An Almost Exact Solution to the Min Completion Time Variance in a Single Machine

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Abstract

We consider a single machine scheduling problem to minimize the completion time variance of n jobs. This problem is known to be NP-hard and our contribution is to establish a novel bounding condition for a characterization of an optimal sequence. Specifically, we prove a necessary and sufficient condition (which can be verified in $O(n \log n)$) for the characterization of five scheduling positions in the optimal sequence. Applying this characterization, we propose a new approach to derive the highest lower bound for the minimal completion time variance, outperforming the existing bounds for this problem. The numerical tests indicate that the novel lower bound is, on average, less than 0.01% far away from the optimal solution, outperforming all the existing lower bounds on the minimum completion time variance.

Keywords: Scheduling; Single machine; Completion time variance; Optimal sequence characterization; Lower bound.

1. Introduction

The problem of minimizing the completion time variance (hereafter referred to as CTV) in a single machine consists of scheduling n non-preemptive jobs in order to minimize the variance of their completion times, where the machine can process at most one job at a time.

Formally, let $\mathcal{N} = \{1, 2, \ldots, n\}$ be a set of jobs, p_i the processing time of job i, and the jobs be numbered such that $p_1 \leq \ldots \leq p_n$. Define a sequence σ as a list of indexes (namely a permutation of n integers), labeled in accordance with the order of processing times. Therefore, jobs and indexes can be used interchangeably. For instance, a sequence $\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(n))$ refers to the case where $\sigma(1)$ is the index of the job processed in the first position, $\sigma(2)$ is the index of the job processed in the second position, $\sigma(n)$ is the index of the job processed in the last position. The dots in the middle refers to the fact that no further specification is given for the inner jobs. For a given sequence of jobs σ, define the completion time of job h under σ as $C_h(\sigma) = \sum_{j=1}^{h} p_{\sigma(j)}$ so that the mean completion time and the completion time variance are

$$\overline{C}(\sigma) = \frac{1}{n}\sum_{h=1}^{n} C_h(\sigma), \qquad \text{and} \qquad CTV(\sigma) = \frac{1}{n}\sum_{h=1}^{n}[C_h(\sigma) - \overline{C}(\sigma)]^2.$$

The problem consists in determining a sequence σ that minimizes $CTV(\sigma)$. This problem was introduced by Merten & Muller (1972a), who motivated the variance performance measure by computer file organization problems, where providing uniform response times to users is a desirable feature. Kanet (1981) later argued that the measure is applicable to any manufacturing process, targeting homogeneity in the level of service.

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After its first appearance, the problem of CTV minimization attracted the attention of the communities of discrete mathematics (Graham et al. 1979, Kubiak 1995) and operations research (Merten & Muller 1972b, Schrage 1975, Eilon & Chowdhury 1977, Kubiak 1993, Al-Turki et al. 2001) over the last half-a-century and resulted in major theoretical progress.

Specifically, in the opening paper, Merten & Muller (1972a) proposed a duality theorem by proving that the sequence minimizing the variance of completion time is antithetical to the sequence minimizing the waiting time variance (hereafter referred to as WTV).[1] Searching for necessary conditions that optimal sequences must satisfy, Eilon & Chowdhury (1977) proved that any CTV minimizer must be V-shaped; this means that jobs scheduled before the job with the shortest processing time p_1 are sorted in the descending order of the processing time while the ones after p_1 are scheduled in the ascending order. The V-shapeness of the optimal sequence has represented and still represents a cornerstone of theoretical research on this problem (it is also invoked in different sections of our undertaking).

Focusing on numerical resolution approaches, Kubiak (1993) proved that the minimum CTV problem is NP-hard, thereby motivating the use of exact pseudopolynomial algorithms, as well as efficient heuristics.[2] Apart from this algorithmic effort, a significant amount of attention has been given, foremost, to the derivation of lower bounds to the minimal CTV as well as to the characterization of dominance properties of the optimal sequence. In this respect, Schrage (1975) characterized the first position of the optimal sequence, by proving that the longest job is always scheduled first. Hall & Kubiak (1991) characterized the second and last positions of the optimal sequence, verifying that the second largest job must be placed last and the third largest job must be situated in the second position. Similarly, bounds for the position of the smallest job in the CTV problem are established by Manna & Prasad (1999).

Building on this stream of literature, the primary goal of this paper is to establish a novel bounding condition for a characterization of an optimal sequence, beyond the well-known positions discovered by Schrage (1975) and Hall & Kubiak (1991). In this vein, Schrage (1975) originally conjectured that there exists an optimal sequence of the form $(n, n-2, n-3, ..., n-1)$. However, Kanet (1981) proved by a counterexample that this sequence is not necessarily optimal.

Consistent with the conjecture of Schrage (1975), the counterexample of Kanet (1981) and the characterization of Hall & Kubiak (1991), we prove a necessary and sufficient condition (which can be verified in $O(n \log n)$) for the optimal sequence to be either of the form $(n, n-2, n-3, ..., n-4, n-1)$ or of the form $(n, n-2, n-4, \dots, n-3, n-1)$. Specifically, we show that when p_{n-1} is smaller or equal than a specific linear combination of $p_1, \dots, p_{n-2}$ (namely, $p_{n-2} + 2p_{n-3} + 0.5 \sum_{i=1}^{n-4} p_i$), there exists an optimal sequence which is either of the form $(n, n-2, n-3, \dots, n-4, n-1)$ or of the form $(n, n-2, n-4, \dots, n-3, n-1)$. By contrast, when p_{n-1} is strictly greater than the aforementioned linear combination, then there exists an optimal sequence which is of the form $(n, n-2, n-3, n-4, \dots, n-1)$. Therefore, this $O(n \log n)$ assessment of the value of p_{n-1} allows establishing five positions of the optimal sequence. As elaborated in the second part of the paper, the characterization of these five positions has a substantial impact on the improvement of the existing lower bounds on the optimal CTV. Next, to fully exploit the knowledge of these five positions, a

[1] For any sequence of the form $\sigma = (i_1, i_2, i_3, ..., i_{n-1}, i_n)$, there is an antithetical sequence of the form $\sigma_A = (i_n, i_{n-1}, ..., i_3, i_2, i_1)$, so that solving the minimum CTV problem indirectly provides a solution to the minimum WTV problem.

[2] A branch and bound algorithm to minimize CTV is given in (Viswanathkumar & Srinivasan 2003) and a tabu search-based solution is developed in (Al-Turki et al. 2001). (Kubiak 1995) formulates the CTV problem as a problem of maximizing a zero-one quadratic function which is a sub-modular function with a special cost structure and derived a pseudo-polynomial algorithm. Further pseudo-polynomial algorithms and fast polynomial approximation schemes are given by Cheng & Kovalyov (1996), Kubiak et al. (2002), Manna & Prasad (1997).

new quadratic programming approach to derive a lower bound for the minimum CTV is proposed, resulting in the highest known lower bound so far.

On the computational side, an extensive numerical experiment involving more than 1000 solved instances with up to 300 jobs is conducted to illustrate the value of the proposed bound and the algorithmic effort to compute it. This shows that the novel lower bound is on average less than 0.01% far away from the variance of the optimal solution.[3]

The rest of this paper is organized as follows. In section 2 we focus on the characterization of new optimal positions, introducing the two fundamental theorems of this paper. In Section 3 we use these novel positions to introduce a new lower bound that improves the current benchmark from Viswanathkumar & Srinivasan (2003). In Section 4, a comprehensive computational analysis is undertaken. Finally, in Section 5, we conclude the paper by proposing different directions of research that can be established based on the proposed methodology. All mathematical proofs are reported in Appendix A to facilitate readability.

2. Characterization of new optimal positions

In this section we present a necessary and sufficient condition for a characterization of new (so far uncovered) positions in the optimal job sequence for the minimization of CTV. Throughout this paper, we use the notation MS to refer to the makespan (*i.e.*, $MS = \sum_{j=1}^{n} p_j$) and σ^* to refer to an optimal sequence for the minimisation of the CTV. The following linear function of the processing times will represent a fundamental quantity of our main results:

$$\overline{p} = p_{n-2} + \frac{4p_{n-3} + p_{n-4} + \overline{\beta}}{2}, \qquad \text{where} \qquad \overline{\beta} = \sum_{i=1}^{n-5} p_i.$$

All the results presented in this paper are valid for the case $n \geq 6$. In fact, for the case $n < 6$, the optimal solution can be analytically computed, as proved by Schrage (1975). Before presenting our main results, some known properties of job sequences are provided in the following Lemmas.

Lemma 1 (Largest positions from Hall & Kubiak (1991)). *There exists an optimal sequence of the form*

$$\sigma^* = \left(\overbrace{(n, n-2)}^{\pi'}, \overbrace{(j_1, j_2, \ldots, j_{n-3})}^{S}, \overbrace{(\ n-1\ )}^{\pi''} \right)$$

where we denoted with $j_1, j_2, \ldots, j_{n-3}$ the $n-3$ unknown jobs in the middle of the sequence. In other words, the largest job is scheduled in the first position, the second largest in the last position and the third largest in the second position.

Lemma 2 (Dual sequence from Schrage (1975)). *Consider a sequence $\sigma = (j_1, j_2, j_3, \ldots, j_{n-1}, j_n)$. The dual sequence of σ is defined as $\sigma^d = (j_1, j_n, j_{n-1}, \ldots, j_3, j_2)$. In other words, the first element of the sequence is kept in the same position, while the remaining $n-1$ elements are inverted from top to bottom. We have the relationships*

$$\overline{C}(\sigma) + \overline{C}(\sigma^d) = MS + p_{j_1} \qquad \text{and} \qquad CTV(\sigma) = CTV(\sigma^d).$$

[3]We used the heuristic solution of Nessah & Chu (2010) (which has been proved to be very close to the optimal solution in the vast majority of instances) to evaluate our lower bounds. This implies that the gap between the proposed lower bound and the optimal solution is potentially even smaller than 0.01%.

Lemma 3 (Sequence decomposition). *Let us consider the following job sequences:*

$$\sigma = \left(\overbrace{(i_1, i_2, \dots, i_r)}^{\pi_1}, \overbrace{(j_1, j_2, \dots, j_s)}^{S}, \overbrace{(i_{r+1}, i_{r+2}, \dots, i_k)}^{\pi_2} \right),$$

where $k + s = n$. We have the following equivalent decompositions of $CTV(\sigma)$:

$$\begin{cases} CTV(\sigma) &= \frac{k}{n} CTV(\pi) + \frac{n-k}{n} CTV(S) + \frac{k(n-k)}{n^2} \left(\overline{C}(S) - \overline{C}(\pi) \right)^2 \\ CTV(\sigma) &= \frac{k}{n} CTV(\pi) + \frac{n-k}{n} CTV(S) + \frac{k}{(n-k)} \left(\overline{C}(\sigma) - \overline{C}(\pi) \right)^2, \end{cases}$$

where π is the subsequence $((i_1, i_2, \dots, i_r), (i_{r+1}, i_{r+2}, \dots, i_k))$ in which job i_{r+1} starts at time $MS - \sum_{h=r+1}^{k} p_{i_h}$ and S is the subsequence $(j_1, j_2, \dots, j_s)$ in which job j_1 starts at time $\sum_{h=1}^{r} p_{i_h}$. The mean completion time of the subsequences π and S is denoted by $\overline{C}(\pi)$ and $\overline{C}(S)$ respectively.

While the proof of Lemma 1 and Lemma 2 is provided by Hall & Kubiak (1991) and Schrage (1975) respectively, the proof of Lemma 3 is in Appendix A.

Using these properties, the following two theorems present the primary results of this section, providing a necessary and sufficient condition for a characterization of new (so far uncovered) positions in the optimal job sequence for the minimization of CTV.

Theorem 1 (First case optimal sequence). *If $p_{n-1} \leq \overline{p}$, then there exists an optimal sequence which is either of the form*

$$\sigma_1^* = (n, n-2, n-3, j_1, \dots, j_{n-5}, n-4, n-1), \qquad \text{(Sequence 1)}$$

or of the form

$$\sigma_2^* = (n, n-2, n-4, j_1, \dots, j_{n-5}, n-3, n-1). \qquad \text{(Sequence 2)}$$

Theorem 2 (Second case optimal sequence). *If $p_{n-1} \geq \overline{p}$, then there exists an optimal sequence which is of the form*

$$\sigma_3^* = (n, \; n-2, \; n-3, \; n-4, \; j_1, \; \dots, \; j_{n-5}, \; n-1). \qquad \text{(Sequence 3)}$$

The proof of Theorem 1 and Theorem 2 are provided in Appendix A.

Note that these theorems set a final conclusion to the open questions that motivated the contributions of Schrage (1975), Kanet (1981) and Hall & Kubiak (1991). For the sake of completeness, the following remark points out the optimal sequence in the frontier case $p_{n-1} = \overline{p}$.

Remark 1 (The case of equality $p_{n-1} = \overline{p}$). *By the proofs of Theorem 1 and Theorem 2, the CTV minimization problem contains at least two alternative optimal solutions when $p_{n-1} = \overline{p}$. Firstly,* (Sequence 2) *cannot be optimal (by proof of Theorem 2, case 3). Using* (Sequence 1)

$$\sigma_1^* = \left(n, n-2, \overbrace{(n-3, j_1, \dots, j_{n-5}, n-4)}^{S}, n-1 \right),$$

construct the following sequence:

$$\omega = \left(n, n-2, \overbrace{(n-3, n-4, j_{n-5}, \dots, j_1)}^{S_d}, n-1 \right),$$

Since ω is of the form (Sequence 3)*, by case (i) of the proof of Theorem 1 (using Lemma 2)), we have that $CTV(\omega) = CTV(\sigma_1^*)$.*

As discussed in the subsequent section, the characterisation of the new positions in (Sequence 1), (Sequence 2) and (Sequence 3) will have a remarkable impact on the construction of lower bounds to the minimum CTV.

3. The best lower bounds

We explore hereafter different alternative procedures to boost the lower bound for $CTV(\sigma^*)$, building on the novel characterization from Theorem 1 and Theorem 2. Throughout this section $LB_{\overline{C}}$ and $UB_{\overline{C}}$ are used to denote lower and upper bounds to the completion time mean of the optimal sequence for the minimization of the CTV.

3.1. ***Basic lower bound***

Returning to considering Theorem 1 and Theorem 2, we know that when $n \geq 6$, an optimal sequence for the minimizing CTV of the form

$$\sigma^* = \left(\pi_1, \overbrace{(j_1, ..., j_{n-6})}^{S}, \pi_2\right)$$

can be found where the sub-sequence $\pi = (\pi_1, \pi_2)$ contains the first and last jobs. This is constructed as

$$(\pi_1^1, \pi_2^1) \;=\; \Big((n, n-2, n-3), (j_{n-5}, n-4, n-1)\Big),$$

and

$$(\pi_1^2, \pi_2^2) \;=\; \begin{cases} \Big((n, n-2, n-4), (j_{n-5}, n-3, n-1)\Big) & \text{if } p_{n-1} \leq \overline{p}, \\ \Big((n, n-2, n-3, n-4), (j_{n-5}, n-1)\Big) & \text{otherwise.} \end{cases}$$

Let $m = \lfloor \frac{n-6}{2} \rfloor$ and $C_{[i]}(S)$ be the completion time of job in position $i = 1, \ldots, n-6$ in the schedule S. Following this, we invoke Lemma 3 to obtain

$$\begin{aligned} CTV(\sigma^*) &= \frac{6}{n} CTV(\pi) + \frac{n-6}{n} CTV(S) + \frac{6(n-6)}{n^2} \left(\overline{C}(S) - \overline{C}(\pi)\right)^2 \\ &\geq \frac{6}{n} CTV(\pi) + \frac{n-6}{n} CTV(S) \\ &= \frac{6}{n} CTV(\pi) + \frac{1}{n} \sum_{i=1}^{n-6} \left(C_{[i]}(S) - \overline{C}(S)\right)^2. \end{aligned}$$

Let $X_i = \frac{1}{2}\left(C_{[m+i]}(S) + C_{[m+1-i]}(S)\right)$ and $Y_i = \frac{1}{2}\left(C_{[m+1+i]}(S) + C_{[m+1-i]}(S)\right)$.
If $n-6$ is even,

$$\begin{aligned} CTV(\sigma^*) &\geq \frac{6}{n} CTV(\pi) + \frac{1}{n} \sum_{i=1}^{m} \left[\left(C_{[m+i]}(S) - \overline{C}(S)\right)^2 + \left(C_{[m+1-i]}(S) - \overline{C}(S)\right)^2\right] \\ &= \frac{6}{n} CTV(\pi) + \frac{1}{n} \sum_{i=1}^{m} \left[\frac{\left(C_{[m+i]}(S) - C_{[m+1-i]}(S)\right)^2}{2} + 2\left(X_i - \overline{C}(S)\right)^2\right] \\ &\geq \frac{6}{n} CTV(\pi) + \frac{1}{n} \sum_{i=1}^{m} \frac{(p_1 + \ldots + p_{2i-1})^2}{2}. \end{aligned}$$

If $n-6$ is odd,

$$\begin{aligned} CTV(\sigma^*) &\geq \frac{6}{n}CTV(\pi) + \frac{1}{n}\sum_{i=1}^{m}\left[\left(C_{[m+1+i]}(S) - \overline{C}(S)\right)^2 + \left(C_{[m+1-i]}(S) - \overline{C}(S)\right)^2\right] \\ &= \frac{6}{n}CTV(\pi) + \frac{1}{n}\sum_{i=1}^{m}\left[\frac{\left(C_{[m+1+i]}(S) - C_{[m+1-i]}(S)\right)^2}{2} + 2\left(Y_i - \overline{C}(S)\right)^2\right] \\ &\geq \frac{6}{n}CTV(\pi) + \frac{1}{n}\sum_{i=1}^{m}\frac{(p_1 + \ldots + p_{2i})^2}{2}. \end{aligned}$$

Hence, we deduce

$$(n-6)CTV(S) \ \geq \ LB_0 \ = \ \frac{1}{2}\begin{cases} \sum\limits_{i=1}^{m}(p_1 + \ldots + p_{2i})^2 & \text{if } n \text{ is odd,} \\ \sum\limits_{i=1}^{m}(p_1 + \ldots + p_{2i-1})^2 & \text{if } n \text{ is even.} \end{cases}$$

and a lower bound can be built as

$$LB_{NN} \ = \ \frac{6}{n}CTV(\pi) + \frac{1}{n}LB_0. \tag{1}$$

To explicitly characterize LB_{NN}, we need the value of $CTV(\pi)$ under the different cases shown in Theorem 1 and Theorem 2.

Corollary 1 (Characterisation of LB_{NN}). *Let* $\bar{L} = 6p_n^2 + 6(p_n + p_{n-2})^2 + 6(p_n + p_{n-2} + p_{n-3})^2 + 6MS^2 + 6(MS - p_{n-1})^2$. *We have that*

$$LB_{NN} \ = \ \frac{1}{6n}(\bar{L} + L_0) + \frac{1}{n}LB_0,$$

where

$$L_0 = \begin{cases} 6(MS - p_{n-1} - p_{n-4})^2 - (3MS + 3p_n - 2p_{n-1} + 2p_{n-2} + p_{n-3} - p_{n-4})^2 & \text{if } p_{n-1} \leq \overline{p}, \\ 6(p_n + p_{n-2} + p_{n-3} + p_{n-4})^2 - (2MS + 4p_n - p_{n-1} + 3p_{n-2} + 2p_{n-3} + p_{n-4})^2 & \text{if } p_{n-1} > \overline{p}. \end{cases}$$

The proof of Corollary 1 is provided in Appendix A.

This value of LB_{NN} is valid for both cases when $p_{n-1} \leq \overline{p}$ and $p_{n-1} > \overline{p}$. From this result, all the elements necessary for the computation of LB_{NN} can be obtained in $O(n)$ time as soon as the jobs are numbered in the nondecreasing order of the processing times. Consequently, the complexity of computing LB_{NN} is $O(n \log n)$.

Based on a different argument (*i.e.*, by defining a problem relaxation), Federgruen & Mosheiov (1996) derived a lower bound for the more general case of completion time squared deviation from a machine specific due date. Building on the result of Hall & Kubiak (1991), Lemma 2.2 of (Kubiak et al. 2002) provided a similar lower bound (which will be referred to as LB_{KCK}):

$$LB_{KCK} \ = \ \begin{cases} \dfrac{1}{2n}\sum\limits_{i=1}^{\lfloor\frac{n-1}{2}\rfloor}(p_1 + \ldots + p_{2i})^2 & \text{if } n \text{ is odd,} \\ \dfrac{1}{2n}\sum\limits_{i=1}^{\lfloor\frac{n-1}{2}\rfloor}(p_1 + \ldots + p_{2i-1})^2 & \text{if } n \text{ is even.} \end{cases} \tag{2}$$

One year later, Viswanathkumar & Srinivasan (2003) deduced a stronger bound (which will be referred to as LB_{VS}), verifying the following relation proved by Nessah & Chu (2010):

$$LB_{VS} = LB_{KCK} + \frac{\left(p_{n-1} - p_{n-2}\right)^2}{4n}. \tag{3}$$

Proposition 1 (New benchmark bound)**.** *We have the following relation:*

$$LB_{NN} \geq LB_{VS} \geq LB_{KCK}.$$

The proof of Proposition 1 is provided in Appendix A.

Therefore, starting from (1), we can build a lower bound which is stronger than the current benchmark. Computational tests to support these inequalities are provided in Section 4.

3.2. ***Improving lower bound***

While LB_{NN} is the best lower bound that can be built only using the information of the five positions characterized by Theorem 1 and Theorem 2 and the variance decomposition in Lemma 3, this bound can still be improved as described in this subsection. Let us consider the following sequence:

$$\sigma = \left(\overbrace{(i_1, \ldots, i_r)}^{\pi_1}, \overbrace{(j_1, \ldots, j_s)}^{S}, \overbrace{(i_{r+1}, \ldots, i_k)}^{\pi_2} \right),$$

where $k + s = n$. By Lemma 3, we have the following decomposition:

$$CTV(\sigma) = \frac{k}{n} CTV(\pi) + \frac{s}{n} CTV(S) + \frac{k}{s} \left(\overline{C}(\sigma) - \overline{C}(\pi) \right)^2. \tag{4}$$

Aiming to improve the lower bound developed in the previous section, we build hereafter an optimization model to find a collection of processing times that resemble the actual ones and verify the previously described dominance properties while minimizing CTV. To do this, let $UB_{\overline{C}}$ and $LB_{\overline{C}}$ be the upper and lower bounds of $\overline{C}(\sigma)$. Let $\ell \in \{LB_{\overline{C}}, UB_{\overline{C}}\}$. Then, by (4) we obtain that

$$\begin{aligned}
CTV(\sigma) &= \tfrac{k}{n} CTV(\pi) + \tfrac{s}{n} CTV(S) + \tfrac{k}{s} \left[\left(\overline{C}(\sigma) - \ell \right) + \left(\ell - \overline{C}(\pi) \right) \right]^2 \\
&= \tfrac{k}{n} CTV(\pi) + \tfrac{s}{n} CTV(S) + \tfrac{k}{s} \left[\left(\overline{C}(\sigma) - \ell \right)^2 - \ell^2 + \overline{C}(\pi)^2 + 2(\ell - \overline{C}(\pi)) \overline{C}(\sigma) \right] \\
&= \tfrac{k}{n} CTV(\pi) + \tfrac{k}{s} \left(\overline{C}(\pi)^2 - \ell^2 \right) + \tfrac{k}{s} \left(\overline{C}(\sigma) - \ell \right)^2 + \tfrac{s}{n} CTV(S) + \tfrac{2k}{s} \left(\ell - \overline{C}(\pi) \right) \overline{C}(\sigma).
\end{aligned}$$

Hence, the variance decomposition (4) can be rewritten as

$$CTV(\sigma) = \begin{cases} \tfrac{k}{n} CTV(\pi) + \tfrac{k}{s} \left(\overline{C}^2(\pi) - LB^2_{\overline{C}} \right) + \tfrac{k}{s} \left(\overline{C}(\sigma) - LB_{\overline{C}} \right)^2 + F^{(LB)}(\sigma), \\ \tfrac{k}{n} CTV(\pi) + \tfrac{k}{s} \left(\overline{C}^2(\pi) - UB^2_{\overline{C}} \right) + \tfrac{k}{s} \left(\overline{C}(\sigma) - UB_{\overline{C}} \right)^2 + F^{(UB)}(\sigma), \end{cases} \tag{5}$$

where

$$F^{(LB)}(\sigma) = \tfrac{s}{n} CTV(S) + \tfrac{2k}{s} \left(LB_{\overline{C}} - \overline{C}(\pi) \right) \overline{C}(\sigma),$$

$$F^{(UB)}(\sigma) = \tfrac{s}{n} CTV(S) + \tfrac{2k}{s} \left(UB_{\overline{C}} - \overline{C}(\pi) \right) \overline{C}(\sigma).$$

From this reformulation, LB_{NN} can be improved in the following two steps: (i) optimizing with respect to the processing times and (ii) combining the best optimized bounds.

Optimizing with respect to the pseudo processing times. In what follows, we establish a relaxation of $F^{(LB)}(\sigma)$ and $F^{(UB)}(\sigma)$ by considering a hypothetical case in which processing times are unknown, while certain statistical properties of their distribution (namely, statistical moments) are known.[4] Specifically, we define a collection of unknown processing times $\beta_{j_1} \ldots \beta_{j_s}$ (that we can call *pseudo* processing times),

[4]Given a collection of real numbers $p_1, \ldots, p_h$, the τ^{th} statistical moment is defined as $(1/h) \sum_{i=1}^{h} p_i^{\tau}$.

verifying certain constraints (statistical moments) that force them to resemble a permutation of $p_1 \dots p_s$, in the same order as in the optimal solution. First, when the processing times are $\beta_{j_1} \dots \beta_{j_s}$, the completion times can be expressed as

$$C_{j_h}(\sigma) = \sum_{x=1}^{r} p_{i_x} + \sum_{x=1}^{h} \beta_{j_x}, \quad h = 1, \dots, s$$

which implies

$$\begin{cases} \overline{C}(\sigma) = \frac{k}{n}\overline{C}(\pi) + \frac{1}{n}\left(s\sum_{x=1}^{r} p_{i_x} + \sum_{x=1}^{s}(s+1-x)\beta_{j_x}\right), \\ \overline{C}(S) = \sum_{x=1}^{r} p_{i_x} + \frac{1}{s}\sum_{x=1}^{s}(s+1-x)\beta_{j_x}. \end{cases}$$

A lower bound of $F^{(\ell)}(\sigma)$, for $\ell \in \{LB,\ UB\}$, is obtained by finding the optimal processing times $\beta_{j_1} \dots \beta_{j_s}$, subject to a constraints structure that captures the statistical moments of the original processing times $p_1 \dots p_s$. The problem is the following:

$$\begin{cases} \min_{\boldsymbol{\beta}} \quad F^{(\ell)}(\boldsymbol{\beta}), & & \text{(6a)} \\ \text{s. to} \quad LB_{\overline{C}} \leq \overline{C}(\boldsymbol{\beta}) \leq UB_{\overline{C}}, & & \text{(6b)} \\ \sum_{i=1}^{s} \beta_i^{\tau} = \sum_{i=1}^{s} p_i^{\tau}, & \tau = 1, \dots, \bar{\tau} & \text{(6c)} \\ \sum_{j=1}^{g(i)} \beta_j \geq \sum_{j=1}^{g(i)} p_j, & i = 1, \dots, n & \text{(6d)} \\ p_1 \leq \beta_i \leq p_s, & i = 1, \dots, s, & \text{(6e)} \end{cases}$$

where $g(i) = 2i - 1$, if n is even, and $g(i) = 2i$, if n is odd. Constraints (6b) set a lower and upper bound to the completion time mean, expressed as a function of $\beta_1, \dots, \beta_s$. Constraints (6c) require the pseudo processing times $\beta_1, \dots, \beta_s$ to mimic the actual processing times $p_1, \dots, p_s$ in terms of the moments. Constraints (6d) enforce the partial summation bounds established by (2). When plugging the optimal value of $F^{(\ell)}(\boldsymbol{\beta})$ from (6a)–(6e), into (5), we get an alternative lower bound to $CTV(\sigma^*)$, whose relationship with LB_{NN}, LB_{VS} and LB_{KCK} is unknown. However, a relationship can be seemingly established in virtue of their optimal convex combination.

Combining best bounds. By letting $\boldsymbol{\beta}^{\ell}$ be the optimal solution of problem (6a)–(6d) and using the best lower bound to the optimal CTV provided in Corollary 1, we deduce three alternative bounds (that express the optimal minimal variance using the decomposition in Lemma 3):

$$\begin{aligned} CTV(\sigma^*) \quad &\geq \quad CTV_{LB} \;=\; \tilde{\mathcal{F}}^{LB}(\boldsymbol{\beta}^{LB}) + \tfrac{6}{n-6}\left(\overline{C}(\sigma^*) - LB_{\overline{C}}\right)^2, \\ CTV(\sigma^*), \quad &\geq \quad CTV_{UB} \;=\; \tilde{\mathcal{F}}^{UB}(\boldsymbol{\beta}^{UB}) + \tfrac{6}{n-6}\left(\overline{C}(\sigma^*) - UB_{\overline{C}}\right)^2, \\ CTV(\sigma^*) \quad &\geq \quad CTV_{NN} \;=\; LB_{NN} + \tfrac{6}{n-6}\left(\overline{C}(\sigma^*) - \overline{C}(\pi)\right)^2, \end{aligned} \tag{7}$$

where

$$\tilde{\mathcal{F}}^{\ell}(\boldsymbol{\beta}^{\ell}) \;=\; F^{(\ell)}(\boldsymbol{\beta}^{\ell}) + \frac{6}{n}CTV(\pi) + \frac{6}{n-6}\left(\overline{C}^2(\pi) - \ell^2\right), \qquad \text{for } \ell \in \{LB, UB\}.$$

It must be noted that building on (7), any form of improvement of the lower bounds $\tilde{\mathcal{F}}^{LB}(\boldsymbol{\beta}^{LB})$, $\tilde{\mathcal{F}}^{UB}(\boldsymbol{\beta}^{UB})$ and LB_{NN} requires estimating the non-negative quantities $\left(\overline{C}(\sigma^*) - LB_{\overline{C}}\right)^2$, $\left(\overline{C}(\sigma^*) - UB_{\overline{C}}\right)^2$ and $(\overline{C}(\sigma^*) - \overline{C}(\pi))^2$. Therefore, an improved lower bound is established in such a way as to minimize the differences $(\overline{C}(\sigma^*) - \ell)^2$, while using the best values between LB_{NN}, $\tilde{\mathcal{F}}^{LB}(\boldsymbol{\beta}^{LB})$ and $\tilde{\mathcal{F}}^{UB}(\boldsymbol{\beta}^{UB})$.

To do so, we search for the optimal convex combination of the three bounds in (7). Let $x, y, z \geq 0$ be such that $x + y + z = 1$ and multiplying the inequalities (7) by x, y, z, respectively, we obtain that $CTV(\sigma^*) \geq xCTV_{LB} + yCTV_{UB} + zCTV_{LB}$. Thus, we state the following corollary.

Corollary 2 (Best lower bound on $CTV(\sigma^*)$)**.** *We have the following relation*

$$CTV(\sigma^*) \;\geq\; LB_{NN}^{IMP} \;=\; \begin{cases} \max\limits_{x,y,z} \quad c_{LB}x + c_{UB}y + c_{NN}z - \dfrac{6}{n-6}Q_1(x,y,z)^2, & \text{(8a)} \\ s.\ to \quad x + y + z = 1, & \text{(8b)} \\ \qquad\quad x, y, z \geq 0, & \text{(8c)} \end{cases}$$

where $Q_1(x,y,z) = xLB + yUB + z\overline{C}(\pi)$*,* $c_{LB} = \tilde{\mathcal{F}}^{LB}(\boldsymbol{\beta}^{LB}) + 6LB^2/(n-6)$*,* $c_{UB} = \tilde{\mathcal{F}}^{UB}(\boldsymbol{\beta}^{UB}) + 6UB^2/(n-6)$ *and* $c_{NN} = LB_{NN} + 6\overline{C}(\pi)^2/(n-6)$*.*

The proof of Corollary 2 is available in Appendix A.

Corollary 2 provides the best lower bound $CTV(\sigma^*)$ as a convex quadratic problem with three variables, one equality constraint and non-negativity constraints. Note that the actual characterisation of this bound requires knowledge of $LB_{\overline{C}}$ and $UB_{\overline{C}}$, whose values can be established for the two different cases of Theorem 1 (*i.e.*, $p_{n-1} \leq \overline{p}$) and Theorem 2 (*i.e.*, $p_{n-1} > \overline{p}$). It must be noted that weak values of $LB_{\overline{C}}$ and $UB_{\overline{C}}$ would simply reduce to the case in which $LB_{NN}^{IMP} = LB_{NN}$ (since the optimal solution of (8a)-(8c) would be such that $z = 1$).

3.3. *On the computation of the improved lower bound*

The results from the previous subsections guarantees the relationship

$$CTV(\sigma^*) \geq LB_{NN}^{IMP} \geq LB_{NN} \geq LB_{VS}. \tag{9}$$

However, the actual computation of LB_{NN}^{IMP} still requires the resolution of problem (6a)–(6d) as well as the determination of $LB_{\overline{C}}$ and $UB_{\overline{C}}$. These aspects are considered in this subsection.

Numerical resolution of problem **(6a)–(6e).** First, it must be noted that the relationship (9) is valid for any relaxation of (6a)–(6d). Second, since model (6a)–(6d) gives rise to a non-linear (non-convex) problem, we can construct a convex relaxation by enlarging its dimensionality to rephrase the non-convex constraints (6c). To do so, let's define variable $\hat{\beta}_{i,\tau} = \beta_i^\tau$ so that $\hat{\beta}_{i,\tau} = \beta_i\hat{\beta}_{i,\tau-1} = \hat{\beta}_{i,1}\hat{\beta}_{i,\tau-1}$. To replace the n variables $\boldsymbol{\beta}$ with the $\overline{\tau}n$ variables $\hat{\boldsymbol{\beta}}$, we combine the following two approaches: (i) the piecewise linear approximation of the τ-power $\hat{\beta}_{i,\tau} = \hat{\beta}_{i,1}^\tau$ and (ii) the McCormick envelope of the bilinear term $\hat{\beta}_{i,1}\hat{\beta}_{i,\tau-1}$. By the first approach, we define a collection of L pivots points $b_0, \ldots b_{L-1}$ (*i.e.*, $b_\ell = p_1 + (p_s - p_1)(\ell/(L-1))$ for $\ell = 0 \ldots L-1$). Therefore, a convex relaxation of the non-convex constraints (6c) is

$$\begin{cases} \displaystyle\sum_{i=1}^{s} \hat{\beta}_{i\tau} = \sum_{i=1}^{s} p_i^\tau, & \tau = 1 \ldots \overline{\tau} & \text{(10a)} \\ p_1\hat{\beta}_{i,\tau} \leq \hat{\beta}_{i,\tau+1}, & i = 1 \ldots s,\ \tau = 1 \ldots \overline{\tau} - 1 & \text{(10b)} \\ p_s\hat{\beta}_{i,\tau} \geq \hat{\beta}_{i,\tau+1}, & i = 1 \ldots s,\ \tau = 1 \ldots \overline{\tau} - 1 & \text{(10c)} \\ \hat{\beta}_{i,\tau} \geq p_1^{\tau-1}\hat{\beta}_{i,1}, & i = 1 \ldots s,\ \tau = 1 \ldots \overline{\tau} & \text{(10d)} \\ \hat{\beta}_{i,\tau} \leq p_s^{\tau-1}\hat{\beta}_{i,1}, & i = 1 \ldots s,\ \tau = 1 \ldots \overline{\tau} & \text{(10e)} \\ \hat{\beta}_{i\tau} \geq (b_\ell)^\tau + \tau(b_\ell)^{\tau-1}(\hat{\beta}_{i,1} - b_\ell), & i = 1 \ldots s, \ell = 1 \ldots L,\ \tau = 1 \ldots \overline{\tau}. & \text{(10f)} \end{cases}$$

To strengthen this convex relaxation, we enforce the quadratic upper limits $\sum_{i=1}^{s}\hat{\beta}_{i,\tau}^2 \le \sum_{i=1}^{s} p_i^{2\tau}$ for $\tau = 1\dots\bar{\tau}$, whose relaxed linear version becomes

$$\sum_{i=1}^{s}\Big((b_\ell)^{2\tau} + 2(b_\ell)^{\tau}(\hat{\beta}_{i,\tau} - (b_\ell)^{\tau})\Big) \le \sum_{i=1}^{s} p_i^{2\tau}, \qquad \tau = 1\dots\bar{\tau},\ \ell = 1\dots L. \tag{11}$$

Finally, bounds for the bilinear term $\hat{\beta}_{i,1}\hat{\beta}_{i,\tau-1}$, can be enforced using the McCormick envelope (Rikun 1997):

$$\begin{cases}
\hat{\beta}_{i,\tau} \ge (p_1^{\tau-1})\hat{\beta}_{i,1} + p_1\hat{\beta}_{i,\tau-1} - p_1^{\tau}, & i = 1\dots s, \tau = 1\dots\bar{\tau} \quad (12a)\\
\hat{\beta}_{i,\tau} \ge (p_s^{\tau-1})\hat{\beta}_{i,1} + p_s\hat{\beta}_{i,\tau-1} - p_s^{\tau}, & i = 1\dots s, \tau = 1\dots\bar{\tau} \quad (12b)\\
\hat{\beta}_{i,\tau} \le (p_s^{\tau-1})\hat{\beta}_{i,1} + p_1\hat{\beta}_{i,\tau-1} - p_1 p_s^{\tau-1}, & i = 1\dots s, \tau = 1\dots\bar{\tau} \quad (12c)\\
\hat{\beta}_{i,\tau} \le (p_1^{\tau-1})\hat{\beta}_{i,1} + p_s\hat{\beta}_{i,\tau-1} - p_1^{\tau-1} p_s, & i = 1\dots s, \tau = 1\dots\bar{\tau}. \quad (12d)
\end{cases}$$

The specification of the objective function $F^{(\ell)}(\sigma)$ for $\ell \in \{LB,\ UB\}$ and the bounds on $\overline{C}(\boldsymbol{\beta})$ (that we called $LB_{\overline{C}}$ and $UB_{\overline{C}}$) depend on two cases $p_{n-1} \le \overline{p}$ and $p_{n-1} > \overline{p}$. In what follows, the notations $F^{(\ell)}(\boldsymbol{\beta})$, $C_i(\boldsymbol{\beta})$ and $\bar{C}(\boldsymbol{\beta})$ are often used instead of $F^{(\ell)}(\sigma)$, $C_i(\sigma)$ and $\bar{C}(\sigma)$ to emphasize the dependency with the decision variables $\beta_1 \dots \beta_{n-5}$ in model (6a)–(6d).

Computing $LB_{\overline{C}}$ ***and*** $UB_{\overline{C}}$. It is important to note that, while the lower bound generated by solving problem (6a)–(6d) is valid for any selection of $LB_{\overline{C}}$ and $UB_{\overline{C}}$ (namely any lower and upper bounds to $\overline{C}(\sigma^*)$), its effectiveness is extremely dependent on a close characterization of $\overline{C}$. We present, hereafter, well-fitted values of $LB_{\overline{C}}$ and $UB_{\overline{C}}$, for the two different cases when $p_{n-1} \le \overline{p}$ and $p_{n-1} > \overline{p}$.

For the case $p_{n-1} \le \overline{p}$, the completion times and mean completion time can be expressed in terms of the decision variables $\beta_1 \dots \beta_{n-5}$ of problem (6a)–(6d) as follows:

$$\begin{cases}
C_{\alpha_1}(\sigma(\boldsymbol{\beta})) = p_n + p_{n-2} + \alpha_1,\\
C_{j_h}(\sigma(\boldsymbol{\beta})) = p_n + p_{n-2} + \alpha_1 + \sum_{i=1}^{h}\beta_i,
\end{cases}$$

and

$$\begin{cases}
\overline{C}(\sigma(\boldsymbol{\beta})) = \frac{6}{n}\overline{C}(\pi) + \frac{1}{n}\left((n-6)(p_n + p_{n-2} + \alpha_1) + \sum_{i=1}^{n-6}(n-5-i)\beta_i\right),\\
\overline{C}(S(\boldsymbol{\beta})) = p_n + p_{n-2} + \alpha_1 + \frac{1}{n-6}\sum_{i=1}^{n-6}(n-5-i)\beta_i,
\end{cases}$$

where $\alpha_1 = p_{n-3}$ and $\alpha_2 = p_{n-4}$. Next, Theorem 1 states that there is an optimal sequence of the form either (Sequence 1) or (Sequence 2) which we can correspondingly rewrite as

$$\sigma^* = \left(\overbrace{(n, n-2, n-3)}^{\pi_1}, \overbrace{(j_1, \dots, j_{n-6})}^{S}, \overbrace{(j_{n-5}, n-4, n-1)}^{\pi_2}\right) \tag{Sequence 1}$$

and

$$\sigma^* = \left(\overbrace{(n, n-2, n-4)}^{\pi_1}, \overbrace{(j_1, ..., j_{n-6})}^{S}, \overbrace{(j_{n-5}, n-3, n-1)}^{\pi_2}\right), \tag{Sequence 2}$$

so that by Lemma 3, we can write

$$CTV(\sigma^*) = \frac{6}{n}CTV(\pi) + \frac{n-6}{n}CTV(S) + \frac{6}{n-6}\left(\overline{C}(\sigma^*) - \overline{C}(\pi)\right)^2.$$

Then, for each ℓ, we have

$$CTV(\sigma^*) \;=\; \frac{6}{n}CTV(\pi) + \frac{6}{n-6}\left(\overline{C}^2(\pi) - \ell^2\right) + \frac{6}{n-6}\left(\overline{C}(\sigma^*) - \ell\right)^2 + F^{(\ell)}(\sigma^*),$$

where

$$F^{(\ell)}(\sigma^*) \;=\; \frac{n-6}{n} CTV(S) + \frac{12}{n-6}\left(\overline{C}(\pi) - \ell\right)\overline{C}(\sigma^*).$$

Specifically, this will be valid for $\ell \in \{LB_{\overline{C}},\, UB_{\overline{C}}\}$. Therefore, the following propositions allow establishing these well-fitted bounds $LB_{\overline{C}}$ and $UB_{\overline{C}}$ to be set at the objective function (6b) and constraint (6b) when $p_{n-1} \leq \overline{p}$.

Proposition 2. *For the case $p_{n-1} \leq \overline{p}$, when the optimal sequence is (Sequence 1) we have*

$$\begin{cases} \overline{C}(\sigma^*) \geq LB_{\overline{C}} &= \frac{MS+p_n}{2} - \frac{n-2}{2n}\left(p_{n-1} - p_{n-2}\right), \\ \overline{C}(\sigma^*) \leq UB_{\overline{C}} &= \frac{1}{2n}\left(2np_n + 2p_{n-1} + 2(n-1)p_{n-2} + 2(n-2)p_{n-3} + (n-1)p_{n-4} + (n-1)\overline{\beta}\right). \end{cases}$$

Instead, when the optimal sequence is (Sequence 2), we have

$$\begin{cases} \overline{C}(\sigma^*) \geq LB_{\overline{C}} &= \frac{MS+p_n}{2} - \frac{n-2}{2n}(p_{n-1} - p_{n-2}), \\ \overline{C}(\sigma^*) \leq UB_{\overline{C}} &= \frac{B(p_{n-5})}{2(\overline{\beta}+5p_{n-5})}, \end{cases}$$

where $B(x) = 5\frac{n-5}{n}x^2 + 2x\left(5p_n + 4p_{n-2} + 3p_{n-4} + 2p_{n-3} + p_{n-1} + \frac{3n-5}{n}\overline{\beta}\right) + 2\overline{\beta}\left(p_n + p_{n-2} + p_{n-4} + \frac{n-1}{2n}\overline{\beta}\right)$.

The proof of Proposition 2 is available in Appendix A.

For the case $p_{n-1} > \overline{p}$, we can express the completion times and the mean completion time in terms of the decision variables $\beta_1 \ldots \beta_{n-5}$ of problem (6a)–(6d) as follows:

$$C_{j_h}(\sigma^*) = p_n + p_{n-2} + p_{n-3} + p_{n-4} + \sum_{i=1}^{h} \beta_i,\; h = 1, \ldots, n-6.$$

Therefore,

$$\begin{cases} \overline{C}(\sigma^*) = \frac{6}{n}\overline{C}(\pi) + \frac{1}{n}\left(\sum_{i=1}^{n-6}(n-5-i)\beta_i\right), \\ \overline{C}(S) = \frac{1}{n-6}\sum_{i=1}^{n-6}(n-5-i)\beta_i, \end{cases}$$

where $\alpha_1 = p_{n-3}$ and $\alpha_2 = p_{n-4}$. Next, Theorem 2 states that there is an optimal sequence of the form (Sequence 3) which can be decomposed as follows:

$$\sigma^* = \left(\overbrace{(n, n-2, n-3, n-4)}^{\pi_1}, \overbrace{(j_1, \ldots, j_{n-6})}^{S}, \overbrace{(j_{n-5}, n-1)}^{\pi_2}\right). \qquad \text{(Sequence 3)}$$

The following proposition allows establishing well-fitted bounds $LB_{\overline{C}}$ and $UB_{\overline{C}}$ to be set at the objective function (6b) and constraints (6b), when $p_{n-1} > \overline{p}$.

Proposition 3. *Let us define $D = 5p_n + 4p_{n-2} + 3p_{n-3} + 2p_{n-4} + p_{n-1} + (n+5)\overline{\beta}/n$, $E = p_n + p_{n-2} + p_{n-3} + p_{n-4} + (n-1)\overline{\beta}/(2n)$ and $F = np_n + p_{n-1} + (n-1)p_{n-2} + (n-2)p_{n-3} + (n-1)(p_{n-4} + \overline{\beta})/2$. For the case $p_{n-1} > \overline{p}$, we have*

$$\begin{cases} \overline{C}(\sigma^*) \geq LB_{\overline{C}} = \max\left(\frac{1}{n}F,\; \frac{1}{2}\min\left(\frac{A(p_{n-5})}{\overline{\beta}+5p_{n-5}}, \frac{A(p_1)}{\overline{\beta}+5p_1}\right)\right), \\ \overline{C}(\sigma^*) \leq UB_{\overline{C}} = \max\left(\frac{B(p_{n-5})}{2(\overline{\beta}+5p_{n-5})}, \frac{B(p_1)}{2(\overline{\beta}+5p_1)}\right), \end{cases}$$

where $A(x) = -5\frac{n-5}{n}x^2 + 2xD + 2\overline{\beta}E$ *and* $B(x) = 5\frac{n-5}{n}x^2 + 2xD + 2\overline{\beta}E$.

The proof of Proposition 3 is presented in Appendix A.

A small illustrative example. Let us consider a small example with $n = 12$. We first focus on Theorem 1 and define the following vector of processing times: $\mathbf{p} = [7, 24, 47, 58, 69, 76, 80, 104, 106, 269, 320]$. We have $p_{11} < \overline{p} = 564.5$, subsequently, the optimal solution is of the form (Sequence 1) or (Sequence 2), for $\alpha_1 = 106$ and $\alpha_1 = 104$ respectively. Table Ex.3.1 reports the main figures of the solution of the min CTV using the procedure described thus far.

	$CTV/\tilde{\mathcal{F}}^\ell$	$\overline{C}(\pi)$	p_i and β_i								
Optimal sequence:	60109.0	750.7	106.0	80.0	69.0	47.0	7.0	24.0	58.0	76.0	104.0
LB(106) = 737.92	60070,0	750.7	106.0	80.0	66.6	42.0	12.5	25.5	55.4	79.0	104.0
LB(104) = 737.92	60091.0	750.0	104.0	80.0	66.6	42.0	11.7	26.2	55.4	79.0	106.0
UB(106) = 753.87	59884.5	750.7	106.0	79.9	60.8	30.8	10.0	38.5	60.8	80.0	104.0
UB(104) = 748.88	60002.1	750.0	104.0	80.0	60.9	38.5	10.0	30.8	60.9	79.9	106.0

Table Ex.3.1: Optimal sequence and β-solution from model (6a)–(6d) for the generation of LB_{NN}^{IMP}. The first row contains the optimal sequence from the 3-rd position to the $(n-1)$-th position (as the first two and the last positions are known). The subsequent rows report the β-solution from model (6a)–(6d). Specifically, the second and third rows correspond to the case when $\ell = LB_{\overline{C}}$, for $\alpha_1 = 106$ and $\alpha_1 = 104$ respectively; the forth and fifth rows correspond to the case when $\ell = UB_{\overline{C}}$, for $\alpha_1 = 106$ and $\alpha_1 = 104$ respectively.

Using the results from Section 3.1 and Corollary 1, we deduce $LB_{NN} = 60030.2$ when $\alpha_1 = 106$ and $LB_{NN} = 60046.7$ when $\alpha_1 = 104$. Therefore, by applying the improving approach described in Section 3.2 and the best lower bound combination from Corollary 2, we get that the best improvement is $LB_{NN}^{IMP} = 60095.0$ when $\alpha_1 = 106$ and $LB_{NN}^{IMP} = 60108.2$ when $\alpha_1 = 104$. This implies that $LB_{NN}^{IMP} = 60095.0$ (with a gap of 0.02%).

Second, to illustrate the case of Theorem 2, we merely need to modify the processing time of p_{n-1}. Therefore, $\mathbf{p} = [7, 24, 47, 58, 69, 76, 80, 104, 106, 120, 566, 566]$. We have $p_{11} > \overline{p} = 564.5$; thus the optimal solution is of the form (Sequence 3). Table Ex.3.2 reports the main figures of the solution of the min CTV using the procedure described thus far.

	$CTV/\tilde{\mathcal{F}}^\ell$	$\overline{C}(\pi)$	p_i and β_i								
Optimal sequence:	93334.2	1003.3	106.0	104.0	69.0	47.0	24.0	7.0	58.0	76.0	80.0
LB = 1024.6	93268.3	1003.3	106.0	104.0	74.9	49.9	19.2	14.2	49.9	73.5	80.0
UB = 1055.7	92485.0	1003.3	106.0	104.0	66.3	33.0	8.0	33.0	64.6	76.0	80.0

Table Ex.3.2: Optimal sequence and β-solution from model (6a)–(6d) for the generation of LB_{NN}^{IMP}. The first row contains the optimal sequence from the 3-rd position to the $(n-1)$-th position (as the first two and the last positions are known). The third and fourth rows report the β-solution from model (6a)–(6d), for the case $\ell = LB_{\overline{C}}$ and $\ell = UB_{\overline{C}}$ respectively.

Using the results from Section 3.1 and Corollary 1, we deduce $LB_{NN} = 92368.5$. Applying the improving approach described in Section 3.2 and the best lower bound combination from Corollary 2, we get $LB_{NN}^{IMP} = 93276.2$. This implies a gap of 0.02% from the optimal solution $CTV(\sigma^*) = 93334.2$.

4. Computational analysis

We provide hereafter the numerical support to the theoretical framework developed thus far. This is done by constructing a large scale computational experiment with several configurations of processing times. To make our numerical test more robust and less sensitive to the specific data, all the computational tests presented hereafter are performed using two types of generators.

The first generator (call it *GEN 0*) proceeds in accordance with the following loop.

i Set $h = 1$; $p_h \sim U(0, b)$.

ii If $h < n$, then $p_{h+1} = p_h + u$, where $u \sim U(0, b)$.

iii Else, stop.

iv Set $h \leftarrow h + 1$.

The second generator (call it *GEN 1*) is instead built as follows,

i Set $h = 1$; $p_h \sim U(0, b)$.

ii If $h < n$, then $p_h = u$, where $u \sim U(0, nb)$.

iii Else, stop and renumber $p_1 \dots p_n$ in increasing order.

iv Set $h \leftarrow h + 1$.

For both generator, the notation $U(0, b)$ refers to the uniform distribution between zero and b.[5] Next, for each generation strategy, p_{n-1} has been overwritten to determine whether the processing times fall within the case of Theorem 1 (*i.e.*, $p_{n-1} \leq \overline{p}$) or within the case of Theorem 2 (*i.e.*, $p_{n-1} > \overline{p}$):[6]

$$\text{either} \qquad p_{n-1} = \left\lfloor \frac{1}{2} \sum_{i=1}^{n-3} p_i \right\rfloor \qquad \text{or} \qquad p_{n-1} = 1 + \lfloor \overline{p} \rfloor.$$

Two groups of instances have been generated and studied in this section:

1. A collection of 1152 small size instances are presented in Subsection 4.1 to study the dependency of LB_{NN} and LB_{NN}^{IMP} on n, $\bar{\tau}$ and b, in comparison with the current benchmarks.
2. A collection of 60 big size instances are presented in Subsection 4.2 to estimate the actual behaviour of LB_{NN}^{IMP} (in terms of relative improvement with respect to LB_{NN} and the computational effort to build it), when the problem size grows large.

All optimization procedures are solved using IBM ILOG CPLEX 12.9 on a R5500 work-station with processor Intel(R) Xeon(R) CPU E5645 2.40 GHz, and 48 Gbytes of RAM, under a Windows Server 2012 operative system.

[5]While *GEN 0* simulates processing times by a Markov Chain, with transaction rule $p_{h+1} = p_h + u$, in *GEN 1* processing times are independent and identically distributed. The latter has been adopted in the main literature (Federgruen & Mosheiov 1996, Viswanathkumar & Srinivasan 2003, Nessah & Chu 2010), as a standard generation approach to test scheduling results. Below we report the expectation and variance of processing times to assess the main characteristics of the generated data:

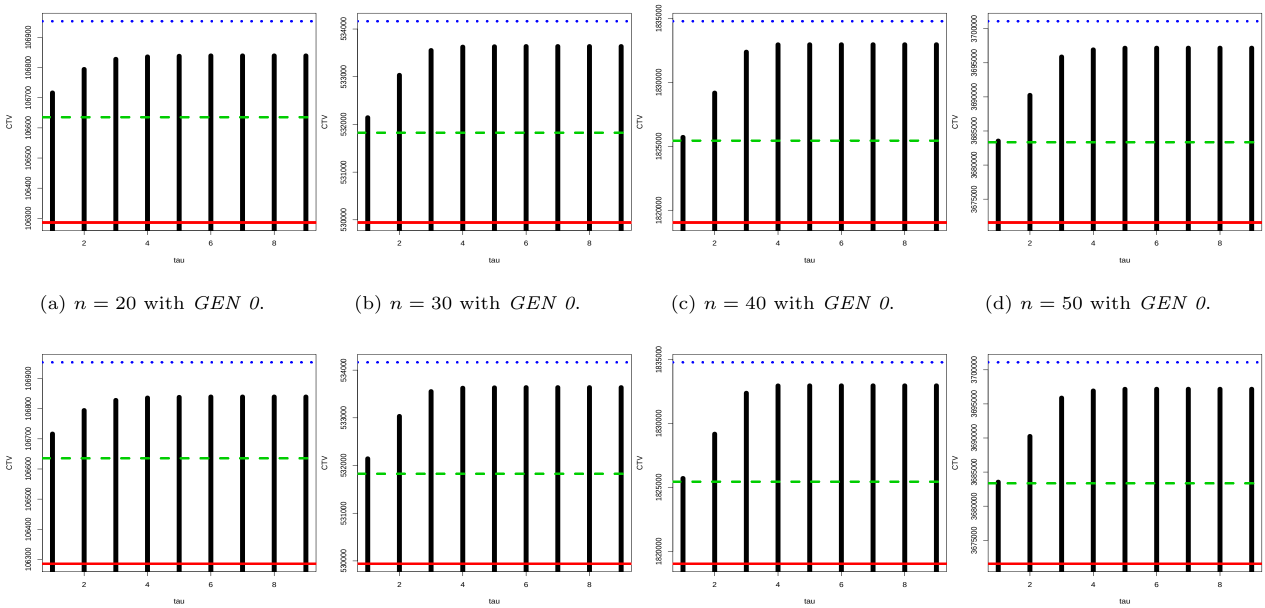
- the i^{th} processing time in *GEN 0* has expectation $E[p_i] = iE[U] = ib/2$ and variance $Var[p_i] = i^2 Var[U] = (ib)^2/12$;
- an arbitrary processing time i has in *GEN 0* the following expectation and variance has expectation $E[p_i] = nb/2$ and variance $Var[p_i] = (nb)^2/12$.

[6]This overwriting approach allows the numerical tests to cover both cases with the same amount of generated instances.

4.1. ***Small instances***

The computational experiment presented in this section involves 1152 small size instances, generated by the cross-combinations of n (at 4 values), $\bar{\tau}$ (at 9 values), b (at 3 values), two types of generators, three realizations and two cases (for $p_{n-1} \leq \overline{p}$ and $p_{n-1} > \overline{p}$). Specifically, we have $n \in \{20, 30, 40, 50\}$, $\bar{\tau} \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $b \in \{5, 7, 10\}$.

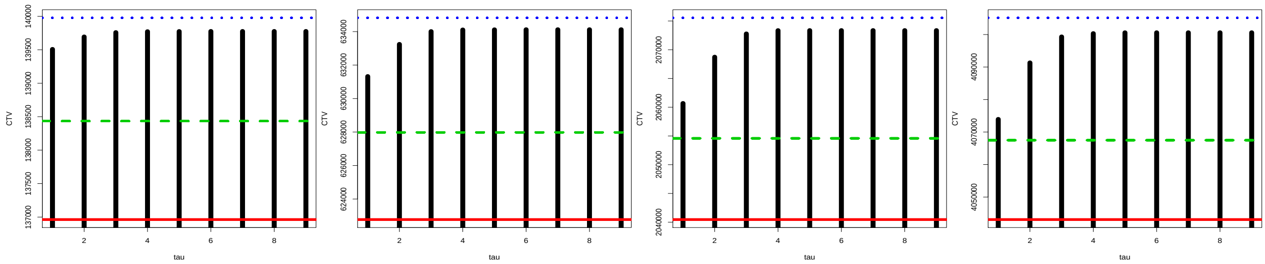
For each of the two respective cases of Theorem 1 (*i.e.*, $p_{n-1} \leq \overline{p}$) and Theorem 2 (*i.e.*, $p_{n-1} > \overline{p}$), figures 1 and 2 illustrate the dependency of LB_{NN}^{IMP} on the number of moments $\bar{\tau}$, characterizing the formulation (6a)–(6d) and its linear reformulation (10a)–(12d). The top and bottom plots correspond to the cases when data are generated using the first and second generators respectively. The solid red line corresponds to the value of LB_{VS}; the dashed green line corresponds to the value of LB_{NN}; the dotted blue line corresponds to the value of of the optimal solution. The vertical bars denote the level of LB_{NN}^{IMP} as a function of $\bar{\tau}$.



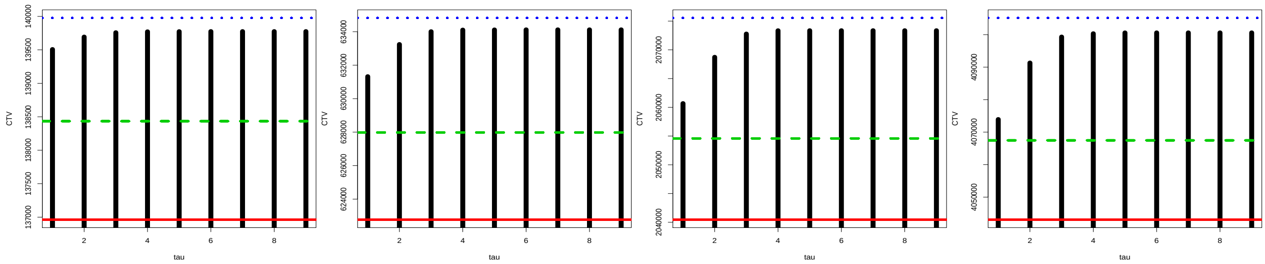
(a) $n = 20$ with *GEN 0*. (b) $n = 30$ with *GEN 0*. (c) $n = 40$ with *GEN 0*. (d) $n = 50$ with *GEN 0*.

(e) $n = 20$ with *GEN 1*. (f) $n = 30$ with *GEN 1*. (g) $n = 40$ with *GEN 1*. (h) $n = 50$ with *GEN 1*.

Figure 1: The impact of the number of included moments $\bar{\tau}$ in problem (6a)–(6e) for the case of Theorem 1 (*i.e.*, $p_{n-1} \leq \overline{p}$), comparing LB_{VS}, LB_{NN}, LB_{NN}^{IMP} and the optimal solution.



(a) $n = 20$ (Generator 0)   (b) $n = 30$ with *GEN 0*.   (c) $n = 40$ with *GEN 0*.   (d) $n = 50$ with *GEN 0*.



(e) $n = 20$ with *GEN 1*.   (f) $n = 30$ with *GEN 1*.   (g) $n = 40$ with *GEN 1*.   (h) $n = 50$ with *GEN 1*.

Figure 2: The impact of the number of included moments $\bar{\tau}$ in problem (6a)–(6e) for the case of Theorem 1 (*i.e.*, $p_{n-1} > \bar{p}$), comparing LB_{VS}, LB_{NN}, LB_{NN}^{IMP} and the optimal solution.

The most immediate insights from these plots is that 5 or 6 moments are sufficient to determine a level of LB_{NN}^{IMP} that substantially outperforms LB_{VS} and LB_{NN}. The second clear insight is that LB_{NN} cuts the distance between LB_{VS} (the current benchmark) and the optimal solution by at least a half in all the analyzed instances. Furthermore, LB_{NN} is outperformed by LB_{NN}^{IMP} even when $\bar{\tau} = 1$.

Using the same collection of instances, Table 3 reports the values of the solutions, their different bounds (*i.e.*, LB_{VS}, the LB_{NN} and the LB_{NN}^{IMP}) and their gaps with respect to the heuristic solution. Specifically, from left to right, the table contains the number of jobs, the generator type, the ratio $p_{n-1}/\bar{p}$, the value of the heuristic solution, the lower bound LB_{VS} (and its gap with respect to the value of the heuristic solution), the lower bound LB_{NN} (and its gap with respect to the value of the heuristic solution) and the lower bounds LB_{NN}^{IMP} for $\bar{\tau} = 2, 3, 4, 5$ (and their gaps with respect to the value of the heuristic solution).

	n	Gen	Ratio	$CTV(\sigma^*)$	LB_{VS} (GAP)	LB_{NN} (GAP)	LB_{NN}^{IMP} (GAP)			
							$\bar{\tau}=2$	$\bar{\tau}=3$	$\bar{\tau}=4$	$\bar{\tau}=5$
For $p_{n-1} \leq \overline{p}$	20	0	0.64	140557	139304 (0.87)	139953 (0.41)	31668032 (0.17)	140269 (0.17)	140328 (0.13)	140341 (0.12)
	20	1	0.50	73350	73268 (0.11)	73318 (0.04)	15944989 (0.02)	73321 (0.02)	73329 (0.03)	73331 (0.02)
	30	0	0.73	730369	722362 (1.08)	725906 (0.60)	153773129 (0.09)	728321 (0.09)	729281 (0.14)	729400 (0.13)
	30	1	0.53	337957	337526 (0.13)	337746 (0.06)	80011233 (0.02)	337750 (0.02)	337829 (0.04)	337853 (0.03)
	40	0	0.79	2548293	2518231 (1.17)	2530391 (0.69)	491183965 (0.07)	2537856 (0.07)	2543789 (0.16)	2544917 (0.13)
	40	1	0.55	1121273	1119847 (0.13)	1120500 (0.07)	248989507 (0.01)	1120516 (0.01)	1120984 (0.03)	1121004 (0.03)
	50	0	0.82	5021169	4964967 (1.11)	4987287 (0.67)	1202564073 (0.04)	5001120 (0.04)	5011607 (0.21)	5013653 (0.18)
	50	1	0.56	2381023	2378182 (0.12)	2379417 (0.07)	602725971 (0.01)	2379417 (0.01)	2380175 (0.04)	2380219 (0.03)
	20	0	0.64	140557	139304 (0.87)	139953 (0.41)	2602763702 (0.10)	140269 (0.10)	140328 (0.13)	140341 (0.12)
	20	1	0.50	73350	73268 (0.11)	73318 (0.04)	1245590701 (0.01)	73321 (0.01)	73329 (0.03)	73331 (0.02)
For $p_{n-1} > \overline{p}$	20	0	1	175850	171693 (2.34)	173730 (1.19)	173730 (0.35)	175349 (0.23)	175408 (0.19)	175419 (0.18)
	20	1	1	144082	141476 (1.82)	142743 (0.93)	142743 (0.26)	143863 (0.15)	143936 (0.10)	143950 (0.09)
	30	0	1	825215	808201 (2.08)	815546 (1.18)	815546 (0.59)	822556 (0.30)	823595 (0.19)	823717 (0.18)
	30	1	1	656309	645790 (1.61)	650343 (0.91)	650343 (0.43)	655118 (0.18)	655785 (0.08)	655849 (0.07)
	40	0	1	2741359	2691322 (1.84)	2711408 (1.10)	2711408 (0.79)	2729272 (0.38)	2735223 (0.21)	2736063 (0.19)
	40	1	1	1990928	1962452 (1.43)	1974027 (0.85)	1974027 (0.52)	1987386 (0.18)	1989662 (0.07)	1989745 (0.06)
	50	0	1	5315451	5228132 (1.63)	5262526 (0.99)	5262526 (0.85)	5292762 (0.43)	5303210 (0.25)	5305054 (0.23)
	50	1	1	4232900	4178590 (1.29)	4199876 (0.78)	4199876 (0.62)	4223989 (0.21)	4230152 (0.07)	4230222 (0.06)

Table 3: Tests with small instances. Each row is averaged over three realizations and three values of $b \in \{5, 7, 10\}$.

4.2. ***Big instances***

The computational experiment presented in this section involves 60 big size instances, generated by the cross-combinations of n (at 5 values, $n \in \{100, 150, 200, 250, 300\}$), two types of generators (with $b = 5$), three realizations and two cases (for $p_{n-1} \leq \overline{p}$ and $p_{n-1} > \overline{p}$). Based on the results from the previous subsection, we set $\bar{\tau} = 5$.

Table 4 reports the values of the solutions, their different bounds (*i.e.*, LB_{VS}, the LB_{NN} and the LB_{NN}^{IMP}), their gaps with respect to the heuristic solution, as well as the CPU time required to perform the computation. Again, from left to right, the table contains the number of jobs, the generator type, the ratio between $p_{n-1}/\overline{p}$, the value of the heuristic solution (with corresponding CPU time), the lower bound LB_{VS}

(and its gap with respect to the value of the heuristic solution), the lower bound LB_{NN} (and its gap with respect to the value of the heuristic solution) and the lower bounds LB_{NN}^{IMP} (and their CPU time and gaps with respect to the value of the heuristic solution).

	n	Gen	Ratio	$CTV(\sigma^*)$ (CPU time)	LB_{VS} (GAP)	LB_{NN} (GAP)	LB_{NN}^{IMP} (CPU time, GAP)
For $p_{n-1} \leq \overline{p}$	100	0	0.90	31928859 (0.211)	31668032 (0.8163)	31763209 (0.5183)	31875023 (113.2 , 0.1710)
	100	1	0.58	15959233 (0.183)	15944989 (0.0893)	15950446 (0.0550)	15955650 (65.1 , 0.0223)
	150	0	0.93	154763610 (0.457)	153773129 (0.6403)	154121006 (0.4153)	154622567 (111.1 , 0.0913)
	150	1	0.59	80066591 (0.394)	80011233 (0.0693)	80031379 (0.0440)	80054316 (127.9 , 0.0153)
	200	0	0.95	493717568 (0.693)	491183965 (0.5133)	492064110 (0.3350)	493390447 (205.0 , 0.0680)
	200	1	0.59	249127456 (0.723)	248989507 (0.0553)	249038671 (0.0357)	249100665 (234.9 , 0.0107)
	250	0	0.96	1207720794 (1.113)	1202564073 (0.4267)	1204340637 (0.2800)	1207253160 (478.0 , 0.0387)
	250	1	0.59	603005848 (1.126)	602725971 (0.0463)	602824326 (0.0300)	602920761 (540.5 , 0.0143)
	300	0	0.97	2612221586 (1.634)	2602763702 (0.3620)	2605999658 (0.2380)	2609539303 (933.4 , 0.0990)
	300	1	0.59	1246088593 (1.650)	1245590701 (0.0400)	1245764209 (0.0260)	1245947640 (970.9 , 0.0113)
For $p_{n-1} > \overline{p}$	100	0	1	32473730 (0.179)	32150196 (0.996)	32267996 (0.633)	32418296 (46.6 , 0.1727)
	100	1	1	17998235 (0.184)	17843977 (0.857)	17899864 (0.546)	17993149 (31.5 , 0.0280)
	150	0	1	156027949 (0.399)	154887687 (0.731)	155288156 (0.474)	155885067 (60.5 , 0.0917)
	150	1	1	87308873 (0.398)	86749038 (0.641)	86946012 (0.415)	87292786 (61.9 , 0.0183)
	200	0	1	496026612 (0.720)	493215879 (0.567)	494192093 (0.370)	495696597 (101.6 , 0.0680)
	200	1	1	266497412 (0.727)	265141791 (0.508)	265612441 (0.332)	266463634 (114.4 , 0.0130)
	250	0	1	1211374635 (1.148)	1205775762 (0.462)	1207704393 (0.303)	1210905229 (236.7 , 0.0387)
	250	1	1	637408880 (1.148)	634707797 (0.424)	635637753 (0.278)	637301291 (265.2 , 0.0170)
	300	0	1	2617626131 (1.705)	2607510309 (0.387)	2610971248 (0.255)	2610971247 (445.9 , 0.2547)
	300	1	1	1306375344 (1.692)	1301623896 (0.364)	1303251188 (0.239)	1306204097 (467.3 , 0.0130)

Table 4: Tests with big instances, with $b = 5$. Each row is averaged over three realizations, of the same size.

Figures reveal that LB_{NN}^{IMP} is less than 0.01% far away from the optimal solution and it can be computed in a couple of minutes even when $n = 300$.

5. Conclusions

In this work, a new sequence characterization for the single machine scheduling problem to minimize the completion time variance is established. In Theorem 1 and Theorem 2, we prove a necessary and sufficient condition (which can be verified in $O(n \log n)$) for the existence of an optimal sequence of the form $(n, n-2, n-3, ..., n-4, n-1)$ or of the form $(n, n-2, n-4, \ldots, n-3, n-1)$. When this condition is not satisfied, there must exist an optimal sequence of the form $(n, n-2, n-3, n-4, \ldots, n-1)$. Building on this theoretical standpoint, in Section 3 we tailor a new approach to derive the highest known lower bound for the minimal completion time variance.

On the computational side, an extensive numerical experiment involving more than 1000 solved instances with up to 300 jobs is conducted, revealing that the proposed bound is on average less than 0.01% far away from an upper bound of the optimal solution (the one computed by the heuristics proposed by Nessah & Chu (2010)). This means that the gap between the proposed lower bound and the optimal solution is potentially even smaller than 0.01%. This constitutes a substantial improvement with respect to the benchmark lower bounds, whose values in the observed instances are approximately 1% far away from the upper bound of Nessah & Chu (2010).

To conclude, we address three further lines of research that can emerge from this work.

1. Firstly, a possible improvement can be achieved by digging into the result of Theorem 1. In fact, we conjecture that there exists a constant $\overline{\overline{p}} \in [p_{n-2}, \overline{p})$ such that

   (i) for each $p_{n-1} \in [p_{n-2}, \overline{\overline{p}}]$, the optimal sequence is of the form (Sequence 2),

   (ii) for each $p_{n-1} \in [\overline{\overline{p}}, \overline{p}]$, the optimal sequence is of the form (Sequence 1).

   An example can be constructed with $n = 9$ to show that $\overline{\overline{p}} = p_{n-2}$ and $\overline{\overline{p}} > p_{n-2}$, respectively. Let $\mathbf{p} = [1, 3, 8, 8, 10, 12, 14, p_8, p_9]$ and note that, for this example, we have $\overline{p} = 53$. The unique value of (Sequence 2) is optimal if $\overline{\overline{p}} = p_7 = 14$. Then, another example can be constructed with $n = 8$: $\mathbf{p} = [1, 2, 8, 9, 11, 12, p_7, p_8]$. For this example, we have $\overline{p} = 44$ and $\overline{\overline{p}} = 18$. Therefore,

   (i) for each $p_7 \in [12, 18]$, the optimal sequence is of the form (Sequence 2),

   (ii) for each $p_7 \in [18, 44]$, the optimal sequence is of the form (Sequence 1).

2. Second, a further direction to be explored is the asymptotic behaviour of the derived lower bound, consistently with the recent results studying asymptotic properties of scheduling problems (Xia et al. 2000, Balseiro et al. 2018, Armony et al. 2019).

3. Finally, the computational resolution of (6a)–(6e) can be further investigated to tackle large scale problems with thousands of jobs as well as their convex reformulations. On this respect, decomposition algorithms (Codato & Fischetti 2006) can be studied and combined with specialized interior-point methods (Castro et al. 2017, Castro & Nasini 2017, 2021) to exploit the separability structure.

Overall, our results establish a new benchmark in the single machine scheduling problem to minimize the completion time variance while opening a rage of new lines of research to be explored in future contributions.

Appendix A: Mathematical proofs

Auxiliary lemmas

Proof of Lemma 3. Note that every time a sequence σ is decomposed in two sub-sequences π (corresponding to jobs $(i_1, i_2, \dots, i_r, i_{r+1}, i_{r+2}, \dots, i_k)$) and S (corresponding to jobs $j_1 \dots j_s$), we can write

$$\overline{C}(\sigma) = \frac{k}{n}\overline{C}(\pi) + \frac{n-k}{n}\overline{C}(S) \quad \text{and} \quad \sum_{h=1}^{n} C_h^2(\sigma) = \sum_{h=1}^{k} C_{i_h}^2(\pi) + \sum_{h=1}^{n-k} C_{j_h}^2(S).$$

Therefore,

$$\begin{aligned} CTV(\sigma) &= \frac{1}{n}\sum_{h=1}^{n} C_h^2(\sigma) - \overline{C}^2(\sigma) \\ &= \frac{k}{n}\left[\sum_{h=1}^{k} \frac{C_{i_h}^2(\pi)}{k}\right] + \frac{n-k}{n}\left[\sum_{h=1}^{n-k} \frac{C_{j_h}^2(S)}{n-k}\right] - \left(\frac{k}{n}\overline{C}(\pi) + \frac{n-k}{n}\overline{C}(S)\right)^2 \\ &= \frac{k}{n}\left[CTV(\pi) + \overline{C}^2(\pi)\right] + \frac{n-k}{n}\left[CTV(S) + \overline{C}^2(S)\right] - \left(\frac{k}{n}\overline{C}(\pi) + \frac{n-k}{n}\overline{C}(S)\right)^2 \\ &= \frac{k}{n}CTV(\pi) + \frac{n-k}{n}CTV(S) + \frac{k(n-k)}{n^2}\left(\overline{C}(S) - \overline{C}(\pi)\right)^2. \end{aligned}$$

For the second decomposition, it sufficient to replace $\overline{C}(S)$ by $\frac{n}{n-k}\overline{C}(\sigma) - \frac{k}{n-k}\overline{C}(\pi)$ in the last equality.

Main theorems and propositions

Proof of Theorem 1. Manna & Prasad (1999) has proved that if p_{n-1} is sufficiently large, the optimal solution is the sequence $(n, n-2, n-3, n-4, n-5, \dots, 1, n-1)$. Consequently, building on the V-shapeness, the following lower bound $\tilde{p}_{n-1}$ is well-defined: for each $p_{n-1} \geq \tilde{p}_{n-1}$, the optimal solution is of the form $(n, n-2, n-3, n-4, \dots, n-1)$ and/or $(n, n-2, \dots, n-3, n-4, n-1)$. (Note that Nessah (2020) proved that if $p_{n-1} \leq p_{n-2}$, then there is an optimal solution of the form (Sequence 1) and/or (Sequence 2)). Then, we need only to prove that $\tilde{p}_{n-1} \geq \overline{p}$.

Assume, by contrary, that $\tilde{p}_{n-1} < \overline{p}$. Since Eilon & Chowdhury (1977) have proved that V-shapedness is a necessary condition for the optimality of a sequence, a contradiction is obtained by showing that if $\tilde{p}_{n-1} < \overline{p}$, the optimal sequence can be neither

$$\sigma_3^* = \left(\overbrace{(n, n-2)}^{\pi_3'}, \overbrace{(n-3,\ n-4,\ j_1,\ j_2,\ \dots,\ j_{n-5})}^{S_3}, \overbrace{n-1}^{\pi_3''}\right) \qquad \text{(Sequence 3)}$$

nor

$$\sigma_4^* = \left(\overbrace{n,\ n-2}^{\pi_4'}, \overbrace{(j_1,\ j_2,\ \dots,\ j_{n-5},\ n-4,\ n-3)}^{S_4}, \overbrace{n-1}^{\pi_4''}\right) \qquad \text{(Sequence 4)}$$

Therefore, by construction, for each $p_{n-1} \geq \tilde{p}_{n-1}$, the optimal solution is of the form (Sequence 3) and/or (Sequence 4) and, we have:

$$\min\{CTV(\sigma_3^*), CTV(\sigma_4^*)\} < \min\{CTV(\sigma_1^*), CTV(\sigma_2^*)\}. \qquad (13)$$

Fix $p_{n-1} = \overline{p}$; we distinguish two cases: case (i) if σ_3^* is optimal and (ii) if σ_4^* is optimal.

(i) Let σ_3^* be optimal and proceed by contradiction. Define the following new sequence, obtained from σ_3^* by replacing S_3 by its dual sequence (see Lemma 2):

$$\omega_3^* = \left(\overbrace{n,\ n-2}^{\pi_3'},\ \overbrace{(n-3, j_{n-5},\ j_{n-4},\ \dots,\ j_1,\ n-4)}^{S_{3,d}},\ \overbrace{n-1}^{\pi_3''} \right)$$

Using Lemma 3, we can write

$$\begin{aligned} CTV(\sigma_3^*) &= \frac{3}{n} CTV(\pi_3) + \frac{n-3}{n} CTV(S_3) + \frac{3(n-3)}{n^2} \left(\overline{C}(S_3) - \overline{C}(\pi_3) \right)^2, \\ CTV(\omega_3^*) &= \frac{3}{n} CTV(\pi_3) + \frac{n-3}{n} CTV(S_{3,d}) + \frac{3(n-3)}{n^2} \left(\overline{C}(S_{3,d}) - \overline{C}(\pi_3) \right)^2 \end{aligned}$$

and we know from Lemma 2 that

$$\begin{cases} CTV(S_3) &= CTV(S_{3,d}), \\ \overline{C}(S_3) + \overline{C}(S_{3,d}) &= 2p_n + 2p_{n-2} + 2p_{n-3} + p_{n-4} + \overline{\beta}, \\ \overline{C}(\pi_3) &= \frac{1}{3}\left(3p_n + 2p_{n-2} + \overline{p} + p_{n-3} + p_{n-4} + \overline{\beta}\right). \end{cases}$$

Note that $p_{n-1} = \overline{p} = p_{n-2} + (4p_{n-3} + p_{n-4} + \overline{\beta})/2$, then

$$\begin{aligned} CTV(\sigma_3^*) - CTV(\omega_3^*) &= \tfrac{3(n-3)}{n^2} \left(\overline{C}(S_3) - \overline{C}(S_{3,d}) \right) \left(\overline{C}(S_3) + \overline{C}(S_{3,d}) - 2\overline{C}(\pi_3) \right) \\ &= \tfrac{n-3}{n^2} \left(\overline{C}(S_3) - \overline{C}(S_{3,d}) \right) \left(6p_n + 6p_{n-2} + 6p_{n-3} + 3p_{n-4} + 3\overline{\beta} - 6\overline{C}(\pi_3) \right) \\ &= \tfrac{n-3}{n^2} \left(\overline{C}(S_3) - \overline{C}(S_{3,d}) \right) \left(-2(\overline{p} - p_{n-2}) + 4p_{n-3} + p_{n-4} + \overline{\beta} \right) \\ &= 0. \end{aligned}$$

Therefore, $CTV(\sigma_3^*) = CTV(\omega_3^*)$. Since ω_3^* is of the form (Sequence 1), (13) contradicts the equation $CTV(\sigma_3^*) = CTV(\omega_3^*)$.

(ii) Let σ_4^* be optimal and proceed by contradiction. We define its dual sequence as

$$\sigma_{4,d}^* = (n,\ n-1,\ n-3,\ n-4,\ j_{n-5},\ j_{n-6},\ \dots,\ j_2,\ j_1,\ n-2).$$

We now denote $x = p_{j_1}$ (namely, the unknown processing time of job j_1) and observe that by (13), we must have $p_{n-4} > x$. Subsequently, to obtain a contradiction, we generate two lower bounds and an upper bound of $\overline{C}(\sigma_4^*)$ and compare them. To generate the upper bound of $\overline{C}(\sigma_{4,d}^*)$, let us consider $\sigma_{4,d}^*$ and the following sequence:

$$\omega = (n,\ n-1,\ n-3,\ j_1,\ j_{n-5},\ j_{n-6},\ \dots,\ j_2,\ n-4,\ n-2).$$

By comparing the variances of $\sigma_{4,d}^*$ and ω, we obtain that:

$$\begin{aligned} CTV(\omega) - CTV(\sigma_{4,d}^*) = & \frac{p_{n-4} - x}{n} \times (-10\overline{C}(\sigma_{4,d}^*) + 10p_n + 8p_{n-1} + 2p_{n-2} + 6p_{n-3} \\ & + 4p_{n-4} + 4\overline{\beta} + 5\tfrac{n-5}{n}(p_{n-4} - x)). \end{aligned}$$

Based on the initial assumption, $CTV(\omega) > CTV(\sigma_{4,d}^*)$, so that with few operations we get

$$\overline{C}(\sigma_{4,d}^*) < \overline{UB}(x)$$

where

$$\overline{UB}(x) = \frac{1}{10} \left(10p_n + 8p_{n-1} + 2p_{n-2} + 6p_{n-3} + 4p_{n-4} + 4\overline{\beta} + 5\frac{n-5}{n}(p_{n-4} - x) \right).$$

Similarly, to generate the lower bound of $\overline{C}(\sigma^*_{4,d})$, let us consider $\sigma^*_{4,d}$ and the following sequence:

$$\omega = (n,\ n-1,\ n-3,\ n-4,\ j_1,\ j_{n-5},\ j_{n-6},\ \ldots,\ j_2,\ n-2)\,.$$

By comparing the variances of $\sigma^*_{4,d}$ and ω, we obtain that:

$$CTV(\omega) - CTV(\sigma^*_{4,d}) = 2\frac{5x+\overline{\beta}}{n}\overline{C}(\sigma^*_{4,d}) + \frac{B(x)}{n}$$

where

$$\begin{aligned} B(x) = \;& 5\frac{n-5}{n}x^2 - 2x\left(5p_n + 4p_{n-1} + p_{n-2} + 3p_{n-3} + 2p_{n-4} + \frac{n+5}{n}\overline{\beta}\right) \\ & -\overline{\beta}\left(2(p_n + p_{n-1} + p_{n-3} + p_{n-4}) + \overline{\beta}\frac{n-1}{n}\right). \end{aligned}$$

Again, based on the initial assumption, $CTV(\omega) > CTV(\sigma^*_4)$. In this case, we can build two lower bounds:

$$\begin{cases} \overline{C}(\sigma^*_4) \geq \overline{LB}_1(x) = -\frac{B(x)}{2(5x+\overline{\beta})} \\ \overline{C}(\sigma^*_4) \geq \overline{LB}_2(x) = \frac{1}{n}\left(np_n + (n-1)p_{n-2} + (3n-4)p_{n-3} + \frac{3n-7}{2}p_{n-4} + \frac{n+5}{2}\overline{\beta} - x\right) \end{cases}$$

where

$$\begin{aligned} \overline{LB}_1(x) = \;& -\frac{n-5}{2n}x + \frac{1}{10}\left(10p_n + 8p_{n-1} + 2p_{n-2} + 6p_{n-3} + 4p_{n-4} + \frac{3n+5}{n}\overline{\beta}\right) \\ & +\overline{\beta}\frac{2(p_{n-1} - p_{n-2}) + 4p_{n-3} + 6p_{n-4} + 2\overline{\beta}}{10(5x+\overline{\beta})}, \end{aligned}$$

and $\overline{LB}_2(x)$ originates from the fact that for each $h = j_{n-5}, \ldots, j_3$, we have $C_h(\sigma^*_4) > C_{n-4}(\sigma^*_4) = p_n + \overline{p} + p_{n-3} + p_{n-4}$. Then, combining $UB_{\overline{C}}(x)$ and $LB_{\overline{C},1}(x)$ by using $x \leq p_{n-5} \leq p_{n-4}$, we obtain that

$$\begin{cases} a\overline{\beta}^2 + b\overline{\beta} - 25c < 0, \text{ where} \\ a = 2n+5 \\ b = 8np_{n-3} - (3n-50)p_{n-4} > 0 \\ c = (n-5)p^2_{n-4}. \end{cases}$$

Since $\overline{\beta} > 0$, we deduce the following

$$\begin{cases} \overline{\beta} < \frac{\sqrt{b^2+100ac}-b}{2a}, \text{ where} \\ a = 2n+5 \\ b = 8np_{n-3} - (3n-50)p_{n-4} > 0 \\ c = (n-5)p^2_{n-4}. \end{cases} \tag{14}$$

Using $\overline{UB}(x)$ and $\overline{LB}_2(x)$, we obtain that

$$8(n-5)p_{n-3} + 2(n-5)p_{n-4} + 5(n-7)x < (3n-25)\overline{\beta}.$$

If $n \leq 8$, then the last inequality implies that $15(n - \frac{85}{15})x < 0$ which contradicts that $n \geq 6$. Hence, $n \geq 9$ and

$$\overline{\beta} > 2\frac{n-5}{3n-25}\left(4p_{n-3} + p_{n-4}\right). \tag{15}$$

Therefore, (14) and (15) imply that after simplification:

$$(3n-25)^2 > 4(2n+5)(n-5) + (n+10)(3n-25)$$

which implies that $n \leq 6$. This is in contradiction with $n \geq 9$.

This completes the proof.

Proof of Theorem 2. Let $\tilde{\tilde{p}}_{n-1}$ be the lower bound of p_{n-1} such that for each $p_{n-1} \geq \tilde{\tilde{p}}_{n-1}$, the optimal solution is of the form (Sequence 3). By Theorem 1, we have $\tilde{\tilde{p}}_{n-1} \geq \overline{p}$. Assume $\tilde{\tilde{p}}_{n-1} > \overline{p}$. Consequently, for each $p_{n-1} < \tilde{\tilde{p}}_{n-1}$, the optimal solution is of the form (Sequence 1) or (Sequence 2) or (Sequence 4), we have:

$$\min\{CTV(\sigma_1^*), CTV(\sigma_2^*), CTV(\sigma_4^*)\} < CTV(\sigma_3^*). \tag{16}$$

Let $p_{n-1} = \overline{p}$; we distinguish the following cases.

1) If σ_1^* is optimal, we have

$$CTV(\sigma_1^*) = CTV\,(n,\ n-2,\ n-3,\ n-4,\ j_{n-5},\ \ldots,\ j_1,\ n-1)$$

which contradicts (16).

2) If σ_4^* is optimal. This case is impossible by case (ii) of Proof of Theorem 1.

3) If σ_2^* is optimal where $\sigma_2^* = (n,\ n-2,\ n-4,\ j_1,\ \ldots,\ j_{n-5},\ n-3,\ n-1)$. Let us consider the following sequence $\omega = (n,\ n-2,\ n-3,\ j_1,\ \ldots,\ j_{n-5},\ n-4,\ n-1)$ (interchanging only the jobs $n-3$ and $n-4$. By comparing the value of CTV, we obtain that

$$\overline{C}(\sigma_2^*) \geq \overline{LB} = \frac{1}{4}\left(4p_n + \overline{p} + 3p_{n-2} + \frac{8}{n}p_{n-3} + 4\frac{n-2}{n}p_{n-4} + 2\overline{\beta}\right). \tag{17}$$

Let $\epsilon > 0$ be a real number and

$$\begin{cases} \overline{p}(\epsilon) = \overline{p} - \epsilon, \\ p_{n-2}(\epsilon) = p_{n-2} + \epsilon, \\ p_{n-3}(\epsilon) = p_{n-3} - \epsilon, \\ p_{n-4}(\epsilon) = p_{n-4} + \epsilon. \end{cases}$$

Denote by $\sigma(\epsilon)$ the following schedule

$$\sigma(\epsilon) = (\overbrace{n,\ n-2,\ n-4}^{\pi_1(\epsilon)}, S, \overbrace{n-3,\ n-1}^{\pi_2(\epsilon)}).$$

Let us consider the following parametric problem:

$$P(\epsilon): \begin{cases} \underset{\omega(\epsilon)}{\text{Minimise}} & CTV(\omega(\epsilon)) \\ \text{s.t.} & i)\ \text{job } n \text{ is to be scheduled on position } 1, \\ & 2i)\ \text{job } n-2 \text{ is to be scheduled on position } 2, \\ & 3i)\ S \text{ are to be scheduled on positions } 4, ..., n-3, \\ & 4i)\ \text{job } n-1 \text{ is to be scheduled on the last position.} \end{cases}$$

Assume that for some $\epsilon > 0$, $\sigma(\epsilon)$ is not optimal of $P(\epsilon)$. Then, the optimal solution is of the form

$$\omega(\epsilon) = (n,\ n-2,\ n-3, S,\ n-4,\ n-1)\,.$$

Let $\overline{\epsilon} > 0$ be defined by

$$\overline{\epsilon} = \sup\{\epsilon > 0 \text{ such that } CTV(\sigma(\epsilon)) < CTV(\omega(\epsilon))\}.$$

By construction, $\overline{\epsilon} < \infty$. Let $x > 0$ be very small such that $x < \frac{p_{n-3}-p_{n-4}}{2} - \overline{\epsilon}$ if $\frac{p_{n-3}-p_{n-4}}{2} - \overline{\epsilon} > 0$. Then, by definition of $\overline{\epsilon}$, we deduce that

$$\begin{cases} CTV(\sigma(\overline{\epsilon})) \leq CTV(\omega(\overline{\epsilon})), \text{ and} \\ CTV(\sigma(\overline{\epsilon}+x)) > CTV(\omega(\overline{\epsilon}+x)). \end{cases} \tag{18}$$

Since $\overline{C}(\sigma(\epsilon)) = \overline{C}(\sigma_2^*) + 2\frac{n-3}{n}\epsilon$ and

$$CTV(\omega(\epsilon)) - CTV(\sigma(\epsilon)) = 8\frac{p_{n-3} - p_{n-4} - 2\epsilon}{n}\left(\overline{C}(\sigma(\epsilon)) - \overline{LB} - \frac{3n-8}{2n}\epsilon\right). \tag{19}$$

Then, system (18) becomes

$$8\frac{p_{n-3} - p_{n-4} - 2\overline{\epsilon}}{n}\left(\overline{C}(\sigma_2^*) - \overline{LB} + \frac{n-4}{2n}\overline{\epsilon}\right) \geq 0, \text{ and} \tag{20}$$

$$8\frac{p_{n-3} - p_{n-4} - 2\overline{\epsilon} - 2x}{n}\left(\overline{C}(\sigma_2^*) - \overline{LB} + \frac{n-4}{2n}(\overline{\epsilon} + x)\right) < 0. \tag{21}$$

Case a) If $p_{n-3} - p_{n-4} - 2\overline{\epsilon} > 0$, then by construction of x, we also have $p_{n-3} - p_{n-4} - 2\overline{\epsilon} - 2x > 0$. Therefore, by (21), we obtain that

$$\overline{C}(\sigma_2^*) < \overline{LB} - \frac{n-4}{2n}(\overline{\epsilon} + x). \tag{22}$$

Hence, by (17) and (22), we obtain that $n < 6$. This is a contradiction with $n \geq 6$.

Case b) If $p_{n-3} - p_{n-4} - 2\overline{\epsilon} < 0$. Therefore, by (20), we obtain that

$$\overline{C}(\sigma_2^*) < \overline{LB} - \frac{n-4}{2n}(\overline{\epsilon} + x). \tag{23}$$

Hence by (17) and (23), we obtain that $n < 6$ which is a contradiction with $n \geq 6$.

Case c) If $p_{n-3} - p_{n-4} - 2\overline{\epsilon} = 0$. Therefore, by (19), we obtain that

$$CTV(\omega(\overline{\epsilon})) = CTV(\sigma(\overline{\epsilon})) = CTV\left(n,\ n-2,\ n-3,\ n-4, S_a,\ n-1\right).$$

Note that $\omega(\overline{\epsilon})$ is of the form σ_1^*, $\sigma(\overline{\epsilon})$ is of the form σ_2^* and $(n,\ n-2,\ n-3,\ n-4, S_d,\ n-1)$ is of the form σ_3^*. Therefore by (16), σ_4^* is optimal for $p_{n-1} = \overline{p}(\overline{\epsilon})$. By the same arguments as Case (ii)-Proof of Theorem 1, we obtain a contradiction. Indeed, in this case, we obtain:

$$\begin{cases} \overline{LB}_1(x) = \dfrac{B(x)}{2(5x + \overline{\beta})} \\ \overline{LB}_2(x) = \frac{1}{n}\left(np_n + (n-1)\overline{p}(\overline{\epsilon}) + p_{n-2}(\overline{\epsilon}) + (n-2)p_{n-3}(\overline{\epsilon}) + (n-3)p_{n-4}(\overline{\epsilon}) + 3\overline{\beta} - x\right) \\ \overline{UB}(x) \ = \ \frac{1}{10}\left(10p_n + 8\overline{p}(\overline{\epsilon}) + 2p_{n-2}(\overline{\epsilon}) + 6p_{n-3}(\overline{\epsilon}) + 4p_{n-4}(\overline{\epsilon}) + 4\overline{\beta} + 5\frac{n-5}{n}(p_{n-4}(\overline{\epsilon}) - x)\right), \end{cases}$$

where

$$\begin{aligned} B(x) = \ & -\frac{n-5}{2n}x + \frac{1}{10}\left(10p_n + 8\overline{p}(\overline{\epsilon}) + 2p_{n-2}(\overline{\epsilon}) + 6p_{n-3}(\overline{\epsilon}) + 4p_{n-4}(\overline{\epsilon}) + \frac{3n+5}{n}\overline{\beta}\right) \\ & + \overline{\beta}\frac{2(\overline{p}(\overline{\epsilon}) - p_{n-2}(\overline{\epsilon})) + 4p_{n-3}(\overline{\epsilon}) + 6p_{n-4}(\overline{\epsilon}) + 2\overline{\beta}}{10(5x + \overline{\beta})}. \end{aligned}$$

By condition $\overline{LB}_1(x) \leq \overline{UB}(x)$, $x \leq \dfrac{p_{n-3} + p_{n-4}}{2}$ and after some simplification, we obtain that:

$$(2n+5)\overline{\beta}^2 + \left[(2n+25)p_{n-3} + (3n+25)p_{n-4}\right]\overline{\beta} - 25(n-5)\left(\frac{p_{n-3} + p_{n-4}}{2}\right)^2 \leq 0. \tag{24}$$

By condition $\overline{LB}_2(x) \leq \overline{UB}(x)$ and after some simplification, we obtain that:

$$(n-5)\left(9p_{n-3} + 11p_{n-4}\right) + 10(n-7)x \leq 2(3n-25)\overline{\beta}. \tag{25}$$

Note that by (25), necessarily $n \geq 9$ because if $n \leq 8$, then $10(3n-17)x < 0$. This implies that $n < 6$, in contradiction with $n \geq 6$. By (24) and(25), we obtain that

$$25(3n-25)^2(p_{n-3}+p_{n-4})^2 \geq (n-5)(2n+5)\left(9p_{n-3}+11p_{n-4}\right)^2$$
$$+(6n-50)\left(9p_{n-3}+11p_{n-4}\right)\left[(2n+25)p_{n-3}+(3n+25)p_{n-4}\right],$$

which implies that $n \leq f(a,b)$, where

$$f(a,b) = \frac{1}{2(9a^2+48ab+43b^2)}\begin{pmatrix} \sqrt{7}(9a+11b)\sqrt{1383a^2+3143ab+1783b^2} \\ -759a^2-1412ab-629b^2 \end{pmatrix}$$

and $a = p_{n-3}$, $b = p_{n-4}$ with $a \geq b > 0$. Since $\max_{a \geq b > 0} f(a,b) \simeq 7.034$, then $n \leq 7$; this is a contradiction with $n \geq 9$.

Proof of Corollary 1. For the case of Theorem 1, we have

$$LB_{NN} = \frac{36}{6n}\min\{CTV(\pi^h) : h \in \{1,2\}\} + \frac{1}{n}LB_0,$$

where $CTV(\pi^h)$ is the completion time variance of the subsequence $\pi^h = (\pi_1^h, \pi_2^h)$. Note that for the case $p_{n-1} \leq \overline{p}$ in Theorem 1, we have

$$(\pi_1^1, \pi_2^1) = ((n, n-2, n-3), (j_{n-5}, n-4, n-1))$$

and

$$(\pi_1^2, \pi_2^2) = ((n, n-2, n-4), (j_{n-5}, n-3, n-1)).$$

After some simplification, we obtain that

$$36CTV(\pi^1) - 36CTV(\pi^2) = -(p_{n-1}-p_{n-2})\frac{p_{n-3}-p_{n-4}}{9} \leq 0.$$

Therefore, $\min\{CTV(\pi^1), CTV(\pi^2)\} = CTV(\pi^1)$. Consequently by (1), we deduce the first case in Corollary 1. The other case is a direct consequence of computing the optimal sequence from Theorem 2.

Proof of Proposition 1. We distinguish two cases.

Case $p_{n-1} \leq \overline{p}$. Let us first show that

$$LB_{NN} = LB_{KCK} + \frac{\left(p_{n-1}-p_{n-2}\right)^2 + \left(p_{n-3}-p_{n-4}\right)^2 + \left(p_{n-1}-p_{n-2}\right)\left(p_{n-3}-p_{n-4}\right)}{3n}, \tag{26}$$

where LB_{KCK} is defined in (2). By Corollary 1, we have

$$LB_{NN} = \frac{6}{n}CTV(\pi) + \frac{1}{2n}L = \frac{6}{n}\left(\frac{\alpha^2+\beta^2+\gamma^2}{12} + \frac{a^2+b^2+ab}{18}\right) + \frac{1}{2n}L$$

where

$$L = \begin{cases} \displaystyle\sum_{i=1}^{r-3}(p_1+\ldots+p_{2i})^2 & \text{if } n = 2r+1 \text{ is odd,} \\ \displaystyle\sum_{i=1}^{r-3}(p_1+\ldots+p_{2i-1})^2 & \text{if } n = 2r+2 \text{ is even} \end{cases} \quad \text{and} \quad \begin{cases} \alpha = \sum_{i=1}^{n-5} p_i \\ \beta = \sum_{i=1}^{n-3} p_i \\ \gamma = \sum_{i=1}^{n-1} p_i \\ a = p_{n-1} - p_{n-2} \\ b = p_{n-3} - p_{n-4}. \end{cases}$$

Therefore, independently of whether n is odd or even, we have that

$$L - 2nLB_{KCK} = -\alpha^2 - \beta^2 - \gamma^2.$$

Consequently,

$$LB_{NN} - LB_{KCK} = \frac{a^2 + b^2 + ab}{3n}.$$

Using (3) and (26), this implies

$$LB_{NN} = LB_{VS} + \frac{\left(p_{n-1} - p_{n-2}\right)^2}{12n} + \frac{\left(p_{n-3} - p_{n-4}\right)^2 + \left(p_{n-1} - p_{n-2}\right)\left(p_{n-3} - p_{n-4}\right)}{3n}.$$

Case $p_{n-1} \geq \overline{p}$**.** In this case, we can express p_{n-1} as follows: $p_{n-1} = p_{n-2} + 2p_{n-3} + p_{n-4}/2 + \alpha/2 + x$, where α is defined in previous case and $x \geq 0$. Then, we have

$$\begin{aligned} 216CTV(\pi) &= \left(\tfrac{5}{2}\alpha + 6p_{n-2} + 6p_{n-3} + \tfrac{7}{2}p_{n-4} + x\right)^2 + \left(\tfrac{5}{2}\alpha + 6p_{n-3} + \tfrac{7}{2}p_{n-4} + x\right)^2 \\ &\quad + \left(\tfrac{5}{2}\alpha + \tfrac{3}{2}p_{n-4} + x\right)^2 + \left(\tfrac{5}{2}\alpha - \tfrac{5}{2}p_{n-4} + x\right)^2 \\ &\quad + \left(\tfrac{13}{2}\alpha + 6p_{n-2} + 12p_{n-3} + \tfrac{11}{2}p_{n-4} + 5x\right)^2 + \left(\tfrac{7}{2}\alpha + \tfrac{5}{2}p_{n-4} - x\right)^2 \\ &= \tfrac{159}{2}\alpha^2 + \alpha\left(108p_{n-2} + 126p_{n-3} + 119p_{n-4}\right) \\ &\quad + \left(6p_{n-2} + 6p_{n-3} + \tfrac{7}{2}p_{n-4} + x\right)^2 + \left(6p_{n-3} + \tfrac{7}{2}p_{n-4} + x\right)^2 \\ &\quad + \left(\tfrac{3}{2}p_{n-4} + x\right)^2 + 2\left(\tfrac{5}{2}p_{n-4} - x\right)^2 \\ &\quad + \left(6p_{n-2} + 12p_{n-3} + \tfrac{11}{2}p_{n-4} + 5x\right)^2. \end{aligned}$$

Since $LB_{NN} = \frac{6}{n}CTV(\pi) + \frac{1}{2n}L$, then after some simplification we obtain:

$$n(LB_{NN} - LB_{VS}) = \tfrac{1}{48}\alpha^2 + \tfrac{\alpha}{72}\left(72(p_{n-2} - p_{n-3}) + 49p_{n-4} + 75x\right) + \tfrac{\Omega}{36}, \tag{27}$$

where

$$\begin{aligned} \Omega &= \left(6p_{n-2} + 6p_{n-3} + \tfrac{7}{2}p_{n-4} + x\right)^2 + \left(6p_{n-3} + \tfrac{7}{2}p_{n-4} + x\right)^2 \\ &\quad + \left(\tfrac{3}{2}p_{n-4} + x\right)^2 + 2\left(\tfrac{5}{2}p_{n-4} - x\right)^2 + \left(6p_{n-2} + 12p_{n-3} + \tfrac{11}{2}p_{n-4} + 5x\right)^2 \\ &\quad - 18(p_{n-3} + p_{n-4})^2 - \tfrac{19}{8}(4p_{n-3} + p_{n-4})^2 \\ &\quad - 18(2p_{n-2} + 2p_{n-3} + \tfrac{3}{2}p_{n-4} + x)^2. \end{aligned}$$

If $\Omega < 0$, then we obtain that

$$48x^2 + 2Ax + B < 0,$$

where $A = (144p_{n-3} + 16p_{n-4}) > 0$ and $B = 35p_{n-4}^2 + 216p_{n-3}p_{n-4} + 360p_{n-3}^2 + 288p_{n-2}p_{n-3} > 0$. This implies that $A^2 - 48B \geq 0$ and consequently

$$0 \leq x < \frac{\sqrt{A^2 - 48B} - A}{48} < 0,$$

which is impossible. Then $\Omega \geq 0$. Hence, by (27) we obtain that

$$LB_{NN} \geq LB_{VS}.$$

Proof of Corollary 2. From the definitions (7), for any $x, y, z \geq 0$ and verifying $x + y + z = 1$, we have

$$\begin{aligned} CTV(\sigma^*) &\geq xCTV_{LB}(\sigma^*) + yCTV_{UB}(\sigma^*) + zCTV_{NN}(\sigma^*) \\ &= x\left[\tilde{\mathcal{F}}^{LB}(\boldsymbol{\beta}^{LB}) + \tfrac{6}{n-6}\left(\overline{C}(\sigma^*) - LB\right)^2\right] + y\left[\tilde{\mathcal{F}}^{UB}(\boldsymbol{\beta}^{UB}) + \tfrac{6}{n-6}\left(\overline{C}(\sigma^*) - UB\right)^2\right] \\ &\quad + z\left[LB_{NN} + \tfrac{6}{n-6}\left(\overline{C}(\sigma^*) - \overline{C}(\pi)\right)^2\right] \\ &= x\left[\tilde{\mathcal{F}}^{LB}(\boldsymbol{\beta}^{LB}) + \frac{6LB^2}{n-6}\right] + y\left[\tilde{\mathcal{F}}^{UB}(\boldsymbol{\beta}^{UB}) + \frac{6UB^2}{n-6}\right] + z\left[LB_{NN} + \frac{6\overline{C}(\pi)^2}{n-6}\right] \\ &\quad + \frac{6}{n-6}\left[Q_0(x,y,z)^2 - Q_1(x,y,z)^2\right] \\ &\geq c_{LB}x + c_{UB}y + c_{NN}z - \frac{6}{n-6}Q_1(x,y,z)^2, \end{aligned}$$

where $Q_0(x,y,z) = \overline{C}(\sigma^*) - xLB - yUB - z\overline{C}(\pi)$ and $Q_1(x,y,z) = xLB + yUB + z\overline{C}(\pi)$. Note that the value of $\overline{C}(\sigma^*)$ is unknown. However, since $Q_0(x,y,z)^2 \geq 0$, the last inequality provides a quadratic expression with respect to x, y and z. Since the inequality is valid for any $x, y, z \geq 0$ verifying $x+y+z=1$, it is also valid for the maximizer of $c_{LB}x + c_{UB}y + c_{NN}z - \frac{6}{n-6}Q_1(x,y,z)^2$.

Proof of Proposition 2. Without loss of generality, we can assume $p_{n-2} - p_{n-1} < 0$. In the opposite case the same result trivially follows from Lemma 2. We first consider the case when the optimal sequence is (Sequence 1). To build $LB_{\overline{C}}$, we note that in this case $\alpha_1 = p_{n-3}$ and $\alpha_2 = p_{n-4}$ and build a new sequence from (Sequence 1) as

$$\sigma = \left(\overbrace{n}^{\pi_1}, \overbrace{(n-2, n-4, j_{n-5}, ..., j_1, n-3)}^{S_d}, \overbrace{n-1}^{\pi_2} \right),$$

where S_d is the dual sequence of $S = (n-2, n-3, j_1, \ldots, j_{n-5}, n-4)$. Then, we have $\overline{C}(\pi) = \frac{MS+p_n}{2}$ and

$$\overline{C}(S) + \overline{C}(S_d) = MS - p_{n-1} + p_n + p_{n-2}.$$

By Lemma 3, we have

$$\begin{aligned} CTV(\sigma^*) &= \tfrac{2}{n}CTV(\pi) + \tfrac{n-2}{n}CTV(S) + \tfrac{2(n-2)}{n^2}\left(\overline{C}(S) - \tfrac{MS+p_n}{2}\right)^2 \\ &\leq \tfrac{2}{n}CTV(\pi) + \tfrac{n-2}{n}CTV(S_d) + \tfrac{2(n-2)}{n^2}\left(\overline{C}(S_d) - \tfrac{MS+p_n}{2}\right)^2. \end{aligned}$$

Lemma 2 establishes that $CTV(S_d) = CTV(S)$, so that with few algebrical operations we conclude that $(\overline{C}(S) - \overline{C}(S_d))(\overline{C}(S) + \overline{C}(S_d) - MS - p_n) \leq 0$. Again, from Lemma 2, we know that $\overline{C}(S) + \overline{C}(S_d) - MS - p_n = p_{n-2} - p_{n-1} < 0$, which implies $\overline{C}(S) \geq \overline{C}(S_d)$. This implies the following:

$$MS - p_{n-1} + p_n + p_{n-2} = \overline{C}(S) + \overline{C}(S_d) \leq 2\overline{C}(S).$$

Hence,

$$\begin{aligned} \overline{C}(\sigma^*) &= \tfrac{1}{n}\left(MS + p_n + (n-2)\overline{C}(S)\right) \\ &\geq \tfrac{1}{2n}\left(2np_n + 2p_{n-1} + 2(n-1)p_{n-2} + 2(n-2)p_{n-3} + (n-1)p_{n-4} + (n-1)\bar{\beta}\right). \end{aligned}$$

This lower bound to $\overline{C}(\sigma^*)$ coincides with $LB_{\overline{C}}$ for the case when the optimal sequence is of the form (Sequence 1). Next, to deduce $UB_{\overline{C}}$, we build the following sequence:

$$\sigma = \left(\overbrace{n, n-2}^{\pi_1}, \overbrace{(n-3, n-4, j_{n-5}, ..., j_1)}^{S_d}, \overbrace{n-1}^{\pi_2} \right),$$

where S_d is the dual sequence of $S = (n-3, j_1, j_2, ..., j_{n-5}, n-4)$. Then, we have $\overline{C}(\pi) = (MS + 2p_n + p_{n-2})/3$ and

$$\overline{C}(S) + \overline{C}(S_d) = MS - p_{n-1} + p_n + p_{n-2} + p_{n-3}.$$

By Lemma 3, we have

$$\begin{aligned} CTV(\sigma^*) &= \tfrac{3}{n}CTV(\pi) + \tfrac{n-3}{n}CTV(S) + \tfrac{3(n-3)}{n^2}\left(\overline{C}(S) - \tfrac{MS+2p_n+p_{n-3}}{3}\right)^2 \\ &\leq \tfrac{3}{n}CTV(\pi) + \tfrac{n-3}{n}CTV(S_d) + \tfrac{3(n-3)}{n^2}\left(\overline{C}(S_d) - \tfrac{MS+2p_n+p_{n-3}}{3}\right)^2. \end{aligned}$$

Invoking Lemma 2 and using the same argument as before, we obtain

$$\overline{C}(S) + \overline{C}(S_d) - (2/3)(MS + 2p_n + p_{n-3}) = -2(p_{n-1} - p_{n-2}) + 4p_{n-3} + p_{n-4} + \bar{\beta} \geq 0,$$

so that $\overline{C}(S) \leq \overline{C}(S_d)$ and

$$\begin{aligned} \overline{C}(\sigma^*) &= \tfrac{1}{n}\left(MS + 2p_n + p_{n-2} + (n-3)\overline{C}(S)\right) \\ &\leq \tfrac{1}{2n}\left(2np_n + 2p_{n-1} + 2(n-1)p_{n-2} + 2(n-2)p_{n-3} + (n-1)p_{n-4} + (n-1)\bar{\beta}\right). \end{aligned}$$

This upper bound to $\overline{C}(\sigma^*)$ coincides with $UB_{\overline{C}}$ for the case when the optimal sequence is of the form (Sequence 1).

We now consider the case when the optimal sequence is (Sequence 2). To build $LB_{\overline{C}}$, we note that in this case $\alpha_1 = p_{n-4}$ and $\alpha_2 = p_{n-3}$ and assume that $p_{n-3} > p_{n-4}$. Then, let σ^* be our sequence (Sequence 2) and construct the following sequence

$$\sigma = (n, n-2, n-3, j_1, j_2, ..., j_{n-5}, n-4, n-1).$$

We define $\delta = p_{n-3} - p_{n-4}$ and compare the variances of σ^* and σ as follows:

$$CTV(\sigma) - CTV(\sigma^*) = \frac{\delta}{n}\left(8\overline{C}(\sigma^*) - 8p_n - 2p_{n-1} - 6p_{n-2} - \frac{16}{n}p_{n-3} - 8\frac{n-2}{n}p_{n-4} - 4\bar{\beta}\right).$$

Since σ^* is optimal, then we obtain the following

$$\overline{C}(\sigma^*) \quad \geq \quad \tfrac{1}{4}\left(4p_n + 2p_{n-1} + 3p_{n-2} + \tfrac{8}{n}p_{n-3} + 4\tfrac{n-2}{n}p_{n-4} + 2\bar{\beta}\right).$$

This lower bound to $\overline{C}(\sigma^*)$ coincides with $LB_{\overline{C}}$ for the case when the optimal sequence is of the form (Sequence 2). Next, to deduce $UB_{\overline{C}}$, let σ^* be again our sequence (Sequence 2) and define $x = p_{i_1}$. We build the following sequence

$$\sigma = (n, n-2, n-4, i_2, ..., i_{n-5}, i_1, n-3, n-1).$$

By comparing their variances, we obtain

$$CTV(\sigma) - CTV(\sigma^*) = -2(5x + \beta)\overline{C}(\sigma^*) + B(x),$$

where

$$B(x) = 5\tfrac{n-5}{n}x^2 + 2x\left(5p_n + 4p_{n-2} + 3p_{n-4} + 2p_{n-3} + p_{n-1} + \tfrac{3n-5}{n}\beta\right) + \beta\left(2p_n + 2p_{n-2} + 2p_{n-4} + \tfrac{n-1}{n}\beta\right).$$

Since σ^* is optimal, we obtain that

$$\overline{C}(\sigma^*) \quad \leq \quad \tfrac{B(x)}{2(5x+\beta)}.$$

The function $x \mapsto \frac{B(x)}{2(5x+\beta)}$ is nondecreasing, then

$$\max_{x\in[p_1,p_{n-5}]}\left(\frac{B(x)}{2(5x+\beta)}\right) = \frac{B(p_{n-5})}{2(\beta + 5p_{n-5})}.$$

Proof of Proposition 3. To build $LB_{\overline{C}}$, when the optimal sequence is of the form (Sequence 3), we first show that $\overline{C}(\sigma^*) \geq \frac{1}{n}F$. Let σ^* be our sequence (Sequence 3), and construct the following sequence

$$\sigma = \left(\overbrace{n, n-2,}^{\pi_1}\overbrace{(n-3, j_{n-5}, ..., j_1, n-4),}^{S_d}\overbrace{n-1}^{\pi_2}\right),$$

where S_d is the dual sequence of $S = (n-3, n-4, j_1, j_2, ..., j_{n-5})$. Then, we have $\overline{C}(\pi) = (MS + 2p_n + p_{n-2})/3$ and

$$\overline{C}(S) + \overline{C}(S_d) = MS - p_{n-1} + p_n + p_{n-2} + p_{n-3}.$$

By Lemma 3, we have

$$\begin{array}{rcl} CTV(\sigma^*) & = & \frac{3}{n}CTV(\pi) + \frac{n-3}{n}CTV(S) + \frac{3(n-3)}{n^2}\left(\overline{C}(S) - \frac{MS+2p_n+p_{n-3}}{3}\right)^2 \\ & \leq & \frac{3}{n}CTV(\pi) + \frac{n-3}{n}CTV(S_d) + \frac{3(n-3)}{n^2}\left(\overline{C}(S_d) - \frac{MS+2p_n+p_{n-3}}{3}\right)^2. \end{array}$$

We have $CTV(S_d) = CTV(S)$. We obtain,

$$\left(\overline{C}(S) - \overline{C}(S_d)\right)\left(\overline{C}(S) + \overline{C}(S_d) - MS - p_n\right) \leq 0.$$

Since $\overline{C}(S) + \overline{C}(S_d) - 2\frac{MS+2p_n+p_{n-3}}{3} = -2(p_{n-1} - p_{n-2}) + 4p_{n-3} + p_{n-4} + \beta < 0$, then $\overline{C}(S) \geq \overline{C}(S_d)$. Hence,

$$\begin{array}{rcl} \overline{C}(\sigma^*) & = & \frac{1}{n}\left(MS + 2p_n + p_{n-2} + (n-3)\overline{C}(S)\right) \\ & \geq & \frac{1}{2n}\left(2np_n + 2p_{n-1} + 2(n-1)p_{n-2} + 2(n-2)p_{n-3} + (n-1)(p_{n-4} + \beta)\right). \end{array}$$

Secondly, to build $LB_{\overline{C}}$, when the optimal sequence is of the form (Sequence 3), we show that $\overline{C}(\sigma^*) \geq \frac{1}{2}\min\left(\frac{A(p_{n-5})}{\beta+5p_{n-5}}, \frac{A(p_1)}{\beta+5p_1}\right)$. To do so, let σ^* be again our sequence (Sequence 3) and consider the following sequence

$$\sigma = (n, n-2, n-3, n-4, j_{n-5}, j_1, j_2, ..., j_{n-6}, n-1).$$

We define $x = p_{j_{n-5}}$ and compare the two variances:

$$CTV(\sigma) - CTV(\sigma^*) = 2(5x+\beta)\overline{C}(\sigma^*) - A(x).$$

Since σ^* is optimal, then we obtain that

$$\overline{C}(\sigma^*) \quad \geq \quad \frac{A(x)}{2(5x+\beta)}.$$

The function $x \mapsto \frac{A(x)}{2(5x+\beta)}$ is not monotone (in general), then

$$\min_{x\in[p_1,p_{n-5}]}\left(\frac{A(x)}{2(5x+\beta)}\right) = \min\left(\frac{A(p_{n-5})}{2(\beta+5p_{n-5})}, \frac{A(p_1)}{2(\beta+5p_1)}\right).$$

To build $UB_{\overline{C}}$, when the optimal sequence is of the form (Sequence 3),we consider again σ^* and the following sequence

$$\sigma = (n, n-2, n-3, n-4, j_2, ..., j_{n-5}, j_1, n-1).$$

We define $x = p_{j_1}$ and compare the two variances:

$$CTV(\sigma) - CTV(\sigma^*) = -2(5x+\beta)\overline{C}(\sigma^*) + B(x).$$

Since σ^* is optimal, then we obtain that

$$\overline{C}(\sigma^*) \quad \leq \quad \frac{B(x)}{2(5x+\beta)}.$$

The function $x \mapsto \frac{B(x)}{2(5x+\beta)}$ is note monotone (in general), then

$$\max_{x\in[p_1,p_{n-5}]}\left(\frac{B(x)}{2(5x+\beta)}\right) = \max\left(\frac{B(p_{n-5})}{2(\beta+5p_{n-5})}, \frac{B(p_1)}{2(\beta+5p_1)}\right).$$

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