Terminal Wealth Maximization under Drift Uncertainty

Kerem Uğurlu

Sunday 13th June, 2021

Department of Mathematics, Nazarbayev University.
e-mail:kerem.ugurlu@nu.edu.kz

Abstract

We study the portfolio optimization problem of an investor seeking to maximize his terminal wealth. The portfolio is composed of one risky asset, a stock, and one riskless asset, a bond. We assume there is Knightian uncertainty on the drift term representing the long-term growth rate of the risky asset. We further assume that the investor has a prior estimate about the drift term and quantify the diffidence of the investor in his prior about the mean. It is assumed that the investor has a logarithmic or power utility. Explicit solutions under this framework have been retrieved. Numerical illustrations are also presented.

Mathematics Subject Classification: 91B16;93E20
Keywords: Knightian uncertainty; Mathematical finance; Utility maximization; Robust optimization; Risk management.

1 Introduction

The basic question in mathematical finance is how to model the stock movements. Starting with pioneering works, [1, 2], the stocks are modelled as geometric Brownian motions, where there exists a fixed underlying reference probability measure $\mathbb{P}$ that is retrieved from historical data of the price movements. However, it is impossible to precisely identify $\mathbb{P}$. Hence, model ambiguity, also called Knightian uncertainty, [3], is inevitably taken into consideration in mathematical finance. Namely, the investor is uncertain about the underlying dynamics of the model, and takes a *robust* approach to the utility maximization problem.

One of the directions in optimization under Knightian uncertainty is the Bayesian approach. In this direction, [4] studies an option pricing problem, where the parameters are learned as information increases. Similarly, [5] studies a utility maximization problem applying Bayesian techniques to learn the uncertain parameters as information increases. In [13], a Markowitz portfolio selection problem with unknown drift vector in the multi-dimensional framework has been studied, where a Bayesian approach from filtering theory is used to learn the posterior distribution about the drift given the observed market data of the assets. Another direction in optimization under Knightian uncertainty is the $\max\min$ approach, where the investor minimizes over the priors, corresponding to different scenarios, and then maximizes over the admissible investment strategies. As described in [9], this $\max\min$ approach can be perceived as a non-cooperative game between two agents, namely the investor and the fictitious agent, nature. The investor tries to maximize his expected utility of terminal wealth by choosing an admissible policy judiciously, while nature is competing with the investor by choosing the parameters of the underlying dynamics to minimize the terminal wealth. We refer the reader to [14] for a thorough discussion on $\max\min$ and Bayesian approaches in utility maximization. Furthermore, an axiomatic approach to Knightian uncertainty has been introduced in [6], where a family of risk evaluation operators, so called coherent and convex risk measures, is introduced. Coherent risk measures can be incorporated into the $\max\min$ framework, where the decision taker behaves according to a family of alternative scenarios, whereas convex risk measures incorporate a penalty factor into the $\max\min$ framework that allow to put more weight into one of the alternative scenarios while keeping the properties of a coherent risk measure except positive homogeneity. Based on [6], [8] applies convex risk measures on a portfolio maximization problem. Another work in this direction has been [15], where Markowitz-type classic portfolio selection problem is studied using convex risk measures. In that direction, the implication of ambiguity aversion for portfolio selection has been derived in [18], which employs scalable relative entropy in the model of ambiguity aversion and shows in a general stochastic differential utility and Black-Scholes framework that the presence of ambiguity aversion is observationally equivalent to escalation of the risk aversion degree. More recently, [7] considers an analogous problem of optimizing coherent risk measures in a controlled Markov chain framework in discrete time, where the decision maker is risk averse against model ambiguity. By appealing to a minimax theorem, the problem along with optimal controls has been solved. [10] incorporates model ambiguity in the context of algorithmic trading. The authors there provide analytical solutions for the robust optimal strategies and show that the resulting dynamic programming equations have classical solutions in some special cases.

In this paper, we study a utility maximization of terminal wealth problem. We assume that the investor has one stock and one bond, riskless asset. The investor has logarithmic or power utility. We take that there is uncertainty on the long-term growth rate of the stock represented by some random process μ, whereas the volatility of the stock σ has been calibrated with no uncertainty assumption on it. We do *not* necessitate μ to be bounded. The boundedness assumption is frequent in the literature due to requirement for compactness in a suitable topology. We refer the reader to recent works in different settings in [16, 17, 12]. The reason for assuming the drift is uncertain but the volatility known with certainty is estimating the drift of a stock with reasonable precision is seriously difficult, if not impossible. It requires extraordinarily long time series corresponding to decades, which is rarely available (see [14] for a discussion on this). On the other hand, estimating the volatility of a single stock is common in practice. Basen on that, we assume the investor has a prior on μ_t but is also diffident about it. The diffidence is quantified by some scalar C_θ utilizing logarithmic or power utility. Under this framework, we derive the optimal value and optimal controls *explicitly* along with numerical illustrations of our results.

The rest of the paper is as follows. In Section 2, we introduce the model dynamics and financial scenario, and present the investor's value function with respect to his utility function. In Section 3, we give the explicit optimal value function along with the optimal controls. In Section 4, we present numerical cases. In Section 5, we discuss our results and conclude the paper.

2 Model Dynamics and Investor's Value Function

2.1 Model Dynamics

We work on a filtered probability space $((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}))$ with $(\mathcal{F}_t)_{t\in[0,T]}$ being the augmentation under $\mathbb{P}$ of the filtration

$$\mathcal{F}_t^{W,\widetilde{W}} = \sigma_s((W_s, \widetilde{W}_s); 0 \le s \le t),\ 0 \le t \le T$$

generated by two independent Brownian motions on $(\Omega, \mathcal{F}, \mathbb{P})$, W and $\widetilde{W}$. It is well-known that $(\mathcal{F}_t)_{t\in[0,T]}$ satisfies the "usual conditions": it is right-continuous and $\mathcal{F}_0$ contains the $\mathbb{P}$-null events in $\mathcal{F}_T^{W,\widetilde{W}}$ (see [19], Section 2.7). We consider a market consisting of a risky asset, whose price at time t is denoted by S_t, and a bond, whose price at time t is denoted by B_t, and assume they satisfy the following dynamics, where the equalities are taken in

$\mathbb{P}-$a.s. sense below and the rest of the paper

$$\begin{aligned} dB_t &= r_t B_t dt,\ B_0 = 1 \\ dS_t &= S_t(\mu_t dt + \sigma_t dW_t) \\ d\mu_t &= b_t dt + \sigma_t^{\mu} d\widetilde{W}_t, \end{aligned} \tag{2.1}$$

Here, $r = (r_t)_{t\in[0,T]}$ is the interest rate of the bond, $\sigma = (\sigma_t)_{t\in[0,T]}$ stands for the volatility, and $\mu = (\mu_t)_{t\in[0,T]}$ stands for the the long-term growth rate of the stock. $b = (b_t)_{t\in[0,T]}$ and $\sigma^{\mu} = (\sigma_t^{\mu})_{t\in[0,T]}$ stand for the appreciation rate and volatility of μ, respectively.

Assumption 2.1. *r, σ, b and σ^{μ} are assumed to be measurable processes having continuous paths, adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ and uniformly bounded in $(t,\omega) \in [0,T]\times\Omega$, whereas μ is assumed to be a measurable process with continuous paths adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$. We further assume that*

$$\sigma_t(\omega) \geq \epsilon > 0, \text{ for all } (t,\omega) \in [0,T]\times\Omega,$$

for some scalar $\epsilon > 0$.

In this framework, we consider a "small investor", i.e. an agent whose actions can not influence the price process, investing in the stock S and the bond B for a given initial endowment $x_0 > 0$. We denote $(\hat{\pi}_t)_{t\in[0,T]}$ as the measurable process adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ process, which stands for the total amount of money invested in the risky asset S_t at time $t \in [0,T]$. We assume $\hat{\pi}$ satisfies

$$\int_0^T \hat{\pi}_t^2 dt < \infty,\ \mathbb{P}-\text{a.s.}$$

such that for initial wealth $X_0 = x_0 > 0$,

$$\begin{aligned} d\hat{X}_t^{\hat{\pi}} &= \hat{\pi}_t \frac{1}{S_t} dS_t dt + (\hat{X}_t^{\hat{\pi}} - \hat{\pi}_t) r_t dt, \\ d\hat{X}_t^{\hat{\pi}} &= \hat{\pi}_t(\mu_t dt + \sigma_t dW_t)dt + (\hat{X}_t^{\hat{\pi}} - \hat{\pi}_t) r_t dt,\ \mathbb{P}-\text{a.s.} \end{aligned}$$

We further represent the amount of money invested in the risky assets as a fraction of current wealth via $\pi_t = \hat{X}_t^{\hat{\pi}} \hat{\pi}_t$ for $t \in [0,T]$, where π_t stands for the corresponding fraction at time $t \in [0,T]$. We define the discounted wealth process $X_t^{\pi} \triangleq e^{-rt}\hat{X}_t^{\hat{\pi}}$. Hence, the dynamics of wealth in this setting are given by

$$dX_t^{\pi} = X_t^{\pi}\pi_t((\mu_t - r_t)dt + \sigma_t dW_t) \tag{2.2}$$

such that the unique solution to (2.2) for $X_t^{\pi} = x$ is

$$X_T^{\pi} = x \exp\left(\int_t^T \pi_u(\mu_u - r_u) - \frac{1}{2}\pi_u^2\sigma_u^2 du + \int_t^T \pi_u \sigma_u dW_u. \right) \tag{2.3}$$

Definition 2.1. *Let $\theta = (\theta_t)_{t\in[0,T]}$ and $\pi = (\pi_t)_{t\in[0,T]}$ denote the measurable processes adapted to $(\mathcal{F}_t)_{t\in[0,T]}$ representing the uncertainty in long-term growth rate $\mu = (\mu_t)_{t\in[0,T]}$ and the cash-value allocated in the risky asset, respectively. We call θ and π admissible and denote it by $\theta \in \Theta_{\text{ad}}$ and $\pi \in \Pi_{\text{ad}}$, respectively, if they satisfy*

(i) $X_t^{\pi} > 0,\ t \in [0,T]$.

(ii) $|\theta_t| \le C_\theta,\ t \in [0,T]$, *for some* $C_\theta > 0$.

Namely, for a fixed θ and π, the wealth of the investor must stay positive, and the uncertainty on the model quantified by θ is uniformly bounded for all $(t,\omega) \in [0,T] \times \Omega$.

Lemma 2.1. Π_{ad} *and* Θ_{ad} *are convex subsets of* $[0,T] \times \Omega$.

Proof. The convexity of Θ_{ad} is immediate by Definition 2.1 i). To show convexity of Π_{ad}, let $\pi^1, \pi^2 \in \Pi_{\text{ad}}$ and consider $\pi^3 \triangleq \alpha\pi^1 + (1-\alpha)\pi^2$. For π^3 to be in Π_{ad}, we need to show $\log(X_t^{\pi^3}) > -\infty$ for $t \in [0,T]$. By Definition 2.1 ii) and by (2.3), we have

$$
\begin{aligned}
\log(X_t^{\pi^3}) &= \alpha\log(x) + (1-\alpha)\log(x) \\
&\qquad + \int_s^t (\alpha\pi_u^1 + (1-\alpha)\pi_u^2)(\mu_u - r_u) - \frac{1}{2}(\alpha\pi_u^1 + (1-\alpha)\pi_u^2)^2\sigma_u^2 du \\
&\qquad + \int_s^t (\alpha\pi_u^1 + (1-\alpha)\pi_u^2)\sigma_u dW_u \\
&\ge \alpha\log(x) + \alpha\int_s^t \pi_u^1(\mu_u - r_u) - \frac{1}{2}(\pi_u^1)^2\sigma_u^2 du \\
&\qquad + \alpha\int_s^t \pi_u^1\sigma_u dW_u \\
&\qquad + (1-\alpha)\log(x) + (1-\alpha)\int_s^t \pi_u^2(\mu_u - r_u) - \frac{1}{2}(\pi_u^2)^2\sigma_u^2 du \\
&\qquad + (1-\alpha)\int_s^t \pi_u^2\sigma_u dW_u \\
&> -\infty.
\end{aligned}
$$

Hence, Π_{ad} is a convex subset of $[0,T] \times \Omega$. □

Let $\theta = (\theta_t)_{t\in[0,T]}$ be as in Definition 2.1. Then, we define a new measure $\mathbb{P}^\theta$ equivalent to $\mathbb{P}$

via

$$
\begin{aligned}
dW_t^\theta &\triangleq dW_t + \theta_t dt \\
\frac{d\mathbb{P}^\theta}{d\mathbb{P}} &\triangleq \exp\bigg(- \int_0^T \theta_t dW_t - \frac{1}{2}\int_0^T \theta_t^2 dt \bigg) \\
&= \exp\bigg(- \int_0^T \theta_t dW_t^\theta + \frac{1}{2}\int_0^T \theta_t^2 dt \bigg) \\
\frac{d\mathbb{P}_t^\theta}{d\mathbb{P}_t} &\triangleq \exp\bigg(- \int_t^T \theta_u dW_u - \frac{1}{2}\int_t^T \theta_u^2 du \bigg) \\
&= \exp\bigg(- \int_t^T \theta_u dW_u^\theta + \frac{1}{2}\int_t^T \theta_u^2 du \bigg).
\end{aligned} \tag{2.4}
$$

In (2.4), we used the Bayes formula for conditional expectation to change from $\mathbb{E}_t^{\mathbb{P}}[\cdot]$ to $\mathbb{E}_t^\theta[\cdot]$ (Lemma 3.5.3 in [19]). Hence, we define

$$
dX_t^{\pi,\theta} \triangleq \pi_t X_t^{\pi,\theta}((\mu_t - r_t - \sigma_t\theta_t)dt + \sigma_t dW_t^\theta)dt, \tag{2.5}
$$

where the additional term $-\sigma_t\theta_t$ gives the uncertainty on drift term. $\mathbb{E}_t^\theta[\cdot] \triangleq \mathbb{E}^\theta[\cdot|\mathcal{F}_t]$ stands for the conditional expectation that is taken with respect to $\mathbb{P}^\theta$ as defined in (2.4).

2.2 Investor's Value Function

The investor wants to maximize his terminal wealth but is uncertain about the underlying dynamics of the drift $(\mu_t)_{t\in[0,T]}$ of the stock. Hence, he takes a robust approach, evaluates his terminal wealth $X_T^{\pi,\theta}$ using the maximum of the admissible policies Π_{ad} among the least favourable Θ_{ad} parameters utilizing $u(x) = \log(x)$ or $u(x) = x^\gamma$ for $0 < \gamma < 1$ by assuming a prior on $\mu = (\mu_t)_{t\geq 0}$. By (2.5), the value function of the investor at $t \in [0,T]$ then reads as

$$
V(t,x) \triangleq \sup_{\pi\in\Pi_{\text{ad}}} \inf_{\theta\in\Theta_{\text{ad}}} \mathbb{E}_t^\theta\Big[u(X_T^{\pi,\theta})\Big]. \tag{2.6}
$$

We continue with the following variant of so called Martingale Optimality Principle (see also Theorem 1.1 of [14]).

Theorem 2.1. *Suppose that the objective is* (2.6) *for* $t \in [0,T]$. *Assume the followings hold:*

(A1) There exists a function $v : [0,T] \times \mathbb{R}^+ \to \mathbb{R}$, *that is* $C^{1,2}([0,T]\times\mathbb{R}^+)$ *with*

$$
v(T,x) = u(x).
$$

(A2) For each fixed $\pi \in \Pi_{\rm ad}$*, there exists an optimal* $(\theta^*_t)_{t\in[0,T]} \in \Theta_{\rm ad}$ *such that*

$$\begin{aligned}&\mathbb{E}^{\theta^*}_t\Big[u(X^{\pi,\theta^*}_T)\Big]\\&\qquad= \inf_{\theta\in\Theta_{\rm ad}} \mathbb{E}^{\theta}_t\Big[u(X^{\pi,\theta}_T)\Big].\end{aligned}$$

Define for $x>0$*,* $\pi \in \Pi_{\rm ad}$ *and* $\theta\in\Theta_{\rm ad}$*,*

$$Y^{\pi,\theta}_t \triangleq v(t, X^{\pi,\theta}_t) \tag{2.7}$$

with $Y^{\pi,\theta}_t$ *being a supermartingale, i.e.*

$$\mathbb{E}^{\theta}_t[Y^{\pi,\theta}_T] \le Y^{\pi,\theta}_t \quad \textit{for } 0\le t\le T. \tag{2.8}$$

(A3) There exists some $\pi^*\in\Pi_{\rm ad}$ *with the corresponding* $\theta^*\in\Theta_{\rm ad}$ *as in (A2) such that for* $0\le s\le t\le T$*,* Y^{π^*,θ^*}_t *is a martingale, i.e.*

$$\mathbb{E}^{\theta^*}_s[Y^{\pi^*,\theta^*}_t] = Y^{\pi^*,\theta^*}_s, \mathbb{P}\text{−a.s.} \tag{2.9}$$

Then, $\pi^*\in\Pi_{\rm ad}$ *in (A3) above is optimal for the problem* (2.6) *such that*

$$\begin{aligned}V(t,x) &= \sup_{\pi\in\Pi_{\rm ad}}\inf_{\theta\in\Theta_{\rm ad}} \mathbb{E}^{\theta}_t\Big[u(X^{\pi,\theta}_T)\Big]\\&= \inf_{\theta\in\Theta_{\rm ad}} \mathbb{E}^{\theta}_t\Big[u(X^{\pi^*,\theta}_T)\Big]\\&= \mathbb{E}^{\theta^*}_t\Big[u(X^{\pi^*,\theta^*}_T)\Big]\\&= v(t,x)\end{aligned}$$

Proof. By (A1) and (A2), we have for $t\in[0,T]$, for any fixed $\pi\in\Pi_{\rm ad}$ and $\theta\in\Theta_{\rm ad}$

$$\mathbb{E}^{\theta}_t[Y^{\pi,\theta}_T] = \mathbb{E}^{\theta}_t\Big[u(X^{\pi,\theta}_T)\Big]$$

Taking supremum over $\pi\in\Pi_{\rm ad}$ and infimum over $\theta\in\Theta_{\rm ad}$, we get

$$\begin{aligned}V(t,x) &= \sup_{\pi\in\Pi_{\rm ad}}\inf_{\theta\in\Theta_{\rm ad}} \mathbb{E}^{\theta}_t\Big[u(X^{\pi,\theta}_T)\Big]\\&\le v(t,x),\end{aligned}$$

where the last inequality follows by (2.7) and (2.8). By (A3), we have $\mathbb{E}^{\theta^*}_t[Y^{\pi^*,\theta^*}_T] = Y^{\pi^*,\theta^*}_t$ for some $\pi^*\in\Pi_{\rm ad}$ and $\theta^*\in\Theta_{\rm ad}$. Then,

$$\begin{aligned}V(t,x) &= \mathbb{E}^{\theta^*}_t[Y^{\pi^*,\theta^*}_T]\\&= Y^{\pi^*,\theta^*}_t\\&= v(t,x).\end{aligned}$$

Hence, we conclude the proof. □

Applying Ito lemma for $t \in [0,T]$ to $Y_t^{\pi,\theta}$ in (2.7), we have by (2.5)

$$dY_t^{\pi,\theta} = \Big(v_t + v_x X_t^{\pi}\big(\pi_t(\mu_t - r_t)\big) + (X_t^{\pi})^2 \frac{1}{2} v_{xx}\pi_t^2\sigma_t^2\Big)dt + X_t^{\pi} v_x \pi_t \sigma_t dW_t, \quad \mathbb{P}\text{−a.s.}$$

By Theorem 2.1, for $x > 0$, (2.9) satisfies the following HJBI PDE:

$$\begin{aligned} &v(T,x) = u(x) \\ &\sup_{\pi\in\Pi_{\mathrm{ad}}} \inf_{\theta\in\Theta_{\mathrm{ad}}} \left[v_t + v_x x\pi_t(\mu_t - r_t - \sigma_t\theta_t) + \frac{1}{2}x^2 v_{xx}\pi_t^2\sigma_t^2 \right] \\ &= v_t + \sup_{\pi\in\Pi_{\mathrm{ad}}} \inf_{\theta\in\Theta_{\mathrm{ad}}} \left[v_x x\pi_t(\mu_t - r_t - \sigma_t\theta_t) + \frac{1}{2}x^2 v_{xx}\pi_t^2\sigma_t^2 \right] \\ &= 0 \end{aligned} \tag{2.10}$$

and the following verification theorem is immediate.

Corollary 2.1. *Suppose there exists a function $v \in C^{1,2}([0,T], \mathbb{R}_+)$ that satisfies the premises of Theorem 2.1 with the HJBI equation* (2.10)*, then $v(t,x)$ is the solution of* (2.6)

The following analogues of Theorem 2.1 and Corollary 2.1 are immediate.

Corollary 2.2. *Suppose that the objective is*

$$\widetilde{V}(t,x) = \inf_{\theta\in\Theta_{\mathrm{ad}}} \sup_{\pi\in\Pi_{\mathrm{ad}}} \mathbb{E}^{\theta}\Big[u(X_T^{\pi,\theta})\Big] \tag{2.11}$$

for $t \in [0,T]$. Assume the followings hold:

(B1) There exists a function $\widetilde{v} : [0,T] \times \mathbb{R}^+ \to \mathbb{R}$, that is $C^{1,2}([0,T] \times \mathbb{R}^+)$ with

$$\widetilde{v}(T,x) = u(x).$$

(B2) For each fixed $\theta \in \Theta_{\mathrm{ad}}$, there exists an optimal $(\pi_t^*)_{t\in[0,T]} \in \Pi_{\mathrm{ad}}$ such that

$$\mathbb{E}_t^{\theta}\Big[u(X_T^{\pi^*,\theta})\Big] = \sup_{\pi\in\Pi_{\mathrm{ad}}} \mathbb{E}_t^{\theta}\Big[u(X_T^{\pi,\theta})\Big].$$

Define for $x > 0$, $\pi \in \Pi_{\text{ad}}$ *and* $\theta \in \Theta_{\text{ad}}$,

$$Y_t^{\pi,\theta} \triangleq \widetilde{v}(t, X_t^{\pi,\theta})$$

with $Y_t^{\pi,\theta}$ *being a submartingale, i.e.*

$$\mathbb{E}_t^{\theta}[Y_T^{\pi,\theta}] \geq Y_t^{\pi,\theta} \quad \text{for } 0 \leq t \leq T.$$

(B3) There exists some $\theta^* \in \Theta_{\text{ad}}$ *with the corresponding* $\pi^* \in \Pi_{\text{ad}}$ *as in (B2) such that for* $0 \leq s \leq t \leq T$, $Y_t^{\pi^*,\theta^*}$ *is a martingale, i.e.*

$$\mathbb{E}_s^{\theta^*}[Y_t^{\pi^*,\theta^*}] = Y_s^{\pi^*,\theta^*}, \mathbb{P}\text{−a.s.}$$

Then, $\theta^* \in \Theta_{\text{ad}}$ *in (B3) above is optimal for the problem* (2.11) *such that*

$$\begin{aligned}
\widetilde{V}(t,x) &= \inf_{\theta \in \Theta_{\text{ad}}} \sup_{\pi \in \Pi_{\text{ad}}} \mathbb{E}_t^{\theta}\Big[u(X_T^{\pi,\theta})\Big] \\
&= \inf_{\theta \in \Theta_{\text{ad}}} \mathbb{E}_t^{\theta}\Big[u(X_T^{\pi^*,\theta})\Big] \\
&= \mathbb{E}_t^{\theta^*}\Big[u(X_T^{\pi^*,\theta^*})\Big] \\
&= \widetilde{v}(t,x)
\end{aligned}$$

Corollary 2.3. *Suppose there exists a function* $\widetilde{v} \in C^{1,2}([0,T], \mathbb{R}_+)$ *that satisfy the premises of Corollary 2.2 with*

$$\begin{aligned}
&\widetilde{v}(T,x) = u(x) \\
&\inf_{\theta \in \Theta_{\text{ad}}} \sup_{\pi \in \Pi_{\text{ad}}} \left[\widetilde{v}_t + \widetilde{v}_x x \pi_t(\mu_t - r_t - \sigma_t \theta_t) + \frac{1}{2} x^2 \widetilde{v}_{xx} \pi_t^2 \sigma_t^2\right] \\
&= \widetilde{v}_t + \inf_{\theta \in \Theta_{\text{ad}}} \sup_{\pi \in \Pi_{\text{ad}}} \left[\widetilde{v}_x x \pi_t(\mu_t - r_t - \sigma_t \theta_t) + \frac{1}{2} x^2 \widetilde{v}_{xx} \pi_t^2 \sigma_t^2\right] \\
&= 0
\end{aligned}$$

Then, $\widetilde{v}(t,x)$ *is the solution of* (2.11).

Lemma 2.2. *The value function* $V(t,x)$ *as defined in Equation* (2.6) *is increasing and concave in* $x > 0$ *for all* $t \in [0,T]$.

Proof. Recall that by assumption, $u(x) = \log(x)$ or $u(x) = x^\gamma$ for $0 < \gamma < 1$ is strictly increasing and strictly concave. For any $x_1 \leq x_2$ with fixed $\theta \in \Theta$ and fixed $\pi \in \Pi_{\text{ad}}$, by monotonicity of $u(\cdot)$, we have

$$
\begin{aligned}
&u(x_1) + \mathbb{E}_t^\theta\Big[u(X_T^{\pi,\theta}) - u(x_1)\Big] \\
&\qquad \leq u(x_2) + \mathbb{E}_t^\theta\Big[u(X_T^{\pi,\theta}) - u(x_2)\Big].
\end{aligned}
$$

Since this holds for any $\pi \in \Pi_{\text{ad}}$ and $\theta \in \Theta$, taking first infimum over $\theta \in \Theta$ for fixed $\pi \in \Pi_{\text{ad}}$, and then taking supremum over $\pi \in \Pi_{\text{ad}}$, we have

$$
V(t, x_1) \leq V(t, x_2).
$$

Similarly, we show concavity of $V(t, x)$. Let $0 < \alpha < 1$ and $x_1, x_2 > 0$. Denote

$$
x_3 \triangleq \alpha x_1 + (1 - \alpha) x_2
$$

Then, we have

$$
\begin{aligned}
V(t, x_3) &= u(x_3) \\
&\qquad + \sup_{\pi \in \Pi_{\text{ad}}} \inf_{\theta \in \Theta_{\text{ad}}} \mathbb{E}_t^\theta\Big[u(X_T^{\pi,\theta}) - u(x_3)\Big] \\
&\geq \alpha u(x_1) + (1 - \alpha) u(x_2) \\
&\qquad + \alpha \sup_{\pi \in \Pi_{\text{ad}}} \inf_{\theta \in \Theta_{\text{ad}}} \mathbb{E}_t^\theta\Big[u(X_T^{\pi,\theta}) - u(x_1)\Big] \\
&\qquad + (1 - \alpha) \sup_{\pi \in \Pi_{\text{ad}}} \inf_{\theta \in \Theta_{\text{ad}}} \mathbb{E}_t^\theta\Big[u(X_T^{\pi,\theta}) - u(x_2)\Big] \\
&= \alpha V(t, x_1) + (1 - \alpha) V(t, x_2).
\end{aligned}
$$

Hence, we conclude the proof. □

We next state the following saddle point property

Lemma 2.3. *Suppose $v \in C^{1,2}([0, T], \mathbb{R}_+)$ satisfies the premises of Theorem 2.1 with the HJBI equation* (2.10) *such that*

$$
v(t, x) = \sup_{\pi \in \Pi_{\text{ad}}} \inf_{\theta \in \Theta_{\text{ad}}} \mathbb{E}_t^\theta\Big[u(X_T^{\pi,\theta})\Big]
$$

Then, we have

$$
\begin{aligned}
&\sup_{\pi \in \Pi_{\text{ad}}} \inf_{\theta \in \Theta_{\text{ad}}} \mathbb{E}_t^\theta\Big[u(X_T^{\pi,\theta})\Big] \\
&\qquad = \inf_{\theta \in \Theta_{\text{ad}}} \sup_{\pi \in \Pi_{\text{ad}}} \mathbb{E}_t^\theta\Big[u(X_T^{\pi,\theta})\Big]
\end{aligned}
$$

Proof. By Theorem 2.1, v satisfies

$$
\begin{aligned}
& v(T,x) = u(x) \\
& 0 = v_t + \sup_{\pi \in \Pi_{\mathrm{ad}}} \inf_{\theta \in \Theta_{\mathrm{ad}}} \left[v_x x \pi_t (\mu_t - r_t - \sigma_t \theta_t) + \frac{1}{2} x^2 v_{xx} \sigma_t^2 \pi_t^2 \right]
\end{aligned}
$$

The mapping

$$
\begin{aligned}
& h : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \\
& h(\pi_t, \theta_t) = v_x x \pi_t (\mu_t - r_t - \sigma_t \theta_t) + \frac{1}{2} x^2 v_{xx} \sigma_t^2 \pi_t^2
\end{aligned}
$$

is concave in π_t and linear, hence convex, in θ_t. Further, by Definition 2.1, π_t and θ_t are convex subsets of $\mathbb{R}$, and θ_t is a compact subset of $\mathbb{R}$ with Π_{ad} and Θ_{ad} being convex subsets of $[0,T] \times \Omega$ by Lemma 2.1. Hence, by Sion's minmax theorem (see [11]), we have

$$
\begin{aligned}
& v(T,x) = u(x) \\
& 0 = v_t + \inf_{\theta \in \Theta_{\mathrm{ad}}} \sup_{\pi \in \Pi_{\mathrm{ad}}} \left[v_x x \pi_t (\mu_t - r_t - \sigma_t \theta_t) + \frac{1}{2} x^2 v_{xx} \sigma_t^2 \pi_t^2 \right].
\end{aligned}
$$

By Corollary 2.2 and Corollary 2.3, this implies that

$$
v(t,x) = \inf_{\theta \in \Theta_{\mathrm{ad}}} \sup_{\pi \in \Pi_{\mathrm{ad}}} \mathbb{E}_t^{\theta} \Big[\int_t^T u(X_T^{\pi,\theta}) \Big].
$$

Hence, we conclude the proof. □

3 Optimal Value Function and Optimal Parameters

In this section, we give explicit representations along with the optimal controls for the optimal control problem with logarithmic and power utility functions. First, we need the following regularity assumption to have explicit solutions.

Assumption 3.1. *For any $\theta \in \Theta_{\mathrm{ad}}$, define*

$$
\begin{aligned}
& \phi_1(t) \triangleq \mathbb{E}_t^{\theta} \Big[\int_t^T (\mu_u - r_u - \sigma_u \theta_u)^2 du \Big] \\
& \phi_2(t) \triangleq \mathbb{E}_t^{\theta} \left[\exp \Big(\int_t^T \frac{\gamma (\mu_u - r_u - \sigma_u \theta_u)^2}{2 \sigma_u^2 (1-\gamma)} \Big) \right]
\end{aligned}
$$

Then, $\phi_1(\cdot)$ and $\phi_2(\cdot)$ are in $C^1([0,T])$ with $\phi_1(0) + \phi_2(0) < \infty$.

3.1 Logarithmic Utility

We have

$$\log(X_T^{\pi,\theta}) = \log(x) + \int_t^T \pi_u(\mu_u - r_u) - \frac{1}{2}\pi_u^2\sigma_u^2 du + \int_t^T \pi_u\sigma_u dW_u$$
$$\log(X_T^{\pi,\theta}) = \log(x) + \int_t^T \pi_u(\mu_u - r_u - \sigma_u\theta_u) - \frac{1}{2}\pi_u^2\sigma_u^2 du + \int_t^T \pi_u\sigma_u dW_u^\theta$$

Hence, the value function $V(t,x)$ reads as

$$\begin{aligned} V(t,x) &\triangleq \log(x) \\ &\quad + \sup_{\pi\in\Pi_{\text{ad}}} \inf_{\theta\in\Theta_{\text{ad}}} \mathbb{E}_t^\theta\left[\int_t^T \pi_s(\mu_s - r_s - \sigma_s\theta_s) - \frac{1}{2}\pi_s^2\sigma_s^2 ds\right] \end{aligned} \tag{3.1}$$

Theorem 3.1. *Let Θ_{ad} and Π_{ad} be as in Definition 2.1 and the value function $V(t,x)$ be as in* (3.1)*. Then, under Assumption 2.1, for $t \in [0,T]$, the optimal parameters are*

$$\theta_t^* = \min\left\{\max\left\{\frac{\mu_t - r_t}{\sigma_t}, -C_\theta\right\}, \min\left\{\frac{\mu_t - r_t}{\sigma_t}, C_\theta\right\}\right\}, \tag{3.2}$$

$$\pi_t^* = \frac{\mu_t - r_t}{\sigma_t^2} - \frac{\theta_t^*}{\sigma_t} \tag{3.3}$$

The optimal value function for $t \in [0,T]$ and $x > 0$ then reads as

$$\begin{aligned} V(t,x) &\triangleq \log(x) + \mathbb{E}_t^{\theta^*}\left[\int_t^T \pi_s^*(\mu_s - r_s - \sigma_s\theta_s^*) - \frac{1}{2}(\pi_s^*)^2\sigma_s^2 ds\right] \\ &= \log(x) + \frac{1}{2}\mathbb{E}_t^{\theta^*}\left[\int_t^T (\mu_s - r_s - \sigma_s\theta_s^*)^2 ds\right] \end{aligned} \tag{3.4}$$

Proof. We assume that $V(t,x) \in C^{1,2}([0,T]\times\mathbb{R}_+)$ such that by Theorem 2.1 and Lemma 2.3, the HJBI equation reads as

$$\begin{aligned} V(T,x) &= \log(x) \\ 0 &= V_t + \sup_{\pi\in\Pi_{\text{ad}}} \inf_{\theta\in\Theta_{\text{ad}}} \left\{xV_x\pi_t(\mu_t - r_t - \sigma_t\theta_t) + \frac{1}{2}x^2V_{xx}\pi_t^2\sigma_t^2\right\} \\ &= V_t + \inf_{\theta\in\Theta}\left\{\sup_{\pi\in\Pi_{\text{ad}}}\left\{xV_x\pi_t(\mu_t - r_t - \sigma_t\theta_t) + \frac{1}{2}x^2V_{xx}\pi_t^2\sigma_t^2\right\}\right\} \end{aligned} \tag{3.5}$$

We make the Ansatz $v(t,x)$ as in (3.4) for (3.5)

$$v(t,x) = \log(x) + \frac{1}{2}\mathbb{E}_t^{\theta^*}\left[\int_t^T (\mu_s - r_s - \sigma_s\theta_s^*)^2 ds\right],$$

where

$$\theta_t^* = \min\left\{ \max\left\{ \frac{\mu_t - r_t}{\sigma_t}, -C_\theta \right\}, \min\left\{ \frac{\mu_t - r_t}{\sigma_t}, C_\theta \right\} \right\}.$$

It is immediate we have $xv_x = 1$ and $x^2 v_{xx} = -1$. Hence, the Ansatz is $C^2(\mathbb{R})$ in x and in $C^{1,2}([0,T],\mathbb{R})$ by Assumption 3.1. Furthermore, plugging v into (3.5), it is easy to see that the optimizers θ^* and π^* are as in (3.7) and (3.8), respectively. Moreover, $v(t,x)$ satisfies (3.5). By Corollary, 2.1, $v(t,x)$ is indeed the solution of (3.5) with $v(t,x) = V(t,x)$. Hence, we conclude the proof. □

Remark 3.1. *The optimal parameters π^* and θ^* in Theorem 3.1 are admissible as in Definition 2.1.*

Example 3.1. *Consider the Black-Scholes stock price model with $\mu_t \equiv \mu$, $r_t \equiv r$ and $\sigma_t \equiv \sigma$ in* (2.1) *and uncertainty parameter C_θ. Then,*

$$V(t,x) = \log(x) + \frac{1}{2}(\mu - r - \sigma\theta^*)(T-t),$$

with

$$\theta^* = \min\left\{ \max\left\{ \frac{\mu - r}{\sigma}, C_\theta \right\}, \min\left\{ \frac{\mu - r}{\sigma}, C_\theta \right\} \right\}$$
$$\pi^* = \frac{\mu - r}{\sigma^2} - \frac{\theta^*}{\sigma}.$$

Here, we note that the value function is indeed in $C^{1,2}([0,T] \times \mathbb{R})$ along with $\theta^* \in \Theta_{\text{ad}}$ and $\pi^* \in \Pi_{\text{ad}}$.

3.2 Power Utility

We are going to use the utility function $U(x) = x^\gamma$ for $x > 0$ and fixed $0 < \gamma < 1$ in this section. Using (2.3), we have

$$(X_T^{\pi,\theta})^\gamma = x^\gamma \mathbb{E}_t^\theta \left[\exp\left(\gamma \int_t^T \left(\pi_u(\mu_u - r_u - \sigma_u\theta_u) - \frac{1}{2}\pi_u^2\sigma_u^2 \right) du + \gamma \int_t^T \pi_u \sigma_u dW_u^\theta \right) \right]$$

Hence, the value function $V(t,x)$ reads as

$$V(t,x) = x^\gamma \sup_{\pi\in\Pi_{\text{ad}}} \inf_{\theta\in\Theta_{\text{ad}}} \mathbb{E}_t^\theta \left[\exp\left(\gamma \int_t^T \left(\pi_u(\mu_u - r_u - \sigma_u\theta_u) - \frac{1}{2}\pi_u^2\sigma_u^2 \right) du + \gamma \int_t^T \pi_u \sigma_u dW_u^\theta \right) \right] \tag{3.6}$$

Theorem 3.2. *Let Θ_{ad} and Π_{ad} be as in Definition 2.1 and the value function $V(t,x)$ be as in* (3.6). *Then, under Assumption 2.1, for $t \in [0,T]$, the optimal parameters are*

$$\theta_t^* = \min\left\{ \max\left\{\frac{\mu_t - r_t}{\sigma_t}, -C_\theta\right\}, \min\left\{\frac{\mu_t - r_t}{\sigma_t}, C_\theta\right\}\right\}, \tag{3.7}$$

$$\pi_t^* = \frac{\mu_t - r_t - \sigma_t\theta_t^*}{(1-\gamma)\sigma_t^2} \tag{3.8}$$

The optimal value function for $t \in [0,T]$ and $x > 0$ then reads as

$$V(t,x) \triangleq x^\gamma \mathbb{E}_t^{\theta^*}\left[\exp\Big(\int_t^T \frac{\gamma(\mu_u - r_u - \sigma_u\theta^*)^2}{2\sigma_u^2(1-\gamma)}\Big)\right]$$

Proof. The proof follows the same lines of Theorem 3.1. We make the Ansatz

$$v(t,x) = x^\gamma \left\{\mathbb{E}_t^{\theta^*}\left[\exp\Big(\int_t^T \frac{\gamma(\mu_u - r_u - \sigma_u\theta_u^*)^2}{2\sigma_u^2(1-\gamma)}du\Big)\right]\right\},$$

where

$$\theta_t^* = \min\left\{ \max\left\{\frac{\mu_t - r_t}{\sigma_t}, -C_\theta\right\}, \min\left\{\frac{\mu_t - r_t}{\sigma_t}, C_\theta\right\}\right\}.$$

Plugging $v(t,x)$ into the HJBI equation

$$\begin{aligned} v(T,x) &= x^\gamma \\ 0 &= v_t + \sup_{\pi\in\Pi_{\text{ad}}} \inf_{\theta\in\Theta_{\text{ad}}} \Big\{ x v_x \pi_t(\mu_t - r_t - \sigma_t\theta_t) \\ &\quad + \frac{1}{2}x^2 v_{xx}\pi_t^2\sigma_t^2 + \frac{1}{2\phi}\theta_t^2 + \log(c_t) - c_t x v_x \Big\} \\ &= v_t + \inf_{\theta\in\Theta}\Big\{ \sup_{\pi\in\Pi_{\text{ad}}} \Big\{ x v_x \pi_t(\mu_t - r_t - \sigma_t\theta_t) \\ &\quad + \frac{1}{2}x^2 v_{xx}\pi_t^2\sigma_t^2 \Big\}\Big\}, \end{aligned}$$

gives $\pi_t^* = \frac{\mu_t - r_t - \sigma_t\theta_t^*}{(1-\gamma)\sigma_t^2}$ and $\theta_t^* = \min\left\{ \max\left\{\frac{\mu_t - r_t}{\sigma_t}, -C_\theta\right\}, \min\left\{\frac{\mu_t - r_t}{\sigma_t}, C_\theta\right\}\right\}$. Moreover, $v(t,x)$ is in $C^{1,2}([0,T],\mathbb{R})$ by Assumption 3.1. Hence, $V(t,x) = v(t,x)$, and we conclude the proof. □

Example 3.2. *Following the Black-Scholes framework in Example 3.1 using x^γ, we have*

$$V(t,x) = x^\gamma \exp\left(\frac{\gamma(\mu - r - \sigma\theta^*)^2(T-t)}{2\sigma^2(1-\gamma)}\right),$$

with the optimal controls

$$\pi^* \equiv \frac{\mu - r - \sigma^*\theta}{(1-\gamma)\sigma^2}$$

$$\theta^* = \min\left\{ \max\left\{\frac{\mu - r}{\sigma}, -C_\theta\right\}, \min\left\{\frac{\mu - r}{\sigma}, C_\theta\right\}\right\}$$

4 Numerical Case Study

In this section, we give numerical examples with explanatory simulations by discussing the effects of the parameters in the model. Let the terminal time be $T = 10$ with $t = 0$. For simplicity, we take that the prior on long- term growth rate is $\mu = 0.3$, and the risk-free interest rate is $r = 0.05$. The volatility of the stock is taken as $\sigma = 0.5$. We further consider the following diffidence levels

$$\mathcal{C}_\theta \in [0.001, 0.01, 0.1, 0.5, 1, 1.5, 2].$$

By (3.8), the optimal portions to invest into the risky asset and the optimal value functions for logarithmic and power utility with $\gamma = 0.3, 0.5, 0.7$ are as in Figure 1. We see that in the logarithmic case as the reliance on the prior increases, namely as C_θ decreases, the optimal π^* converges to the value $\frac{\mu_t - r_t}{\sigma_t^2} = 1$. Similarly, as the reliance on the prior decreases, namely as C_θ increases, the optimal π^* converges to 0 and with $C_\theta = \frac{\mu_t - r_t}{\sigma_t}$, the optimal portion to invest into risky asset is 0. The corresponding figures for power utility with $\gamma = 0.3, 0.5, 0.7$ are to be seen in Figure 2. We see the same pattern as in log utility for the power utility for optimal portion and value function.

5 Concluding Remarks

In this paper, we have considered an investor having a logarithmic or power utility in a Merton problem of deciding the optimal consumption flow and the share in the risky asset in finite horizon interval. The investor has a portfolio composed of one stock and one bond, and is diffident about the long-term growth rate of the stock. In this framework, we provide explicit solutions for the optimal value function along with the optimal share ratio in the risky asset. We show that the diffidence range C_θ indeed determines how much the investor takes his prior on the long-term growth μ_t into consideration while making decisions. If $C_\theta \geq |\frac{\mu_t - r_t}{\sigma_t}|$, then the investor refrains from investing into risky asset completely and invests all his wealth into his bond. Analogously, as $C_\theta = 0$, the investor solves the classical Merton's terminal wealth maximization with the optimal parameters and value function, in particular,

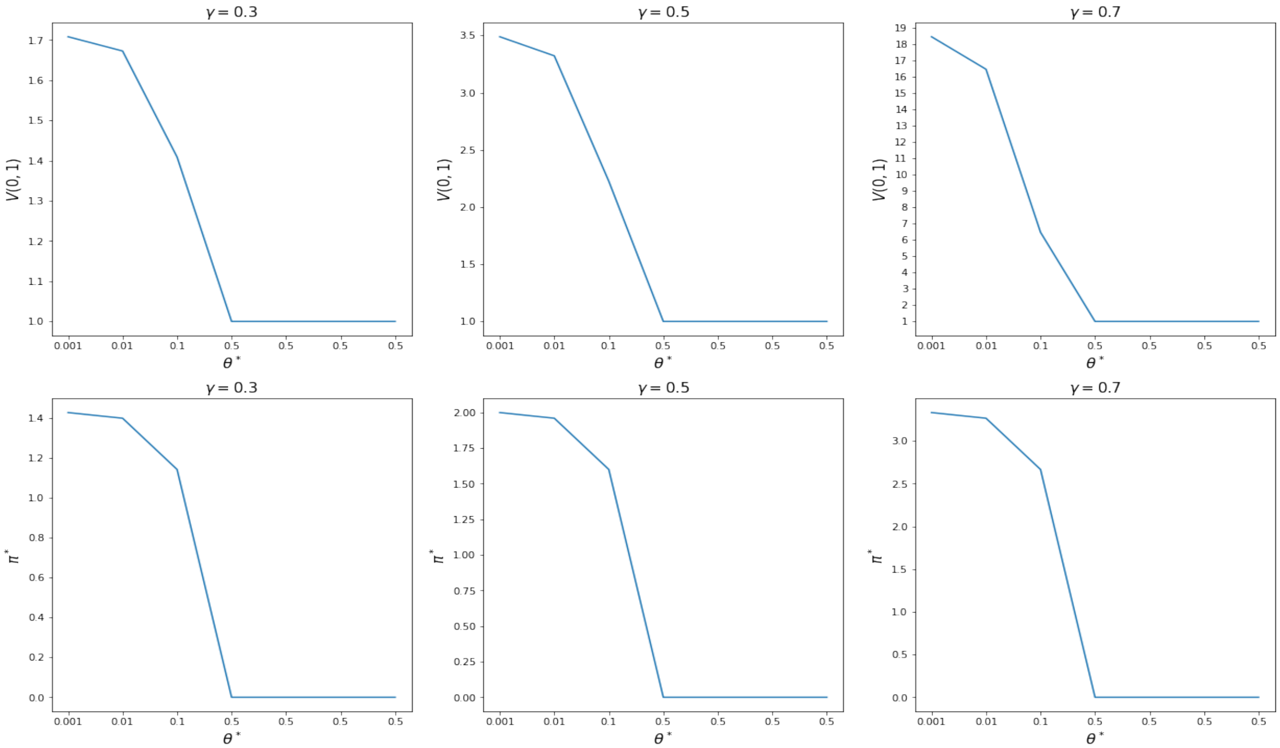

Figure 1: Optimal Value and Optimal Portion with Power Utility

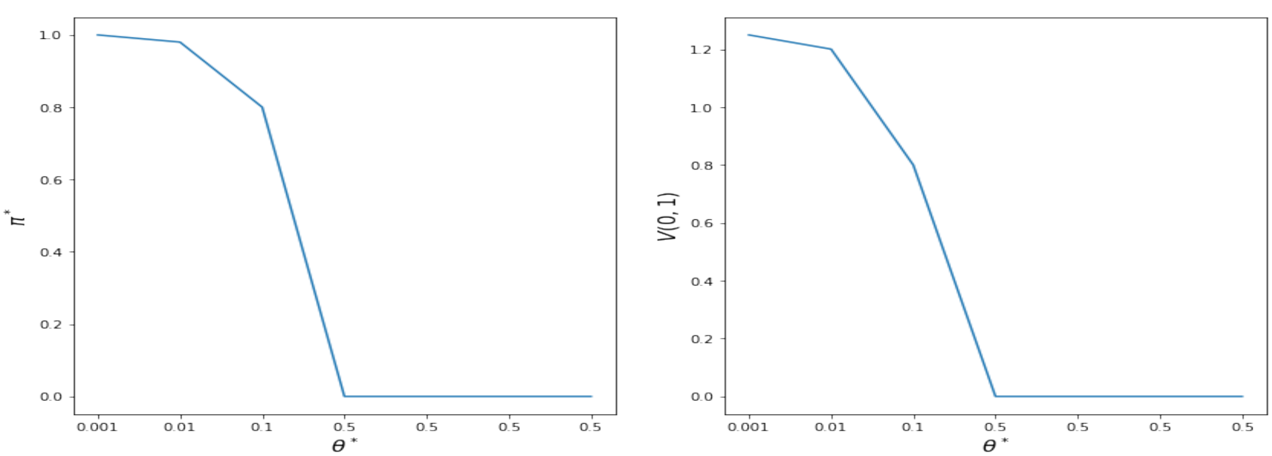

Figure 2: Optimal Value and Optimal Portion with Logarithmic Utility

there is no diffidence on the underlying model. Namely, depending on $\frac{\mu_t - r_t}{\sigma_t}$, the investor refrains from the risky asset completely with $\pi_t = 0$, or takes a position with $\pi_t^* = \frac{\mu_t - r_t - \sigma_t \theta^*}{\sigma_t^2}$, where $\theta^* = \pm C_\theta$.

References

[1] Merton, R. C. Lifetime portfolio selection under uncertainty: the continuous time case. *Rev. Econom. Statist.*, 51:247-257, 1969.

[2] Fischer, B., and Scholes, M. The Pricing of Options and Corporate Liabilities. *The Journal of Political Economy*, Vol. 81, No. 3, pp. 637-654, 1973.

[3] Knight, F. H. Risk, Uncertainty, and Profit. *Boston, MA: Hart, Schaffner & Marx; Houghton Mifflin Company*, 1921.

[4] Cvitanic, J. and Karatzas, I. On dynamic measures of risk. *Finance and Stochastics*, 3, 451-482 1999.

[5] Karatzas, I. and Zhao, X. Bayesian adaptive portfolio optimization. Handbook of mathematical finance. *Cambridge University Press, Cambridge*, pp 632–670, 2001.

[6] Artzner, P., F. Delbaen, J.-M. Eber, and D. Heath Coherent Measures of Risk. *Mathematical Finance* 9, 203-228. 1999.

[7] Bauerle, N., and Rieder, U. Markov Decision Processes Under Ambiguity. *Banach Center Publications*, 122, 25-39, 2020.

[8] Schied, A. Risk measures and robust optimization problems. *Stochastic Models* 22(4):753-831, 2006.

[9] Hansen, L., Sargent, T. Robust control and model uncertainty, *The American Economic Review*, 91: 60-66, 2001.

[10] Cartea, A., Donnelly R., and Jaimungal, S. Algorithmic trading with model uncertainty, *SIAM Journal on Financial Mathematics*, 8(1), 635–671, 2017.

[11] Sion, M. On general minimax theorems. *Pacific Journal of Mathematics*, 8:171-176, 1958.

[12] Neufeld, A., and Nutz, M. Robust utility maximization with Levy Processes. *Mathematical Finance*, Vol. 28-1, 82-105, 2018.

[13] De Franco, C., Nicole, J., and Pham, H.. Bayesian learning for the Markovitz portfolio selection problem. *International Journal of Theoretical and Applied Finance*, 22-7, 2019.

[14] Rogers, L.C.G., Optimal investment, Springer-Verlag, Berlin, 2013.

[15] Kountzakis, C.E.. On efficient portfolio selection using convex risk measures. *Mathematics and Financial Economics*, 4:223–252, 2011.

[16] Ekstrom, E., and Vannestal, M. American Options and Incomplete Information. *International Journal of Theoretical and Applied Finance*, Vol.22-4, 2019.

[17] Bartl, D., Kupper, M., and Vannestal, M. Duality Theory for Robust Utility Maximization. *Finance and Stochastics*, to appear.

[18] Maenhout, P. J. Robust portfolio rules and asset pricing. *Review of Financial Studies*, 17(4): 951-983, 2004.

[19] Karatzas I, and Shreve, S.E. Brownian Motion and Stochastic Calculus. Springer-Verlag, New York, 1987.