Analysis of Energy Markets Modeled as Equilibrium Problems with Equilibrium Constraints

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Abstract

Equilibrium problems with equilibrium constraints are challenging both theoretically and computationally. However, they are suitable/adequate modeling formulations in a number of important areas, such as energy markets, transportation planning, and logistics. Typically, these problems are characterized as bilevel Nash-Cournot games where there is a game at the top level and possibly one at the bottom level too. For instance, determining the equilibrium price in an energy market involves top-level decisions of the generators that feed the bottom-level problem of an independent system operator. In this example, there is a Nash game at the top level with a shared system operator at the bottom level. Taking the Karush-Kuhn-Tucker conditions of the lower-level optimization problems and inserting them into each upper-level player's problem is one popular approach, but it has both numerical and theoretical difficulties. To tackle the resulting highly nonlinear model we propose a primal-dual regularization with the remarkable property of yielding equilibrium prices of minimal norm. This theoretical feature can be seen as a stabilizing mechanism for price signals. It proves also useful in guiding the solution process when solving such problems computationally mixed complementarity formulations. For a general energy market model we prove existence theorems for a specific equilibrium, and convergence of the proposed regularization scheme. Our numerical results using the PATH solver illustrate the proposal. In particular, we exhibit that, in the given context, PATH with the regularization approach computes for most of our computational results a true equilibrium while without regularization the outcome is quite the opposite.

1 Introduction

The issue of obtaining clearing prices in energy markets is a topic of intensive research at many levels, from economic modeling to actual calculations in practice. The excellent surveys [25] and [2], covering more than thirty years of work on the subject, confirm that generating adequate and clear-to-interpret price signals is fundamental for the energy sector, to ensure the financing of operational and investment costs; see also [38]. For a recent computational proposal in this direction (in one specific setting), see [28]. Both [34] and [3] point out the need to continue to evolve in the design of energy markets, to achieve cost-effective policies for resource adequacy. This task is by no means straightforward, as illustrated by the many different mechanisms that have been proposed to provide capacity remuneration and ensure future investment [6], [39], [5]. In the past years, traditional concerns mentioned in [35], such as market welfare, generation adequacy, and security of supply, have been complemented with targets of renewables integration and sustainability; see [21]. Furthermore, the world-wide trend in incentivizing flexibility added another complicating layer to the problem, particularly regarding the operation of the network.

One goal of energy markets is to provide market-based price signals for energy operations and investments– important for market flexibility. Typically, prices can be realistically modeled as the output of an equilibrium problem involving the interaction between the system operator and the agents. A resulting equilibrium price later on may be augmented with compensation that the regulator determines to correct market distortions, protecting consumers from abuses of market power [38] and agents from missing-money problems [33].

From a modeling point of view, initial mathematical programming models [17] evolved to more sophisticated equilibrium problems, cast as Nash games or complementarity formulations [20], [16]; see also [32] and [14]. The research on equilibrium models for energy systems is very rich. We mention a few works that show the large spectrum of related literature. Market-power issues and bidding strategies for the Brazilian power system were investigated in [26], and [7]. Bilevel game-theoretic models for electricity markets are discussed in [23], particularly regarding the effect of congestion and network transmission on the existence of equilibria; see also [8]. In [15], a bilevel approach with potentially discrete variables is presented with an application to the stylized Benelux power network using a mathematical program with equilibrium constraints formulation varying market leaders. Other theoretical studies are [10] and [12], analyzing multi-leader Cournot equilibria, and Walrasian and non-cooperative Nash games, respectively. Economic implications with respect to market design can be found in [11] and [22]. In [30] and [29] stochastic games and relations with complementarity models are explored, for risk-neutral and risk-averse agents. For more studies on the impact of risk models on mathematical optimization problems with equilibrium constraints, we refer to [37] and [13]. Finally, [4] considers bilevel optimization for analyzing investments in power markets and develops several algorithms to efficiently solve the resulting equilibrium problem.

As commented in [2], given the unusual characteristics of electricity generation and transmission, the efficient allocation of resources in an energy market is a highly challenging task. Our work confirms this assertion by pinpointing pitfalls that arise at the very basis of the pricing mechanism, namely when solving equilibrium problems to define a price signal. For such a market we consider an equilibrium problem with equilibrium constraints (EPEC) that arises when agents optimize their revenue in a bilevel setting. Specifically, in the upper level each agent decides both the bidding price and the corresponding generation, subject to operational constraints and a shared constraint ensuring the dispatch clears the market. The optimal dispatch is defined in the lower level by the independent system operator (ISO), who solves the same problem for all the agents, taking into account demand satisfaction and other system-wide constraints. For a general EPEC model of this type, we develop a theoretical study identifying configurations leading to equilibrium prices that are larger than the maximal dispatched bid. This undesirable situation may induce agents to manipulate their offers in order to reach those critical configurations. Another problematic aspect refers to the *lack of uniqueness* of the price signal: if the mathematical model does not provide a unique answer, the pricing policy is not well-defined. Under such circumstances, reproducibility becomes an issue too, as different solvers or different computational methods may produce different output, and find different prices. As exhibited in Proposition 1 below, even in the more simple instances, an interval of equilibrium prices is not unusual. These are precisely the market configurations that give the so-called marginal agents an opportunity to artificially increase the price, see the discussion below (3) for more details and Figure 5 for a numerical illustration.

The drawbacks mentioned above are inherent to EPEC models. Yet, formulating the equilibrium problem in a bilevel setting represents the market behavior well. In order to mitigate the these drawbacks, we introduce a sequence of approximating EPECs, depending on certain regularization parameters. This novel approach, based on a scheme that controls the largest marginal rent, ensures that in the limit the *minimal norm* price signal will be found, see Theorem 5 below. Our analysis of the ISO lower-level problem complements a primal view with a dual perspective that provides an interesting economic interpretation. With our approach the ISO maintains control of the agents' marginal rents by having access to certain energy reserves; see Section 4.

The rest of this work is organized as follows. In Section 2 we introduce some notation and the considered EPEC model, resulting from including the optimality conditions for the ISO problem in the upper-level agents' problems. Section 3 presents our new regularized formulation that discourages the undesirable situation in which, rather than choosing the minimum possible value for the marginal price, the highest one is chosen by the EPEC model. In Section 4, by means of convex analysis techniques, the material introduced in Section 3 is interpreted from the perspective of the ISO, adopting a primal point of view. In Section 5, using the insights from the simpler configuration, the more complex models with network and transmission constraints are considered. To illustrate the type of difficulties that must be addressed when solving EPECs, the section also includes several numerical experiments with the PATH solver [9] using GAMS. The benchmark provides insights on how the approach helps in obtaining solutions to the complementarity system that are solutions for the EPEC. In the last section, some concluding remarks are provided.

2 Main features considered for the energy market

In what follows, we introduce the notation for the static, deterministic EPECs described in this paper. We note that neither dynamic relations nor uncertainty are included in the study. Instead, the stylized model is used to highlight difficulties inherent to EPEC models and to propose a methodology that addresses those difficulties. The approach is applicable in the more general setting, as illustrated by the instances considered in Section 5.

2.1 The general setting

Suppose there are N agents that generate energy-bids in the market for one time period. For the ith agent, $i \in \{1, \ldots, N\}$, the *bid* $0 \leq (p_i, g_i) \in \mathbb{R}^2$ consists of a selling price p_i and the amount of energy g_i, that the agent is willing to generate for this price. The unit cost of generation is φ_i.

The ISO receives bids from all the agents and, taking into account the system demand, decides both the *market price* $P(p, g, l)$ that will be paid for the energy and the *dispatch* $l = (l_1, l_2, \ldots, l_N)$. Here, we define the bid and generation vectors, respectively as $p = (p_1, p_2, \ldots, p_N)$ and $g = (g_1, g_2, \ldots, g_N)$. The dispatch is the output of an optimization problem solved by the ISO, noting that for some agents there may be a difference between offer and dispatch. The market price, a specific function of the bids, generation and dispatch, depends on the ISO's policy. For example, if the aim is to give a preference to less expensive bidders, it is sound to remunerate the generators with the highest price, among all the dispatched agents:

$$P(p, g, l) := \max \{p_j : l_j > 0\} \tag{1}$$

(this price function is not explicit, but results from the actions of the agents in the market). This pricing mechanism, in principle sound and intuitive, turns out to be difficult to implement for equilibrium problems; see the comments after Proposition 1 below.

In a competitive framework, each agent behaves strategically by maximizing its own profit, taking into account possible actions of the other agents. For convenience, below we consider the objective function for the ith agent:

$$f_i(p_i, g_i, l_i, P(p, g, l)),$$

representing a disutility, such as the negative of profit or some function to hedge against downside profit risk. The ith-agent determines bids by solving the minimization problem:

$$\begin{cases} \min\limits_{(p_i,g_i,l_i)} & f_i(p_i, g_i, l_i, P(p,g,l)) \\ \text{s.t.} & (p_i, g_i) \in S_i^{\text{OP}} \\ & g \in S_{\text{shared}}^{\text{OP}} \\ & (p,g,l) \in S_{\text{shared}}^{\text{ISO}} \,, \end{cases} \tag{2}$$

where the notation describing the feasible set is discussed below. First, regarding the objective function, the remuneration of agent i typically involves the term $l_i P(p,g,l)$, and this product introduces a *bilinearity* that poses a challenge in the optimization process. Second, the feasible region in (2) is split into three sets, S_i^{OP}, $S_{\text{shared}}^{\text{OP}}$, and $S_{\text{shared}}^{\text{ISO}}$, with different structures. Operational constraints that are specific to the technology employed by the ith agent to generate energy are included in the first set, S_i^{OP}, that is specific to each agent. If the maximum generation capacity is $g_i^{\max}$ and there is a maximum price $p_i^{\max}$, this set could be as follows with just upper and lower bounds:

$$S_i^{\text{OP}} := \{(p,g) : \varphi_i \le p \le p_i^{\max}, 0 \le g \le g_i^{\max}\} \,. \tag{3}$$

In this equation, the upper bound could be a price cap set by the regulator, e.g., the value of lost load. As for the lower bound, the marginal cost of generation is a natural one only in a static formulation. In a dynamic setting considered in operational models, ramping and warm-up/shut-down constraints reduce the generation and, to encourage the dispatch in those circumstances, agents may sometimes drive the minimum bidding price below cost [36]. Regarding generation, it is important to note that the value $g_i^{\max}$ may not coincide with the actual maximum capacity of generation. The agent rather sets as upper bound the maximum level that it is willing to bid. This distinction is particularly important for the so-called marginal agent, who can strategically play with its own generation to increase the equilibrium price. We refer to Proposition 1 and the comments that follow for more details.

The second set, $S_{\text{shared}}^{\text{OP}}$, includes operational constraints that are *shared* by several agents. Typically, this situation arises for a group of agents generating hydropower from a set of cascaded reservoirs. Like the first set of constraints, this second set is explicitly described by equality and inequality constraints involving components of the generation vector g (e.g. stream-flow balance constraint, exchange limits, etc.). The third set $S_{\text{shared}}^{\text{ISO}}$ is also shared by all the agents but describes how the dispatch l is determined by the ISO in an *implicit* manner, by means of another optimization problem. In an abstract formulation,

$$S_{\text{shared}}^{\text{ISO}} := \{(p,g,l) : l \in D(p,g)\} \,, \tag{4}$$

where the *dispatch multifunction* $D : \mathbb{R}^N \times \mathbb{R}^N \rightrightarrows \mathbb{R}^N$ returns, for a given bid (p,g), the set of $D(p,g)$ of quantities dispatched according to (1), for example.

2.2 A simple ISO problem

Consider the following simple, but illustrative, formulation representing the ISO decision process. Given the bidding prices p and quantities g, the ISO minimizes the total expense, $p^\top l$, in a manner that satisfies the demand d. The corresponding optimization problem is

$$\begin{cases} \min\limits_{l} & p^\top l \\ \text{s.t.} & 0 \le l \\ & l \le g & (\lambda) \\ & \sum\limits_{j=1}^{N} l_j = d & (\pi)\,, \end{cases} \tag{5}$$

where the right-most variables in parentheses, λ and π, denote the Lagrange multipliers associated with the constraints on the left. The meaning of these multipliers (also known as dual variables) as *shadow prices* is well-known. An economic interpretation of the optimal multiplier λ^* as *marginal rent* will be discussed in Section 3 below. An optimal multiplier π^* is the *marginal price*, representing the infinitesimal change in the expense arising from an infinitesimal change in the demand.

Since π^* measures the price for producing one more unit of energy, its value can be seen by the operator as revealing the true price of energy for the electricity system as a whole. From this point of view, instead of using (1), the ISO could set $P(p,g,l) = \pi^*$ as pricing policy. Even though this definition looks simple and natural at first sight, Proposition 1 shows that in some configurations it is not a well-defined function of the decision variables.

The interest of the simple formulation (5) is that the corresponding optimal primal and dual solutions can be expressed in a closed form. To do so, the statement and its proof below use the following notation:

- the index $j_{\mathtt{mg}}$ refers to the *marginal* agent, that is, the one whose bid completes the demand:
$$\sum_{k<\mathtt{mg}} g_{j_k} < d \le \sum_{k\le\mathtt{mg}} g_{j_k}\,.$$
- The N-dimensional vector with all components equal to 1 is denoted by $\mathbf{1}$.
- For two vectors of matching dimensions, $u \perp v$ means that $u^\top v = 0$.
- The positive-part function of a scalar s is defined by $[s]^+ := \max(s,0)\,.$

Proposition 1 (Solution to ISO problem (5)**)** *Re-ordering the indices if necessary, suppose in problem* (5) *the bidding prices satisfy the relation*

$$p_{j_1} \le p_{j_2} \le \cdots \le p_{j_N}\,,$$

and set $p_{j_{N+1}} := \infty$. The following holds:

(i) The dispatch defined by

$$l_j^* = \begin{cases} g_j & \text{if } j \in \{j_k : 1 \le k < \mathtt{mg}\} \\ d - \sum\limits_{k<\mathtt{mg}} l^*_{j_k} = d - \sum\limits_{k<\mathtt{mg}} g_{j_k} & \text{if } j = j_{\mathtt{mg}} \\ 0 & \text{otherwise,} \end{cases}$$

solves (5) *and, hence, $l^* \in D(p,g)$, the set-valued mapping from* (4).

(ii) The optimal Lagrange multipliers associated with the demand constraint in (5) *are*

$$\Pi^* = \begin{cases} \{p_{j_{\mathtt{mg}}}\} & \text{if } l^*_{j_{\mathtt{mg}}} < g_{j_{\mathtt{mg}}} \\ \left[p_{j_{\mathtt{mg}}}, \min\limits_{k>\mathtt{mg}, g_{j_k}>0} p_{j_k}\right] & \text{if } l^*_{j_{\mathtt{mg}}} = g_{j_{\mathtt{mg}}}, \text{ and } j_{\mathtt{mg}} < N \\ \left[p_{j_{\mathtt{mg}}}, +\infty\right) & \text{if } l^*_{j_{\mathtt{mg}}} = g_{j_{\mathtt{mg}}}, \text{ and } j_{\mathtt{mg}} = N\,. \end{cases}$$

possibly non-unique.

(iii) For each π^* in the set-valued mapping Π^*, the vector λ^{π^*} with components

$$\lambda_j^{\pi^*} = [\pi^* - p_j]^+ \text{ for } 1,\dots,N$$

is an optimal multiplier for the capacity constraint $l \le g$ in (5).

Proof. The vector l^* is clearly feasible for (5) and $\lambda^{\pi^*} \ge 0$. Also, we have that

$$\lambda_j^{\pi^*} = \begin{cases} \pi^* - p_j & \text{if } j \in \{j_k : 1 \le k \le \mathtt{mg}\} \\ 0 & \text{otherwise} \end{cases}$$

and if $l^*_{j_{\mathtt{mg}}} < g_{j_{\mathtt{mg}}}$, then $\pi^* = p_{j_{\mathtt{mg}}}$ and so $\lambda^{\pi^*}_{j_{\mathtt{mg}}} = 0$. Taking into account all these relations and the definition of l^*, yields the complementarity relation

$$0 \le g - l^* \perp \lambda^{\pi^*} \ge 0.$$

On the other hand, note that

$$p + \lambda^{\pi^*} - \pi^*\mathbf{1} = \left([\pi^* - p_j]^+ - (\pi^* - p_j)\right)_{j=1}^N \ge 0$$

and, since $l^*_{j_k} = 0$ for $k > \mathtt{mg}$, we have also that

$$0 \le l^* \perp p + \lambda^{\pi^*} - \pi^*\mathbf{1} \ge 0\,,$$

which shows that the primal-dual point $(l^*, \pi^*, \lambda^{\pi^*})$ is optimal for (5). Finally, taking another pair of multipliers $(\bar{\pi}, \bar{\lambda})$, since $l^*_{j_{\tt mg}} > 0$, from $l^*_{j_{\tt mg}}(p_{j_{\tt mg}} + \bar{\lambda}_{j_{\tt mg}} - \bar{\pi}) = 0$, we have that $0 \leq \bar{\lambda}_{j_{\tt mg}} \leq \bar{\pi} - p_{j_{\tt mg}}$. Also, when $l^*_{j_{\tt mg}} < g_{j_{\tt mg}}$ it holds that $\bar{\lambda}_{j_{\tt mg}} = 0$ which implies $p_{j_{\tt mg}} = \bar{\pi}$. In case $l^*_{j_{\tt mg}} = g_{j_{\tt mg}}$, and $k > {\tt mg}$, with $g_{j_k} > 0$. Since $l^*_{j_k} = 0$ and $(g_{j_k} - l^*_{j_k})\bar{\lambda}_{j_k} = 0$ we have that $\bar{\lambda}_{j_k} = 0$ and using that $p_{j_k} + \bar{\lambda}_{j_k} - \bar{\pi} \geq 0$ yields $p_{j_k} \geq \bar{\pi}$, as stated. ∎

Proposition 1 reveals the role of the marginal agent in the determination of π^*. The marginal agent can play strategically, bringing $g^{\max}_{\tt mg}$ down to the residual demand, thus triggering the second situation in Proposition 1(ii), and making the optimal multiplier not unique. Our numerical results reported in Section 5.3 confirm that this behavior renders the equilibrium price much higher. By contrast, the regularization scheme proposed in this work prevents the marginal agent from playing such a strategic game; see Figure 5.

In our development, we focus on the EPEC that results from simultaneously considering all the agents' problems, that is (2) written for $i = 1, \ldots, N$. Then, if there are no coupling operational constraints ($S^{\rm OP}_{\rm shared}$ is empty), and the ISO behavior is given by (4) and (5), we have:

$$\begin{array}{c}\text{Find an}\\ \text{equilibrium,}\\ \text{solving}\\ \text{for } i=1,\ldots,N\\ \text{the } \textbf{bilevel}\\ \text{problems}\end{array}\left\{\begin{array}{lll}\min\limits_{g_i,p_i,l_i} & f_i(p_i, g_i, l_i, P(p,g,l)) & \\ \text{s.t.} & 0 \leq g_i \leq g_i^{\max} & \\ & \varphi_i \leq p_i \leq p_i^{\max} & \\ & l \in \arg\min(S^{\rm ISO}_{\rm shared}) & = \left\{\begin{array}{lll}\min\limits_{l} & p^\top l & \\ \text{s.t.} & 0 \leq l & \\ & l \leq g & (\lambda) \\ & \sum_{j=1}^{N} l_j = d & (\pi)\end{array}\right\}\end{array}\right. \tag{6}$$

This is an equilibrium problem with shared constraints $S^{\rm ISO}_{\rm shared}$, given *implicitly* by the lower-level problem.

Remark 2 (On the lack of uniqueness) Regarding (4), notice that if there is more than one minimizer l^*, the dispatch function

$$D^1(p, g) = \{l^* \text{ solving (5)}\}$$

can be multi-valued (this is the reason for writing (4) as an inclusion). By Proposition 1, such is the case whenever the highest dispatched price is bid by more than one agent, the same $p_{j_{\tt mg}}$ is associated with different generation offers $g_{j_{{\tt mg},1}}, g_{j_{{\tt mg},2}}, \ldots$. This creates an indifference set for the ISO regarding the optimal dispatch l^*, as any distribution of the marginal dispatch $d - \sum_{k<{\tt mg}} g_{j_k}$ among the marginal agents yields the same output from the ISO point of view. The ISO problem (4) is a model without transmission constraints, so all the different $g_{j_{{\tt mg},l}}$ are the same when it comes to satisfaction of the demand. With a network representation in (4), there are indifference sets when two agents that inject power in the same bus bid the same price and their price is the marginal one. Since the ISO problem is part of an EPEC, we observed that this phenomenon of indifference can lead to numerical difficulties and cycling in the solution process. □

Proposition 1 also confirms that for the optimal multiplier of problem (5) to be unique, the marginal agent should be dispatched at a level that is *strictly smaller* than the marginal bid on generation : $l^*_{j_{\tt mg}} < g_{j_{\tt mg}}$ if there is only one marginal agent. Suppose $j_{\tt mg} < N$, i.e., the marginal agent's bid is not the most expensive one in the market. If the offer is manipulated to force $l^*_{j_{\tt mg}} = g_{j_{\tt mg}}$, there will be a whole interval of marginal prices, $[p_{j_{\tt mg}}, p_{j_{{\tt mg}+1}}]$ (when $j_{\tt mg} = N$, the interval is the half-line $[p_{j_{\tt mg}}, \infty)$). In this setting the choice of the specific ISO's policy plays a fundamental role. With rule (1) the market price would be the left extreme of the interval, the smallest possible choice. If, instead, the policy of the ISO were to remunerate the energy at marginal price, the rule would be $P(p, g, l) = \pi^*$. This gives an opportunity for the marginal agent to act strategically, trying to bid an amount $g_{j_{\tt mg}} = l^*_{j_{\tt mg}}$. This feature has a crucial impact on practical efficiency, in our numerical experiments we noticed that the multi-valuedness of π^* results in higher market prices, see Figure 5 in Section 5.3.

To handle this difficulty, we write the dual of the dispatch problem (5):

$$\begin{cases} \max\limits_{\pi,\lambda} & \pi d - \lambda^\top g \\ \text{s.t.} & \pi - \lambda_j \le p_j,\ j = 1,\dots,N \\ & \lambda \ge 0\,, \end{cases} \tag{7}$$

and replace the problem by its optimality conditions. Primal-dual feasibility and strong duality yield the equivalent formulation:

$$\text{Find an equilibrium, solving for } i = 1,\dots,N, \text{ the } \textbf{bilinear} \text{ problems} \quad \begin{cases} \min\limits_{g_i,p_i,l_i,\pi,\lambda} & f_i(p_i, g_i, l_i, P(p,g,l)) \\ \text{s.t.} & 0 \le g_i \le g_i^{\max} \\ & \varphi_i \le p_i \le p_i^{\max} \\ & 0 \le l \le g \\ & \sum_{j=1}^{N} l_j = d \\ & \pi - \lambda_j \le p_j,\ j = 1,\dots,N \\ & \lambda \ge 0 \\ & p^\top l = \pi d - \lambda^\top g \end{cases} \tag{8}$$

This rewriting eliminates the bilevel setting, at the expense of adding bilinear terms in the constraints, that are not simple to tackle, but have the merit of being explicit. By contrast, the bilinear term in the objective, $l_i P(g,p,l)$, remains implicit and is harder to deal with in an algorithm. Replacing $P(g,p,l)$ in the objective by the individual bidding price p_i leads to equilibria with high prices. A common practice is to use the dual variable as an approximation,

$$P(p,g,l) \approx \pi\,, \tag{9}$$

and look for an equilibrium for problems that replace the remuneration $l_i P(g,p,l)$ in the objective function in (8) by the bilinear term $l_i\pi$:

$$\varphi_i g_i - l_i \pi\,.$$

However, keeping in mind the result stated in Proposition 1, in some configurations the marginal price is a full interval. Ideally, the ISO pricing preference should be the smallest possible value, as in (1). This is also reflected in the ISO objective function, which is being minimized. But with the replacement (9) what happens is exactly the opposite: since agents maximize revenue (or minimize disutility), using the proxy in the new objective function will always favor the *largest* possible choice for π, making the approximation a gross overestimation. This phenomenon is confirmed by our numerical experiments.

To address this issue, we introduce a new regularization scheme that provides a mechanism of selection in the multiplier set. The alternative approach depends on a parameter β than, when driven to zero, ensures that the minimum price signal will be found.

3 A dual view for the ISO problem

We now explore how to modify the ISO problem in (8) to prevent the EPEC model from taking the highest possible marginal price, if the proxy (9) is used. To do so, we adopt the viewpoint of a *dual* ISO, that could be thought as being more concerned with prices than with dispatch (a dispatch-oriented ISO would directly solve the primal problem). In view of the constraints in (7) (and also by the relations in Proposition 1),

$$\lambda_j = \max(\pi - p_j, 0)\,.$$

If the marginal price replaces the market price as in (9), then any dispatched agent that bids below the market price has a multiplier λ_j in (7) that represents an inframarginal rent. The wording *rent*, sometimes termed as surplus, refers to an amount that is received by the agent without any effort (revenue, by contrast, involves some work of the agent, to generate energy that may be dispatched). The rent λ_j is positive only when $p_i < \pi$, that is, when the agent gets paid more than the bid. Generators typically rely upon such rent to cover fixed costs.

By Proposition 1, all the dispatched agents except the marginal ones are dispatched at their bidding level $l_i^* = g_i$, so for those agents it holds that $\lambda_i g_i = \lambda_i l_i^*$. If for a marginal agent the dispatch satisfies $l^*_{j_{\mathtt{mg}}} < g_{j_{\mathtt{mg}}}$, the rent will be zero, because $\pi^* = p_{j_{\mathtt{mg}}}$. Moreover, since in the dual problem (7) the objective function is $\pi d - \lambda^\top g$, we conclude that the dual ISO is in fact maximizing the overall payment to the agents *net from any rent*.

On the other hand, the unfavorable situation $l^*_{j_{\mathtt{mg}}} = g_{j_{\mathtt{mg}}}$ leads to a positive rent for the marginal agent and an overall increase in the rent of all the other dispatched agents. In order to maintain control of the rent, instead of solving (7), we define a *regularized* problem for the dual ISO that discourages large values of marginal rent. This is done penalizing λ through a convex function $h(\lambda)$ satisfying $h(\lambda) > 0$ for $\lambda > 0$, and $h(0) = 0$. Given a penalty parameter $\beta \geq 0$, the new dual ISO problem is

$$\begin{cases} \max\limits_{\pi,\lambda} & \pi d - \lambda^\top g - \beta h(\lambda) \\ \text{s.t.} & \pi - \lambda_j \leq p_j\,, \text{ for } j = 1, \ldots, N \\ & \lambda_j \geq 0\,, \text{ for } j = 1, \ldots, N\,. \end{cases} \qquad (10)_\beta$$

Notice that when taking $\beta = 0$, problem $(10)_0$ recovers the original dual linear program (7). One possibility for the penalizing function h is the ℓ_∞-norm, i.e.,

$$h(\lambda) := \|\lambda\|_\infty = \max\{|\lambda_j| : j = 1, \ldots, N\}\,.$$

Among other features, the ℓ_∞-norm maintains the regularized ISO problem as a linear program and also yields more specific convergence results when β goes to zero, as shown in Corollary 12. However, this is not the only choice that might be useful, and keeping a general function h gives better insight into the properties and consequences of regularization. For this reason, to study the asymptotic behavior of $(10)_\beta$ and its relation with (7), we require the penalty to satisfy the following properties (which hold for any ℓ_r-norm $h(x) = \|x\|_r$ with $1 \leq r \leq \infty$).

Assumption 3 (Conditions on the penalizing function) *The convex function $h : \mathbb{R}^N \to \mathbb{R}$ satisfies $h \geq 0$ with $h(0) = 0$ and the conditions:*

1. *If $0 \leq x \leq y$, then $h(x) \leq h(y)$.*
2. *If $0 \leq x \leq y$ and $x_j < y_j$ whenever $y_j > 0$, then $h(x) < h(y)$.*
3. *For any $M \in \mathbb{R}$, the sublevel set $\{x : h(x) \leq M\}$ is bounded.* □

Lemma 4 (Properties of the regularized dual ISO problem) *Let $(\pi(\beta), \lambda(\beta))$ be any solution to $(10)_\beta$, where the function h satisfies Assumption 3. Then the following hold.*

(i) The marginal price is non-negative, $\pi(\beta) \geq 0$, and the marginal rent defined as

$$\lambda^\pi_j(\beta) := [\pi(\beta) - p_j]^+ \text{ for } j = 1, \ldots, N\,, \qquad (11)$$

satisfies $\lambda^\pi(\beta) \leq \lambda(\beta)$.

(ii) The pair $(\pi(\beta), \lambda^\pi(\beta))$ is also a solution to problem $(10)_\beta$.

(iii) Given any two solutions $(\pi^1(\beta), \lambda^{1,\pi}(\beta))$ and $(\pi^2(\beta), \lambda^{2,\pi}(\beta))$ to problem $(10)_\beta$,

$$\pi^1(\beta) \leq \pi^2(\beta) \iff \lambda^{1,\pi}(\beta) \leq \lambda^{2,\pi}(\beta).$$

If, in addition, $\pi^1(\beta) < \pi^2(\beta)$ and $\lambda^{2,\pi}(\beta) \neq 0$, then $h(\lambda^{1,\pi}(\beta)) < h(\lambda^{2,\pi}(\beta))$.

Proof. Given a solution $(\pi(\beta), \lambda(\beta))$ of $(10)_\beta$, the fact that $\pi(\beta) \geq 0$ is clear. Also, it is easy to see that the pair $(\pi(\beta), \lambda^\pi(\beta))$ is feasible in $(10)_\beta$ and that, as stated in item (i), $\lambda^\pi(\beta) \leq \lambda(\beta)$.

To show item (ii), notice that by Assumption (3) we have that $h(\lambda^\pi(\beta)) \leq h(\lambda(\beta))$; and since $g \geq 0$ we have $g^\top \lambda^\pi(\beta) \leq g^\top \lambda(\beta)$. Thus $\pi(\beta)d - g^\top \lambda(\beta) - \beta h(\lambda(\beta)) \leq \pi(\beta)d - g^\top \lambda^\pi(\beta) - \beta h(\lambda^\pi(\beta))$, which shows that $(\pi(\beta), \lambda^\pi(\beta))$ is also a solution.

For item (iii), first note that $\pi^1(\beta) \leq \pi^2(\beta)$, combined with the facts that $[\cdot]^+$ is monotonically non-decreasing and (11), implies that $\lambda^{1,\pi}(\beta) \leq \lambda^{2,\pi}(\beta)$. The converse statement assumes that $\lambda^{1,\pi}(\beta) \leq \lambda^{2,\pi}(\beta)$. Now, let us consider the following two cases. Suppose first that $\lambda^{2,\pi}(\beta) = 0$. Then $\lambda^{1,\pi}(\beta) = 0$ and, by $(10)_\beta$, this forces

$$\pi^1(\beta) = \min_j \{p_j\} = \pi^2(\beta)\,.$$

In the second case, when $\lambda^{2,\pi}(\beta) \neq 0$, from (11), there exists some index j such that

$$\pi^2(\beta) - p_j = [\pi^2(\beta) - p_j]^+ = \lambda_j^{2,\pi}(\beta) \geq \lambda_j^{1,\pi}(\beta) = [\pi^1(\beta) - p_j]^+ \geq \pi^1(\beta) - p_j \,,$$

from which the desired relation follows. Finally, assuming $\pi^1(\beta) < \pi^2(\beta)$, we have that $\lambda^{1,\pi}(\beta) \leq \lambda^{2,\pi}(\beta) \neq 0$, and for all j,

$$\pi^2(\beta) - p_j > \pi^1(\beta) - p_j \,.$$

As a result, for all the components j for which $\lambda_j^{2,\pi}(\beta) > 0$, we have that

$$\lambda_j^{2,\pi}(\beta) = \pi^2(\beta) - p_j > \max\{\pi^1(\beta) - p_j, 0\} = \lambda_j^{1,\pi}(\beta) \,,$$

and, by Assumption (3), we have that $h(\lambda^{2,\pi}(\beta)) > h(\lambda^{1,\pi}(\beta))$, which concludes the proof. ■

Lemma 4 is useful when considering convergence of a sequence of approximations, as the regularization parameter tends to zero. In (iii) and (iv) below we show that the approach converges to a price with minimal norm. The statement in item (iv), in particular, states that the limit price will always be the price bid by the marginal agent.

Theorem 5 (Asymptotic behavior of regularized dual ISO problems) *Consider any sequence of solutions to $(10)_\beta$ $\{(\pi(\beta), \lambda^\pi(\beta))\}$, parameterized by β. Under the assumptions in Lemma 4, the following hold.*

(i) *As $\beta \to 0$, the sequence $\{(\pi(\beta), \lambda^\pi(\beta))\}$ converges to a point $(\bar{\pi}, \lambda^{\bar{\pi}})$.*

(ii) *The limit point $(\bar{\pi}, \lambda^{\bar{\pi}})$ solves problem $(10)_0$, that is, the original dual ISO problem (7).*

(iii) *For any other solution to (7), say (π^0, λ^0), it holds that*

$$\bar{\pi} \leq \pi^0 \quad \textit{and} \quad \lambda^{\bar{\pi}} \leq \lambda^0.$$

(iv) *The limit price $\bar{\pi}$ coincides with the marginal price $p_{j_{\text{mg}}}$ in Proposition 1.*

Proof. For item (i), we start with proving that the sequence is bounded, and then show that all its accumulation points are the same.

Note that problems $(10)_\beta$ have the same feasible set for all β. Let (π^0, λ^{π^0}) be any solution to (7) By the optimality of $(\pi(\beta), \lambda^\pi(\beta))$ in $(10)_\beta$, for all $\beta \geq 0$ it holds that

$$\begin{aligned} \pi^0 d - (\lambda^{\pi^0})^\top g - \beta h(\lambda^{\pi^0}) \quad &\leq \quad \pi(\beta) d - (\lambda^\pi(\beta))^\top g - \beta h(\lambda^\pi(\beta)) \\ &\leq \quad \pi^0 d - (\lambda^{\pi^0})^\top g - \beta h(\lambda^\pi(\beta)) \,, \end{aligned} \tag{12}$$

where the last inequality follows from $\pi(\beta) d - (\lambda^\pi(\beta))^\top g \leq \pi^0 d - (\lambda^{\pi^0})^\top g$.

By Assumption 3,

$$h(\lambda^{\pi^0}) \geq h(\lambda^\pi(\beta)) \,, \tag{13}$$

and the boundedness of the sublevel sets $\{\lambda : h(\lambda) \leq h(\lambda^{\pi^0})\}$ implies that the sequence $\{\lambda^\pi(\beta)\}$ is bounded.

Next, combining (11), the constraints in $(10)_\beta$, and the fact that $\pi(\beta) \geq 0$, implies that the sequence $\{\pi(\beta)\}$, is bounded as well.

Consider any accumulation point $(\pi^{\text{acc}}, \lambda^{\text{acc}})$ of $\{(\pi(\beta), \lambda^\pi(\beta))\}$, i.e., let $\beta_k \to 0$ and let $\{(\pi(\beta_k), \lambda(\beta_k))\} \to (\pi^{\text{acc}}, \lambda^{\text{acc}})$. It is easy to check that $(\pi^{\text{acc}}, \lambda^{\text{acc}})$ is feasible in $(10)_0$ and $\lambda^{\text{acc}} = \lambda^{\pi^{\text{acc}}}$. Passing onto the limit in (12) yields

$$\lambda^{\pi^0 \top} g - \pi^0 d = \lambda^{\text{acc}\top} g - \pi^{\text{acc}} d \,,$$

from which we conclude that $(\pi^{\text{acc}}, \lambda^{\text{acc}})$ solves $(10)_0$.

Before continuing with item (i), we next show that item (iii) holds for $(\pi^{\text{acc}}, \lambda^{\text{acc}})$ in place of $(\bar{\pi}, \lambda^{\bar{\pi}})$. Passing onto the limit in (13), as $\beta_k \to 0$, it gives that $h(\lambda^{\pi^0}) \geq h(\lambda^{\text{acc}})$. Suppose, for contradiction purposes, that $\pi^{\text{acc}} > \pi^0$. Lemma 4 ensures that $\lambda^{\text{acc}} \geq \lambda^{\pi 0} \geq 0$, which implies $h(\lambda^{\pi 0}) \leq h(\lambda^{\text{acc}})$. Therefore, $h(\lambda^{\pi^0}) = h(\lambda^{\text{acc}})$. At this point, we need to consider two cases. If $\lambda^{\text{acc}} = 0$ then $\lambda^{\text{acc}} = \lambda^{\pi^0} = 0$. By Lemma 4, we have that $\pi^{\text{acc}} = \pi^0$, which contradicts our assumption. On the other hand, assuming $\lambda^{\text{acc}} \neq 0$, implies, using Lemma 4(ii), that

$h(\lambda^{\pi^0}) < h(\lambda^{\text{acc}})$, which also contradicts the assumption $\pi^{\text{acc}} > \pi^0$. Thus, $\pi^{\text{acc}} \leq \pi^0$, which also yields $\lambda^{\text{acc}} \leq \lambda^{\pi^0} \leq \lambda^0$.

Considering any other accumulation point $(\hat{\pi}^{\text{acc}}, \hat{\lambda}^{\text{acc}})$ of $\left\{(\pi(\beta), \lambda^{\pi}(\beta))\right\}$, because we have that $(\hat{\pi}^{\text{acc}}, \hat{\lambda}^{\text{acc}})$ solves (7), it holds that

$$\pi^{\text{acc}} \leq \hat{\pi}^{\text{acc}} \quad \text{and} \quad \lambda^{\text{acc}} \leq \hat{\lambda}^{\text{acc}}.$$

By a similar argument we can show that

$$\hat{\pi}^{\text{acc}} \leq \pi^{\text{acc}} \quad \text{and} \quad \hat{\lambda}^{\text{acc}} \leq \lambda^{\text{acc}}.$$

Hence, $(\pi^{\text{acc}}, \lambda^{\text{acc}}) = (\hat{\pi}^{\text{acc}}, \hat{\lambda}^{\text{acc}})$. We have therefore established that all accumulation points of the bounded sequence $\left\{(\pi(\beta), \lambda^{\pi}(\beta))\right\}$ coincide, i.e., the sequence converges. And since $\lambda^{\text{acc}} = \lambda^{\pi^{\text{acc}}}$, we have that the limit point can be written as $(\bar{\pi}, \lambda^{\bar{\pi}})$. This concludes item (i). Then, items (ii) and (iii) follow in a straightforward way.

To see the final item (iv), recall from Proposition 1 that $p_{j_{\mathtt{mg}}} \leq \bar{\pi}$ and since $(p_{j_{\mathtt{mg}}}, \lambda_{p_{j_{\mathtt{mg}}}})$ solves (7), from item (iii), this means that $\bar{\pi} \leq p_{j_{\mathtt{mg}}}$. This concludes the proof. ■

Recall from Proposition 1(iii) that when the marginal agent is dispatched up to the bid, that is when $l^*_{j_{\mathtt{mg}}} = g_{j_{\mathtt{mg}}}$, the equilibrium price lies in the interval $\Pi^* = \left[p_{j_{\mathtt{mg}}}, \min_{k>\mathtt{mg}, g_{j_k}>0} p_{j_k}\right]$. A remarkable feature of Theorem 5(iv) is that the property $\bar{\pi} = p_{j_{\mathtt{mg}}}$ holds *independently* of the dispatch. This ensures that, as announced, the multiplier $\pi(\beta)$ acts as a *selection mechanism* that in the limit provides the smallest possible value for the price, among all the (infinite) choices in the multiplier set Π^*.

4 Back to the primal ISO problem

Since solving EPECs is far from straightforward, when considering successive equilibrium problems with diminishing regularization parameters, in the numerical experiments we take a small value for β and use the corresponding ISO problem $(10)_\beta$ as an approximation for (7). Corollary 12, given at the end of this section, summarizes all the results for the ℓ_∞-regularization and provides the theoretical background for our numerical assessment. In particular, solving the regularized EPEC results in equilibrium price of minimal norm.

In order to guide the choice of the bound for the regularization parameter, and determine its impact (or interference) in the bidding process, we now play the reverse game and formulate the dual of the regularized dual ISO problem $(10)_\beta$. The resulting problem is called bi-dual, because it is the dual of the dual problem. The bi-dual is a primal ISO problem that regularizes (5) and clarifies the role of the parameter β in determining certain *reserve* that the ISO can access to complete the dispatch and keep controlled both the price and the rent.

Before presenting our theoretical results we review briefly two well-known concepts from convex analysis: the *subgradient* and the *Fenchel conjugate*. Detailed explanations can be found in any book of Convex Analysis, for instance [1], [19].

Given a convex function $h : \mathbb{R}^N \to \mathbb{R}$ and $x \in \mathbb{R}^N$, the subgradient of h at x is the set denoted and defined by

$$\partial h(x) = \{s : h(y) \geq h(x) + s^\top (y - x) \quad \forall y\}. \tag{14}$$

The Fenchel conjugate $h^* : \mathbb{R}^N \to \mathbb{R} \cup \{-\infty, \infty\}$ of h is defined by

$$h^*(s) = \sup_{x \in \mathbb{R}^N} \{x^\top s - h(x)\}. \tag{15}$$

The subgradient generalizes the concept of gradient from a vector to a set, when the convex function h is not differentiable (if h is differentiable, the set is a singleton – the usual derivative). On the other hand, the Fenchel conjugate h^* is a convex function with an interesting economic interpretation. Suppose $h(x)$ represents the cost of production of a good x that can be sold at a price s. Then, the Fenchel conjugate at s defined in (15) represents the optimal profit that can be achieved for the price s by choosing the quantity x to be produced and sold.

A relation between these two concepts that we use frequently below states that

$$s \in \partial h(x) \quad \Longleftrightarrow \quad h(x) + h^*(s) = x^\top s\,. \tag{16}$$

Continuing with the economic interpretation when h represents a cost, the subdifferential of h at x is the set of all the prices that ensure an optimal profit when x is the production level.

The stage is now set to present the theoretical results in this section.

Proposition 6 (Regularized primal ISO problem) *Given a convex function h satisfying $h(\lambda) \geq 0$ and $h(0) = 0$, the primal problem associated with the regularized dual $(10)_\beta$ is*

$$\left\{\begin{array}{ll} \min\limits_{l,w} & l^\top p + \beta h^*\Big(\dfrac{l+w-g}{\beta}\Big) \\ s.t. & \mathbf{1}^\top l = d \\ & l\,, w \geq 0\,, \end{array}\right. \tag{17$_\beta$}$$

where h^* denotes the conjugate function of h.

When h is the ℓ_∞-norm, the problem above reduces to the linear program

$$\left\{\begin{array}{ll} \min\limits_{l\,,s} & l^\top p \\ s.t. & \mathbf{1}^\top l = d \\ & l \leq g + \beta s \\ & \mathbf{1}^\top s \leq 1 \\ & l, s \geq 0\,. \end{array}\right. \tag{18}$$

Proof. The Lagrangian of $(10)_\beta$ is given by

$$\begin{aligned} L(\pi, \lambda, l, w) =& \pi d - \lambda^\top g - \beta h(\lambda) - l^\top(\pi\mathbf{1} - \lambda - p) + w^\top \lambda \\ =& l^\top p + \pi(d - \mathbf{1}^\top l) + \lambda^\top(-g + l + w) - \beta h(\lambda)\,, \end{aligned}$$

where $l\,, w \geq 0$ are the multipliers associated to the respective constraints. Then, the optimality conditions for $(10)_\beta$ state that

$$\begin{aligned} 0 =& d - \mathbf{1}^\top l \\ 0 =& -g + l + w - \beta s\,, \text{ for } s \in \partial h(\lambda) \\ 0 \leq& \, l \perp \pi\mathbf{1} - \lambda - p \leq 0 \\ 0 \leq& \, w \perp \lambda \geq 0\,. \end{aligned}$$

In the second equality, the subgradient $s = \frac{l+w-g}{\beta} \in \partial h(\lambda)$ satisfies relation (16) written with x replaced by λ:

$$s \in \partial h(\lambda) \iff s^\top \lambda = h(\lambda) + h^*(s)\,.$$

Therefore, multiplying by β,

$$(l + w - g)^\top \lambda = \beta h(\lambda) + \beta h^*(s)\,.$$

Next, $w^\top \lambda = 0$ implies that $l^\top \lambda = g^\top \lambda + \beta h(\lambda) + \beta h^*(s)$, and using $l^\top(\pi\mathbf{1} - \lambda - p) = 0$ we obtain that

$$l^\top(\pi\mathbf{1} - p) = l^\top \lambda = g^\top \lambda + \beta h(\lambda) + \beta h^*(s)\,.$$

Finally, the identity $l^\top(\pi\mathbf{1}) = \pi d$ yields

$$\pi d - l^\top p = \lambda^\top g + \beta h(\lambda) + \beta h^*(s)\,.$$

The expression for the bi-dual problem follows, using the expression for s.

To specialize the result to the ℓ_∞-norm note that, directly form definition of the conjugate function,

$$h(\lambda) = \|\lambda\|_\infty \iff h^*(s) = \left\{\begin{array}{ll} 0 & \text{if } \sum_{j=1}^N |s_j| \leq 1 \\ +\infty & \text{otherwise.} \end{array}\right.$$

As a result, $(17)_\beta$ writes down as

$$\begin{cases} \min\limits_{l,w} & l^\top p \\ \text{s.t.} & \mathbf{1}^\top l = d \\ & \sum_{j=1}^{N} |l_j + w_j - g_j| \leq \beta \\ & l, w \geq 0\,, \end{cases}$$

which is equivalent to problem (18). ■

Problem (18) is a linear program because the conjugate of the ℓ_∞-norm is defined in terms of the ℓ_1-norm. In the family of ℓ_r-norms, the only ones yielding linear programs in $(17)_\beta$ are the *polyhedral* norms, $r = 1$ and $r = +\infty$. The latter gives the primal ISO problem (18), whereas the former can be found in Example 7 below.

We now turn our attention to the interpretation of the regularization in the primal problem. In the original ISO problem, the marginal rent is the multiplier associated to the capacity constraint $l \leq g$, that is no longer explicit in $(17)_\beta$. At first sight, this makes less clear the meaning of the variable λ in the regularized problem. In fact, such meaning depends on the penalizing function.

The choice $h(\cdot) = \frac{1}{2}\|\cdot\|^2$, which by definition of the conjugate implies that $h^* = h$, gives in $(17)_\beta$ the objective function

$$l^\top p + \beta h^*\Big(\frac{l + w - g}{\beta}\Big) = l^\top p + \frac{1}{2\beta}\|l + w - g\|^2\,.$$

This specific regularization is a quadratic penalization of the capacity constraint.

If the penalizing function is the ℓ_∞-norm, the capacity constraint appears explicitly, and λ plays the role of a genuine marginal rent, in the sense that it is the multiplier associated to the capacity constraint. The primal format (18) reveals the regularized primal ISO as disposing of a *generation reserve* equal to βs. Indeed, the capacity constraint $l_j - \beta s_j \leq g_j$ allows for a value $l_j > g_j$ to be optimal, seemingly allowing the ISO dispatch the jth agent *beyond* the bid. Of course, this is not possible. Rather, this situation, that leads to positive values for an optimal $s_j^*(\beta)$, is to be understood as the ISO having access to an additional source of energy, out of the market – a battery perhaps.

In view of Theorem 5, the ISO uses this reserve to control the marginal rent and ensure the price π will be the minimal one. This interpretation can be extended to any penalty in the family of norms.

Example 7 (Regularizing with a norm) To each norm $h(\cdot) = \|\cdot\|$ we associate a dual norm, defined by

$$h_D(x) = \max_{h(y)=1} x^\top y\,.$$

Since by definition of the conjugate,

$$h^*(x) = \begin{cases} 0 & \text{if } h_D(x) \leq 1 \\ +\infty & \text{otherwise,} \end{cases}$$

the regularized primal problem $(17)_\beta$ can be expressed as

$$\begin{cases} \min\limits_{l,w} & l^\top p \\ \text{s.t.} & \mathbf{1}^\top l = d \\ & h_D(l + w - g) \leq \beta \\ & l, w \geq 0\,. \end{cases}$$

Notice that the meaning of $\lambda^\pi(\beta)$ as marginal rent is preserved, because

$$0 \leq \lambda^\pi(\beta) \perp g(\beta) + \beta s(\beta) - l(\beta) \geq 0.$$

For the ℓ_∞-norm considered in Proposition 6, the particular instance (18) uses as h_D the ℓ_1-norm, because

$$h(\cdot) = \|\cdot\|_\infty \text{ and } h_D(\cdot) = \|\cdot\|_1\,.$$

The other situation yielding a linear program for $(17)_\beta$ is when

$$h(\cdot) = \|\cdot\|_1 \text{ and } h_D(\cdot) = \|\cdot\|_\infty\,.$$

In this case, the ISO regularized primal problem is equivalent to

$$\begin{cases} \min\limits_{l\,,s} & l^\top p \\ \text{s.t.} & \mathbf{1}^\top l = d \\ & l \leq g + \beta s \\ & s \leq \mathbf{1} \\ & l, s \geq 0\,. \end{cases}$$

Like in (18), the capacity constraint is explicit and involves a reserve on generation. For these two polyhedral norms, $\|\beta s\| \leq \beta$, so β together with the chosen norm determines the amount of reserve. For this reason, in our numerical experiments β is bounded by a small percentage of the demand. The regularized ISO can slightly alter the dispatch, passing from g_j to $g_j + \beta s_j^*(\beta) > g_j$, using the reserve to complete the generation and attend the demand (alternatively, the difference can be thought of as real-time corrections of the dispatch). □

In order to relate the solutions to the original problem (5), given in Proposition 1, with the solutions of the regularized primal ISO problem $(17)_\beta$, the penalty must verify one more relation, stated below.

Assumption 8 (Additional condition on the penalty) *For any $x \in \mathbb{R}^N$, it holds that*

$$h(x) = h(abs(x)) \textit{ where } abs(x) := (|x_1|, |x_2|, \ldots, |x_N|)\,. \qquad \square$$

Once again, Assumption 8 is satisfied by any ℓ_r-norm with $1 \leq r \leq \infty$. We state several technical properties related with the new assumption.

Lemma 9 (Consequences of Assumption 8) *The following holds for a function h satisfying Assumption 8.*

(i) *The conjugate h^* satisfies Assumption 8.*

(ii) *For any $\lambda \in \mathbb{R}^N$ and a subgradient $s \in \partial h(\lambda)$,*

$$\lambda \geq 0 \textit{ with } \lambda \neq 0 \Longrightarrow s_j \geq 0 \textit{ for any component } j \textit{ for which } \lambda_j > 0.$$

(iii) *Suppose, in addition, that h is a norm whose dual norm h_D has the property that, for any $0 \leq x \leq y$,*

$$\exists j \in \{1, \ldots, N\} \textit{ such that } x_j < y_j \implies h_D(x) < h_D(y)\,.$$

Then $s \in \partial h(\lambda)$ in item (ii) is such that $s_j = 0$ whenever $\lambda_j = 0$.

Proof. To show (i), denote

$$\text{sign}(x) := (\text{sign}(x_1), \text{sign}(x_2), \ldots, \text{sign}(x_N)).$$

Let $\circ$ represent the Hadamard product between vectors, that is

$$\text{sign}(x) \circ y := (\text{sign}(x_1)y_1, \text{sign}(x_2)y_2, \ldots, \text{sign}(x_N)y_N)\,.$$

Then the identity below holds for any $x, y \in \mathbb{R}^N$, showing the assertion:

$$\begin{aligned} x^\top y - h(y) &= \text{abs}(x)^\top(\text{sign}(x) \circ y) - h(y) \\ &= \text{abs}(x)^\top(\text{sign}(x) \circ y) - h(\text{sign}(x) \circ y)\,. \end{aligned}$$

By definition of conjugate function, the subgradient $s \in \partial h(\lambda)$ solves the problem

$$h^{**}(\lambda) = \sup_z \{\lambda^\top z - h^*(z)\}\,.$$

Since $h = h^{**}$ by convexity of h, this means that

$$h(\lambda) = \lambda^\top s - h^*(s) \geq \lambda^\top z - h^*(z) \quad \text{for all } z \in \mathbb{R}^N\,. \tag{19}$$

To show item (ii), we proceed by contradiction and suppose there exists j such that $\lambda_j > 0$ and $s_j < 0$. Then we have that $\lambda^\top \text{abs}(s) > \lambda^\top s$ and since $h^*(s) = h^*(\text{abs}(s))$ by item (i),

$$\lambda^\top \text{abs}(s) - h^*(\text{abs}(s)) > \lambda^\top s - h^*(s),$$

which contradicts (19). Thus $s_j \geq 0$ whenever $\lambda_j > 0$, as stated.

We proceed with item (iii). When h is a norm, as explained in Example 7, having $\lambda \neq 0$ implies that $h_D(s) = 1$. This gives in (19) the following:

$$h(\lambda) = \lambda^\top s \geq \lambda^\top z \quad \text{for all } z \text{ with } h_D(z) = 1\,. \tag{20}$$

We proceed again by contradiction, supposing that for some component j_0 it holds that $s_{j_0} \neq 0$ and $\lambda_{j_0} = 0$. Defining $\hat{s}$ by

$$\hat{s}_j = \begin{cases} 0 & \text{if } j = j_0 \\ |s_j| & \text{otherwise,} \end{cases}$$

we have that $\hat{s} \neq 0$, $0 \leq \hat{s} \leq \text{abs}(s)$ and $\hat{s}_{j_0} < \text{abs}(s)_{j_0}$. Then

$$0 < h_D(\hat{s}) < h_D(\text{abs}(s)) = h_D(s) = 1.$$

This contradicts (20), because

$$\frac{\lambda^\top \hat{s}}{h_D(\hat{s})} > \lambda^\top \hat{s} = \lambda^\top \text{abs}(s) \geq \lambda^\top s\,.$$

This completes the proof. ∎

Thanks to the properties stated in Lemma 9, we now characterize the dispatch of the regularized ISO.

Proposition 10 (Solution to regularized ISO problem (17)$_\beta$) *Suppose h in (10)$_\beta$ satisfies Assumptions 3 and 8. Let $(\pi(\beta), \lambda^\pi(\beta))$ be the marginal pair from Lemma 4(i), solving the regularized dual ISO problem (10)$_\beta$. The following holds for $(l(\beta), w(\beta))$, a solution to the regularized primal problem (17)$_\beta$.*

(i) $s = \dfrac{l(\beta) + w(\beta) - g}{\beta} \in \partial h(\lambda^\pi(\beta))\,.$

(ii) *For $j = 1, 2, \ldots, N$,*

$$\begin{array}{llll} \textit{if } p_j < \pi(\beta), & \textit{then} \quad s_j \geq 0\,, & l_j(\beta) = g_j + \beta s_j\,, & w_j = 0 \\ \textit{if } p_j > \pi(\beta), & \textit{then} & l_j(\beta) = 0 & w_j = g_j + \beta s_j\,. \end{array}$$

(iii) *If, in addition h is a norm and $\lambda^\pi(\beta) \neq 0$, then s has length 1 in the dual norm. Furthermore, if the dual norm h_D satisfies the conditions in Lemma 9(iii), then the statement in (ii) also includes that $s_j = 0$ whenever $p_j \geq \pi(\beta)$.*

Proof. The first item is straightforward from the optimality conditions of (17)$_\beta$ and the identity

$$l_\pi(\beta) = g - w^\pi(\beta) + \beta s\,.$$

Consider $j = 1, 2, \ldots, N$. If $p_j < \pi(\beta)$, then $\lambda^\pi_j(\beta) = \pi(\beta) - p_j > 0$. The complementarity condition between λ and w implies that $w^\pi_j = 0$ and, hence, $l_j(\beta) = g_j + \beta s_j$.

For the case $p_j > \pi(\beta)$, we have that $\pi(\beta) - p_j < 0 \leq \lambda^\pi_j(\beta)$. Again, the complementarity condition between this constraint and the Lagrange multiplier l implies that $l_j(\beta) = 0$ and $w_j(\beta) = g_j + \beta s_j$, showing item (ii).

Finally, when h is a norm and $\lambda^\pi(\beta) \neq 0$, from Lemma 9, we have that $h_D(s) = 1$, and this concludes the proof. ∎

It is worth noting that the property required for the norm in Lemma 9(iii) is satisfied by the ℓ_∞-norm, but *not by the ℓ_1-norm*. This is the reason why in our numerical results the regularized EPECs are defined using the former option.

Like in Theorem 5, we now consider convergence of a sequence of approximations as the parameter β tends to zero, now from the primal point of view.

Theorem 11 (Behavior of regularized primal ISO problems) *Given the marginal price $p_{j_{\mathtt{mg}}}$ from Proposition 1, consider the index-sets*

$$J^- = \{j : p_j < p_{j_{\mathtt{mg}}}\} \quad \text{and} \quad J^+ = \{j : p_j > p_{j_{\mathtt{mg}}}\}.$$

Let $\{(l(\beta), w(\beta), s(\beta))\}$ be any sequence parameterized by $\beta > 0$, where $(l(\beta), w(\beta))$ solves (17)$_\beta$ and $s(\beta) = \frac{l(\beta)+w(\beta)-g}{\beta}$. Under the assumptions in Proposition 10, the following holds.

(i) The sequence $\{(l(\beta), w(\beta), s(\beta))\}$ is bounded.

(ii) Any accumulation point l^{acc} of $\{l(\beta)\}$ solves the primal ISO problem (5).

(iii) There exists $M > 0$ such that for $\beta > 0$ sufficiently small,

$$\begin{array}{llll} j \in J^- \implies & l_j(\beta) = g_j + \beta s_j(\beta), & w_j(\beta) = 0, & |l_j(\beta) - g_j| \leq M\beta \\ j \in J^+ \implies & l_j(\beta) = 0, & w_j(\beta) = g_j + \beta s_j(\beta), & |w_j(\beta) - g_j| \leq M\beta. \end{array}$$

(iv) If, in addition h is a norm satisfying the conditions in Lemma 9(iii), then the statement above can be refined by taking $M = 1$, which implies that $w_j(\beta) = g_j$ for $j \in J^+$.

Furthermore, the sequences $\{l_j(\beta)\}$ and $\{w_j(\beta)\}$ converge for any $j \in J^- \cup J^+$.

Proof. Consider a sequence of dual solutions $\Big\{(\pi(\beta), \lambda^\pi(\beta))\Big\}$, shown to be convergent in Proposition 10. Then $p_{j_{\mathtt{mg}}} = \lim_{\beta\to 0} \pi(\beta)$, with $s(\beta) \in \partial h(\lambda^\pi(\beta))$. Since the sequence $\{\lambda^\pi(\beta)\}$ is bounded and h is convex, we have that the family of subdifferentials $\{\partial h(\lambda^\pi(\beta))\}$ is uniformly bounded, and so, the sequence $\{s(\beta)\}$ is bounded: there exists $M > 0$ such that

$$\|s(\beta)\| \leq M.$$

On the other hand, it is clear that $l(\beta)$ is feasible for problem (5) and since the feasible set of this problem is bounded, we have that the sequence $\{l(\beta)\}$ is bounded. Then, from $w(\beta) = g - l(\beta) + \beta s(\beta)$, we have that the sequence $\{w(\beta)\}$ is also bounded. Item (i) is established.

In order to prove item (ii), note that $h^*(s(\beta)) = s(\beta)^\top \lambda^\pi(\beta) - h(\lambda^\pi(\beta))$ implies that the sequence $\{h^*(s(\beta))\}$ is bounded, and hence, $\beta h^*(s(\beta)) \to 0$, as $\beta \to 0$. Now, considering the strong duality condition for (17)$_\beta$ – (10)$_\beta$

$$\pi(\beta)d - \lambda^\pi(\beta)^\top g - \beta h(\lambda^\pi(\beta)) = p^\top l(\beta) + \beta h^*(s(\beta))$$

and passing to the limit as $\beta \to 0$, taking a convergent subsequence if necessary, we have that

$$p_{j_{\mathtt{mg}}} d - \lambda_{p_{j_{\mathtt{mg}}}}^\top g = p^\top l^{\mathrm{acc}}.$$

This shows that the strong duality condition also holds for (5) – (7), and since l^{acc} is feasible for (5), item (ii) follows.

Finally, note that, letting $p^- = \max_{j\in J^-} p_j$ and $p^+ = \min_{j\in J^+} p_j$, we have that

$$p^- < p_{j_{\mathtt{mg}}} < p^+.$$

Therefore, for $\beta > 0$ small enough,

$$p^- < \pi(\beta) < p^+.$$

The final statement is straightforward from Proposition 10, Lemma 9, and the fact that $0 \leq s_j(\beta) \leq h_D(s_j(\beta) = 1$, for $j \in J^-$. ■

Most of the items in the theorem above are of asymptotic nature. A remarkable exception is item (iii), that characterizes the optimal dispatch for all small β. The characterization does not involve the marginal agents because, similarly to the situation pointed out in Remark 2 for the original primal problem ($\beta = 0$), there is an ambiguity created by the ISO's indifference that arises when more than one agent bids the same marginal price.

We conclude our theoretical analysis gathering the results specific for the ℓ_∞-norm, which is polyhedral and satisfies not only Assumptions 3 and 8, but also the condition given in Lemma 9(iii).

Corollary 12 (Summary of theory for the ℓ_∞-regularization) *With the notation and assumptions in Proposition 1, Theorems 5 and 11, consider the regularized dual ISO problem obtained with $h = \|\cdot\|_\infty$. The sequence $\{(\pi(\beta), \lambda^\pi(\beta))\}$ of solutions to the dual version (10)$_\beta$ satisfies the following:*

$$\left.\begin{array}{l}\lim_{\beta\to 0} \pi(\beta) = p_{j_{\mathtt{mg}}} \\ \lim_{\beta\to 0} \|\lambda^\pi(\beta)\|_\infty = \max(p_{j_{\mathtt{mg}}} - p_{j_1}, 0)\end{array}\right\} \text{ provide minimal-norm solutions to (7).}$$

In addition, for any $\beta > 0$ sufficiently small, the pair $(l(\beta), w(\beta))$ solving the primal version (18) *is such that*

$$l_j(\beta) = \begin{cases} g_j & \text{if } p_j > p_{\min}, j \in J^- \\ 0 & \text{if } j \in J^+, \end{cases}$$

where $p_{\min} = \min_j p_j$. Finally, The marginal dispatch completes the demand, following an arbitrary distribution among the marginal agents, if there is more than one bidding $p_{j_{\mathtt{mg}}}$, as noted in Remark 2. □

Proof. The statements follow from the previous results. Then only exception concerns the value of $l_j(\beta)$ for indices j such that $p_j > p_{\min}$ and $j \in J^-$. For such j-indices, Theorem 11(iii) states that

$$l_j(\beta) = g_j + \beta s_j.$$

Then, from (18), it is easy to check, by optimality arguments, that $s_j = 0$ for any j such that $p_j > p_{\min}$. ■

Remark 13 In the proofs above, it is easy to see that $\sum s_j = 1$. Then, in case that there exists only one index j such that $p_j > p_{\min}$, this would force $s_j = 1$, which in turn implies that $l_j(\beta) = g_j + \beta$. When the index is not unique, there is no unique solution to (18), yet we can always choose one j that assigns a dispatch $g_j + \beta$ to one of the least expensive units.

5 Putting the method in perspective and model extensions

The formulation considered so far sheds light on the difficulties that need to be addressed when solving EPECs. Thanks to the insight gained with the stylized ISO problem, we can now present and analyze the regularization scheme in a general setting.

5.1 Minimality properties of regularized multipliers

It is convenient to adopt a compact notation, in which each agent i has variables x^i, and minimizes a function f_i that depends on the ISO's primal and dual decisions, denoted by y and μ, respectively. Operational constraints in the upper level are denoted by $G^i(x^i) \le 0$. Gathering all the agents decisions in the vector x, the lower level problem solved by the ISO is an abstract linear program, parameterized by x. Accordingly, consider the general EPEC that results from agents $i = 1, \ldots, N$ solving the following problem:

$$\left\{\begin{array}{lll} \min\limits_{x^i} & f^i(x^i, y, \mu) & \\ \text{s.t.} & G^i(x^i) \le 0 & \\ & y \text{ solves} & \left\{\begin{array}{ll} \min\limits_{y} & c(x)^\top y \\ \text{s.t.} & y \ge 0 \\ & By = b(x) \quad (\mu)\,. \end{array}\right. \end{array}\right. \tag{21}$$

To see that our initial problem (6) is a particular instance of (21), it suffices to make the following identifications for the upper and lower level variables:

$$x^i = (p_i, g_i)\,, \quad \text{and} \quad y = (\ell, w) \text{ for a slack variable } w \ge 0.$$

Identification for the ith agent's objective and constraints functions are straightforward, while the ISO problem amounts to taking

$$c(x) = (p, 0)\,, \quad B = \begin{bmatrix} -I_N & -I_N \\ \mathbf{1}^\top & 0 \end{bmatrix}\,, \quad \text{and} \quad b(x) = \begin{pmatrix} -g \\ d \end{pmatrix}\,.$$

Similarly to the procedure in Proposition 6, we modify the lower-level problem by means of the conjugate h^* of a convex function h. In order to regularize only some of the multipliers, we make use of a row-matrix P to project the dual variable μ onto some of its components (for the stylized problem, P has null components except for those indices corresponding to λ, where the the entry is 1, so that $P\mu = \lambda$).

Proposition 14 (Regularized primal problem) *Consider the linear program*

$$\begin{cases} \min\limits_{y \geq 0} & c^\top y \\ \text{s.t.} & By = b\,. \end{cases}$$

Then, under the assumptions in Proposition 6, to the dual regularization

$$\begin{cases} \max\limits_{\mu} & b^\top \mu - \beta h(P\mu) \\ \text{s.t.} & B^\top \mu \leq c\,, \end{cases} \tag{22}$$

corresponds the primal problem

$$\begin{cases} \min\limits_{y \geq 0, s} & c^\top y + \beta h^*(s) \\ \text{s.t.} & By + \beta P^\top s = b\,. \end{cases} \tag{23}$$

Proof. The Lagrangian $L(\mu, y) = -b^\top \mu + \beta h(P\mu) + y^\top (B^\top \mu - c)$ gives for (22) the optimality conditions

$$\exists\Big(\mu, y, s \in \partial h(P\mu)\Big) \text{ s.t. } \begin{cases} 0 = -b + \beta P^\top s + By \\ 0 = y^\top (B^\top \mu - c)\,, \qquad y \geq 0\,, B^\top \mu \leq c\,. \end{cases} \tag{24}$$

Using the equivalence $s \in \partial h(z) \iff s^\top z = h(z) + h^*(s)$, for the objective function in (22), we then obtain that

$$\begin{aligned} -b^\top \mu + \beta h(P\mu) &= -b^\top \mu + \beta s^\top (P\mu) - \beta h^*(s) \\ &= (-b + \beta P^\top s)^\top \mu - \beta h^*(s) \\ &= -(By)^\top \mu - \beta h^*(s) \\ &= -c^\top y - \beta h^*(s)\,, \end{aligned}$$

where the third equality is by the first relation in (24), and the last is by the complementarity relation in (24). Taking into accont the change in sign, this completes the proof that (23) is the primal problem that corresponds to (22). ■

In our numerical experience we observed that, rather than solving the single problem (21), it is preferable to solve a sequence of regularized EPECs, driving to zero the values of the regularization parameter β. In this approach, a run is warm-started taking as initial values for the upper- and lower-level variables the output of the run with the previous β. This technique has the effect of stabilizing the solutions found by PATH in a manner that is beneficial in all cases, not only when the multiplier in (21) is not unique (for the simple ISO problem, this situation arises when the marginal agent is dispatched at its bid, recall Proposition 1 and comments that follow).

Along the lines of Corollary 12, our next result justifies the empirical observation stated above. It exhibits that the regularization induced by Proposition 14 provides a selection mechanism that is continuous in β and converges to a special equilibrium, specified below.

Theorem 15 (Asymptotic properties of regularized multipliers) *Consider the family of regularized EPECs*

$$\text{for decreasing } \beta > 0, \quad \begin{cases} \min\limits_{x^i} & f^i(x^i, y, \mu) \\ \text{s.t.} & g^i(x^i) \leq 0 \\ & y \text{ solves } \begin{cases} \min\limits_{y,s} & c(x)^\top y + \beta h^*(s) \\ \text{s.t.} & y \geq 0\,, \\ & By + \beta P^\top s = b(x) \quad (\mu)\,. \end{cases} \end{cases}$$

Let $\mu(\beta)$ be a dual solution to the (regularized) lower-level problem above. Then, the following holds.

1. *As $\beta \to 0$, the sequences $\{h(P\mu(\beta))\}$ and $\{b^\top \mu(\beta)\}$ are convergent. Furthermore, for any dual solution μ^* of*

$$\begin{cases} \min\limits_{y} & c(x)^\top y \\ \text{s.t.} & y \geq 0 \\ & By - b(x) = 0 \quad (\mu)\,, \end{cases} \tag{25}$$

   it holds that $\lim_{\beta\to 0} h(P\mu(\beta)) \leq h(P\mu^*)$ and $\lim_{\beta\to 0} b^\top \mu(\beta) = b^\top \mu^*$.

2. *Every accumulation point $\hat{\mu}$ of $\{\mu(\beta)\}$ is a dual solution to (25) satisfying*

$$h(P\hat{\mu}) = \min\{h(P\mu^*) : \mu^* \textit{dual solution to } (25)\} \tag{26}$$

Proof. We drop the dependence on x in what follows.

Note that the dual of the regularized lower level problem

$$\begin{cases} \min\limits_{(y,s)} & c^\top y + \beta h^*(s) \\ \text{s.t.} & y \geq 0\,, s \text{ free} \\ & By + \beta P^\top s - b = 0\,, \end{cases} \tag{27}$$

is given by

$$\begin{cases} \max\limits_{\mu} & b^\top \mu - \beta h(P\mu) \\ \text{s.t.} & B^\top \mu \leq c. \end{cases} \tag{28}$$

Taking $\beta = 0$, (28) also gives the dual problem to (25).

Let μ^* be any dual solution in (25), fixed for now. The fact that μ^* is a dual solution in (25) implies that it is feasible in (28). Hence,

$$b^\top \mu^* - \beta h(P\mu^*) \leq b^\top \mu(\beta) - \beta h(P\mu(\beta)).$$

On the other hand, by the optimality of μ^* in (28) for $\beta = 0$ and the fact that $\mu(\beta)$ is feasible for it (since it solves (28) for $\beta > 0$), it holds that

$$b^\top \mu(\beta) \leq b^\top \mu^*\,.$$

Combining the last two relations, we obtain that for all $\beta > 0$,

$$b^\top \mu^* - \beta h(P\mu^*) \leq b^\top \mu(\beta) - \beta h(P\mu(\beta)) \leq b^\top \mu^* - \beta h(P\mu(\beta)). \tag{29}$$

Then, from the first and last terms of (29), we conclude that

$$0 \leq h(P\mu(\beta)) \leq h(P\mu^*),$$

which shows that the sequence $\{h(P\mu(\beta))\}$ is bounded. Then, taking the limit as $\beta \to 0$ in (29), we also conclude that

$$\lim_{\beta\to 0} b^\top \mu(\beta) = b^\top \mu^*.$$

For showing convergence of $\{h(P\mu(\beta))\}$ as $\beta \to 0$, consider any subsequences $\beta_k \to 0$, and $\{h(P\mu(\beta_k)\} \to h(P\mu(\beta))$, passing onto subsequence if necessary (recall that $\{h(P\mu(\beta))\}$ had been proven to be bounded). For any $\beta > 0$, by the optimality of $\mu(\beta)$ in the corresponding problem, and as explained in more detail in related considerations above, we have that

$$b^\top \mu(\beta_k) - \beta h(P\mu(\beta_k)) \leq b^\top \mu(\beta) - \beta h(P\mu(\beta)) \leq b^\top \mu^* - \beta h(P\mu(\beta)).$$

Then, taking the limit in k, we obtain that

$$b^\top \mu^* - \beta \lim_k h(P\mu(\beta_k)) \leq b^\top \mu^* - \beta h(P\mu(\beta)),$$

which implies that

$$h(P\mu(\beta)) \leq \liminf_{\beta\to 0} h(P\mu(\beta)).$$

Thus,

$$\limsup_{\beta\to 0} h(P\mu(\beta)) \leq \liminf_{\beta\to 0} h(P\mu(\beta)) \leq h(P\mu^*),$$

which shows the convergence of $\{h(P\mu(\beta))\}$.

The second item follows directly from the results established above. ■

Remark 16

1. An issue not discussed in Theorem 15 is the existence of accumulation points of $\{\mu(\beta)\}$. This depends on the structure of the particular model. For example, whenever B has full rank, as in the case discused in Section 2, it is clear that the dual feasible set of (25) is bounded and thus so is $\{\mu(\beta)\}$. For other cases, the boundedness of $\{h(P\mu(\beta))\}$ and $\{b^\top\mu(\beta)\}$ combined with some additional assumptions, like $b > 0$, might imply the boundedness of $\{\mu(\beta)\}$. Generally, this is clearly model-dependent.
2. Item (ii) of Theorem 15 implies that every accumulation point $\hat{\mu}$ of $\{\mu(\beta)\}$ has the minimal projection $P\hat{\mu}$ among all dual solution to (25). This result, with Assumption 3 and taking into account the special structure of the model in Section 2, reproduces the minimal price result shown in Theorem 5, item (iii).
3. Theorem 15, item (i), shows that the sequence $\{h(P\mu(\beta))\}$ is convergent. However, under additional assumptions, like h being strictly convex and Assumption 3, it is possible to show that actually the projected sequence $\{P\mu(\beta)\}$ is convergent as well.

The general format of problem (25) makes it possible for the ISO to include network constraints when defining the dispatch. Such would be the case in the US, where day-ahead markets rely on the ISO solving very detailed security-constrained models. It is explained in the thourough discussion in [18] that such a feature originated in a pre-existing integrated structure of Regional Transmission Operators. Europe focused instead on a single-market implementation supported by power exchanges (for such a market, (25) may not include transmission constraints). The level of detail introduced in the lower-level problem depends on the market configuration. This is the reason for our abstract setting (25), which encompasses various market formats, including future ones, with market bids and clearing systems adapted to new agents dealing with storage and renewable intermittent sources of energy.

5.2 Bidding to more than one market

A more complex configuration appears when agents place bids in several markets, handled by separate ISOs, but connected by some transmission line. Instead of (25), in the lower level there are different ISOs' optimization problems and a constraint coupling all the markets through a transmission line. The resulting EPEC, which has in the lower level a variational inequality and no longer one optimization problem, can still be tackled following our approach developed above.

Suppose there are K markets, handled by separate ISOs. Then, for $k \in K$, ISO$_k$ determines the market price μ_k, to be paid for the k-th market demand d_k, as well as the dispatch, denoted by y_k. Given matrices T_k suitably defined to represent how agents connect to the transmission line, with capacity κ, then

$$\sum_{k=1}^{K} T_k y_k \leq \kappa \quad (\eta)$$

is the constraint coupling the markets. The value of the dual variable η represents the unit fee to be paid for transporting energy through the transmission line.

In this setting, instead of (21), we have the following multi-market EPEC

$$\left\{\begin{array}{lll} \min\limits_{x^i} & f^i(x^i, y, \mu, \eta) & \\ \text{s.t.} & G^i(x^i) \leq 0 & \\ & \text{for } k = 1, \dots, K \quad y_k \text{ solves} & \left\{\begin{array}{ll} \min\limits_{y_k} & c_k(x)^\top y_k \\ \text{s.t.} & y_k \geq 0 \\ & B_k y_k = b_k(x) \quad (\mu_k) \end{array}\right. \\ & \text{and } \displaystyle\sum_{k=1}^{K} T_k y_k \leq \kappa \quad (\eta) & . \end{array}\right. \tag{30}$$

The corresponding family of regularized EPECs is obtained by replacing the K ISO problems by their regularized formulations, as in (23).

The upper-level variables and functions in (30) are defined to represent the multi-market setting. As an illustration, consider the situation depicted by the diagram in Figure 1. There are two markets, with respective demands d_1 and d_2, so $K = 2$. The N generating companies are distributed into two sets, I_1 and I_2, gathering agents in each market. Bids can be placed in both markets, noting that if, for instance, agent $i \in I_1$ bids to market 2, the exported energy goes through the transmission line, and incurs an additional expense, depending on η. The generation g_i is distributed into two parts, a fraction $\theta_i \in [0, 1]$ of g_i will be sold locally, in the market where the agent is located. The remaining generation $(1 - \theta_i)g_i$ is offered to the other market, to be exported through the transmission line, of capacity κ.

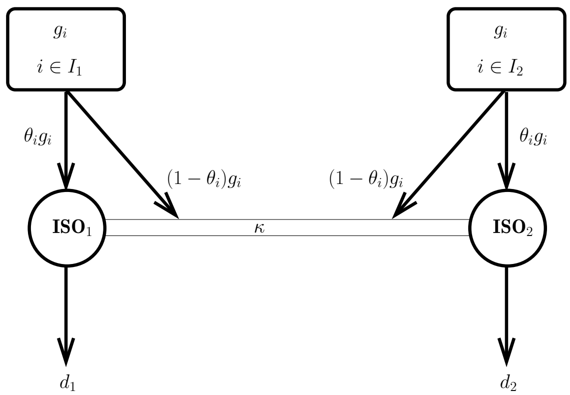

Figure 1: Two markets with one transmission line. For $\theta_i \in [0, 1]$, agent i in market 1 bids locally the generation $\theta_i g_i$ at price p_i, and offers to export to market 2 the fraction $(1 - \theta_i)g_i$ at price q_i. Exchanges between markets are limited by the capacity κ of the transmission line.

The agent decision variables are $x^i := (g_i, \theta_i, p_i, q_i)$, given that quantities and prices are specified as follows:

– in its local market, agent i bids $\theta_i g_i$ at price p_i

– to the other market, agent i offers to export $(1-\theta_i)g_i$ at price q_i through the transmission network.

Accordingly, when for example $i \in I_1$, the objective function in the upper level is

$$f^i(x^i, y, \mu) = (\pi_1 - \varphi_1)l_1 + (\pi_2 - \varphi_1 - \eta)l_2 .$$

With respect to the formulation with only one market, the only difference is that in the lower level now there is a variational inequality, grouping the optimality conditions of both markets and the capacity constraint of the transmission line,

$$\sum_{k\in\{1,2\}} T_k y_k \le \kappa \quad (\eta) ,$$

The lower-level decision variables are $\left((y_k, \mu_k)_{k=1}^2, \eta\right)$, where the multiplier η defines the charge that agents have to pay for each unit of energy transmitted through the line.

5.3 Numerical assessment

In the single-market EPEC model (21), the proxy approximation (9) for the price function $P(g, p, l)$ yields the upper objective function $f^i(l_i, \pi) = (\pi - \varphi_i)l_i$ for each agent's problem. In the bottom level, the regularized ISO solves problem (18), with the ℓ_∞-norm in the dual.

There are a number of ways to tackle the non-convexity that appears in EPECs. With our proposal, the EPEC can be handled computationally, for example by the PATH solver [9]. A direct solution of (21) leads to a computationally challenging problem with severely non-convex/disjunctive complementarity constraints, [31], [24, Chapter 7.3]. It is important to keep in mind that, while being a stationary point for the complementarity system residual, the solution provided by PATH may not be an equilibrium (having a bilinear upper objective and bilevel formulation makes the resulting mixed complementarity problem non-monotone). One of our conclusions is that the output is very sensitive to the initial input. Experimentation and the resulting appropriate tuning of the regularization parameter, implementing the mechanism of warm starts for decreasing values of β, leads to useful, optimal/equilibrium solutions. The results below are meant to illustrate these conclusions for the given problem.

5.3.1 Benchmark information and values at equilibrium

The family of regularized EPECs considers decreasing values of the parameter,

$$\beta = (0.2 - 0.02j)d\,, \text{for } j = 0, 1, \ldots, 10\,, \qquad \text{and a given demand } d\,, \tag{31}$$

which amounts to the ISO having access to a "battery" that covers 20%, 18%,. . . , 2%, and 0% of the market's demand. The results provided by PATH for $\beta = 0$ (after the sequence of regularizations and warm starts with $\beta > 0$) are then compared to the output obtained by PATH for the original EPEC (21), without regularization. Both approaches (without and with regularization) use the same starting point, randomly taken in the generators' bid feasible sets.

Thanks to the theoretical results shown for the stylized model in Proposition 1 and Corollary 12, we can make a thorough assessment of the output and gauge its quality. In particular we show empirically, over thousands of starting points in several experiments, that the output of PATH after regularizations succeeds in finding genuine equilibria much more often than the direct approach.

Given as input the number of players N, the marginal cost as well as the maximum bidding price and generation increase with the index of the player:

$$\text{for } j = 1, \ldots, N \text{ we set }\,, \quad \varphi_j = \frac{j}{2}\,, \quad p_j^{\max} = 2\varphi_j = j\,, \quad g_j^{\max} = j\,.$$

The index of the marginal agent $j_{\mathtt{mg}} \le N$ is also an input, and we consider the following two values for the demand:

$$D := \sum_{j=1}^{j_{\mathtt{mg}}-1} g_j^{\max} + 0.5 g_{j_{\mathtt{mg}}}^{\max} \quad \text{and} \quad \tilde{D} := \sum_{j=1}^{j_{\mathtt{mg}}-1} g_j^{\max} + 1.0 g_{j_{\mathtt{mg}}}^{\max}\,.$$

To understand the consequences of this setting, recall from Proposition 1 that generators bidding a price cheaper than the marginal one are dispatched by the ISO at their maximal capacity. The value chosen for the demand D absorbs all the generation capacity of infra-marginal agents, but not that of the marginal agent. With $\tilde{D}$, by contrast, the marginal agent is dispatched up to its bid. Since by Proposition 1, in the latter case the price is not unique, we expect the runs with demand $\tilde{D}$ to be more challenging.

At least for sufficiently small β, by Corollary 12 and Proposition 1. the ISO values at equilibrium are

$$\begin{array}{lll} \text{dispatch} & l_{j_k}(\beta) = \left\{ \begin{array}{ll} g_{j_k}^{\max} & \text{for } k = 1, \ldots, \mathtt{mg}-1 \\ 0.5(\text{or } 1) g_{j_{\mathtt{mg}}}^{\max} & \text{for } k = \mathtt{mg} \text{ (if the demand is } D \text{ or } \tilde{D}) \\ 0 & \text{for } k = \mathtt{mg}+1, \ldots, N\,, \text{and} \end{array} \right. \\ \text{price} & \pi(\beta) \in [j_{\mathtt{mg}}/2, j_{\mathtt{mg}+1}/2]\,, \text{ the interval of bidding prices of agent } j_{\mathtt{mg}}. \end{array} \tag{32}$$

The result is also valid for the original EPEC (21). Accordingly, both without and with regularization, runs providing a nonzero dispatch for an agent with index larger than the marginal one, or with a price too large cannot correspond to an equilibrium.

We coded the model in GAMS and PATH to directly solve the EPEC (21). Thanks to the Extended Mathematical Programming (EMP) extension available in GAMS, instead of manually writing down the complementarity system, replicating variables to make the system square, it suffices to write the model on a high level, indicating which variables are owned by the agents and which ones by the ISO.

The EMP framework vastly facilitates a direct formulation of multiple optimization problems with equilibrium constraints such as (21). For full details on the different features of the extension, we refer to [27]. Here we just give an overview for (21). We consider generating companies act as optimization agents in the upper level, with objective function `OBJ(i)` representing f^i, and variable `X(pn, i)` representing x^i, where `pn` refers to name of the component (price p_i or generation g_i in the vector x^i). Similarly, in the lower level, the so-called "equilibrium agent" solves a variational inequality derived from the optimality conditions of (25), defined in GAMS as `defVI`, on variables `ZZ(dualv,i)`, representing the dispatch $DIS(i)$ of each agent (l_i), their marginal rent (λ_i) and the market price (π). After defining the corresponding functions and variables in GAMS, the EMP code section is simply

```
equilibrium
implicit  DISPATCH(i),PRICE,defDISPATCH(i),defPRICE
min OBJ(i) s.t. X(pn,i),defOBJ(i)
vi defVI(dualv,i),ZZ(dualv,i)
```

where the `implicit` line contains lower level variables that appear in the upper level problem.

All the experiments were performed on a notebook running under Ubuntu 18.04.4 LTS, with i7 CPU 1.90GHzx8 cores and 31.3GiB of memory. The parameter settings for PATH is the default provided by GAMS.

5.3.2 Single-market results

We start considering a setting where $N = 3$ agents bid to one market and set the marginal agent to be $j_{\mathtt{mg}} = 2$. Even for this very simple market instance, to obtain a genuine equilibrium when solving a complementarity system with PATH is very delicate. The proposed regularization is clearly beneficial, even though it also fails sometimes.

As the output is very dependent on the initial point, the experiment repeats the runs with 2500 different starting bids $\left(p_j^0, g_j^0\right)$, randomly generated in $[\varphi_j, p_j^{\max}] \times [0, g_j^{\max}]$. Each run takes a starting point and first calls PATH to solve the original EPEC (21). Then, from the same starting point, PATH is called eleven times, for decreasing values of β, as in (31), with $d \in \{D, \tilde{D}\}$. Barring the first run ($\beta = 0.2d$), the regularization procedure warm-starts each run using as initial point the output of the run with the previous value of β. Altogether, without and with regularization, obtaining the output for the 2500 starting points involves running PATH 30000 times ($30000 = 2500 \times 12$ values of β). In general, PATH runs were very fast, taking less than 15 minutes to complete the full experiment.

As already mentioned, not all solutions obtained by PATH are an equilibrium. In view of (32), a run is declared a failure if an agent j with $j > \mathtt{mg}$ is dispatched, or if the computed price is larger than $1.05 j_{\mathtt{mg}+1}/2$. A run can also be declared a failure by PATH itself, if a local solution was not found. Over 5000 runs, the decisions over the non-dispatched agents was always correct with both approaches. By contrast, the original (21) miscalculated the equilibrium price more than half of the runs, while with regularization the output was incorrect about 25% of the runs. Failures occured mostly with the demand set to $\tilde{D}$ (80% and 25% of the runs without and with regularization, respectively).

Figure 2 shows the dispatch and generation bids computed in mean for each value of β.

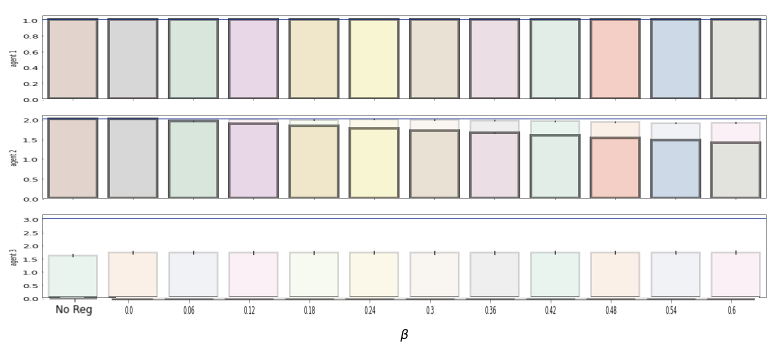

Figure 2: Mean dispatch per agent, computed by PATH for different values of β when $d = \tilde{D} = 3$. The shaded areas in the background illustrate the generation bids of the agents and the left-most column corresponds to the runs without regularization. Agent 3, the most expensive one, is never dispatched. The ISO dispatches agent 1 at maximum capacity, and completes the demand with generation from agent 2, using its reserve from the "battery", which decreases with β.

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The graph in Figure 2 confirms the results from (32), represented in a condensed form explained below. The ith row of plots shows the output for the ith agent, with bars in different colors corresponding to different values of β. The leftmost bar shows the variant without regularization ($\beta = 0$). In each bar, the bright color indicates the dispatch while the shaded area is the generation bid by the agent. For instance, we see in the rightmost columns of bars, with the largest value of β, that agent 1 is dispatched to its bid (there is no shaded area), while agent 2 is dispatched below its bid (there is a shaded area). The shaded area of agent 2 becomes smaller as β decreases, as expected (the "reserve" available to complete the demand is smaller). For all values of β, the third row of bars confirms that agent 3 is never dispatched, as all bars are shaded, there is no bright color. The behaviour with $d = D = 2$ was similar.

In Figure 2 notice that, as β decreases, the regularized dispatch of agent 2 progressively increases to the optimal value ($l_2^* = d - 1$). The "battery" provided by the regularization accounts for the difference. The impact of the regularization in the decision making process is perceptible in Figure 3, with the mean composition of the dispatch, for each value of β.

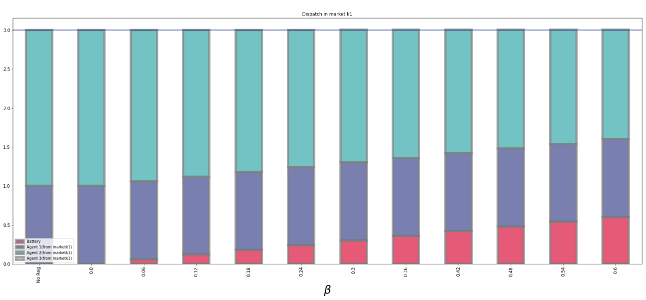

Figure 3: Mean dispatch composition when $d = \tilde{D} = 3$. Agent 1 in dark blue is dispatched to the maximum and agent 2 in light blue completes the demand (the battery in displayed in red). The left-most column corresponds to the runs without regularization, followed for increasing values of β (the behaviour with $d = D = 2$ was similar).

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Other than failing less often, there is not a noticeable difference in the dispatch computed with and without regularization. The beneficial effect of our proposal is perceptible on the dual variables, where it acts as a stabilizing mechanism. Figure 4 gives an illustration, with the values of the prices output by PATH with both approaches. The points correspond to 10000 runs, repeating the experiment 4 times, considering the maximum bidding price of the marginal agent is 1.25 or 1.5, and varying the demand in $\{D, \tilde{D}\}$.

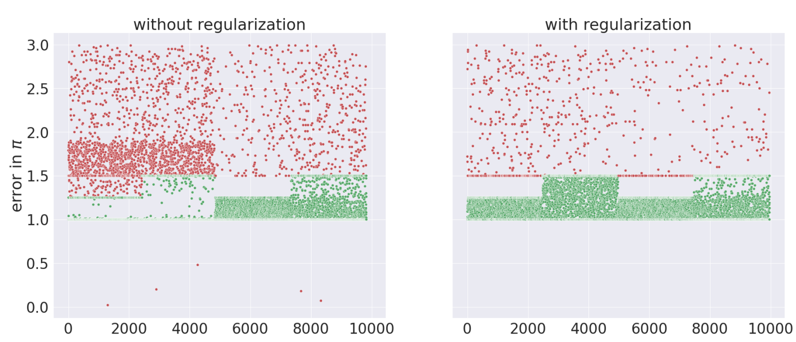

Figure 4: Prices found with both approaches (only runs with $\pi^* \leq 3$ are displayed, for better visibility). Each graph reports the results of four experiments, running for 2500 different starting points a problem with $(p_2^{\max}, d) \in \{(1.25, \tilde{D}), (1.50, \tilde{D}), (1.25, D), (1.50, D)\}$. Points in red cannot correspond to an equilibrium. We notice many more green points on the right graph, indicating the benefits of the regularization approach

Recall from (32) that the marginal agent, $j_{\mathtt{mg}} = 2$, could bid any price between 1 and its maximum price. The green points in the Figure 4 indicate that PATH yielded prices in the allowed range. Such was the case most of the times with the regularization, but not with the direct solution approach.

To illustrate the role of the marginal agent in the price determination, we consider the setting in Figure 4 having $p_2^{\max} = 1.25$ and demand $D = 2$. Since agent 1 is dispatched at value 1, agent 2 covers the residual demand, equal to 1 and the equilibrium price should be in the interval $[1, 1.25]$. We repeated 2500 runs, varying the marginal agent capacity

$$g_2^{\max} = 2 - 0.1j \quad \text{for} \quad j \in \{0, 1, \dots, 10\}\,.$$

The statistic for the equilibrium price is reported in Figure 5. There is a sharp increase in the price computed without regularization when the capacity coincides with the residual demand.

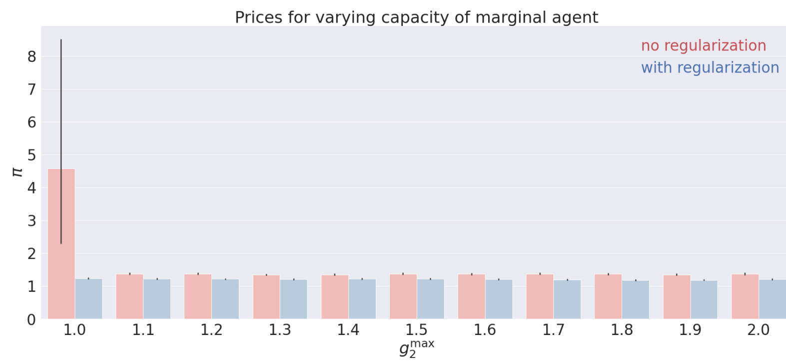

Figure 5: Prices found with both approaches, as the marginal agent maximum generation drops down to the value of the residual demand. Without regularization, equilibrium prices are always larger than without regularization. Notice the significant increase when $g_2^{\max} = 1$: without regularization, the computed price is on average larger than 4, and can reach values equal to 8, while with regularization the equilibrium price stays at its correct value, 1.25.

Table 1 informs on a final set of experiments for the single-market model, varying the number of agents between 3 and 30, selecting randomly the marginal agent, and with the two demand values, D and $\tilde{D}$. Each market configuration was run with both approaches using 2500 starting points.

Table 1: Type of output obtained with PATH when $N \in [3, 30]$ (over 40000 runs)

`PATH fails`		`Too small price`		`Too large price`	
original	regularized	original	regularized	original	regularized
224	175	3	2	15372	10202

The conclusion is similar to the case with $N = 3$ agents. More precisely, PATH appears to be very sensitive to the starting point, and the regularization increases the number of runs that can provide an equilibrium (because at the very least the computed price is within the bidding interval of the marginal agent, see (32)).

5.3.3 Two markets

In order to assess the multi-market EPEC (30) we consider the same three agents, but now distributed in two markets. Agent 1 is in market 1, while agents 2 and 3 are in market 2, so in Figure 1, $I_1 = \{1\}$ and $I_2 = \{2, 3\}$. Both markets face a demand $d_k = 2$. In market 1, as agent 1 generating capacity is 1, agents 2 and 3 must export to market 1 the missing energy. Also, since the total demand is 4 and agents 1 and 2 can generate 3 in total, agent 3 is always dispatched. We expect the equilibrium price not to exceed the $p_3^{\max} = 3$. The transmission line has a capacity varying in $\kappa = \{1.5, 3, 6\}$. In particular, $\kappa = 6$ is sufficiently large to render free the energy transmission between markets.

We repeated the same procedure, comparing for 2500 starting points the output of a direct solution of (30) with the regularization approach, solving the regularized EPEC for decreasing values of β. The total number of PATH solves was 90000 ($= 3 \times 2500 \times 12$), out of which PATH failed in 7780 runs, most of them when solving directly (30), without regularization (6672 times). The regularization proves again beneficial, as illustrated by Figures 6 and 7, with the prices π and η computed in the experiment.

Figure 6: Prices with both approaches (top and bottom) for markets 1 and 2 (left and right). Each graph reports the results of three experiments, running for 2500 different starting points a problem with $\kappa \in \{1.5, 3, 6\}$. Points in red cannot correspond to an equilibrium. The different length between the top and bottom graphs measures the larger number of failures in the top, without regularization, most of them occur when the line is congested ($\kappa = 1.5$). With regularization, there are many more green points, confirming once more the benefits of the regularization approach.

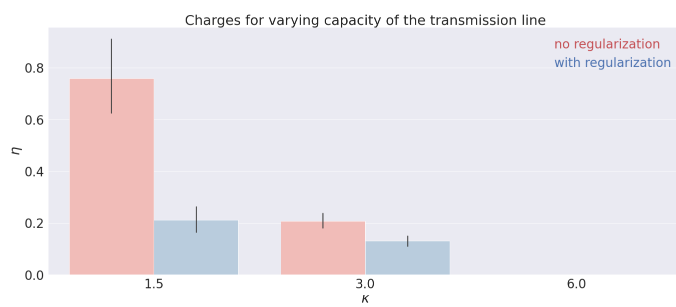

Figure 7: Statistics for the values found with both approaches for the transmission charge η, when the capacity line is $\kappa \in \{1.5, 3, 6\}$. Without regularization, the fee is larger and exhibits more variability (the length of the vertical black line in each bar), particularly when κ is small. As expected, for the larger value of κ, the transmission constraint is inactive and the value of η is zero with both approaches.

Concluding Remarks

We presented theoretical analysis pointing out some downsides of EPEC models in a general bilevel setting. While the considered instance is simple, it serves well to provide a clear insight that the model can choose the highest possible value for the marginal price, in reasonable market configurations (a situation certainly undesirable).

Our proposal of replacing the lower level ISO problem by a new regularization addresses this issue, as it ensures that in the limit the *minimal norm* price signal is produced. The regularized problem, which remains a linear programming problem if a polyhedral norm is employed, has interesting economic interpretations derived from analyzing both its dual and primal versions. Namely, the regularization can be seen as endowing the ISO with a small reserve that allows to control the marginal rent of the dispatched agents, and indirectly discourages the marginal agent from behaving strategically to increase the equilibrium price. The reserve can be thought of as being available out of the market, or simply being incorporated in the corrections that modify the generation when operating the system in real time.

Our theoretical analysis is complemented with a thorough numerical assessment. The experiments, designed to shed a light on the numerical difficulty inherent to solving EPECs, show the interest of the proposal as a stabilizing device that helps guiding the process towards an output that is usually an equilibrium, even if the mixed complementarity formulation resulting from EPEC is not monotone.

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