+8d\mu_\tau)\frac{(1-3d\mu_\tau)^2}{1+d\mu_\tau}\leq 0.
\end{aligned}
$$

Because of the requirement $\mu_\tau<1/3sd$, $\mu_\tau<1/6d$, thereupon,

$$
\begin{aligned}
& [1-(6+\alpha^2)d\mu_\tau+(9-\alpha^2)d^2\mu_\tau^2]\|h\|_2^2-4\alpha\sqrt{d}(\epsilon+\eta)(1-3d\mu_\tau)\|h\|_2 \\
& -19d(\epsilon+\eta)^2\frac{(1-3d\mu_\tau)^2}{1+d\mu_\tau}\leq 0.
\end{aligned}
$$

It is not difficult to examine that

$$
\begin{aligned}
\|h\|_2 \leq & \frac{1}{2[1-(6+\alpha^2)d\mu_\tau+(9-\alpha^2)d^2\mu_\tau^2]}\Bigg\{4\alpha\sqrt{d}(\epsilon+\eta)(1-3d\mu_\tau)+\bigg\{[4\alpha\sqrt{d}(\epsilon+\eta)(1-3d\mu_\tau)]^2 \\
& + 76[1-(6+\alpha^2)d\mu_\tau+(9-\alpha^2)d^2\mu_\tau^2]d(\epsilon+\eta)^2\frac{(1-3d\mu_\tau)^2}{1+d\mu_\tau}\bigg\}^{1/2}\Bigg\} \\
\leq & \frac{\sqrt{d}(1-3d\mu_\tau)(\epsilon+\eta)}{[1-(6+\alpha^2)d\mu_\tau+(9-\alpha^2)d^2\mu_\tau^2]}\left\{4\alpha+\sqrt{\frac{19[1-(6+\alpha^2)d\mu_\tau+(9-\alpha^2)d^2\mu_\tau^2]}{1+d\mu_\tau}}\right\} \\
\leq & \frac{\sqrt{d}(1-3d\mu_\tau)(4\alpha+\sqrt{19})}{[1-(6+\alpha^2)d\mu_\tau+(9-\alpha^2)d^2\mu_\tau^2]}(\epsilon+\eta).
\end{aligned}
$$

In the situation of $s\geq 3$, from (3.21) and (3.23), one can get

$$
(1+d\mu_\tau)\|h\|_2^2-d\mu_\tau[6s\sqrt{d}(\epsilon+\eta)+3\alpha\|h\|_2]^2\leq\sqrt{d}(\epsilon+\eta)[6s\sqrt{d}(\epsilon+\eta)+3\alpha\|h\|_2].
$$

We can recast the above equation as

$$
\begin{aligned}
& [1+(1-9\alpha^2)d\mu_\tau]\|h\|_2^2-3\alpha\sqrt{d}(\epsilon+\eta)(1+12sd\mu_\tau)\|h\|_2 \\
& -6sd(\epsilon+\eta)^2(1+6sd\mu_\tau)\leq 0.
\end{aligned}
$$

Due to the condition $\mu_\tau<1/3sd$, $1+(1-9\alpha^2)d\mu_\tau>0$ and $sd\mu_\tau<1/3$ when $s\geq 3$, so,

$$
[1+(1-9\alpha^2)d\mu_\tau]\|h\|_2^2-15\alpha\sqrt{d}(\epsilon+\eta)\|h\|_2-18sd(\epsilon+\eta)^2\leq 0.
$$

One can easily check that

$$\begin{aligned}\|h\|_2 \leq & \frac{1}{2[1+(1-9\alpha^2)d\mu_\tau]}\left\{15\alpha\sqrt{d}(\epsilon+\eta)+\left\{[15\alpha\sqrt{d}(\epsilon+\eta)]^2+72sd[1+(1-9\alpha^2)d\mu_\tau](\epsilon+\eta)^2\right\}^{1/2}\right\} \\ \leq & \frac{\sqrt{d}(\epsilon+\eta)}{1+(1-9\alpha^2)d\mu_\tau}\left\{15\alpha+3\sqrt{2s[1+(1-9\alpha^2)d\mu_\tau]}\right\} \\ \leq & \frac{\sqrt{d}(15\alpha+3\sqrt{2s})}{1+(1-9\alpha^2)d\mu_\tau}(\epsilon+\eta).\end{aligned}$$

□

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