

Randomized Assortment Optimization

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When a firm selects an assortment of products to offer to customers, it uses a choice model to anticipate their probability of purchasing each product. In practice, the estimation of these models is subject to statistical errors, which may lead to significantly suboptimal assortment decisions. Recent work has addressed this issue using robust optimization, where the true parameter values are assumed unknown and the firm chooses an assortment that maximizes its worst-case expected revenues over an uncertainty set of likely parameter values, thus mitigating estimation errors. In this paper, we introduce the concept of *randomization* into the robust assortment optimization literature. We show that the standard approach of deterministically selecting a single assortment to offer is not always optimal in the robust assortment optimization problem. Instead, the firm can improve its worst-case expected revenues by selecting an assortment randomly according to a prudently designed probability distribution. We demonstrate this potential benefit of randomization both theoretically in an abstract problem formulation as well as empirically across three popular choice models: the multinomial logit model, the Markov chain model, and the preference ranking model. We show how an optimal randomization strategy can be determined exactly and heuristically. Besides the superior in-sample performance of randomized assortments, we demonstrate improved out-of-sample performance in a data-driven setting that combines estimation with optimization. Our results suggest that more general versions of the assortment optimization problem—incorporating business constraints, more flexible choice models and/or more general uncertainty sets—tend to be more receptive to the benefits of randomization.

1. Introduction

Selecting an assortment of products to offer to customers is a central problem across business operations, with manifold applications in the travel, hospitality, and retail industries. A firm solving this *assortment optimization problem* seeks to maximize expected revenues or a related objective by selecting a subset of possible products to carry, often subject to constraints such as shelf or display space. With the growth of novel online marketing and e-commerce settings where firms can quickly and flexibly adjust displayed assortments, this classic operations problem is attracting increasing research interest.

The difficulty in this problem lies in accounting for customer behavior: If a prospective customer's preferred product is not part of the offered assortment, she may either substitute it for another one or leave with no purchase. In order to capture this demand substitution,

the firm must specify and estimate a customer *choice model*, which describes a customer’s probability of choosing each product for any assortment the firm could offer. Since data on customer behavior in the face of numerous possible assortments is invariably sparse, the estimation of such models is difficult, and estimation errors can lead to significantly suboptimal assortment decisions.

The specification of the choice model involves the familiar bias-variance tradeoff. Most common choice models, such as the multinomial logit (MNL) model (see, *e.g.*, Talluri and van Ryzin 2004), capture customer behavior by imposing strict parametric assumptions. Although this approach results in tractable model estimation and assortment optimization problems that require relatively small amounts of data, imposing restrictive assumptions can introduce a significant bias into estimates (*i.e.*, under-fitting) as well as raise theoretical concerns (such as the independence from irrelevant alternatives, or IIA, property). These pitfalls can be avoided through richer models, such as parametric generalizations of the MNL model, the Markov chain (MC) model (Blanchet et al. 2016), or non-parametric models based on customer preference rankings (Honhon et al. 2012, Bertsimas and Mišić 2019). However, as a result of their added flexibility, more complex models are in turn prone to variance (*i.e.*, over-fitting) unless large amounts of data are available.

The risk of over-fitting the choice model is particularly pernicious when estimates are used to select the best assortment to offer, due to the well-known error-maximization effect of optimization (Smith and Winkler 2006). To address this issue, recent assortment optimization papers have embraced the *robust optimization* paradigm (Rusmevichientong and Topaloglu 2012, Bertsimas and Mišić 2017, Désir et al. 2021). Robust optimization acknowledges the fact that the parameters of the choice model are not known exactly, which is particularly important for complex models with many parameters to be estimated. Instead of point estimates, a decision maker adopting the robust approach specifies an uncertainty set that contains the unknown true parameter values or preferences with a pre-specified confidence. She then chooses an assortment in view of the worst parameter setting within this uncertainty set, thus hedging against over-fitting in model estimation.

In this paper, we study *randomized* robust assortment optimization. That is, we show that the standard approach of deterministically selecting a single assortment to offer is not always optimal in the robust assortment optimization problem. Instead, the decision maker can improve her worst-case expected revenues by selecting an assortment randomly

according to a prudently designed probability distribution. We demonstrate this potential benefit of randomization both theoretically in an abstract problem formulation as well as empirically across three popular choice models: the MNL model, the MC model, and the preference ranking model.

We first study an abstract robust assortment optimization problem whose choice model can represent any (reasonable or unreasonable) customer behavior. We show that for randomly drawn instances of this problem, the decision maker benefits from randomizing her choice with probability approaching 1 when the number of admissible assortments and/or the uncertainty about the problem parameters increase. That is, the problem becomes *randomization-receptive*. Our analysis, while stylized in some of its elements in order to ease exposition, allows us to conclude that in the context of robust assortment optimization, randomization-receptiveness can be expected to be the norm rather than the exception.

For the ubiquitous MNL model, we show that its randomization-receptiveness hinges on whether the assortment choice is constrained. In the absence of constraints, the decision maker never benefits from randomizing her choice, that is, the model is *randomization-proof*. By contrast, when the problem is subjected to a cardinality constraint, the MNL model becomes randomization-receptive. We show that not only can the decision maker benefit from choosing an assortment randomly, but the gain from using a randomized strategy over a deterministic one can be arbitrarily large.

The MC model can be randomization-receptive even in the absence of cardinality constraints, but this depends on the characterization of the uncertainty faced by the firm. We show that the MC model is randomization-proof for uncertainty sets that exhibit a specific rectangularity property, which we call product-wise substitution sets. We establish this result through a novel interpretation of the associated assortment optimization problem as a robust Markov decision process (MDP), which allows us to directly apply results from the robust MDP literature. By contrast, for general uncertainty sets the MC model becomes randomization-receptive. Here, the benefits of randomization are bounded in the unconstrained case, and they can again become unbounded in the cardinality-constrained setting.

The preference ranking model is also randomization-receptive even in the absence of cardinality constraints. Here again, the benefits of randomization can be unbounded in the constrained setting, whereas they remain bounded in the unconstrained case. We also

establish the existence of parsimonious randomization strategies in the preference ranking model: Even though there are exponentially many possible assortments, there are always optimal strategies that randomize between at most $K + 1$ assortments, where K is the number of considered preference rankings. This parsimony is absent in the constrained MNL and MC models, where all optimal randomization strategies can become arbitrarily complex.

For all three choice models, we also show how an optimal randomization can be determined exactly (by a column generation scheme) and heuristically (through a local search algorithm). We illustrate the runtimes of these solution schemes, as well as the potential benefits of randomization, on synthetic instances of the cardinality-constrained MNL, MC and preference ranking models. For the MNL model, we also demonstrate how the superior in-sample (worst-case) performance of randomized assortments can translate into an improved out-of-sample (expected) performance in a data-driven setting that combines estimation with optimization. Our synthetic experiments are complemented by a data-driven study on a previously published real-life data set, where we show that randomized assortments significantly outperform their deterministic counterparts. A key insight that emerges from our analysis is that more general versions of the assortment optimization problem—where the generality can be owed to the presence of business constraints, more flexible choice models and/or more general uncertainty sets—tend to be more receptive to the benefits of randomization.

It is important to recognize that randomization may not be practical in all settings, such as for a small brick-and-mortar retailer with a limited number of stores. The natural application for randomized assortment decisions is in e-commerce and other online settings, where different forms of randomization are already being applied (*e.g.*, in A/B testing various aspects of the sales experience). In these contexts, it is easy for a firm to quickly and flexibly vary the assortment displayed to a customer. Even in conventional retail settings, firms are experimenting with novel strategies adopted from online settings such as A/B testing different aspects of the store design.¹ In a similar vein, a randomized assortment choice could be implemented across a chain of otherwise homogeneous retail stores. Our results show how, in such settings, a firm may benefit from choosing an assortment randomly according to a prudently designed probability distribution.

¹ See, *e.g.*, <https://www.bain.com/insights/successful-a-b-tests-in-retail-hinge-on-these-design-considerations/>.

The remainder of the paper proceeds as follows. We review the relevant related literature in Section 2. We next introduce the nominal, deterministic robust and randomized robust assortment optimization problems, as well as our notions of randomization-receptiveness/-proofness, in Section 3, where we also study the randomization-receptiveness of an abstract robust assortment optimization problem. Sections 4–6 study the benefits of randomization, as well as exact and heuristic schemes to determine randomized assortments, under the MNL, MC and preference ranking models. Section 7 presents numerical results, and Section 8 concludes the paper. All proofs, the algorithms to compute randomized assortments, as well as details of the numerical experiments are relegated to the appendix.

Notation. We refer to the sets of non-negative and strictly positive real numbers by \mathbb{R}_+ and \mathbb{R}_{++} , respectively. We refer to the vector of all ones and the i -th canonical basis vector as \mathbf{e} and \mathbf{e}_i , respectively; in both cases, the context will dictate the dimension of these objects. For a finite index set \mathcal{X} , we let $\Delta(\mathcal{X}) = \{\mathbf{p} \in \mathbb{R}_+^{|\mathcal{X}|} : \sum_{x \in \mathcal{X}} p_x = 1\}$ denote the associated probability simplex. The Hadamard (element-wise) product is denoted by ‘ \circ ’.

2. Related Literature

This paper is related to the extensive literature on assortment optimization and choice models: Talluri and van Ryzin (2006), Kök et al. (2015), and Gallego and Topaloglu (2019) provide comprehensive overviews of this field. This literature has sought to resolve the twin problems of accurately capturing customer demand substitution using discrete choice models and efficiently finding the corresponding optimal assortment. The most popular choice model is the MNL model dating back to the work of Luce (1959) and Plackett (1975). Although the MNL model is liable to under-fitting data and suffers from the IIA property, it remains popular as both its estimation and the resulting assortment optimization problem can be solved efficiently (Talluri and van Ryzin 2004), even under a cardinality constraint on the size of the offered assortment (Rusmevichientong et al. 2010, Davis et al. 2013).

The literature has proposed a number of richer choice models to account for the MNL model’s shortcomings. These include generalizations of the MNL model, such as the nested logit (Williams 1977, Davis et al. 2014) and mixture of MNL models (Rusmevichientong et al. 2014), which however come at the cost of more difficult estimation and optimization. Recently, two more general classes of choice models have been proposed. Blanchet et al. (2016) develop a tractable Markov chain model that approximates a number of parametric

models; a similar idea was used in a simulation study in [Zhang and Cooper \(2005\)](#). [Feldman and Topaloglu \(2017\)](#) consider the MC model in network revenue management, [Désir et al. \(2020\)](#) study the constrained assortment optimization problem, and [Şimşek and Topaloglu \(2018\)](#) propose a method to estimate its parameters. The second class of models is based on preference rankings, early examples of which include [Mahajan and van Ryzin \(2001\)](#) and [Rusmevichientong et al. \(2006\)](#). This approach considers customer preferences through distributions over preference lists, which allows very general preference structures without imposing a parametric model. [Farias et al. \(2013\)](#) and [van Ryzin and Vulcano \(2015, 2017\)](#) study the estimation of preference ranking models. Although the assortment selection problem is intractable for general preference ranking models ([Aouad et al. 2018](#)), special cases ([Honhon et al. 2012](#), [Paul et al. 2018](#), [Aouad et al. 2021](#)) can be solved efficiently. [Bertsimas and Mišić \(2019\)](#) consider the closely related problem of product line design under this model and propose a mixed-integer optimization based solution approach.

While more complex choice models reduce the bias in estimates, their added flexibility conversely tends to make them prone to variance (over-fitting). This concern is particularly acute when the estimate feeds into the assortment optimization problem due to the error-maximization effect of optimization ([Smith and Winkler 2006](#)), which is well known in finance ([Michaud 1989](#), [DeMiguel and Nogales 2009](#)) and machine learning (see, *e.g.*, [Bishop 2006](#), [Hastie et al. 2009](#)). The robust optimization approach ([Ben-Tal et al. 2009](#), [Bertsimas et al. 2011](#)) explicitly recognizes that estimation should not produce a single point estimate for the choice model parameters but rather an uncertainty set in which the parameters lie with a pre-specified confidence. By selecting the optimal assortment in view of the worst parameter setting deemed plausible, the robust approach hedges against over-fitting and can thus be seen as a form of regularization ([El Ghaoui and Le Bret 1997](#), [Xu et al. 2009](#)). Robust optimization has been applied to a wide array of operational problems in revenue management ([Birbil et al. 2009](#), [Perakis and Roels 2010](#)), portfolio selection ([Goldfarb and Iyengar 2003](#), [Bertsimas and Sim 2004](#)), inventory management ([Bertsimas and Thiele 2006](#)), facility location ([Baron et al. 2011](#)), and appointment scheduling ([Mak et al. 2015](#)).

The robust approach has proved successful in accounting for parameter uncertainty in choice models. The estimation procedure of [Farias et al. \(2013\)](#) is based on obtaining a worst-case revenues estimate among preference distributions. [Rusmevichientong and](#)

Topaloglu (2012), Désir et al. (2021), and Bertsimas and Mišić (2017) study the robust assortment optimization problem under the MNL, MC, and preference ranking models, respectively. Rusmevichientong and Topaloglu (2012) introduce uncertainty sets over MNL valuations and show that this robust MNL model preserves the feature of the nominal model that revenue-ordered assortments are optimal. Désir et al. (2021) extend this robust approach to the MC model and develop efficient algorithms to solve it. Bertsimas and Mišić (2017) consider the related problem of robust product line design under a preference ranking model with both parameter and structural uncertainty.

In all the aforementioned models, the decision maker deterministically chooses a single assortment to offer. We extend these models by allowing the decision maker to instead randomly choose an assortment according to a probability distribution, and showing when this may benefit her. From a mathematical perspective, the potential benefit from randomization arises from applying robust optimization to a discrete optimization problem (Bertsimas et al. 2016, Delage et al. 2019, Delage and Saif 2021). We show, however, that the superior performance of randomized assortments under this worst-case objective can translate into improved results under the original expected value objective if the model parameters are estimated from data, as is typically the case in practice. Through this insight, as well as our theoretical analysis of the randomization receptiveness of an abstract assortment optimization problem, we also contribute to the robust optimization literature.

While randomized strategies appear to be new in the context of assortment optimization, they have been successfully applied in a growing number of related revenue management contexts. The most prominent example of this are heuristic admission control strategies employed in network revenue management (Reiman and Wang 2008, Jasin and Kumar 2012, Jasin 2015, Bumpensanti and Wang 2020) to decide which assortment (*e.g.*, of flight legs) to offer to customers over time (or which classes of customers to admit) under uncertainty and resource constraints. These papers show that judiciously resolving a deterministic approximation of the problem can provide guarantees on expected revenues losses, when the firm probabilistically decides which assortment to offer, which is referred to as the allocation control (PAC) heuristic. Ferreira et al. (2018) study a retailer’s dynamic pricing problem with inventory constraints and learning about customer demand. The retailer balances an exploration-exploitation tradeoff by selecting a distribution over a discrete set of prices to offer. In a recent paper, Ma (2021) compares the decision to offer an assortment

with the use of lotteries where participating buyers do not know which product they will eventually receive. Drawing on the mechanism design literature, he provides conditions for offering an assortment to be optimal compared to such randomized allocation mechanisms. We add to this emerging stream of literature by showing the value of optimal randomized strategies in the classic assortment optimization problem.

3. The Robust Assortment Optimization Problem

In this section, we define the generic robust assortment optimization problem, and we study whether using a randomized strategy may benefit a firm solving this problem. We show that under specific assumptions, random instances of the problem benefit from randomization with high probability. In subsequent sections, we study three subclasses of this problem that are characterized by prevalent choice models.

In the *nominal assortment optimization problem*, a firm chooses an assortment to offer to customers out of N candidate products. A customer then either buys one of the offered products or makes no purchase, according to a discrete choice model. Formally, an instance of the problem is a tuple $(\mathcal{N}, \mathcal{S}, \mathfrak{C}, \mathbf{r})$ defined as follows. Let $\mathcal{N} = \{1, \dots, n\}$ denote the set of products, and let $\mathcal{N}_0 = \mathcal{N} \cup \{0\}$ be the extended set that contains the no-purchase alternative indexed 0. We denote by $\mathcal{S} \subseteq \{S : S \subseteq \mathcal{N}\}$ the set of admissible assortments, which may exclude some subsets of \mathcal{N} due to (*e.g.*, cardinality) constraints. When the problem is unconstrained, \mathcal{S} is the power set of \mathcal{N} . The choice model is a mapping $\mathfrak{C} : \mathcal{S} \rightarrow \Delta(\mathcal{N}_0)$ such that $\mathfrak{C}(i|S) = 0$ for all $i \in \mathcal{N} \setminus S$, $S \in \mathcal{S}$. The price of product $i \in \mathcal{N}$ is $r_i > 0$; we also include the price of the no-purchase option, $r_0 = 0$, in the vector of product prices $\mathbf{r} \in \mathbb{R}_+^{n+1}$. The firm chooses an assortment $S^* \in \mathcal{S}$ that maximizes its expected revenues $R(S) = \sum_{i \in \mathcal{N}} r_i \cdot \mathfrak{C}(i|S)$ among all admissible assortments $S \in \mathcal{S}$:

$$R_{\text{nom}}^* = \max_{S \in \mathcal{S}} R(S). \quad (\text{NOMINAL})$$

In reality, the choice model is typically estimated from data and hence uncertain. The *robust assortment optimization problem* takes this uncertainty into account. An instance of this problem is a tuple $(\mathcal{N}, \mathcal{S}, \mathfrak{C}, \mathcal{U}, \mathbf{r})$, where the set of products \mathcal{N} and the set of admissible assortments \mathcal{S} are defined as before. The choice model $\mathfrak{C} : \mathcal{S} \times \mathcal{U} \rightarrow \Delta(\mathcal{N}_0)$, however, now depends on a realization $u \in \mathcal{U}$ from an uncertainty set \mathcal{U} (*e.g.*, over choice model parameters). To avoid technicalities, we assume that \mathcal{U} is a compact subset of a

finite-dimensional space. We stipulate that $\mathfrak{C}(i|S, u) = 0$ for all $i \in \mathcal{N} \setminus S$, $S \in \mathcal{S}$ and $u \in \mathcal{U}$. The *deterministic robust assortment optimization problem* (see, e.g., Rusmevichientong and Topaloglu 2012) then chooses the assortment that maximizes the worst-case expected revenues

$$R_{\text{det}}^*(\mathcal{U}) = \max_{S \in \mathcal{S}} R^*(S), \quad (\text{DETERMINISTIC ROBUST})$$

where $R^*(S) = \min_{u \in \mathcal{U}} R(S, u)$ and $R(S, u) = \sum_{i \in S} r_i \cdot \mathfrak{C}(i|S, u)$. When the uncertainty set $\mathcal{U} = \{u^0\}$ is a singleton, the choice model is known exactly and **DETERMINISTIC ROBUST** recovers the classical **NOMINAL** problem.

Instead of selecting a single assortment to offer, the firm could offer *each* assortment $S \in \mathcal{S}$ with a probability $p_S \geq 0$. In this paper, we propose this *randomized robust assortment optimization problem*:

$$R_{\text{rand}}^*(\mathcal{U}) = \max_{\mathbf{p} \in \Delta(\mathcal{S})} R^*(\mathbf{p}), \quad (\text{RANDOMIZED ROBUST})$$

where $R^*(\mathbf{p}) = \min_{u \in \mathcal{U}} R(\mathbf{p}, u)$ and $R(\mathbf{p}, u) = \sum_{S \in \mathcal{S}} p_S \cdot R(S, u)$. Note that the deterministic assortments $S \in \mathcal{S}$ correspond to degenerated randomized assortments $\mathbf{e}_S \in \Delta(\mathcal{S})$ that place all probability mass onto individual assortments, and thus the feasible region of **RANDOMIZED ROBUST** contains the feasible solutions of **DETERMINISTIC ROBUST**.

3.1. Can Randomization Be Beneficial?

We now study whether the decision maker in the robust assortment optimization problem can benefit from randomized strategies for generic choice models and uncertainty sets \mathfrak{C} and \mathcal{U} . In later sections, we consider the value of randomization under three common choice models: the MNL model, the MC model, and the preference ranking model, each of which implies different definitions for \mathfrak{C} and \mathcal{U} .

Since the **RANDOMIZED ROBUST** problem subsumes the **DETERMINISTIC ROBUST** problem, clearly $R_{\text{rand}}^*(\mathcal{U}) \geq R_{\text{det}}^*(\mathcal{U})$. Our goal is to show under what conditions randomization can benefit the firm, that is, the strict inequality $R_{\text{rand}}^*(\mathcal{U}) > R_{\text{det}}^*(\mathcal{U})$ is satisfied. We say the problem is then receptive to randomization, as per the following definition.

DEFINITION 1 (RANDOMIZATION-RECEPTIVENESS/PROOFNESS). An instance of the robust assortment optimization problem is *randomization-receptive* if $R_{\text{rand}}^*(\mathcal{U}) > R_{\text{det}}^*(\mathcal{U})$ and *randomization-proof* otherwise. Likewise, we say that the (un-)constrained robust assortment optimization problem under a particular choice model is randomization-proof if all of its instances are randomization-proof, and it is randomization-receptive otherwise.

We note that **NOMINAL** is randomization-proof by construction. Indeed, for singleton uncertainty sets $\mathcal{U} = \{u^0\}$, **RANDOMIZED ROBUST** reduces to a linear program that attains its optimal value at an extreme point of the probability simplex $\Delta(\mathcal{S})$. This in turn corresponds to a randomization that places unit probability onto a single assortment $S \in \mathcal{S}$. The next example illustrates that the robust assortment optimization problem may indeed be randomization-receptive, but this is not always the case.

EXAMPLE 1. Suppose there are three assortments to offer and two possible uncertainty realizations $\mathcal{U} = \{u, u'\}$. Let the expected revenues for offering each assortment under \mathcal{U} be $(R(S_1, u), R(S_2, u), R(S_3, u)) = (2, 0, 1)$ and $(R(S_1, u'), R(S_2, u'), R(S_3, u')) = (1/2, 3/2, 1)$. The left-hand graph in Figure 1 shows that this problem is *randomization-proof*. The figure depicts the expected revenues for all possible randomizations over the three assortments for u and u' ; worst-case expected revenues are shown as solid regions. The optimal deterministic solution is $\mathbf{e}_{S_3} = (0, 0, 1)$ (along the vertical axis) with $R_{\text{det}}^*(\mathcal{U}) = 1$. No randomization achieves strictly higher worst-case expected revenues: $R_{\text{rand}}^*(\mathcal{U}) = R_{\text{det}}^*(\mathcal{U})$.

The problem in the right-hand graph, on the other hand, is *randomization-receptive*. The expected revenues are now $(R(S_1, u), R(S_2, u), R(S_3, u)) = (1, 1/2, 3/2)$ and $(R(S_1, u'), R(S_2, u'), R(S_3, u')) = (1/2, 2, 1)$. Moving from the deterministic solution \mathbf{e}_{S_3} to an alternative solution with $p_2 > 0$ clearly improves the worst-case expected revenues, and hence we have $R_{\text{rand}}^*(\mathcal{U}) > R_{\text{det}}^*(\mathcal{U})$.

Example 1 establishes that a randomized strategy can benefit the decision maker in the robust assortment optimization problem. The following result formalizes this idea, providing a necessary and sufficient condition for when an instance of the problem is randomization-proof.

THEOREM 1. *Fix a robust assortment optimization instance $(\mathcal{N}, \mathcal{S}, \mathfrak{C}, \mathcal{U}, \mathbf{r})$. The instance is randomization-proof if and only if there is $\mathbb{Q} \in \mathcal{P}(\mathcal{U})$ such that $\mathbb{E}_{\mathbb{Q}}[R(S, \tilde{u})] \leq R_{\text{det}}^*(\mathcal{U})$ for all $S \in \mathcal{S}$, where $\mathcal{P}(\mathcal{U})$ denotes the set of all probability distributions supported on \mathcal{U} .*

In the special case where the uncertainty set \mathcal{U} is finite, the condition of **Theorem 1** simplifies to the existence of a vector $\boldsymbol{\kappa} \in \Delta(\mathcal{U})$ such that $\sum_{u \in \mathcal{U}} \kappa_u \cdot R(S, u) \leq R_{\text{det}}^*(\mathcal{U})$ for all $S \in \mathcal{S}$. The condition in **Theorem 1** guarantees that at an optimal deterministic assortment, there is no direction to move without compromising worst-case expected revenues.

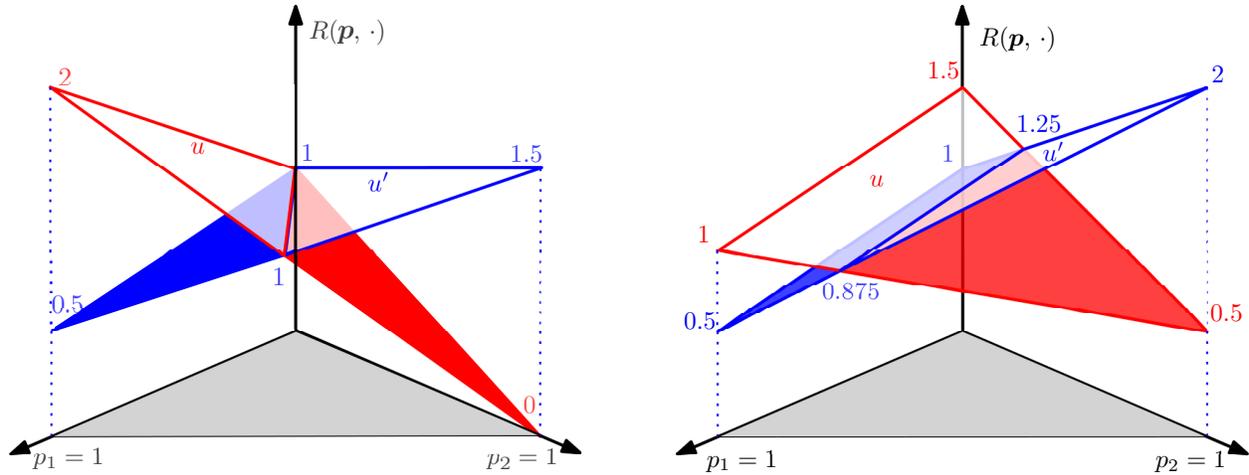


Figure 1 Randomization-proof (left) and randomization-receptive (right) instances of **RANDOMIZED ROBUST**. In both cases, we show two-dimensional projections of the three-dimensional probability simplex $\Delta(S)$ with $S = \{S_1, S_2, S_3\}$ (grey triangles). The red and blue planes illustrate the expected revenues under the two uncertainty realizations u and u' , respectively; solid (hollow) areas indicate regions where the respective realization constitutes the worst (best) case.

In other words, it is necessary and sufficient for the concave maximization problem **RANDOMIZED ROBUST** to be optimized by an extreme point \mathbf{e}_{S^*} , $S^* \in S$. Indeed, \mathbf{e}_{S^*} optimizes **RANDOMIZED ROBUST** if and only if its super-differential

$$\partial R^*(\mathbf{e}_{S^*}) = \text{conv} \{ \mathbf{R} \in \mathbb{R}^{|\mathcal{S}|} : [R_S = R(S, u) \forall S \in \mathcal{S}] \text{ for } u \in \mathcal{U} \text{ such that } R(S^*, u) = R_{\text{det}}^*(\mathcal{U}) \}$$

contains a super-gradient $\mathbf{g} \in \partial R^*(\mathbf{e}_{S^*})$ satisfying $\mathbf{g}^\top (\mathbf{p} - \mathbf{e}_{S^*}) \leq 0$ for all $\mathbf{p} \in \Delta(S)$, and this condition can be shown to be equivalent to the requirement in [Theorem 1](#).

Figure 1 illustrates the condition. In the left graph, the optimality of the deterministic solution \mathbf{e}_{S_3} is certified by the distribution $\mathbb{Q} = \frac{1}{3} \cdot \delta_u + \frac{2}{3} \cdot \delta_{u'}$, so that the expected value $\mathbb{E}_{\mathbb{Q}}[R(S, \tilde{u})]$ equals $R_{\text{det}}^*(\mathcal{U}) = 1$ for all three assortments. Thus, there is no direction at the optimal deterministic solution in which we can move to improve the worst-case expected revenues. In the right graph, such a direction exists. In terms of [Theorem 1](#), any distribution \mathbb{Q} with $\mathbb{Q}[\tilde{u} = u'] < 1$ implies that $\mathbb{E}_{\mathbb{Q}}[R(S_3, \tilde{u})] > R_{\text{det}}^*(\mathcal{U})$, while the distributions satisfying $\mathbb{Q}[\tilde{u} = u'] = 1$ imply $\mathbb{E}_{\mathbb{Q}}[R(S_2, \tilde{u})] = 2 > R_{\text{det}}^*(\mathcal{U})$. Note that the super-differential ∂R^* simplifies to the singleton set containing the gradient when **RANDOMIZED ROBUST** is smooth.

The condition in [Theorem 1](#) is closely related to strong duality, which we will use in subsequent sections when discussing common choice models. Specifically, strong duality

implies the condition, but the reverse is not true. Consequently, as the next result shows, strong duality implies randomization-proofness.

COROLLARY 1. *Suppose that strong duality holds for the problem $(\mathcal{N}, \mathcal{S}, \mathfrak{C}, \mathcal{U}, \mathbf{r})$:*

$$\max_{S \in \mathcal{S}} \min_{u \in \mathcal{U}} R(S, u) = \min_{u \in \mathcal{U}} \max_{S \in \mathcal{S}} R(S, u).$$

Then the problem is randomization-proof.

Intuitively, any optimal deterministic assortment achieves the objective value of the left-hand side of the equation, while the right-hand side of this equation constitutes an upper bound on the worst-case expected revenues achievable by any (deterministic or randomized) assortment strategy: It records the expected revenues that can be achieved if the decision maker knew the uncertainty realization u prior to making an assortment choice. Since this ‘crystal ball’ upper bound is achieved by an optimal deterministic assortment, the decision maker cannot improve by randomizing between assortments.

The reverse statement, however, is not true: A problem instance may be randomization proof even when strong duality does not hold. Indeed, the left-hand example in Figure 1 shows this: we can verify that the ‘crystal ball’ solution is u' with expected revenues 1.5 (at S_2), which is greater than $R_{\text{det}}^*(\mathcal{U}) = 1$. The equivalence between strong duality and the condition in Theorem 1 holds in the special case where the worst-case uncertainty realization is unique. Theorem 1 then simplifies to the following result: The problem is randomization-proof if and only if, under the worst-case realization, the decision maker cannot improve her expected revenues by moving away from the the optimal deterministic assortment S^* .

COROLLARY 2. *Fix a robust assortment optimization problem $(\mathcal{N}, \mathcal{S}, \mathfrak{C}, \mathcal{U}, \mathbf{r})$ and let S^* be an optimal deterministic assortment. If the worst-case parameter set $\arg \min_{u \in \mathcal{U}} R(S^*, u)$ is a singleton $\{u^*\}$, then the problem is randomization-proof if and only if*

$$R^*(S^*) \geq \max_{S \in \mathcal{S}} R(S, u^*).$$

3.2. Should We Expect Randomization to Be Beneficial?

Having established that randomization can benefit the decision maker, we now investigate whether this randomization-receptiveness should be considered an exception or the expected behavior. To this end, we study *random robust assortment optimization problems*

$(\mathcal{N}, \mathcal{S}, \tilde{\mathcal{C}}, \mathcal{U}, \mathbf{r})$ where \mathcal{N} , \mathcal{S} , \mathcal{U} and \mathbf{r} are fixed, but where the choice model $\tilde{\mathcal{C}}$ is random. For simplicity of exposition, our analysis in this section assumes that \mathcal{U} is a finite set and that each choice vector $\tilde{\mathcal{C}}(\cdot|S, u)$, $u \in \mathcal{U}$, is an i.i.d. random vector generated by the probability distribution \mathbb{P}_S , $S \in \mathcal{S}$.²

ASSUMPTION 1. The problem $(\mathcal{N}, \mathcal{S}, \tilde{\mathcal{C}}, \mathcal{U}, \mathbf{r})$ satisfies the following two conditions:

- (i) For all $S \in \mathcal{S}$, the distribution \mathbb{P}_S admits a density f_S satisfying $f_S(\mathbf{x}) \in [\underline{f}, \bar{f}]$ for all $\mathbf{x} \in \Delta(\mathcal{N}_0) : x_i = 0$ for $i \in \mathcal{N} \setminus S$, where $\underline{f}, \bar{f} > 0$.
- (ii) There is $\alpha > \frac{1}{|\mathcal{U}|}$ such that at least $|\mathcal{S}|^\alpha$ of the assortments $S \in \mathcal{S}$ contain a product with price $\max\{r_i : i \in \mathcal{N}\}$.

The first condition in [Assumption 1](#) ensures that for each assortment $S \in \mathcal{S}$, the choice models $\tilde{\mathcal{C}}(\cdot|S, u)$ exhibit sufficient variability across the different uncertainty realizations $u \in \mathcal{U}$. This is intuitive since we noted earlier that the nominal assortment optimization problem is randomization-proof. The second condition in [Assumption 1](#) prevents the existence of a single or a few assortments $S \in \mathcal{S}$ that dominate all other assortments in terms of worst-case expected revenues, as in that case, too, randomization-receptiveness would be less likely. We emphasize that [Assumption 1](#) only provides *sufficient* conditions for randomization-receptiveness, and their choice was guided by simplicity of exposition rather than the aim to provide the tightest possible mathematical characterization.

THEOREM 2. *Under [Assumption 1](#), for every $\delta \in (0, 1)$ there is $N \in \mathbb{N}$ such that the random robust assortment optimization problem $(\mathcal{N}, \mathcal{S}, \tilde{\mathcal{C}}, \mathcal{U}, \mathbf{r})$ is randomization-proof with probability at most δ whenever $\max\{|\mathcal{S}|, |\mathcal{U}|\} \geq N$ and $\min\{|\mathcal{S}|, |\mathcal{U}|\} \geq 2$.*

[Theorem 2](#) shows that under [Assumption 1](#), the random robust assortment optimization problem becomes almost surely randomization-receptive as the number of admissible assortments and/or the cardinality of the uncertainty set increase. Either way of increasing the size of the problem increases the chance of the existence of a randomized assortment with worst-case expected revenues larger than those of any optimal deterministic assortment, leading to randomization-receptiveness in the spirit of [Corollary 2](#).

² Under mild regularity conditions, robust assortment optimization problems with (un)countably infinite uncertainty sets \mathcal{U} can be approximated arbitrarily closely by problems with finite yet sufficiently large sets \mathcal{U} . One can use this reasoning to show that our results extend to general uncertainty sets \mathcal{U} . Since this more general case does not provide further intuition and the analysis becomes obfuscated by technicalities, we omit this generalization.

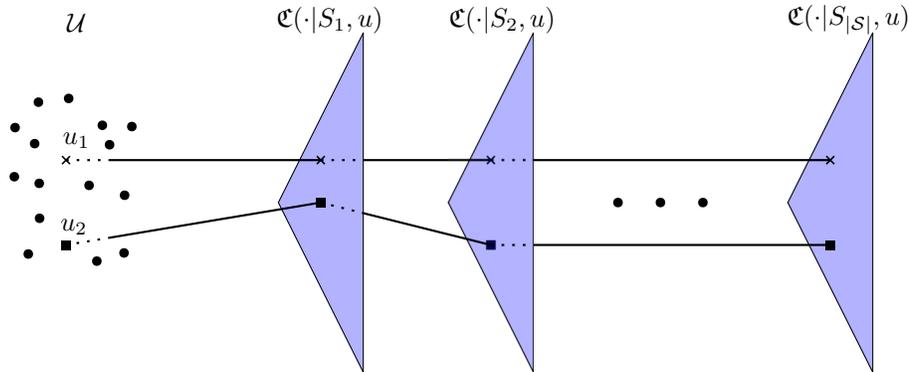


Figure 2 Randomization-receptive choice models are dense in the space of all choice models. A choice model \mathcal{C} specifies for each uncertainty realization $u \in \mathcal{U}$ (black dots on the left) a choice vector $\mathcal{C}(\cdot|S, u)$ for each assortment $S \in \mathcal{S}$ (black dots inside the blue probability simplices on the right). While individual choice models may be randomization-proof, almost all choice models are randomization-receptive once the instance size (as measured by $|\mathcal{S}|$ or $|\mathcal{U}|$) becomes sufficiently large.

In the next sections, we investigate the randomization-receptiveness of the robust assortment optimization problem under the MNL model, the MC model, and the preference ranking model. It is interesting to reconcile the findings of [Theorem 2](#) with our observation that in the absence of business constraints, the problem is in fact randomization-proof under, *e.g.*, the MNL model. The choice models that can be generated under the first condition of [Assumption 1](#) are dense in the space of all choice models in the following sense: The choice models that imply randomization-proofness of the robust assortment optimization problem, independent of the instance size, have measure 0 in the space of all possible choice models (*cf.* [Figure 2](#)). That is, the neighborhood of each randomization-proof choice model comprises almost exclusively of randomization-receptive choice models, assuming sufficiently large instance sizes. In the next sections, we will see that the robust assortment optimization problem is randomization-receptive, as could be expected from the intuition provided in this section, under the constrained MNL model, general MC models, as well as the preference ranking model.

4. Multinomial Logit Model

We first study the robust assortment optimization problem under the MNL model. We show that whether the firm can benefit from randomization depends on the existence of a cardinality constraint on the size of the assortment. The unconstrained problem is randomization-proof ([Section 4.1](#)), while the cardinality-constrained problem is not

only randomization-receptive, but the resulting benefit can also be arbitrarily large (Section 4.2). Section 4.3 develops algorithms for optimally solving the latter problem.

The MNL model is parameterized by a vector of customer product valuations $\mathbf{v} = (v_0, v_1, \dots, v_n) \in \mathbb{R}_{++}^{n+1}$. Given these valuations and an assortment $S \in \mathcal{S}$, a customer purchases product $i \in \mathcal{N}$ with probability

$$\psi_i(S, \mathbf{v}) = \begin{cases} \frac{v_i}{v_0 + \sum_{j \in S} v_j} & \text{if } i \in S, \\ 0 & \text{otherwise,} \end{cases} \quad (1)$$

and the corresponding no-purchase probability is $\psi_0(S, \mathbf{v}) = 1 - \sum_{i \in S} \psi_i(S, \mathbf{v})$. The corresponding expected revenues amount to

$$R(S, \mathbf{v}) = \sum_{i \in S} r_i \cdot \psi_i(S, \mathbf{v}) = \frac{\sum_{i \in S} r_i \cdot v_i}{v_0 + \sum_{i \in S} v_i}, \quad (2)$$

where we identify the parameter vector \mathbf{u} with the product valuations \mathbf{v} . In the corresponding robust assortment optimization problems, we assume that \mathbf{u} is only known to be contained in a compact uncertainty set $\mathcal{U} = \mathcal{V} \subseteq \mathbb{R}_{++}^{n+1}$.

4.1. The Unconstrained Multinomial Logit Problem

Désir et al. (2021) show that the robust assortment optimization problem under the unconstrained MNL model satisfies strong duality, that is, we have

$$\max_{S \in \mathcal{S}} \min_{\mathbf{v} \in \mathcal{V}} R(S, \mathbf{v}) = \min_{\mathbf{v} \in \mathcal{V}} \max_{S \in \mathcal{S}} R(S, \mathbf{v}).$$

Corollary 1 then implies that the problem is randomization-proof.

COROLLARY 3. *The robust assortment optimization problem under the unconstrained MNL model is randomization-proof.*

4.2. The Cardinality-Constrained Multinomial Logit Problem

The randomization-proofness of the unconstrained robust assortment optimization problem under the MNL model relies on the strong duality of the problem, which ceases to hold if we impose a cardinality constraint $|S| \leq C$ on the size of the offered assortment. The next illustrative example demonstrates that as a result of this, the cardinality-constrained robust MNL problem is randomization-receptive.

Table 1 Expected revenues of different assortments in [Example 2](#) (worst-case scenarios highlighted in bold).

Scenario	{1}	{2}	{3}	{1, 2}	{1, 3}	{2, 3}	$\frac{1}{3}\{1, 2\} + \frac{1}{3}\{1, 3\} + \frac{1}{3}\{2, 3\}$	{1, 2, 3}
\mathbf{v}^1	5	5	6.7	6.7	7.5	7.5	7.2	8
\mathbf{v}^2	5	6.7	5	7.5	6.7	7.5	7.2	8
\mathbf{v}^3	6.7	5	5	7.5	7.5	6.7	7.2	8

EXAMPLE 2. Consider the robust MNL problem with three products, $r_1 = r_2 = r_3 = 10$, the cardinality constraint $|S| \leq 2$, and an uncertainty set \mathcal{V} that comprises the three valuation vectors $\mathbf{v}^1 = (1, 1, 1, 2)$, $\mathbf{v}^2 = (1, 1, 2, 1)$, and $\mathbf{v}^3 = (1, 2, 1, 1)$, where the first coordinate of the valuation vector is the valuation of the no-purchase alternative.

We can calculate the worst-case expected revenues of any deterministic assortment using (2). For example, for $S = \{1, 2\}$ under valuation \mathbf{v}^1 , the expected revenues are $R(S, \mathbf{v}^1) = \frac{r_1 \cdot v_1 + r_2 \cdot v_2}{v_0 + v_1 + v_2} = \frac{10 \cdot 1 + 10 \cdot 1}{1 + 1 + 1} = \frac{20}{3} \approx 6.7$. Repeating the calculation for \mathbf{v}^2 and \mathbf{v}^3 shows that \mathbf{v}^1 is the worst case for this assortment. Table 1 shows expected revenues for all deterministic assortments under all valuations, with optimal worst-case expected revenues $R_{\text{det}}^*(\mathcal{V}) = 6.7$. However, randomizing between $\{1, 2\}$, $\{1, 3\}$ and $\{2, 3\}$ with equal probabilities $1/3$ results in $R_{\text{rand}}^*(\mathcal{V}) = \frac{1}{3} \times (6.7 + 7.5 + 7.5) \approx 7.2 > R_{\text{det}}^*(\mathcal{V})$ under any valuation vector. The table also illustrates the randomization-proofness condition in [Corollary 2](#). Each optimal deterministic assortment has a unique worst-case parameter (*e.g.*, \mathbf{v}^1 for $\{1, 2\}$) under which other assortments yield strictly higher expected revenues of $7.5 > 6.7$. Thus, the necessary and sufficient condition in [Corollary 2](#) does not hold, and the problem is randomization-receptive. Note that the unconstrained problem is optimized by the assortment $\{1, 2, 3\}$, which results in worst-case expected revenues of 8 that are higher than any other deterministic revenues and any possible randomization; for the unconstrained problem, the condition of [Corollary 2](#) is therefore again satisfied.

Why does the firm benefit from randomizing the assortment choice once under a cardinality constraint? We can interpret the perhaps surprising finding of [Example 2](#) from the complementary perspectives of diversification in the presence of estimation error and game theory. Under the first interpretation, each of the deterministic assortments exposes the decision maker to significant estimation risk: while the assortment $\{1, 2\}$, say, produces high expected revenues under the favorable valuation scenarios \mathbf{v}^2 and \mathbf{v}^3 , it results in substantially lower expected revenues under the adverse scenario \mathbf{v}^1 . By randomizing between

the three assortments, the decision maker can diversify this risk and alleviate the error maximizing effect of optimization. Notice that such hedging is only valuable under a cardinality constraint $|S| \leq C$. In the unconstrained MNL model, strong duality precludes any diversification benefits: It ensures the existence of a parameter realization $\mathbf{v}^* \in \mathcal{V}$ such that the worst-case expected revenues of an optimal deterministic assortment weakly dominate the expected revenues of all other assortments $S \in \mathcal{S}$ under \mathbf{v}^* .

Under the second interpretation, we can regard the robust assortment optimization problem as a Stackelberg leader-follower game. In this game, the decision maker (leader) selects an assortment S , after which ‘nature’ (follower) responds with the most adverse valuation scenario from within the uncertainty set \mathcal{V} . A Stackelberg leader may benefit from randomized strategies when the follower is *oblivious* (as shown in the literature on security games, see [Korzhyk et al. 2011](#), [Mastin et al. 2015](#), [An et al. 2016](#) and [Bertsimas et al. 2016](#)). That is, rather than observing the actually implemented decision (*i.e.*, the assortment S), the follower can only observe the probabilities with which different decisions are selected (*i.e.*, the randomization weights \mathbf{p}). Under each deterministic assortment in [Example 2](#), the follower can choose an adverse parameter realization \mathbf{v} under which the customer is likely to exercise her no-purchase option, resulting in low worst-case expected revenues. Anticipating these actions, the leader can mitigate them by using a randomized strategy to a significant worst-case benefit.

We now show that the benefit from randomization can indeed be arbitrarily large—but the optimal strategy may require randomization between many different assortments.

THEOREM 3. *For any number of products $n \geq 2$ and any restriction $|S| \leq C$, $C \in \{1, \dots, n-1\}$, there are instances of the cardinality-constrained robust MNL problem where*

1. $R_{\text{det}}^*(\mathcal{V}) = 0$ while $R_{\text{rand}}^*(\mathcal{V}) > 0$;
2. *the unique optimal randomized assortment strategy places equal (positive) probability on each assortment S satisfying $|S| = C$ and zero probability on all other assortments.*

The proof of [Theorem 3](#), which is relegated to the appendix, considers a class of robust constrained MNL instances where all products have equal price and the uncertainty set \mathcal{V} comprises all valuation vectors \mathbf{v} in which exactly B valuations are zero and the remaining valuations are 1. For $B \geq C$, where C is the admissible assortment cardinality, any deterministic assortment S results in worst-case expected revenues of zero since the uncertainty

set contains a vector \mathbf{v} in which $v_i = 0$ for all $i \in S$. The optimal randomized strategy, on the other hand, places equal (positive) probability on all assortments with exactly C products, and zero probability on all other assortments. As long as $B < n$, this randomized strategy raises strictly positive worst-case revenues since there is a strictly positive probability that the offered assortment contains products whose values are strictly positive even under the worst-case valuation realization.

An immediate consequence of the first statement in [Theorem 3](#) is the following.

COROLLARY 4. *For any number of products $n \geq 2$ and any restriction $|S| \leq C$, $C \in \{1, \dots, n-1\}$, there are instances of the cardinality-constrained robust MNL problem where the benefits $R_{rand}^*(\mathcal{V})/R_{det}^*(\mathcal{V})$ from randomization are arbitrarily large.*

Setting $C = n/2$ in the second statement in [Theorem 3](#), we arrive at the following result.

COROLLARY 5. *For any number of products $n \geq 2$, there are instances of the cardinality-constrained robust MNL problem where the unique optimal randomized assortment strategy randomizes between $\Theta(2^n/\sqrt{n})$ many assortments.*

[Corollary 5](#) shows that the optimal randomization strategy may be very complex in that it requires randomization between an exponentially large number of assortments. We next turn to the problem of finding the optimal strategy.

4.3. Solving the Randomized Constrained Multinomial Logit Problem

[Appendix A](#) presents two algorithms to compute randomization strategies for the constrained robust MNL problem with a binary representable uncertainty set of the form

$$\mathcal{V} = \{\mathbf{v} = \mathbf{F}\boldsymbol{\xi} : \mathbf{A}\boldsymbol{\xi} \leq \mathbf{b}, \boldsymbol{\xi} \in \{0, 1\}^m\}, \quad (3)$$

where $\mathbf{F} \in \mathbb{R}^{(n+1) \times m}$, $\mathbf{A} \in \mathbb{R}^{l \times m}$ and $\mathbf{b} \in \mathbb{R}^l$. While any discrete uncertainty set is binary representable, we list some popular representatives that enjoy compact representations.

(i) **Budget uncertainty sets.** For lower and upper valuation bounds $\underline{\mathbf{v}}, \bar{\mathbf{v}} \in \mathbb{R}^{n+1}$ and an uncertainty budget $\Gamma \in \mathbb{N}$, we define the uncertainty set

$$\mathcal{V} = \{\mathbf{v} = \bar{\mathbf{v}} - (\bar{\mathbf{v}} - \underline{\mathbf{v}}) \circ \boldsymbol{\xi} : \mathbf{e}^\top \boldsymbol{\xi} \leq \Gamma, \boldsymbol{\xi} \in \{0, 1\}^{n+1}\}.$$

Under the budget uncertainty set, up to Γ valuations v_i can attain their lower bounds \underline{v}_i , whereas the remaining valuations v_i attain their upper bounds \bar{v}_i .

(ii) **Factor model uncertainty sets.** For a nominal valuation vector $\mathbf{v}^0 \in \mathbb{R}^{n+1}$ and a factor loading matrix $\Phi \in \mathbb{R}^{(n+1) \times m}$, we define the uncertainty set

$$\mathcal{V} = \{ \mathbf{v} = \mathbf{v}^0 + \Phi(2\xi - \mathbf{e}) : \xi \in \{0, 1\}^m \}.$$

Here, the valuations \mathbf{v} differ from their nominal values \mathbf{v}^0 by $\Phi\mathcal{B}_\infty$, where $\mathcal{B}_\infty = \{-1, 1\}^m$ contains the extreme points of the unit ∞ -norm ball in \mathbb{R}^m .

(iii) **Norm uncertainty sets.** For a nominal valuation vector $\mathbf{v}^0 \in \mathbb{R}^{n+1}$ and lower and upper valuation bounds $\underline{\mathbf{v}}, \bar{\mathbf{v}} \in \mathbb{R}^{n+1}$, we define the uncertainty set

$$\mathcal{V} = \left\{ \mathbf{v} = \mathbf{v}^0 + (\bar{\mathbf{v}} - \mathbf{v}^0) \circ \xi^+ + (\underline{\mathbf{v}} - \mathbf{v}^0) \circ \xi^- : \begin{array}{l} \|(\bar{\mathbf{v}} - \mathbf{v}^0) \circ \xi^+ + (\underline{\mathbf{v}} - \mathbf{v}^0) \circ \xi^-\|_p \leq \theta \\ \xi^+ + \xi^- \leq \mathbf{e}, \xi^+, \xi^- \in \{0, 1\}^{n+1} \end{array} \right\}$$

where $p \in \{0, 1, \infty\}$ and $\theta \in \mathbb{R}_+$. Norm uncertainty sets hedge against valuations \mathbf{v} that reside in a θ -neighbourhood of the nominal valuations \mathbf{v}^0 , as measured by the p -norm.

The randomized constrained MNL model can be formulated as a robust linear program. To this end, denote by $\mathcal{S} = \{S \in \mathcal{N} : |S| \leq C\}$ the set of assortments whose cardinality is less or equal to the size restriction C . The problem can then be written as

$$\begin{aligned} & \text{maximize} && \min_{\mathbf{v} \in \mathcal{V}} \sum_{S \in \mathcal{S}} p_S \cdot \frac{\sum_{i \in S} r_i \cdot v_i}{v_0 + \sum_{i \in S} v_i} \\ & \text{subject to} && \sum_{S \in \mathcal{S}} p_S = 1 \\ & && p_S \geq 0, S \in \mathcal{S}. \end{aligned} \tag{4}$$

This problem is computationally challenging since it typically comprises an exponential number of decision variables, and its objective function contains an embedded optimization problem that minimizes a non-convex function over a discrete uncertainty set \mathcal{V} . For later reference, we note that the dual of problem (4) amounts to the robust linear program

$$\begin{aligned} & \text{minimize} && \max_{\mathbf{v} \in \mathcal{V}} \sum_{S \in \mathcal{S}} \kappa_S \cdot \frac{\sum_{i \in S} r_i \cdot v_i}{v_0 + \sum_{i \in S} v_i} \\ & \text{subject to} && \sum_{S \in \mathcal{S}} \kappa_S = 1 \\ & && \kappa_S \geq 0, S \in \mathcal{S}. \end{aligned} \tag{5}$$

Strong duality between (4) and (5) holds since (4) is feasible by construction. Similar to problem (4), problem (5) is computationally challenging due to the typically exponential number of decision variables as well as the embedded maximization over a discrete ‘uncertainty set’ \mathcal{S} whose decision variables S appear in a non-convex objective function.

In Appendix A, we propose an exact column generation scheme as well as a heuristic local search method to solve the randomized constrained MNL problem.

5. Markov Chain Model

Under the MC model proposed by Blanchet et al. (2016), the choice behavior of a customer is described through a vector $\boldsymbol{\lambda} \in \mathbb{R}_+^{n+1}$, where λ_i , $i = 0, \dots, n$, denotes the probability of product i being the most preferred choice, as well as a matrix $\boldsymbol{\rho} = (\rho_{ij})_{i,j} \in \mathbb{R}_+^{(n+1) \times (n+1)}$, where ρ_{ij} , $i, j = 0, \dots, n$, characterizes the probability of the customer substituting product i by product j if product i is not available. In other words, an arriving customer attempts to purchase each product $i = 0, \dots, n$ with probability λ_i . If the preferred product, say i , is not offered, then she attempts to purchase each product $j = 0, \dots, n$ with probability ρ_{ij} , and the process continues as if j had been her preferred choice. We require that $\rho_{00} = 1$ as well as $\rho_{ii} = 0$ and $\rho_{i0} > 0$ for all $i \neq 0$.

Désir et al. (2021) study a robust assortment optimization problem under the MC model where the substitution matrices $\boldsymbol{\rho}$ are only known to reside in an uncertainty set

$$\mathcal{U} \subseteq \left\{ \boldsymbol{\rho} \in \mathbb{R}_+^{(n+1) \times (n+1)} : \sum_{j=0}^n \rho_{ij} = 1 \quad \forall i = 0, \dots, n \right\},$$

which we assume to be compact in order to avoid technicalities. Under this setting, the substitution behavior of the customers can be determined by any substitution matrix $\boldsymbol{\rho} \in \mathcal{U}$, and the decision maker optimizes the expected revenues in view of the worst such matrix. We require that all $\boldsymbol{\rho} \in \mathcal{U}$ satisfy $\rho_{00} = 1$ as well as $\rho_{ii} = 0$ and $\rho_{i0} > 0$ for all $i \neq 0$. In addition, we require that there exists a $\boldsymbol{\rho} \in \mathcal{U}$ and a product $i \neq 0$ satisfying that $\rho_{i0} < 1$.

In the following, Section 5.1 first expresses the unconstrained robust assortment optimization problem under the MC model as a robust Markov decision process (MDP). We then study the benefits of randomization under two different classes of uncertainty sets for the MC model: product-wise substitution sets where no information is available about the dependence of ρ_{ij} and ρ_{kl} whenever $i \neq k$ (Section 5.2) and general substitution sets (Section 5.3). Section 5.4, finally, discusses the solution of the (un-)constrained robust assortment optimization problem under the MC model.

5.1. A Robust MDP Reformulation for the Unconstrained Markov Chain Problem

Our objective is to reformulate the robust assortment optimization problem under the MC model as an instance of a robust MDP as per the following definition.

DEFINITION 2 (ROBUST MDP). A robust MDP is defined by the tuple $(\mathcal{X}, \mathcal{A}, \mathcal{P}, q, c, \gamma)$, where \mathcal{X} denotes the state space, \mathcal{A} represents the action space, $q(x)$, $x \in \mathcal{X}$, characterize the initial state distribution, $c(x, a)$, $x \in \mathcal{X}$ and $a \in \mathcal{A}$, denote the immediate rewards, and $\gamma \in (0, 1)$ is the discount factor. The ambiguity set

$$\mathcal{P} \subseteq \left\{ p : \mathcal{X} \times \mathcal{A} \times \mathcal{X} \rightarrow \mathbb{R}_+ : \sum_{x' \in \mathcal{X}} p(x'|x, a) = 1 \quad \forall x \in \mathcal{X}, \forall a \in \mathcal{A} \right\}$$

contains all transition kernels p that are deemed plausible by the decision maker.

A robust MDP starts in state $x \in \mathcal{X}$ with known probability $q(x)$. The decision maker can then select any action $a \in \mathcal{A}$, upon which an immediate reward of $c(x, a)$ is earned and the MDP transitions to state $x' \in \mathcal{X}$ with probability $p(x'|x, a)$, where p can be any element of the ambiguity set \mathcal{P} . The process then continues in the same fashion, governed by the same transition kernel p , for an infinite length of time. The decision maker wishes to determine a policy $\pi : \mathcal{X} \rightarrow \mathcal{A}$, which declares for each state $x \in \mathcal{X}$ which action $a \in \mathcal{A}$ is to be taken in state x , that maximizes the worst-case expected total discounted reward:

$$\max_{\pi \in \Pi} \min_{p \in \mathcal{P}} \mathbb{E} \left[\sum_{t=0}^{\infty} \gamma^t \cdot r(x_t, a_t) \mid x_0 \sim q \right]$$

Here, Π denotes the set of all deterministic, memoryless policies $\pi : \mathcal{X} \rightarrow \mathcal{A}$, $\{(x_t, a_t)\}_{t=0}^{\infty}$ is the stochastic process induced by the initial probabilities q , the transition probabilities p and the policy π , and \mathbb{E} is the expectation operator with respect to this process.

We next construct a robust MDP for the unconstrained robust assortment optimization problem.

DEFINITION 3 (ROBUST MDP REPRESENTATION). For a given MC model, we define the robust MDP $(\mathcal{X}, \mathcal{A}, \mathcal{P}, q, c, \gamma)$ via the state space $\mathcal{X} = \mathcal{N}_0$, the action space $\mathcal{A} = \{\top, \perp\}$, the ambiguity set \mathcal{P} that contains all transition kernels p satisfying

$$p(x'|x, \top) = \begin{cases} 1 & \text{if } x' = x, \\ 0 & \text{otherwise,} \end{cases} \quad p(x'|x, \perp) = \begin{cases} \rho_{xx'}/\gamma & \text{if } x, x' \neq 0, \\ 1 - \sum_{x \in \mathcal{N}} \rho_{x\chi}/\gamma & \text{if } x' = 0, \\ 0 & \text{otherwise} \end{cases} \quad \forall x, x' \in \mathcal{X} \quad (6)$$

for some $\boldsymbol{\rho} \in \mathcal{U}$, the initial state probabilities $q(x) = \lambda_x$, $x \in \mathcal{X}$, the immediate rewards $c(x, \top) = (1 - \gamma)r_x$ and $c(x, \perp) = 0$, $x \in \mathcal{X}$, and $\gamma = 1 - \min \{\rho_{i0} : \boldsymbol{\rho} \in \mathcal{U}, i \in \mathcal{N}\}$.

Intuitively, the states of the robust MDP describe the different purchase options $x \in \mathcal{N}_0$ of the customer, and the actions \top and \perp characterize the options of the decision maker to include (exclude) any of the products $x \in \mathcal{N}$ in/from the assortment. Note that the virtual product 0 is always available, and hence the selected action does not matter in state 0. If the customer enters a state whose associated purchase option is part of the assortment, then this state keeps generating revenues and is never left. Otherwise, the next purchase option considered by the customer is selected randomly according to some $\rho \in \mathcal{U}$.

We next show that [Definition 3](#) indeed describes a valid robust MDP.

OBSERVATION 1. *The robust MDP from [Definition 3](#) is well-defined.*

We are now ready to establish the equivalence between the unconstrained robust assortment optimization problem under the MC model and our robust MDP from [Definition 3](#).

THEOREM 4. *The unconstrained robust assortment optimization problem under the MC model is equivalent to the robust MDP from [Definition 3](#) in the following sense:*

- (i) *For every $S \in \mathcal{S}$, the worst-case expected total discounted reward of any $\pi_S \in \Pi$ satisfying $\pi_S(i) = \top$, $i \in S$, and $\pi_S(i) = \perp$, $i \in \mathcal{N} \setminus S$, coincides with the revenues $R^*(S)$.*
- (ii) *For every $\pi \in \Pi$, the revenues $R^*(S)$ of $S = \{i \in \mathcal{N} : \pi(i) = \top\}$ coincide with the worst-case expected total discounted reward of π .*

[Theorem 4](#) shows that there is a one-to-many relationship between the assortments $S \in \mathcal{S}$ of the assortment optimization problem and the policies $\pi_S : \mathcal{X} \rightarrow \mathcal{A}$ through the relation $i \in S \Leftrightarrow \pi_S(i) = \top$, $i \in \mathcal{N}$. The relationship is not one-to-one since the no-purchase option 0 is always available, irrespective of whether $\pi_S(0) = \top$ or $\pi_S(0) = \perp$. [Theorem 4](#) allows us to apply the rich arsenal of solution methods for robust MDPs to the unconstrained deterministic robust assortment optimization problem under the MC model with product-wise substitution sets. In the following two subsections, we will leverage the established theory for robust MDPs to investigate under which conditions the decision maker may benefit from randomizing between multiple assortments, that is, when $R_{\text{rand}}^*(\mathcal{U}) > R_{\text{det}}^*(\mathcal{U})$.

REMARK 1 (NOMINAL ASSORTMENT OPTIMIZATION). If the uncertainty set of the robust assortment optimization problem is a singleton, say $\mathcal{U} = \{\rho^0\}$, then the robust MDP from [Definition 3](#) reduces to a nominal MDP. In that case, the conclusions from [Observation 1](#) and [Theorem 4](#) continue to apply.

5.2. Product-Wise Substitution Sets

We say that the uncertainty set \mathcal{U} of a robust assortment optimization problem under the MC model has *product-wise substitution sets* whenever

$$\mathcal{U} = \left\{ \boldsymbol{\rho} \in \mathbb{R}_+^{(n+1) \times (n+1)} : \exists \boldsymbol{\rho}^0, \dots, \boldsymbol{\rho}^n \in \mathcal{U} \text{ such that } \rho_{ij} = \rho_{ij}^i \ \forall i, j \in \mathcal{N}_0 \right\}.$$

One readily verifies that this condition is equivalent to requiring that

$$\mathcal{U} = \bigtimes_{i \in \mathcal{N}_0} \mathcal{U}_i, \quad \text{where } \mathcal{U}_i = \{ \boldsymbol{\rho}_i = (\rho_{i0}, \dots, \rho_{in}) \in \mathbb{R}_+^{n+1} : \boldsymbol{\rho} \in \mathcal{U} \}.$$

Intuitively speaking, the uncertainty set \mathcal{U} has product-wise substitution sets when knowledge of the uncertain substitution probabilities $\boldsymbol{\rho}_i = (\rho_{i0}, \dots, \rho_{in})$ for product $i \in \mathcal{N}_0$ does not allow the decision maker to infer anything about the uncertain substitution probabilities $\boldsymbol{\rho}_j = (\rho_{j0}, \dots, \rho_{jn})$ for any other product $j \in \mathcal{N}_0$, $j \neq i$, beyond the fact that $\boldsymbol{\rho}_j \in \mathcal{U}_j$.

THEOREM 5. *If the uncertainty set \mathcal{U} of the robust assortment optimization problem under the MC model has product-wise substitution sets, the problem is randomization-proof.*

The proof of [Theorem 5](#) shows that for product-wise substitution sets, the robust MDP from [Definition 3](#) has an (x, a) -rectangular ambiguity set, which implies strong duality:

$$\max_{\pi \in \Pi} \min_{p \in \mathcal{P}} \mathbb{E} \left[\sum_{t=0}^{\infty} \gamma^t \cdot r(x_t, a_t) \mid x_0 \sim q \right] = \min_{p \in \mathcal{P}} \max_{\pi \in \Pi} \mathbb{E} \left[\sum_{t=0}^{\infty} \gamma^t \cdot r(x_t, a_t) \mid x_0 \sim q \right].$$

[Corollary 1](#) then implies that the problem is randomization-proof.

[Désir et al. \(2021\)](#) derive a strong duality result for the unconstrained robust assortment optimization problem under the MC model with product-wise substitution sets (termed *row-wise uncertainty* in that work). In contrast to our result, which establishes a connection between the unconstrained assortment optimization problem and robust MDPs and subsequently leverages existing results for robust MDPs, [Désir et al. \(2021\)](#) prove strong duality *ab initio*.

While the *unconstrained* robust assortment optimization problem is randomization-proof whenever \mathcal{U} has product-wise substitution sets, the problem becomes randomization-receptive if we impose a cardinality constraint on the size of the admissible assortments. This is further illustrated in the following example.

Table 2 Expected revenues of different assortments in **Example 3** (worst-case scenarios highlighted in bold).

Scenario	($\uparrow, \uparrow, \uparrow$)	($\uparrow, \uparrow, \downarrow$)	($\uparrow, \downarrow, \uparrow$)	($\uparrow, \downarrow, \downarrow$)	($\downarrow, \uparrow, \uparrow$)	($\downarrow, \uparrow, \downarrow$)	($\downarrow, \downarrow, \uparrow$)	($\downarrow, \downarrow, \downarrow$)
{1}	7.3	7.8	7.8	8.1	7.3	7.8	7.8	8.1
{2}	7.8	7.3	7.8	7.3	8.1	7.8	8.1	7.8
{3}	8.1	8.1	7.8	7.8	7.8	7.8	7.3	7.3
$\sum_i \frac{1}{3} \cdot \{i\}$	7.7	7.7	7.8	7.7	7.7	7.8	7.7	7.7

EXAMPLE 3. Consider an MC model with three products, product-wise prices of $r_i = 10$, $i \in \mathcal{N}$, initial choice probabilities $\lambda = (0, 1/3, 1/3, 1/3)$ and the uncertainty set

$$\mathcal{U} = \left\{ \rho \in \mathbb{R}_+^{4 \times 4} : \rho_0 = (1, 0, 0, 0) \text{ and } \begin{array}{l} \rho_1 \in \{\rho^\uparrow = (0.2, 0, 0.3, 0.5), \rho^\downarrow = (0.2, 0, 0.5, 0.3)\}, \\ \rho_2 \in \{\rho^\uparrow = (0.2, 0.3, 0, 0.5), \rho^\downarrow = (0.2, 0.5, 0, 0.3)\}, \\ \rho_3 \in \{\rho^\uparrow = (0.2, 0.3, 0.5, 0), \rho^\downarrow = (0.2, 0.5, 0.3, 0)\} \end{array} \right\}.$$

The uncertainty set \mathcal{U} contains $2^3 = 8$ substitution matrices, and one readily verifies that \mathcal{U} has product-wise substitution sets. Assume that only assortments with a single product are allowed, that is, $|S| \leq C = 1$. Table 2 lists the expected revenues of all admissible deterministic assortments, as well as one randomized assortment, under all scenarios of \mathcal{U} . The table shows that the randomized assortment outperforms the eligible deterministic assortments in terms of worst-case expected revenues. The randomization-receptiveness of the instance can also be confirmed using [Theorem 1](#), which states that the instance is randomization-proof if and only if there is $\kappa \in \Delta(\mathcal{U})$, $\mathcal{U} = \{\uparrow, \downarrow\}^3$, such that $\sum_u \kappa_u \cdot R(S, u) \leq 7.3$ for all $S \in \mathcal{S}$. For any $u \in \mathcal{U} \setminus \{(\uparrow, \uparrow, \uparrow), (\downarrow, \uparrow, \uparrow)\}$, $\kappa_u > 0$ will lead to $\sum_u \kappa_u \cdot R(\{1\}, u) > 7.3$. When $\kappa_u > 0$ for any $u \in \{(\uparrow, \uparrow, \uparrow), (\downarrow, \uparrow, \uparrow)\}$, on the other hand, we have $\sum_u \kappa_u \cdot R(\{2\}, u) > 7.3$. Hence, there is no $\kappa \in \Delta(\mathcal{U})$ such that the condition holds, which implies that the problem is indeed randomization receptive.

We next show that, similar to the constrained MNL problem (*cf.* [Theorem 3](#)), the potential benefits of randomization in the cardinality-constrained MC problem can be arbitrarily large.

THEOREM 6. *For any number of products $n \geq 3$ and any restriction $|S| \leq C$, $C \in \{1, \dots, n-2\}$, there are instances of the cardinality-constrained robust MC problem with product-wise substitution sets where $R_{\text{det}}^*(\mathcal{U}) = 0$ while $R_{\text{rand}}^*(\mathcal{U}) > 0$.*

Similar to Section 4.2, an immediate consequence of [Theorem 6](#) is the following.

COROLLARY 6. *For any number of products $n \geq 3$ and any restriction $|S| \leq C$, $C \in \{1, \dots, n-2\}$, there are instances of the cardinality-constrained robust MC problem with product-wise substitution sets where the benefits $R_{\text{rand}}^*(\mathcal{U})/R_{\text{det}}^*(\mathcal{U})$ from randomization are arbitrarily large.*

5.3. General Substitution Sets

While product-wise substitution sets may appear to constitute an intuitive choice for the uncertainty set \mathcal{U} , they are unlikely to arise as a result of a statistical estimation from historical data. Instead, a data-driven estimation approach based on a confidence region formed, for example, by a maximum likelihood estimation, would exhibit an asymptotically elliptical shape under which all rows ρ_i of the uncertain substitution matrix ρ are dependent (*cf.* Billingsley 1961). Under such general substitution sets, strong duality no longer holds for the associated robust MDPs (*cf.* Wiesemann et al. 2013), which opens up the possibility for the problem to be randomization-receptive. This does not automatically imply, however, that randomization is beneficial, as the next result shows.

PROPOSITION 1. *Any instance of the unconstrained robust MC problem with two products, irrespective of the geometry of the uncertainty set \mathcal{U} , is randomization-proof.*

On the other hand, we can find instances with three or more products where the unconstrained robust MC problem with general substitution sets is receptive to randomization.

EXAMPLE 4. Consider an MC model with three products, prices $r_1 = 4.66$, $r_2 = 1$ and $r_3 = 10$, initial choice probabilities $\lambda = (0, 0.37, 0.62, 0.01)$, and the uncertainty set

$$\mathcal{U} = \left\{ \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0.01 & 0 & 0.05 & 0.94 \\ 0.26 & 0.69 & 0 & 0.05 \\ 0.90 & 0.05 & 0.05 & 0 \end{array} \right), \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0.90 & 0 & 0.05 & 0.05 \\ 0.01 & 0.44 & 0 & 0.55 \\ 0.01 & 0.94 & 0.05 & 0 \end{array} \right) \right\}.$$

One readily verifies that \mathcal{U} does not have product-wise substitution sets. Table 3 lists the expected revenues of all deterministic assortments, as well as one randomized assortment, under both scenarios in \mathcal{U} . The table shows that the randomized assortment outperforms the deterministic assortments in terms of worst-case expected revenues. Since the optimal deterministic assortment $\{1, 3\}$ has a unique worst-case parameter realization under which the expected revenues of the assortments $\{3\}$ and $\{2, 3\}$ are strictly larger, the randomization-receptiveness is also immediately certified by Corollary 2.

Table 3 Expected revenues of different assortments in [Example 4](#) (worst-case scenarios highlighted in bold).

Assortment	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}	$0.374 \cdot \{3\} + 0.626 \cdot \{1, 3\}$
Scenario 1	3.72	0.66	7.93	2.35	4.13	4.22	2.44	5.68
Scenario 2	4.62	0.64	4.02	2.39	6.57	0.92	2.44	5.61

Intuitively speaking, the adversary ‘nature’ can benefit from knowing the decision maker’s assortment choice if the uncertainty set in the MC model does not have product-wise substitution sets. Since randomizing between different assortments ensures that nature only observes the randomization strategy but not the actual assortment shown to each customer, the decision maker can increase her worst-case expected revenues.

OBSERVATION 2. For any number $n \geq 3$ of products, the unconstrained MC problem is randomization-receptive.

A natural question is whether, similar to the constrained robust MNL and MC problems, the potential benefits of randomization can be unbounded in the unconstrained MC problem under general substitution sets. As the following result shows, this is not the case.

OBSERVATION 3. Under the unconstrained MC problem with general substitution sets, the benefits of randomization are bounded from above by

$$\frac{R_{\text{rand}}^*(\mathcal{U})}{R_{\text{det}}^*(\mathcal{U})} \leq \frac{\max\{r_i : i \in \mathcal{N}\}}{\min\{r_i : i \in \mathcal{N}\}}.$$

We next discuss two immediate consequences of [Observation 3](#).

COROLLARY 7. As long as all products $i \in \mathcal{N}$ carry strictly positive prices $r_i > 0$, the benefits of randomization under the unconstrained MC problem with general substitution sets are bounded. Moreover, the problem is randomization-proof if all products have the same price, that is, if $r_i = r_j$ for all $i, j \in \mathcal{N}$.

In the cardinality-constrained problem with general substitution sets, the benefits are again unbounded: Since general substitution sets encompass product-wise substitution sets as a special case, the findings of [Theorem 6](#) and [Corollary 6](#) immediately apply to the problem with general substitution sets as well. Moreover, similar to [Corollary 5](#), the optimal randomization strategy for the cardinality-constrained robust MC problem may be very complex.

PROPOSITION 2. *For any number of products $n \geq 2$, irrespective of the substitution set, there are instances of the cardinality-constrained robust MC problem where the unique optimal randomized assortment strategy randomizes between $\Theta(2^n/\sqrt{n})$ many assortments.*

5.4. Solving the Randomized Markov Chain Problem

To solve the unconstrained and constrained randomized assortment optimization problem under the MC model with general substitution sets, we stipulate that the uncertainty set $\mathcal{U} \subseteq \mathbb{R}_+^{(n+1) \times (n+1)}$ is a finite set of substitution matrices. In the special case of product-wise substitution sets, we furthermore stipulate that $\mathcal{U} = \times_{i \in \mathcal{N}_0} \mathcal{U}_i$ for finite sets $\mathcal{U}_i \subseteq \mathbb{R}^{n+1}$ of substitution vectors for each product $i \in \mathcal{N}_0$. Popular examples of such uncertainty sets include projections of factor model uncertainty sets onto the cross-product of $n + 1$ probability simplices $\Delta(\mathcal{N}_0)$. We present in Appendix A a two-layer column generation framework to solve the aforementioned assortment optimization problems.

6. Preference Ranking Model

The preference ranking model (see, *e.g.*, [Bertsimas and Mišić 2019](#)) is parameterized by K bijective preference rankings $\sigma_k : \mathcal{N}_0 \rightarrow \{1, \dots, n + 1\}$, $k \in \mathcal{K} = \{1, \dots, K\}$, with occurrence probabilities $\lambda_k \in \mathbb{R}_+$ satisfying $\mathbf{e}^\top \boldsymbol{\lambda} = 1$. The probability that a customer is characterized by the preference ranking σ_k , $k \in \mathcal{K}$, is λ_k . Faced with the assortment S as well as the no-purchase option 0, such a customer purchases the product $i \in S \cup \{0\}$ that has the smallest rank in σ_k , that is, $i \in \arg \min \{\sigma_k(j) : j \in S \cup \{0\}\}$. Hence, the expected revenues of the assortment $S \in \mathcal{S}$ under the preference ranking model amount to

$$R(S, \boldsymbol{\lambda}) = \sum_{k \in \mathcal{K}} \lambda_k \cdot R_k(S), \quad (7)$$

where $R_k(S) = r_i$ for the unique $i \in S \cup \{0\}$ that satisfies $\sigma_k(i) < \sigma_k(j)$ for all $j \in S \cup \{0\}$, $j \neq i$. In the corresponding robust assortment optimization problem, we assume that the vector of occurrence probabilities $\boldsymbol{\lambda}$ is only known to be contained in a compact ambiguity set $\mathcal{U} = \Lambda \subseteq \{\boldsymbol{\lambda} \in \mathbb{R}_+^K : \mathbf{e}^\top \boldsymbol{\lambda} = 1\}$, and we seek to maximize the worst-case expected revenues.

In the following, we study the potential benefits of randomization under the unconstrained (Section 6.1) and cardinality-constrained (Section 6.2) robust preference ranking problem. Section 6.3, finally, discusses the exact and heuristic solution of the (un-)constrained robust assortment optimization problem under the preference ranking model.

Table 4 Ranking prevalences and expected revenues of different assortments in [Example 5](#).

Scenario	λ_1	λ_2	λ_3	$\{1\}$	$\{2\}$	$\{1,2\}$	$\frac{1}{3} \cdot \{2\} + \frac{2}{3} \cdot \{1,2\}$
1	1	0	0	1	2	1	1.33
2	0	0.5	0.5	1	1	1.5	1.33

6.1. The Unconstrained Preference Ranking Problem

We first illustrate that, similar to the MC problem under general uncertainty sets, the robust assortment optimization problem under the preference ranking model is randomization-receptive even in the absence of any further constraints (such as cardinality constraints).

EXAMPLE 5. Consider the unconstrained robust preference ranking problem with two products, $r_1 = 1$ and $r_2 = 2$ and the following three preference rankings:

$$1 \rightarrow 2 \rightarrow 0, \quad 2 \rightarrow 1 \rightarrow 0, \quad 1 \rightarrow 0 \rightarrow 2.$$

The associated ranking prevalences and the expected revenues of the deterministic assortments as well as one randomized assortment are listed in [Table 4](#). As the table shows, randomizing between the assortments $\{2\}$ and $\{1,2\}$ results in a 33% improvement of the worst-case expected revenues over the best deterministic assortment. Note that irrespective of the distribution $\mathbb{Q} \in \Delta(\{1,2\})$ over the scenarios 1 and 2, we always have $\mathbb{E}_{\mathbb{Q}}[R(\{2\}, \tilde{u})] > 1$ or $\mathbb{E}_{\mathbb{Q}}[R(\{1,2\}, \tilde{u})] > 1$. Hence, [Theorem 1](#) certifies that the instance is indeed randomization-receptive.

In fact, randomization can be beneficial for any number of products, as we show next.

OBSERVATION 4. *For any number of products $n \geq 2$, the preference ranking problem is randomization-receptive.*

Contrary to the constrained MNL and MC problems (*cf.* [Theorems 3](#) and [6](#)), however, and similar to the unconstrained MC problem with general substitution sets (*cf.* [Observation 3](#)), the benefits of randomization are bounded.

OBSERVATION 5. *For any instance of the robust preference ranking problem with $n = 2$ products, the benefits of randomization are bounded from above by $R_{\text{rand}}^*(\mathcal{U}) / R_{\text{det}}^*(\mathcal{U}) \leq 2$. Moreover, for any instance the benefits of randomization are bounded from above by*

$$\frac{R_{\text{rand}}^*(\mathcal{U})}{R_{\text{det}}^*(\mathcal{U})} \leq \frac{\max\{r_i : i \in \mathcal{N}\}}{\min\{r_i : i \in \mathcal{N}\}}.$$

Example 6 in the appendix shows that the bound for the two-product case is asymptotically tight. Similar to Section 5.3, two consequences of **Observation 5** are immediate.

COROLLARY 8. *If all products carry strictly positive prices $r_i > 0$, the benefits of randomization under the robust preference ranking model are bounded. Moreover, the problem is randomization-proof if all products have the same price, that is, if $r_i = r_j$ for all $i, j \in \mathcal{N}$.*

We have seen that in the cardinality-constrained MNL and MC problems, optimal policies may require randomization between an exponentially large number of assortments (cf. **Corollary 5** and **Proposition 2**). By contrast, under the unconstrained preference ranking model, there is always a parsimonious optimal randomization strategy.

THEOREM 7. *Under the preference ranking model, there always exists an optimal randomization strategy which places strictly positive weight on no more than $K + 1$ assortments. Moreover, when the uncertainty set is an affine map of a polyhedral subset of \mathbb{R}^m , there exists an optimal randomization strategy which places strictly positive weight on no more than $m + 1$ assortments.*

6.2. The Cardinality-Constrained Preference Ranking Problem

We now consider the preference ranking problem under a cardinality constraint on assortment size, where any admissible assortment $S \in \mathcal{S}$ must satisfy $|S| \leq C$. We first show that, similar to the cardinality-constrained MNL and MC problems (cf. **Theorems 3** and **6**), the potential benefits of randomization in the cardinality-constrained preference ranking problem can be arbitrarily large.

THEOREM 8. *For any number of products $n \geq 2$ and any restriction $|S| \leq C$, $C \in \{1, \dots, n - 1\}$, there are instances of the cardinality-constrained robust preference ranking problem where $R_{\text{det}}^*(\mathcal{U}) = 0$ while $R_{\text{rand}}^*(\mathcal{U}) > 0$.*

As before, an immediate consequence of **Theorem 8** is the following.

COROLLARY 9. *For any number of products $n \geq 2$ and any restriction $|S| \leq C$, $C \in \{1, \dots, n - 1\}$, there are instances of the cardinality-constrained robust preference ranking problem where the benefits $R_{\text{rand}}^*(\mathcal{U})/R_{\text{det}}^*(\mathcal{U})$ from randomization are arbitrarily large.*

Interestingly, the existence of parsimonious optimal randomization strategies carries over to the preference ranking model with arbitrary constraints on the set of admissible assortments. This parsimony is in contrast to the constrained MNL and MC models, where all optimal randomization strategies can become arbitrarily complex.

COROLLARY 10. *Under the preference ranking model with arbitrary constraints on the set of admissible assortments, there exists an optimal randomization strategy which places strictly positive weight on no more than $K + 1$ assortments. Moreover, when the uncertainty set is an affine map of a polyhedral subset of \mathbb{R}^m , there exists an optimal randomization strategy which places strictly positive weight on no more than $m + 1$ assortments.*

6.3. Solving the Randomized Preference Ranking Problem

Theorem 7 and Corollary 10 allow us to formulate the unconstrained and constrained randomized assortment optimization problem under the preference ranking model as robust K -adaptability problems that determine $K + 1$ assortments and their randomization weights (Bertsimas and Caramanis 2010, Hanasusanto et al. 2015). While the resulting problems can be expressed as mixed-integer linear programs of compact size that are amenable to solution via standard solvers, our numerical experiments indicate that this approach does not scale to problems of interesting size. Instead, we develop an exact and a heuristic column generation scheme to compute randomization strategies for the preference ranking model. To this end, we assume that the ambiguity set Λ is a polyhedral set of the form

$$\Lambda = \{ \boldsymbol{\lambda} = \mathbf{F}\boldsymbol{\xi} : \mathbf{A}\boldsymbol{\xi} \leq \mathbf{b}, \boldsymbol{\xi} \in \mathbb{R}_+^m \}, \quad (8)$$

where $\mathbf{F} \in \mathbb{R}^{K \times m}$, $\mathbf{A} \in \mathbb{R}^{l \times m}$ and $\mathbf{b} \in \mathbb{R}^l$. We list below two popular choices of such sets.

(i) **Norm ambiguity sets.** For a nominal ranking prevalence vector $\boldsymbol{\lambda}^0 \in \mathbb{R}_+^K$ and a radius $\theta \in \mathbb{R}_+$, we define the ambiguity set

$$\Lambda = \left\{ \boldsymbol{\lambda} : \mathbf{e}^\top \boldsymbol{\lambda} = 1, \|\boldsymbol{\lambda} - \boldsymbol{\lambda}^0\|_p \leq \theta, \boldsymbol{\lambda} \in \mathbb{R}_+^K \right\},$$

where $p \in \{1, \infty\}$. Norm ambiguity sets hedge against all perturbations of the ranking prevalence vector $\boldsymbol{\lambda}$ that are contained in a p -ball of radius θ around the nominal vector $\boldsymbol{\lambda}^0$. In particular, the choice $p = 1$ recovers the total variation distance, a popular ϕ -divergence (Bayraksan and Love 2015), which allows us to choose the radius θ based on statistical bounds.

(ii) **Approximate ellipsoidal ambiguity sets.** For a nominal ranking prevalence vector $\boldsymbol{\lambda}^0 \in \mathbb{R}^K$ and a symmetric and positive definite matrix $\mathbf{P} \in \mathbb{R}^{K \times K}$, we set

$$\Lambda = \left\{ \boldsymbol{\lambda} : \mathbf{e}^\top \boldsymbol{\lambda} = 1, \boldsymbol{\lambda} = \boldsymbol{\lambda}^0 + \mathbf{P}\boldsymbol{\xi}, \|\boldsymbol{\xi}\|_1 \leq \sqrt{K}, \|\boldsymbol{\xi}\|_\infty \leq 1, \boldsymbol{\lambda} \in \mathbb{R}_+^K, \boldsymbol{\xi} \in \mathbb{R}^K \right\}.$$

Note that $\{\boldsymbol{\xi} \in \mathbb{R}^K : \|\boldsymbol{\xi}\|_2 \leq 1\} \subseteq \{\boldsymbol{\xi} \in \mathbb{R}^K : \|\boldsymbol{\xi}\|_1 \leq \sqrt{K}\} \cap \{\boldsymbol{\xi} \in \mathbb{R}^K : \|\boldsymbol{\xi}\|_\infty \leq 1\}$, and thus the ambiguity set constitutes an outer (conservative) approximation of an ellipsoid with center $\boldsymbol{\lambda}^0$ and semi-axes defined by \mathbf{P} . Ellipsoidal ambiguity sets recover the Pearson χ^2 -divergence, another popular ϕ -divergence (Bayraksan and Love 2015), and they emerge asymptotically as confidence regions of a maximum likelihood estimation.

The randomized robust assortment optimization problem under the preference ranking model can be formulated as the following robust linear program:

$$\begin{aligned} & \text{maximize} && \min_{\boldsymbol{\lambda} \in \Lambda} \sum_{S \in \mathcal{S}} \sum_{k \in \mathcal{K}} p_S \cdot \lambda_k \cdot R_k(S) \\ & \text{subject to} && \sum_{S \in \mathcal{S}} p_S = 1 \\ & && p_S \geq 0, S \in \mathcal{S} \end{aligned} \tag{9}$$

Problem (9) is computationally challenging as it involves exponentially many decision variables p_S , $S \in \mathcal{S}$. For our solution schemes, it is useful to consider the dual of problem (9):

$$\begin{aligned} & \text{minimize} && \max_{S \in \mathcal{S}} \sum_{k \in \mathcal{K}} \lambda_k \cdot R_k(S) \\ & \text{subject to} && \boldsymbol{\lambda} \in \Lambda \end{aligned} \tag{10}$$

Strong duality between (9) and (10) holds since (9) is feasible by construction. Although problem (10) contains only polynomially many decision variables, the optimization problem embedded in its objective function maximizes over a combinatorial set $S \in \mathcal{S}$. Thus, neither the primal problem (9) nor the dual problem (10) is amenable to a solution with an off-the-shelf solver. We present in Appendix A.3 a column generation scheme for problem (9) that can be interpreted as a cutting plane approach for problem (10).

7. Numerical Results

This section presents numerical results for both synthetic and real-world data. The results in Sections 7.1–7.3 aim to elucidate when randomization can help to improve the worst-case expected revenues on synthetic data sets under the cardinality-constrained MNL problem, the MC problem, and the unconstrained preference ranking problem. Moreover, for the MNL problem, we demonstrate how the improvement of the worst-case expected revenues can translate into improvements of the out-of-sample expected revenues in a data-driven setting. We also investigate the computational price to be paid for exact and heuristic solutions to the deterministic and randomized robust assortment optimization problems.

Finally, in Section 7.4, we demonstrate the value of randomization on a real-world data set. All solution schemes are implemented in C++ and run on Intel Xeon 2.20GHz cluster nodes with 16 GB dedicated main memory in four-core mode. Our data sets and detailed results, together with the source codes of all our algorithms, can be found online.³

7.1. The Cardinality-Constrained Multinomial Logit Problem

We first consider the cardinality-constrained MNL model where the product prices r_i are selected uniformly at random from the interval $[0, 10]$. We use a budget uncertainty set (*cf.* Section 4.3) where the lower and upper product valuations \underline{v}_i and \bar{v}_i , $i \in \mathcal{N}$, are chosen uniformly at random from the intervals $[0, 4]$ and $[6, 10]$, respectively, whereas the valuation of the no-purchase option is fixed at $v_0 = 5$.

Table 5 presents results for $n = 20$ products where the uncertainty budgets Γ (rows) and the assortment cardinalities $|S| \leq C$ (columns) are set to various percentages of n . For each table entry, the first (upper) value denotes the percentage of 250 randomly generated instances in which the optimal randomized assortment outperformed the optimal deterministic robust assortment in terms of worst-case expected revenues, while the second (lower) value reports the average outperformance on those instances. The table shows that the benefits of randomization are most significant when Γ is close to C and both quantities are small relative to the number of products. The latter is intuitive as $C = n$ recovers the randomization-proof unconstrained robust MNL problem (*cf.* Corollary 3) while $\Gamma = n$ recovers the randomization-proof nominal MNL problem under the valuations $\mathbf{v}^0 = \underline{\mathbf{v}}$ (*cf.* Example 3.2 of Rusmevichientong and Topaloglu 2012).

We next investigate the benefits of randomization when the number n of products varies. To this end, we select $C = \Gamma = \lfloor \frac{1}{2}\sqrt{n} \rfloor$, which is in line with the central limit theorem-type uncertainty budget sets proposed by Bandi and Bertsimas (2012). The ‘randomized exact objective’ column of Table 6 shows that the benefits of randomization (measured here in terms of the average outperformance over the deterministic robust assortment over 250 random problem instances), while decreasing with problem size, is significant for all considered instance sizes. Interestingly, our exact column generation scheme for the randomized problem is actually faster than the cutting plane technique that we implemented for the deterministic robust problem, as the columns ‘deterministic exact runtime’ and

³ www.doc.ic.ac.uk/~wwiesema/assortment_opt.zip

		C							
		5%	10%	15%	20%	25%	30%	50%	75%
Γ	5%	100.00%	98.00%	83.60%	52.40%	20.40%	6.80%	0.40%	0.40%
		33.50%	4.17%	0.97%	0.35%	0.20%	0.10%	0.00%	0.00%
	10%	96.40%	100.00%	96.40%	79.60%	48.40%	17.20%	0.00%	0.00%
		20.20%	19.99%	5.09%	1.60%	0.54%	0.24%	0.00%	0.00%
	15%	74.00%	95.20%	99.60%	92.00%	68.00%	35.60%	0.00%	0.00%
		13.94%	12.18%	13.12%	4.48%	1.50%	0.54%	0.00%	0.00%
	20%	50.00%	80.00%	92.40%	97.20%	82.40%	56.80%	0.00%	0.00%
		9.49%	7.19%	7.63%	8.67%	3.42%	1.12%	0.00%	0.00%
25%	26.40%	50.40%	70.40%	84.40%	90.00%	70.40%	1.60%	0.80%	
	6.87%	4.84%	4.56%	4.92%	5.72%	2.36%	0.04%	0.01%	
30%	11.20%	27.60%	41.60%	61.20%	74.80%	80.00%	1.20%	0.40%	
	5.27%	3.29%	2.85%	2.68%	3.00%	3.46%	0.15%	0.01%	
50%	0.00%	0.00%	0.00%	0.40%	1.20%	1.60%	8.40%	0.40%	
	0.00%	0.00%	0.00%	0.26%	0.28%	0.26%	1.07%	0.01%	
75%	0.00%	0.00%	0.00%	0.00%	0.40%	0.40%	0.80%	0.40%	
	0.00%	0.00%	0.00%	0.00%	0.01%	0.01%	0.00%	0.01%	

Table 5 Benefits of randomization in the cardinality-constrained MNL problem. In the table, the rows (columns) correspond to different uncertainty budgets Γ (assortment cardinalities C).

‘randomized exact runtime’ in Table 6 reveal. This is caused by the fact that the cutting plane technique requires significantly more iterations (median 14 for $n = 25$ and median 47 for $n = 50$, for example) than the column generation scheme (median 5 for $n = 25$ and median 7 for $n = 50$, with 2-5 primal and dual iterations per main iteration). Since each (main) iteration adds a fractional linear term to the problems that requires an individual reformulation resulting in additional auxiliary variables and big-M constraints, the number of (main) iterations is a key performance indicator for both algorithms.

The columns ‘deterministic heuristic’ and ‘randomized heuristic’, finally, report the runtimes and objective values of the ADXOpt heuristic for the deterministic and randomized robust problems, respectively. In both cases, the objective values are again measured relative to the worst-case expected revenues of the exact deterministic robust problem. We see that for both problems, the ADXOpt heuristic performs very well, particularly on larger instance sizes. For the deterministic robust problem, the optimality gaps approach 0% as the instance sizes grow, while for the randomized robust problem, the outperformance of the ADXOpt heuristic approaches the outperformance of the exact solution approach. We thus conclude that one should solve the deterministic and randomized robust problems exactly for small problem sizes, in which case the cutting plane and column generation schemes are fast, while one may want to resort to heuristic solutions for larger problem sizes, where the optimality gap of ADXOpt decreases rapidly.

n	deterministic			randomized			
	exact runtime	heuristic objective	runtime	exact objective	runtime	heuristic objective	runtime
5	0.02	-6.80%	0.00s	37.95%	0.01s	24.97%	0.00s
10	0.05	-1.83%	0.00s	35.15%	0.03s	32.03%	0.00s
15	0.13	-0.96%	0.00s	35.39%	0.08s	34.01%	0.00s
20	2.30	-0.84%	0.00s	21.30%	1.80s	20.11%	0.03s
25	6.94	-0.41%	0.01s	22.72%	3.58s	21.96%	0.05s
30	18.38	-0.38%	0.01s	21.99%	5.88s	21.15%	0.09s
35	27.18	-0.40%	0.02s	21.63%	8.10s	21.04%	0.12s
40	299.18	-0.23%	0.05s	9.41%	64.21s	8.58%	0.76s
45	592.28	-0.12%	0.07s	7.29%	97.56s	6.71%	1.09s
50	937.29	-0.11%	0.09s	6.09%	171.07s	5.66%	1.57s

Table 6 Exact and heuristic solutions for the cardinality-constrained MNL problem. All percentages are reported relative to the optimal solutions of the deterministic robust problem.

We close this section with a data-driven experiment. To this end, we fix $n = 10$ products and consider different cardinalities $C \in \{1, 2, 3, 4\}$ for the admissible assortments. The product prices r_i are generated randomly by the same procedure as before. The true customer valuations are unknown and satisfy $v_i^0 = e^{\beta_i^0}$, where β_i^0 is drawn uniformly at random from -3 to 3 , $i \in \mathcal{N}$; we fix $v_0^0 = e^0$ and assume that this quantity is known. We assume that 5, 10, \dots , 95 historical samples of random assortments of cardinality C are available, together with the purchase choice that each customer made. In the nominal model, we then estimate the valuations $\hat{\mathbf{v}}$ from the historical data using a maximum likelihood estimation, and we solve the resulting nominal assortment optimization problem. In the deterministic and randomized robust models, we employ the budget uncertainty set where both $\Gamma \in \{0, \dots, n\}$ and $(\underline{\mathbf{v}}, \bar{\mathbf{v}}) = ([1 - \gamma]\hat{\mathbf{v}}, [1 + \gamma]\hat{\mathbf{v}})$, $\gamma \in \{0, 0.025, \dots, 0.5\}$ are selected using 7-fold cross-validation on the available historical data. Since the validation data reports the customer choices for assortments that differ from the assortment computed using the training data, each iteration of our cross-validation proceeds in two steps. We first use the validation set to perform a maximum likelihood estimation of the valuations. We subsequently use the nominal MNL model associated with these estimated valuations to approximate the out-of-sample revenues of the assortment computed using the training data. We then compare the actual out-of-sample expected revenues of the three models under newly generated data using the true valuations \mathbf{v}^0 .

For each cardinality $C \in \{1, 2, 3, 4\}$ and sample size 5, 10, \dots , 95, Figure 3 reports the average optimality gaps (over 100 randomly generated instances) of the out-of-sample expected revenues of each model relative to the expected revenues of the nominal model under the true valuations. The figure shows that these optimality gaps decrease with sample

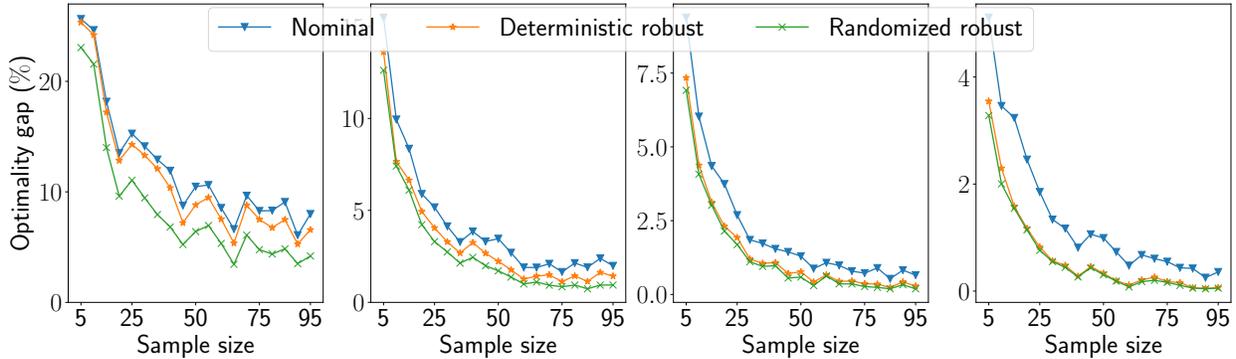


Figure 3 Data-driven experiment for the cardinality-constrained MNL problem. All optimality gaps are reported relative to the expected revenues of the clairvoyant model that knows the true customer valuations.

size (*i.e.*, from left to right in each graph) as well as the cardinality of the assortment (*i.e.*, from the leftmost to the rightmost graph). This is expected as in both cases, the estimation problem can rely on more data and hence becomes easier. In all cases, the randomized robust model outperforms the deterministic robust model, which in turn outperforms the nominal model. We emphasize that this is not *a priori* obvious as the deterministic and randomized robust models use the available historical data for both estimation and parameter selection, and hence have less data than the nominal model to estimate $\hat{\mathbf{v}}$. Interestingly, for small cardinalities—where the estimation problem is most challenging—the randomized robust model significantly outperforms both the nominal and the deterministic robust model. For larger cardinalities, the performance of the deterministic and randomized robust models are becoming more similar. It is noteworthy, however, that the randomized robust model is never performing worse than the deterministic robust model (in terms of average optimality gap), while it is at the same time easier to solve (*cf.* Table 6).

7.2. The Markov Chain Problem

We next consider the MC model where the product prices are again selected uniformly at random. We study the randomization-receptive cases of this problem identified in Section 5, namely the cardinality-constrained problem under product-wise substitution sets, and both the unconstrained and the constrained problem under general substitution sets. We employ the factor model uncertainty sets outlined in Section 5.4 with K factors leading to uncertainty sets with 2^K realizations under general substitution sets and $(2^K)^n$ realizations under product-wise substitution sets, respectively.

		K						K			
		5	8	10	12			5	8	10	12
n	20	93.60%	97.60%	98.80%	99.60%	20	99.20%	100.00%	100.00%	100.00%	
		0.91%	1.29%	1.35%	1.48%		4.51%	7.53%	8.70%	9.20%	
	30	95.60%	98.80%	99.20%	98.80%	30	100.00%	100.00%	100.00%	100.00%	
		0.65%	0.81%	0.96%	1.05%		6.35%	9.93%	12.01%	13.48%	
	40	95.20%	96.76%	98.80%	99.60%	40	100.00%	100.00%	100.00%	100.00%	
		0.45%	0.60%	0.63%	0.73%		7.85%	12.52%	15.20%	16.94%	
	50	93.20%	99.60%	97.20%	99.20%	50	100.00%	100.00%	100.00%	100.00%	
		0.37%	0.44%	0.47%	0.55%		9.17%	14.82%	17.45%	17.80%	
	60	95.20%	97.60%	98.00%	98.80%	60	100.00%	100.00%	100.00%	100.00%	
		0.29%	0.32%	0.37%	0.41%		9.99%	15.90%	18.22%	18.27%	
	70	95.20%	98.80%	98.80%	99.19%	70	100.00%	100.00%	100.00%	100.00%	
		0.22%	0.26%	0.32%	0.32%		11.40%	16.60%	17.82%	16.93%	
80	94.40%	98.00%	100.00%	99.59%	80	100.00%	100.00%	100.00%	100.00%		
	0.20%	0.23%	0.27%	0.27%		11.92%	17.56%	16.79%	17.70%		
90	97.60%	99.20%	100.00%	99.02%	90	100.00%	100.00%	100.00%	n/a		
	0.17%	0.17%	0.22%	0.22%		12.94%	17.15%	16.19%	n/a		
100	97.20%	98.80%	98.33%	100.00%	100	100.00%	100.00%	100.00%	n/a		
	0.17%	0.17%	0.18%	0.21%		13.46%	16.63%	15.24%	n/a		

Table 7 Benefits of randomization under the MC problem with general substitution sets. The rows and columns correspond to the number of products n and the number of factors K , respectively.

Left table: unconstrained problem. Right table: cardinality constraint $|S| \leq C = 5$.

We first study the MC model under general substitution sets. To this end, we compare the deterministic and randomized robust solutions for $K \in \{5, 8, 10, 12\}$ factors, $n \in \{20, 30, \dots, 90\}$ products and a cardinality constraint, when used, of $|S| \leq C = 5$. We determine the optimal randomized assortments using the method described in Appendix A.2. The left-hand side of Table 7 (left) shows the results for the unconstrained problem. The rows and columns in the table correspond to the numbers of products n factors K of the uncertainty set, respectively. For each entry, the two numbers again denote the fraction of 250 randomly generated instances that were randomization receptive (top) and the average outperformance of the randomized robust solution over the deterministic robust one on these instances (bottom). The table shows that the benefits of randomization are highest when the number of products is relatively low while the number of factors is high. The right side of the same table shows the corresponding results for the cardinality-constrained problem. Introducing the cardinality constraint significantly increases both the likelihood and the magnitude of randomization gains. The cardinality-constrained problem is, however, also more difficult to solve, and we were unable to obtain solutions for $N \geq 90$ when $K = 12$. Tables 12 and 13 in Appendix F compare the solution times for the deterministic robust problem with those of the randomized robust problem.

	n					
	10	12	14	16	18	20
lower bound (%)	1.15	2.24	4.70	6.94	7.11	2.45
upper bound (%)	1.15	2.24	4.70	6.99	8.61	12.24

Table 8 Outperformance of the randomized assortments over the deterministic ones under the MC model with product-wise substitution sets and the cardinality constraint $|S| \leq C = 5$.

We next consider the cardinality-constrained MC problem with product-wise substitution sets. This problem variant turns out to be substantially harder to solve due to the large cardinality $(2^K)^n$ of the uncertainty set \mathcal{U} . We therefore fix $K = 2$ and limit the runtime of Step 2(b) in each iteration of our column generation scheme (see Appendix A.2) to 100 seconds. Table 8 reports the relative outperformance of our (heuristic, due to the premature termination) randomized assortments over the best deterministic assortments. The outperformance is again reported in terms of lower (top entry) and upper (bottom entry) bounds since the worst-case evaluation problem is terminated prematurely. The results represent averages over 100 randomly generated instances. While it is difficult to draw definite conclusions due to the widening gap of the bounds, the benefits of randomized assortments appear to grow with the number of products.

7.3. The Preference Ranking Problem

We next consider the preference ranking model where the product prices are again selected uniformly at random from the interval $[0, 10]$. We use a 1-norm ambiguity set (*cf.* Section 6.3) where the nominal ranking prevalence vector λ^0 is selected uniformly at random from the K -dimensional probability simplex.

In our first experiment, we fix the number of products to $n = 20$ and vary the radius $\theta \in \{0.25, 0.5, \dots, 2\}$ of the ambiguity set as well as the number of preference rankings $K \in \{10, 100, 200, 500, 1,000\}$. Table 9 presents the percentages of instances (across 250 randomly generated instances) where the randomized robust model improved upon the deterministic robust one in terms of worst-case expected revenues (first number) as well as the average relative outperformance across those instances (second number). The table shows that the benefits of randomization are most pronounced when the ambiguity is large but not covering the entire probability simplex (which is the case when $\theta = 2$). This is intuitive since under full ambiguity, nature can choose a Dirac distribution that places all probability mass on the least favorable (assortment-dependent) preference ranking, in which case randomization does not help. Likewise, if the ambiguity is small, we approach a non-robust problem where randomization does not help either.

		K				
		10	100	200	500	1,000
θ	0.00	0.00%	0.00%	0.00%	0.00%	0.00%
		0.00%	0.00%	0.00%	0.00%	0.00%
	0.25	3.00%	13.30%	13.30%	15.45%	12.88%
		0.60%	0.12%	0.06%	0.06%	0.03%
	0.50	30.04%	60.09%	58.80%	50.64%	62.23%
		1.21%	0.29%	0.21%	0.19%	0.23%
	0.75	25.75%	78.11%	80.69%	83.26%	78.97%
		0.63%	0.55%	0.51%	0.49%	0.46%
	1.00	33.48%	88.41%	94.85%	93.56%	94.85%
		1.99%	1.06%	1.00%	1.08%	0.96%
	1.25	29.18%	95.71%	97.00%	96.14%	97.85%
		3.08%	2.37%	2.50%	1.91%	1.84%
	1.50	19.74%	98.71%	98.71%	100.00%	100.00%
		2.63%	3.91%	3.47%	3.38%	3.04%
	1.75	0.00%	93.56%	99.57%	100.00%	99.57%
		0.00%	10.81%	9.22%	6.49%	7.17%
	2.00	0.00%	0.00%	0.00%	0.00%	0.00%
		0.00%	0.00%	0.00%	0.00%	0.00%

Table 9 Benefits of randomization in the preference ranking problem. The rows (columns) correspond to different ambiguity radii θ (numbers of preference rankings K).

We now investigate the benefits of randomization when the number n of products varies. To this end, Table 10 fixes the radius of ambiguity to $\theta = 1.5$ and reports the outperformance of the exact (column ‘randomized exact objective’) and heuristic (column ‘randomized heuristic objective’) solutions to our randomized robust model relative to the optimal solution to the deterministic robust model. The table shows that while the benefits of randomization decrease with the number of products n , they remain non-trivial for all considered problem sizes. Moreover, the performance of the ADXOpt heuristic is more or less independent of the number of products n as well as improving with the number of preference rankings K .

Table 10 also reports the runtimes of the mixed-integer programs for the deterministic robust model (column ‘deterministic exact runtime’) as well as our column generation scheme (column ‘randomized exact runtime’) and the ADXOpt heuristic (column ‘randomized heuristic runtime’) for the randomized robust model. The table shows that the randomized exact problem is significantly more challenging to solve than deterministic one. Indeed, while the exact deterministic robust problem can be solved as a monolithic MILP, the exact randomized robust problem is solved in a cutting plane fashion and thus suffers from similar scalability issues as the cutting plane scheme for the deterministic robust MNL problem (*cf.* Table 6). The number of cutting plane iterations grows with the

n	K	deterministic	randomized			
		exact runtime	exact objective	exact runtime	heuristic objective	heuristic runtime
10	10	0.01s	0.54%	0.00s	0.51%	0.00s
10	100	0.05s	4.03%	0.12s	3.64%	0.03s
10	200	0.10s	4.95%	0.32s	4.10%	0.07s
10	500	0.41s	3.11%	0.90s	2.74%	0.18s
10	1000	0.68s	3.44%	2.57s	2.92%	0.49s
20	10	0.01s	0.44%	0.01s	-0.15%	0.00s
20	100	0.16s	3.83%	0.81s	3.16%	0.08s
20	200	0.34s	5.32%	2.48s	4.32%	0.18s
20	500	1.41s	3.38%	7.49s	2.94%	0.54s
20	1000	5.02s	2.99%	20.59s	2.61%	1.09s
30	10	0.01s	0.41%	0.02s	0.11%	0.00s
30	100	0.33s	3.82%	3.11s	2.89%	0.16s
30	200	0.78s	3.69%	7.38s	3.13%	0.40s
30	500	4.43s	2.97%	31.24s	2.61%	1.03s
30	1000	16.30s	2.91%	102.33s	2.62%	2.16s
40	10	0.03s	0.14%	0.03s	-0.01%	0.00s
40	100	0.49s	4.83%	6.43s	3.81%	0.33s
40	200	1.66s	3.23%	22.06s	2.52%	0.71s
40	500	8.73s	2.40%	82.31s	2.13%	1.81s
40	1000	43.61s	2.32%	362.83s	2.07%	3.83s
50	10	0.03s	0.08%	0.03s	-0.08%	0.01s
50	100	0.92s	2.17%	8.19s	1.74%	0.45s
50	200	2.56s	3.95%	53.55s	3.20%	1.22s
50	500	17.28s	2.71%	219.30s	2.33%	2.91s
50	1000	75.45s	2.46%	1011.82s	2.24%	6.99s

Table 10 Exact and heuristic solutions for the preference ranking problem. All percentages are reported relative to the optimal solutions of the deterministic robust problem.

problem size (median 6 for $n = 20$, $K = 10$ vs. median 49 for $n = 50$, $K = 1,000$), and each subproblem of the algorithm requires the solution of a MILP whose number of decision variables scales in K . The ADXOpt heuristic, on the other hand, provides close-to-optimal solutions while requiring substantially less computational resources.

We have also attempted to solve the randomized robust assortment optimization problem under the preference ranking model as a K -adaptability problem (*cf.* Section 6.3). It turns out, however, that the resulting mixed-integer optimization problems grow quickly in size and become difficult to solve for instances with more than 10 products. We therefore do not report computational results for this solution scheme.

7.4. Experiments with Real-World Data

In this section, we demonstrate the value of randomized strategies on real data. To this end, we use the data set of [Bertsimas and Mišić \(2017, 2019\)](#), which is derived from a field study on consumer preferences for hypothetical Timbuk2 laptop bag designs by [Toubia](#)

et al. (2003) (see also Belloni et al. 2008). The data set contains 330 preference rankings for 3,584 bag designs, as well as the revenues associated with each design.⁴ Bertsimas and Mišić (2017) use the data to study the value of accounting for uncertainty in deterministic robust assortment optimization under the preference ranking model (among others). Here, we extend their approach to the randomized robust assortment optimization problem.

Bertsimas and Mišić (2017) solve the nominal and deterministic robust problems heuristically for the entire data set. By contrast, we use the approach described in Section 6.3 to solve the nominal as well as the deterministic and randomized robust problems exactly for a subset of available products. Specifically, we pick the $n = 300$ products generating the highest average consumer utilities in the data set and trim the preference ranking lists to only include these products. Since the rankings do not contain an explicit no-purchase option, we insert this choice (*i.e.*, product 0) randomly according to a geometrically distributed random variable (capped at the upper bound $n + 1$) with success probability 0.1. Following Bertsimas and Mišić (2017), we assume that each customer type is equally probable, that is, $\lambda_k = 1/K$ for all $k \in \mathcal{K}$. For the deterministic and randomized robust problems, we employ a 1-norm ambiguity set with varying radii θ (*cf.* Section 6.3).

Following Bertsimas and Mišić (2017), we use two metrics to quantify the value of the deterministic and randomized robust approaches: the worst-case loss and the relative improvement. The worst-case loss (WCL) measures the largest possible loss of the optimal nominal assortment S^N over the uncertainty set \mathcal{U} :

$$\text{WCL} = 100\% \cdot \frac{R_{\text{nom}}^* - R^*(S^N)}{R_{\text{nom}}^*},$$

where R_{nom}^* is the expected revenues of S^N under the nominal occurrence probabilities λ_0 and $R^*(S^N)$ denotes the worst-case expected revenues of S^N over the uncertainty set \mathcal{U} . The second metric is the relative improvement (RI) in worst-case expected revenues when using the deterministic or randomized robust approach instead of the nominal assortment:

$$\begin{aligned} \text{RI}^D &= 100\% \cdot \frac{R_{\text{det}}^*(\mathcal{U}) - R^*(S^N)}{R^*(S^N)} \\ \text{and } \text{RI}^R &= 100\% \cdot \frac{R_{\text{rand}}^*(\mathcal{U}) - R^*(S^N)}{R^*(S^N)}, \end{aligned}$$

⁴ The data set is available at <https://github.com/vvmisic/optimalPLD>.

	WCL(%)	RI ^D (%)	RI ^R (%)	RI RD (%)
θ 0.1	11.58	0.08	0.49	0.41
0.2	22.66	0.15	1.36	1.21
0.3	33.44	0.20	3.25	3.05
0.4	43.97	0.25	6.24	5.97
0.5	54.42	0.45	10.82	10.32
0.6	64.42	1.29	19.48	17.93

Table 11 Benefits of randomization in the real-world case study over $n = 300$ products. The rows correspond to different radii θ of the uncertainty set.

where $R_{\text{det}}^*(\mathcal{U})$ and $R_{\text{rand}}^*(\mathcal{U})$ denote the optimal values of the deterministic and randomized robust assortment optimization problems, respectively.

Table 11 reports the worst-case losses and relative improvements for different radii $\theta \in \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6\}$, averaged over 50 runs where the positions of the no-purchase option are updated each time. The table shows that the worst-case loss from solving the nominal problem grows with the radius of the uncertainty set, as expected. The deterministic robust approach (column RI^D), consistently with Bertsimas and Mišić (2017), benefits the decision maker in terms of the relative improvement, and the magnitude of the improvement increases with the radius of the uncertainty set. The relative improvement achieved by the randomized robust strategy (RI^R) is significantly higher, however, demonstrating the additional benefit of randomization in hedging against uncertainty in the preference rankings. The difference in relative improvements between the two robust approaches is reported in the column RIRD, which shows that the additional gains of the randomized approach are substantial and grow with the size of the uncertainty set. The results thus indicate the potential promise of employing randomization in robust assortment optimization on real-world data.

8. Concluding Remarks

Our results call for more research into the importance of randomization for revenue management. We have seen that a firm can benefit from using randomization—instead of the standard approach of deterministically offering a single assortment—in the robust assortment optimization problem. However, it is not *a priori* obvious which problem settings may benefit from randomization: two similar models, such as the MC model with product-wise and general uncertainty sets, may lead to starkly different conclusions. While our analysis suggests that more general versions of the assortment optimization problem tend to be more receptive to the benefits of randomization, more research is needed into this potential

benefit under other choice models and constraints, as well as other revenue-management problems. Also, all of the computational approaches developed in this paper study the static problem formulation where the decision maker does not update her knowledge about the uncertain problem parameters. As a next step, it would be instructive to study optimal dynamic randomization strategies where the decision maker learns about the uncertain problem parameters over time.

Acknowledgments

The first author gratefully acknowledges funding from the Imperial College President's PhD Scholarship programme. The last author gratefully acknowledges funding from the Engineering and Physical Sciences Research Council under the grants EP/R045518/1, EP/T024712/1 and EP/W003317/1. All three authors are indebted to the area editor (Gustavo Vulcano) as well as the anonymous associate editor and referees, whose insightful comments have led to substantial improvements of the manuscript.

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Appendix A: Exact and Heuristic Solution Schemes for the Randomized Robust Assortment Optimization Problem

We first describe a generic solution scheme for instances $(\mathcal{N}, \mathcal{S}, \mathcal{C}, \mathcal{U}, \mathbf{r})$ of the randomized robust assortment optimization problem **RANDOMIZED ROBUST**. Afterwards, Sections A.1, A.2 and A.3 provide the specific implementation details for the MNL, MC and preference ranking models, respectively.

Column generation scheme for **RANDOMIZED ROBUST**

1. **Initialization.** Set $\hat{\mathcal{S}}$ to any non-empty subset of \mathcal{S} (currently considered assortments). Set $\hat{\mathcal{U}}$ to any non-empty subset of \mathcal{U} (currently considered uncertainty realizations). Set $\text{LB} = -\infty$ and $\text{UB} = +\infty$ (current lower and upper bounds on the optimal value).

2. **Primal Step.** Solve the restricted primal problem

$$\text{maximize } \min_{\mathbf{p} \in \Delta(\hat{\mathcal{S}})} \sum_{\substack{u \in \mathcal{U} \\ S \in \hat{\mathcal{S}}} p_S \cdot R(S, u) : \quad (11)$$

- (a) Set $\text{LB}_p = -\infty$ and $\text{UB}_p = \text{UB}$ (lower and upper bounds on the optimal value of the restricted primal problem). Set \mathbf{p} to any point in $\Delta(\hat{\mathcal{S}})$ (current restricted randomization strategy).
- (b) Solve the subproblem

$$\tau_{\mathbf{p}}^* = \text{minimize } \sum_{\substack{u \in \mathcal{U} \\ S \in \hat{\mathcal{S}}} p_S \cdot R(S, u). \quad (12)$$

Update $\text{LB}_p = \max\{\text{LB}_p, \tau_{\mathbf{p}}^*\}$ and $\hat{\mathcal{U}} = \hat{\mathcal{U}} \cup \{u^*\}$, where u^* is an optimal solution of the subproblem.

- (c) Solve the doubly restricted primal problem (11) that replaces \mathcal{U} with $\hat{\mathcal{U}}$. Update $\text{UB}_p = \min\{\text{UB}_p, \tau^*\}$, where τ^* is the optimal value of the doubly restricted problem, and set \mathbf{p} to an optimal solution of the problem.
- (d) If $\text{UB}_p - \text{LB}_p < \epsilon/2$, set $\text{LB} = \text{LB}_p$ and go to Step 3. Otherwise, go to Step 2(b).

3. **Dual Step.** Solve the restricted dual problem

$$\text{minimize } \max_{\boldsymbol{\kappa} \in \Delta(\hat{\mathcal{U}})} \sum_{\substack{S \in \mathcal{S} \\ u \in \hat{\mathcal{U}}} \kappa_u \cdot R(S, u) : \quad (13)$$

- (a) Set $\text{LB}_d = \text{LB}$ and $\text{UB}_d = +\infty$ (lower and upper bounds on the optimal value of the restricted dual problem). Set $\boldsymbol{\kappa}$ to any point in $\Delta(\hat{\mathcal{U}})$ (convex combination of the currently considered valuations).
- (b) Solve the subproblem

$$\tau_{\boldsymbol{\kappa}}^* = \text{maximize } \sum_{\substack{S \in \mathcal{S} \\ u \in \hat{\mathcal{U}}} \kappa_u \cdot R(S, u). \quad (14)$$

Update $\text{UB}_d = \min\{\text{UB}_d, \tau_{\boldsymbol{\kappa}}^*\}$ and $\hat{\mathcal{S}} = \hat{\mathcal{S}} \cup \{S^*\}$, where S^* is an optimal solution of the subproblem.

- (c) Solve the doubly restricted dual problem (13) that replaces \mathcal{S} with $\hat{\mathcal{S}}$. Update $\text{LB}_d = \max\{\text{LB}_d, \tau^*\}$, where τ^* is the optimal value of the doubly restricted problem, and set $\boldsymbol{\kappa}$ to an optimal solution of the problem.
- (d) If $\text{UB}_d - \text{LB}_d < \epsilon/2$, set $\text{UB} = \text{UB}_d$ and go to Step 4. Otherwise, go to Step 3(b).

4. **Termination.** Iterate between Steps 2 and 3 until the gap between LB and UB is zero or sufficiently small. Then terminate with the primal-dual pair $(\mathbf{p}^*, \boldsymbol{\kappa}^*) \in \mathbb{R}^{|\mathcal{S}|} \times \mathbb{R}^{|\mathcal{U}|}$, where $p_S^* = p_S$, $S \in \hat{\mathcal{S}}$, $= 0$ otherwise and $\kappa_u^* = \kappa_u$, $u \in \hat{\mathcal{U}}$, $= 0$ otherwise.

The ‘outer layer’ of the algorithm iterates between conservative approximations to the primal and dual problems (11) and (13) that consider all uncertainty realizations $u \in \mathcal{U}$ (in the primal problem) and assortments $S \in \mathcal{S}$ (in the dual problem) but that only consider subsets of the admissible decisions p_S , $S \in \hat{\mathcal{S}} \subseteq \mathcal{S}$ (in the primal problem) and κ_u , $u \in \hat{\mathcal{U}} \subseteq \mathcal{U}$ (in the dual problem). The resulting restrictions remain computationally challenging as they involve embedded non-convex optimization problems over $u \in \mathcal{U}$ and $S \in \mathcal{S}$, respectively, and they are solved iteratively up to a pre-specified tolerance of $\epsilon \geq 0$. To this end, the ‘inner layers’ of the algorithm iterate between (i) determining an optimal solution to the embedded optimization problem while fixing the solution to the outer optimization problems in Steps 2(b) and 3(b) and (ii) solving the outer optimization problems for the refined approximations of the embedded optimization problems in Steps 2(c) and 3(c).

The correctness of the algorithm is summarized in the following statement.

THEOREM 9. *The two-layer column generation algorithm presented above converges in a finite number of iterations and terminates with an optimal primal-dual solution pair $(\mathbf{p}^*, \boldsymbol{\kappa}^*)$ to problems (11) and (13).*

Proof of Theorem 9. The proof largely follows similar arguments as the proof of Theorem 2 of Delage and Saif (2021) and is thus omitted for the sake of brevity. \square

Our column generation scheme is reminiscent to, but differs in several important aspects from, the two-layer algorithm proposed in Section 5.3 of Delage and Saif (2021). While Delage and Saif (2021) study a *two-stage* robust optimization problem with a *linear* objective function, our randomized assortment optimization problem constitutes a *single-stage* robust optimization problem that comprises a *non-convex* objective function. To solve their problem, Delage and Saif (2021) equivalently express their primal and dual subproblems through Karush-Kuhn-Tucker reformulations, whereas our subproblems require a different case-by-case treatment for each choice model that we elaborate upon in the following subsections.

A.1. Exact and Heuristic Solution of the Randomized Constrained MNL Problem

Recall from equation (2) that for the MNL choice model, the revenues $R(S, u)$ evaluate to

$$R(S, u) = R(S, \mathbf{v}) = \sum_{i \in S} r_i \cdot \psi_i(S, \mathbf{v}) = \frac{\sum_{i \in S} r_i \cdot v_i}{v_0 + \sum_{i \in S} v_i},$$

and recall from equation (3) that we consider the binary representable uncertainty set

$$\mathcal{V} = \{\mathbf{v} = \mathbf{F}\boldsymbol{\xi} : \mathbf{A}\boldsymbol{\xi} \leq \mathbf{b}, \boldsymbol{\xi} \in \{0, 1\}^m\}.$$

In the following, we present mixed-integer linear programs (MILPs) for the subproblems (12) and (14) of our column generation scheme that resemble the MILP formulations for sum-of-ratio problems, see Li (1994).

For the subproblem (12) in the primal step of our column generation scheme, we first replace the decision variables $\mathbf{v} \in \mathcal{V}$ with their definition and introduce auxiliary variables d_S , $S \in \hat{\mathcal{S}}$, that equate to the (inverses of the) denominators in the objective function:

$$\begin{aligned} & \text{minimize} && \sum_{S \in \hat{\mathcal{S}}} p_S \cdot d_S \cdot \sum_{i \in S} r_i \cdot \mathbf{F}_i^\top \boldsymbol{\xi} \\ & \text{subject to} && d_S \cdot \left[\sum_{i \in S \cup \{0\}} \mathbf{F}_i^\top \boldsymbol{\xi} \right] = 1 && \forall S \in \hat{\mathcal{S}} \\ & && \mathbf{A}\boldsymbol{\xi} \leq \mathbf{b} \\ & && \boldsymbol{\xi} \in \{0, 1\}^m, d_S \in \mathbb{R}_+, S \in \hat{\mathcal{S}}. \end{aligned}$$

Here, \mathbf{F}_i^\top denotes the i -th row of the matrix \mathbf{F} (where the counting starts from zero). For all $S \in \hat{\mathcal{S}}$ and $j \in \{1, \dots, m\}$, we now apply the exact linearization

$$z_{Sj} = d_S \cdot \xi_j \iff \begin{bmatrix} z_{Sj} \leq d_S, & z_{Sj} \leq M \cdot \xi_j \\ z_{Sj} \geq 0, & z_{Sj} \geq d_S - M \cdot (1 - \xi_j) \end{bmatrix},$$

where $z_{Sj} \in \mathbb{R}$, $S \in \hat{\mathcal{S}}$ and $j = 1, \dots, m$, constitute auxiliary decision variables and M denotes a sufficiently large positive constant that can be determined from the geometry of the uncertainty set (Li 1994). The resulting formulation is an MILP that can be solved with off-the-shelf software.

The subproblem (14) in the dual step of our algorithm constitutes a nominal assortment optimization problem under the mixture of multinomial logits (MMNL) choice model. MILP reformulations for this problem have been proposed, among others, by Bront et al. (2009), Méndez-Díaz et al. (2014) and Şen et al. (2018). To solve this problem, we first introduce the binary decision variables $x_i \in \{0, 1\}$, $i \in \mathcal{N}$, to denote whether product i is contained in the assortment S and introduce auxiliary variables d_v , $\mathbf{v} \in \hat{\mathcal{V}}$, that equate to the (inverses of the) denominators in the objective function:

$$\begin{aligned} & \text{maximize} && \sum_{\mathbf{v} \in \hat{\mathcal{V}}} \kappa_{\mathbf{v}} \cdot d_{\mathbf{v}} \cdot \sum_{i \in \mathcal{N}} r_i \cdot v_i \cdot x_i \\ & \text{subject to} && d_{\mathbf{v}} \cdot \left[v_0 + \sum_{i \in \mathcal{N}} v_i \cdot x_i \right] = 1 \quad \forall \mathbf{v} \in \hat{\mathcal{V}} \\ & && \sum_{i \in \mathcal{N}} x_i \leq C \\ & && \mathbf{x} \in \{0, 1\}^n, d_{\mathbf{v}} \in \mathbb{R}_+, \mathbf{v} \in \hat{\mathcal{V}} \end{aligned}$$

For all $\mathbf{v} \in \hat{\mathcal{V}}$ and $i \in \mathcal{N}$, we now apply the exact linearization

$$z_{vi} = d_{\mathbf{v}} \cdot x_i \iff \begin{bmatrix} z_{vi} \leq d_{\mathbf{v}}, & z_{vi} \leq M \cdot x_i \\ z_{vi} \geq 0, & z_{vi} \geq d_{\mathbf{v}} - M \cdot (1 - x_i) \end{bmatrix},$$

where $z_{vi} \in \mathbb{R}$, $\mathbf{v} \in \hat{\mathcal{V}}$ and $i \in \mathcal{N}$, constitute auxiliary decision variables and M denotes a sufficiently large positive constant that can again be determined from the geometry of the uncertainty set (Li 1994). The resulting formulation is again an MILP.

The main computational challenge in the column generation scheme of the previous section arises from the subproblems of Steps 2(b) and 3(b), which amount to discrete sums-of-ratios problems. As an alternative to solving those problems exactly, we can also adapt the ADXOpt heuristic of Jagabathula (2014), which was originally designed for nominal assortment optimization problems under general choice models, to these subproblems. Since the subproblem of Step 3(b) is already in the form considered by Jagabathula (2014), we focus on the subproblem of Step 2(b) and consider the budget uncertainty set of Section 4.3:

$$\mathcal{V} = \{ \mathbf{v} = \bar{\mathbf{v}} - (\bar{\mathbf{v}} - \underline{\mathbf{v}}) \circ \boldsymbol{\xi} : \mathbf{e}^\top \boldsymbol{\xi} \leq \Gamma, \boldsymbol{\xi} \in \{0, 1\}^{n+1} \}$$

The adaptations to other uncertainty sets are straightforward.

Local search heuristic for problem (12)

1. **Initialization.** Set $\boldsymbol{\xi} = \mathbf{0}$ (current worst-case valuation).

2. **Additions.** If $\mathbf{e}^\top \boldsymbol{\xi} < \Gamma$, evaluate the objective function of problem (12) for all $\boldsymbol{\xi}' = \boldsymbol{\xi} + \mathbf{e}_i$, $i \in \mathcal{N}_0$, for which $\boldsymbol{\xi}' \in \{0, 1\}^{n+1}$. If any such $\boldsymbol{\xi}'$ improves the objective function of (12), update $\boldsymbol{\xi}$ to the best such $\boldsymbol{\xi}'$ and repeat Step 2.
3. **Deletions and exchanges.** Evaluate the objective function of problem (12) for all $\boldsymbol{\xi}' = \boldsymbol{\xi} - \mathbf{e}_i$, $i \in \mathcal{N}_0$, and $\boldsymbol{\xi}' = \boldsymbol{\xi} + \mathbf{e}_i - \mathbf{e}_j$, $i, j \in \mathcal{N}_0$ and $i \neq j$, for which $\boldsymbol{\xi}' \in \{0, 1\}^{n+1}$. If any such $\boldsymbol{\xi}'$ improves the objective function of (12), update $\boldsymbol{\xi}$ to the best such $\boldsymbol{\xi}'$ and go to Step 2. Otherwise, go to Step 4.
4. **Termination.** Terminate and return $\boldsymbol{\xi}$ as worst-case valuation.

Being a local search heuristic, the worst-case valuation returned by our adaptation of ADXOpt may not represent a global minimizer of the primal subproblem (12). As such, we may increase LB_p and, subsequently, LB , excessively. Likewise, the application of ADXOpt to the dual subproblem (14) may decrease UB_d and, subsequently, UB , excessively. Either of these occurrences may cause the two-layer column generation scheme to terminate early with a feasible but suboptimal primal-dual solution pair $(\mathbf{p}^*, \boldsymbol{\kappa}^*)$. Our numerical experiments in Section 7 show, however, that the local search heuristic performs remarkably well and returns solutions that are very close—and often identical—to the optimal ones.

A.2. Exact and Heuristic Solution of the Randomized MC Problem

For the MC choice model, the revenues $R(S, u)$ can be expressed as

$$R(S, u) = R(S, \boldsymbol{\rho}) = \sum_{i \in \mathcal{N}} \lambda_i \cdot g_i \quad \text{with} \quad g_i = \begin{cases} r_i & \text{if } i \in S, \\ \sum_{j \in \mathcal{N}} \rho_{ij} \cdot g_j & \text{if } i \in \mathcal{N} \setminus S. \end{cases}$$

Also, recall that we consider finite uncertainty sets $\mathcal{U} \subseteq \mathbb{R}_+^{(n+1) \times (n+1)}$ for the general substitution sets as well as $\mathcal{U} = \prod_{i \in \mathcal{N}_0} \mathcal{U}_i$ with $\mathcal{U}_i \subseteq \mathbb{R}^{n+1}$, $i \in \mathcal{N}_0$, for the product-wise substitution sets.

To solve the subproblem (12) in the primal step of our column generation scheme, we first consider the case of general substitution sets. In that case, we can explicitly evaluate $R(S, \boldsymbol{\rho})$ for each assortment $S \in \hat{\mathcal{S}}$ and each substitution matrix $\boldsymbol{\rho} \in \mathcal{U}$ by solving a separate linear program

$$\begin{aligned} & \text{minimize} && \sum_{i \in \mathcal{N}} \lambda_i \cdot g_i \\ & \text{subject to} && g_i = r_i && \forall i \in S \\ & && g_i \geq \sum_{j \in \mathcal{N}} \rho_{ij} \cdot g_j && \forall i \in \mathcal{N} \setminus S \\ & && g_i \in \mathbb{R}, i \in \mathcal{N} \end{aligned}$$

and subsequently select a substitution matrix $\boldsymbol{\rho} \in \mathcal{U}$ that minimizes $\sum_{S \in \hat{\mathcal{S}}} p_S \cdot R(S, \boldsymbol{\rho})$. A similar approach would be inefficient for product-wise substitution sets, however, as $\prod_{i \in \mathcal{N}_0} |\mathcal{U}_i|$ many linear programs would have to be solved. Instead, we first formulate the subproblem (12) as the mixed-integer bilinear program

$$\begin{aligned}
& \text{minimize} && \sum_{S \in \hat{\mathcal{S}}} p_S \cdot \left[\sum_{i \in \mathcal{N}} \lambda_i \cdot g_i^S \right] \\
& \text{subject to} && g_i^S = r_i && \forall i \in S, \forall S \in \hat{\mathcal{S}} \\
& && g_i^S \geq \sum_{\rho_i \in \mathcal{U}_i} x_i^{\rho_i} \cdot y_i^{\rho_i S} && \forall i \in \mathcal{N} \setminus S, \forall S \in \hat{\mathcal{S}} \\
& && y_i^{\rho_i S} = \sum_{j \in \mathcal{N}} \rho_{ij}^{\rho_i} \cdot g_j^S && \forall i \in \mathcal{N} \setminus S, \forall S \in \hat{\mathcal{S}}, \forall \rho_i \in \mathcal{U}_i \\
& && \sum_{\rho_i \in \mathcal{U}_i} x_i^{\rho_i} = 1 && \forall i \in \mathcal{N} \\
& && g_i^S \in \mathbb{R}, i \in \mathcal{N} \text{ and } S \in \hat{\mathcal{S}} \\
& && x_i^{\rho_i} \in \{0, 1\}, i \in \mathcal{N} \text{ and } \rho_i \in \mathcal{U}_i \\
& && y_i^{\rho_i S} \in \mathbb{R}, i \in \mathcal{N}, \rho_i \in \mathcal{U}_i \text{ and } S \in \hat{\mathcal{S}},
\end{aligned}$$

where $x_i^{\rho_i}$ is 1 if $\rho_i \in \mathcal{U}_i$ is selected as worst-case substitution vector for product $i \in \mathcal{N}$, and 0 otherwise. Note that the variables $x_i^{\rho_i}$ satisfying $i \in \bigcap_{S \in \hat{\mathcal{S}}} S$ do not appear in the first three constraints and can thus be eliminated from the problem. For all $\rho_i \in \mathcal{U}_i$, $i \in \mathcal{N} \setminus S$ and $S \in \hat{\mathcal{S}}$, we now apply the exact linearization

$$z_i^{\rho_i S} = x_i^{\rho_i} \cdot y_i^{\rho_i S} \iff \begin{bmatrix} z_i^{\rho_i S} \leq x_i^{\rho_i}, & z_i^{\rho_i S} \leq y_i^{\rho_i S} \\ z_i^{\rho_i S} \geq 0, & z_i^{\rho_i S} \geq x_i^{\rho_i} + y_i^{\rho_i S} - 1 \end{bmatrix},$$

where $z_i^{\rho_i S} \in \mathbb{R}$, $i \in \mathcal{N} \setminus S$, $S \in \hat{\mathcal{S}}$ and $\rho_i \in \mathcal{U}_i$, constitute auxiliary decision variables. The resulting formulation is an MILP that can be solved with off-the-shelf software.

In view of the subproblem (14) in the dual step of our algorithm, we introduce the binary decision variables $x_i \in \{0, 1\}$, $i \in \mathcal{N}$, to denote whether product i is contained in the assortment S , as well as the continuous auxiliary decision variables g_i^ρ , $i \in \mathcal{N}$ and $\rho \in \hat{\mathcal{U}}$, to obtain the following mixed-integer bilinear program:

$$\begin{aligned}
& \text{maximize} && \sum_{\rho \in \hat{\mathcal{U}}} \kappa_\rho \cdot \left[\sum_{i \in \mathcal{N}} \lambda_i \cdot g_i^\rho \right] \\
& \text{subject to} && g_i^\rho \leq r_i \cdot x_i + \sum_{j \in \mathcal{N}} \rho_{ij} \cdot g_j^\rho \cdot (1 - x_i) && \forall i \in \mathcal{N}, \rho \in \hat{\mathcal{U}} \\
& && g_i^\rho \in \mathbb{R}, i \in \mathcal{N} \text{ and } \rho \in \hat{\mathcal{U}} \\
& && \mathbf{x} \in \mathcal{X}
\end{aligned}$$

Here, $\mathcal{X} \subseteq \{0, 1\}^n$ reflects the business constraints encoded by the choice of \mathcal{S} . The first constraint enforces that under the substitution matrix $\rho \in \hat{\mathcal{U}}$, g_i^ρ attains the value r_i if product i is selected, and g_i^ρ is upper bounded by $\sum_{j \in \mathcal{N}} \rho_{ij} \cdot g_j^\rho$ otherwise. We apply the exact linearization

$$z_i^\rho = x_i \cdot y_i^\rho \iff \begin{bmatrix} z_i^\rho \leq x_i, & z_i^\rho \leq y_i^\rho \\ z_i^\rho \geq 0, & z_i^\rho \geq x_i + y_i^\rho - 1 \end{bmatrix},$$

where $z_i^\rho \in \mathbb{R}$, $i \in \mathcal{N}$ and $\rho \in \hat{\mathcal{U}}$, are auxiliary decision variables, to obtain an equivalent MILP reformulation.

As in the MNL choice model, the main computational challenge in the column generation scheme arises from the subproblems of Steps 2(b) and 3(b). Instead of solving those problem exactly, we can principally solve them with local search heuristics akin to the ADXOpt heuristic of Jagabathula (2014). In our numerical experiments, however, such local search heuristics did not perform well as the evaluation of each candidate solution requires the solution of a linear program for each $S \in \hat{\mathcal{S}}$ (for the primal subproblem, to determine the values of g_i^S) or $\rho \in \hat{\mathcal{U}}$ (for the dual subproblem, to determine the values of g_i^ρ). This is in contrast to the MNL and preference ranking choice models, where the evaluation of each candidate solution only requires the computation of a closed-form expression.

A.3. Exact and Heuristic Solution of the Randomized Preference Ranking Problem

Recall from equation (7) that for the preference ranking choice model, the revenues $R(S, u)$ evaluate to

$$R(S, u) = R(S, \boldsymbol{\lambda}) = \sum_{k \in \mathcal{K}} \lambda_k \cdot R_k(S),$$

and recall from equation (8) that we consider the polyhedral uncertainty set

$$\Lambda = \{ \boldsymbol{\lambda} = \mathbf{F}\boldsymbol{\xi} : \mathbf{A}\boldsymbol{\xi} \leq \mathbf{b}, \boldsymbol{\xi} \in \mathbb{R}_+^m \}.$$

The randomized assortment optimization problem under the preference ranking choice model is simpler than its counterparts under the MNL and MC models since in the present case, the revenues are linear in the uncertainty realization. In the following, we provide a simplified column generation scheme that is reminiscent of the single-layer algorithm proposed in Section 5.4 of [Delage and Saif \(2021\)](#).

Column generation scheme for problem (9)

1. **Initialization.** Set $\hat{\mathcal{S}}$ to any non-empty subset of \mathcal{S} (currently considered assortments). Set $\text{LB} = -\infty$ and $\text{UB} = +\infty$ (current lower and upper bounds on the optimal value).
2. **Master problem.** Solve the restricted dual problem (10) that replaces $S \in \mathcal{S}$ with $S \in \hat{\mathcal{S}}$ in the objective function. Update $\text{LB} = \max\{\text{LB}, \tau^*\}$, where τ^* is the optimal value of the restricted dual problem, and set $\boldsymbol{\lambda}$ to an optimal solution of the problem.
3. **Subproblem.** Solve the subproblem

$$\begin{aligned} & \text{maximize} && \sum_{k \in \mathcal{K}} \lambda_k \cdot R_k(S) \\ & \text{subject to} && S \in \mathcal{S}. \end{aligned}$$

Update $\text{UB} = \min\{\text{UB}, \tau^*\}$, where τ^* is the optimal value of the subproblem, and $\hat{\mathcal{S}} = \hat{\mathcal{S}} \cup \{S^*\}$, where S^* is an optimal solution of the problem.

4. **Termination.** Iterate between Steps 2 and 3 until the gap between LB and UB is zero or sufficiently small. Then terminate with the primal-dual pair $(\mathbf{p}^*, \boldsymbol{\lambda}^*) \in \mathbb{R}^{|\mathcal{S}|} \times \mathbb{R}^K$, where p_S^* , $S \in \hat{\mathcal{S}}$, can be derived from the shadow prices of the epigraph reformulation of the restricted dual problem (10), and $p_S^* = 0$ for $S \notin \hat{\mathcal{S}}$.

The algorithm iterates between solving two problems: Step 2 solves an increasingly accurate progressive approximation of the dual problem (10) to determine a candidate solution $\boldsymbol{\lambda} \in \Lambda$. Step 3 evaluates the exact objective function of problem (10) for $\boldsymbol{\lambda}$ and identifies a cutting plane to add to the approximation from Step 2. Both problems can be expressed as finite (MI)LPs that can be solved with standard software ([Bertsimas and Mišić 2019](#)).

The correctness of the algorithm is summarized next.

THEOREM 10. *The column generation algorithm presented above converges in a finite number of iterations and terminates with an optimal primal-dual solution pair $(\mathbf{p}^*, \boldsymbol{\lambda}^*)$ to problems (9) and (10).*

Proof of Theorem 10. The proof largely follows similar arguments as the proof of Theorem 2 of [Delage and Saif \(2021\)](#) and is thus omitted for the sake of brevity. \square

The main computational challenge in the above column generation scheme arises from the subproblem of Step 3, which amounts to a nominal assortment optimization problem over the preference ranking model. Instead of solving this problem exactly, we can solve it heuristically with the ADXOpt heuristic of [Jagathula \(2014\)](#). In doing so, we may miss the global maximizer of the subproblem and therefore decrease UB excessively, which in turn can cause the column generation scheme to terminate early with a feasible but suboptimal primal-dual solution pair $(\mathbf{p}^*, \boldsymbol{\lambda}^*)$. Similar to the cardinality-constrained MNL model, however, our numerical experiments in Section 7 show that the local search heuristic performs remarkably well and generally returns solutions of excellent quality.

Appendix B: Auxiliary Results and Proofs for Section 3

Throughout this section, \mathbb{P} refers to the product distribution generated by the assortment-wise distributions \mathbb{P}_S , $S \in \mathcal{S}$, that are characterized in [Assumption 1](#).

Proof of Theorem 1. Denote by $\mathcal{R} = \{\mathbf{R} \in \mathbb{R}^{|\mathcal{S}|} : [R_S = R(S, u) \forall S \in \mathcal{S}] \text{ for some } u \in \mathcal{U}\}$ the set of all expected revenues vectors for the different uncertainty realizations $u \in \mathcal{U}$. The statement of the theorem then claims that the instance $(\mathcal{N}, \mathcal{S}, \mathcal{C}, \mathcal{U}, \mathbf{r})$ is randomization-proof if and only if there is $\mathbf{R}' \in \text{conv } \mathcal{R}$ such that $\mathbf{R}' \leq R_{\text{det}}^*(\mathcal{U}) \cdot \mathbf{e}$, where $\text{conv } \mathcal{R}$ denotes the convex hull of \mathcal{R} .

We first note that

$$R^*(\mathbf{p}) = \min_{u \in \mathcal{U}} R(\mathbf{p}, u) = \min_{\mathbf{R} \in \mathcal{R}} \mathbf{p}^\top \mathbf{R} = \min_{\mathbf{R} \in \text{conv } \mathcal{R}} \mathbf{p}^\top \mathbf{R}, \quad (15)$$

where the last identity follows from the fact that the minimization over $\mathbf{R} \in \mathcal{R}$ has a linear objective, which allows us to replace its feasible region \mathcal{R} by its convex hull $\text{conv } \mathcal{R}$.

Assume first that there is $\mathbb{Q} \in \mathcal{P}(\mathcal{U})$ such that $\mathbb{E}_{\mathbb{Q}}[R(S, \tilde{u})] \leq R_{\text{det}}^*(\mathcal{U})$ for all $S \in \mathcal{S}$. As discussed before, this is equivalent to assuming that there is $\mathbf{R}' \in \text{conv } \mathcal{R}$ such that $\mathbf{R}' \leq R_{\text{det}}^*(\mathcal{U}) \cdot \mathbf{e}$. In that case, any randomized assortment $\mathbf{p} \in \Delta(\mathcal{S})$ satisfies

$$R^*(\mathbf{p}) = \min_{\mathbf{R} \in \text{conv } \mathcal{R}} \mathbf{p}^\top \mathbf{R} \leq \mathbf{p}^\top \mathbf{R}' \leq R_{\text{det}}^*(\mathcal{U}),$$

where the identity follows from (15), the first inequality holds since $\mathbf{R}' \in \text{conv } \mathcal{R}$, and the second inequality is due to the fact that $\mathbf{p} \in \Delta(\mathcal{S})$ and $\mathbf{R}' \leq R_{\text{det}}^*(\mathcal{U}) \cdot \mathbf{e}$.

Assume now that there is no $\mathbb{Q} \in \mathcal{P}(\mathcal{U})$ such that $\mathbb{E}_{\mathbb{Q}}[R(S, \tilde{u})] \leq R_{\text{det}}^*(\mathcal{U})$ for all $S \in \mathcal{S}$. This is equivalent to assuming that there is no $\mathbf{R}' \in \text{conv } \mathcal{R}$ such that $\mathbf{R}' \leq R_{\text{det}}^*(\mathcal{U}) \cdot \mathbf{e}$. In that case, we note that

$$\max_{\mathbf{p} \in \Delta(\mathcal{S})} R^*(\mathbf{p}) = \max_{\mathbf{p} \in \Delta(\mathcal{S})} \min_{\mathbf{R} \in \text{conv } \mathcal{R}} \mathbf{p}^\top \mathbf{R} = \min_{\mathbf{R} \in \text{conv } \mathcal{R}} \max_{\mathbf{p} \in \Delta(\mathcal{S})} \mathbf{p}^\top \mathbf{R} \geq \min_{\mathbf{R} \in \text{conv } \mathcal{R}} \max_{S \in \mathcal{S}} R_S > R_{\text{det}}^*(\mathcal{U}),$$

where the first identity follows from (15), the second one is due to strong duality, the third one holds since $\mathbf{e}_S \in \Delta(\mathcal{S})$ for all $S \in \mathcal{S}$, and the inequality holds by assumption. \square

Proof of Corollary 1. Fix $u' \in \arg \min_{u \in \mathcal{U}} \max_{S \in \mathcal{S}} R(S, u)$ and $S' \in \arg \max_{S \in \mathcal{S}} R(S, u')$. For any $\mathbf{p} \in \Delta(\mathcal{S})$ in ([RANDOMIZED ROBUST](#)), we then have

$$R^*(\mathbf{p}) = \min_{u \in \mathcal{U}} \sum_{S \in \mathcal{S}} p_S \cdot R(S, u) \leq \sum_{S \in \mathcal{S}} p_S \cdot R(S, u') \leq \sum_{S \in \mathcal{S}} p_S \cdot R(S', u') = R(S', u') = \max_{S \in \mathcal{S}} \min_{u \in \mathcal{U}} R(S, u),$$

where the first inequality holds since $u' \in \mathcal{U}$ and the second inequality holds by construction of S' , respectively. The first identity follows from the fact that $\sum_{S \in \mathcal{S}} p_S = 1$. The second identity, finally, follows from the construction of S' and u' as well as the assumed strong duality. The statement now follows since the choice of the probability vector \mathbf{p} was arbitrary. \square

Proof of Corollary 2. Assume that the worst-case parameter set is indeed a singleton. If the inequality from the statement of the corollary holds, then the choice $\mathbb{Q} = \delta_{u^*}$ in [Theorem 1](#), which is the Dirac distribution that places unit mass on the realization $\tilde{u} = u^*$, implies that the problem is randomization-proof. If the problem is randomization-proof, on the other hand, then any distribution \mathbb{Q} satisfying the condition in

Theorem 1 has to satisfy $\mathbb{Q}[\tilde{u} = u^*] = 1$ since $R(S^*, u) > R^*(S^*)$ for all $u \in \mathcal{U} \setminus \{u^*\}$. In that case, however, we have $R(S, u^*) \leq R^*(S^*)$ for all $S \in \mathcal{S}$, that is, the inequality from the statement of the corollary holds. \square

In the remainder of this section, we denote by $\underline{r}(S) = \min\{r_i : i \in S\}$ and $\bar{r}(S) = \max\{r_i : i \in S\}$ the minimum and maximum product price over any subset $S \subseteq \mathcal{N}$, respectively. The proof of **Theorem 2** relies on several auxiliary results that we state and prove first.

LEMMA 1. *Under Assumption 1, we have that:*

- (i) *The random expected revenues $\tilde{R}(S, u)$, $S \in \mathcal{S}$ and $u \in \mathcal{U}$, are supported on $[0, \bar{r}(S)]$, and the cumulative distribution function of $\tilde{R}(S, u)$ is continuous and strictly increasing over this support.*
- (ii) *The random worst-case expected revenues $\tilde{R}^*(S)$ of any assortment $S \in \mathcal{S}$ are \mathbb{P} -a.s. attained by a single uncertainty realization $u \in \mathcal{U}$, that is $|\arg \min_{u \in \mathcal{U}} \tilde{R}(S, u)| = 1$ \mathbb{P} -a.s. for all $S \in \mathcal{S}$.*

Proof of Lemma 1. The first statement follows immediately from the first condition of **Assumption 1**. In view of the second statement, we note that for all $S \in \mathcal{S}$ and $u_1, u_2 \in \mathcal{U}$, we have that

$$\mathbb{P} \left[\tilde{R}(S, u_1) = \tilde{R}(S, u_2) \right] = \mathbb{P} \left[\sum_{i \in S} r_i \cdot \left(\tilde{\mathfrak{C}}(i|S, u_1) - \tilde{\mathfrak{C}}(i|S, u_2) \right) = 0 \right] = 0,$$

where the last equality is due to the fact that the product prices r_i are strictly positive and the two i.i.d. random vectors $\tilde{\mathfrak{C}}(\cdot|S, u_1)$ and $\tilde{\mathfrak{C}}(\cdot|S, u_2)$ are both governed by the atomless distribution \mathbb{P}_S . The statement then follows from the fact that $|\mathcal{U}|$ is finite. \square

LEMMA 2. *Fix a random robust assortment optimization problem $(\mathcal{N}, \mathcal{S}, \tilde{\mathfrak{C}}, \mathcal{U}, \mathbf{r})$ satisfying Assumption 1. For any $\delta \in (0, 1)$, there is a conjugate pair (p^*, q^*) such that for every $S \in \mathcal{S}$, there exists $\theta_S \in \mathbb{R}$ satisfying:*

- (i) $\mathbb{P} \left[\tilde{R}(S, u) \geq \theta_S \right] \leq |\mathcal{U}| \sqrt{\frac{\delta}{p^* \cdot |\mathcal{S}|}}$ for all $u \in \mathcal{U}$.
- (ii) *There are at least $|\mathcal{S}|^\alpha - 1$ other assortments $S' \in \mathcal{S} \setminus \{S\}$, $\alpha > \frac{1}{|\mathcal{U}|}$, such that $\mathbb{P} \left[\tilde{R}(S', u) \geq \theta_S \right] \geq |\mathcal{U}| \sqrt{\frac{\delta}{q^* \cdot |\mathcal{S}|}}$ for all $u \in \mathcal{U}$.*

Recall that $(p^*, q^*) \in [\mathbb{R} \cup \{+\infty\}]^2$ is a conjugate pair if $1/p^* + 1/q^* = 1$.

Proof of Lemma 2. We prove the statement in three steps. In the first step, we show that for any fixed $S \in \mathcal{S}$ and any $p \geq 1$, there are $\theta_S(p)$ and sufficiently many assortments $S' \in \mathcal{S}$ for which the two conditions of the statement hold for all q sufficiently large. In the second step, we restrict the range of p such that the result from Step 1 holds for a conjugate q of p . In the third step, finally, we restrict the range of p further to ensure that the result from Step 2 holds simultaneously for all $S \in \mathcal{S}$.

In view of the first step, we define for any $S \in \mathcal{S}$ and $p \in [1, \infty)$ the function

$$\theta_S(p) = F_S^{-1} \left(1 - |\mathcal{U}| \sqrt{\frac{\delta}{p \cdot |\mathcal{S}|}} \right),$$

where F_S^{-1} is the inverse cumulative distribution function common to all $\tilde{R}(S, u)$, $u \in \mathcal{U}$. The value $\theta_S(p)$, which constitutes the common $1 - |\mathcal{U}| \sqrt{\frac{\delta}{p \cdot |\mathcal{S}|}}$ quantile of the random variables $\tilde{R}(S, u)$, $u \in \mathcal{U}$, will serve as our

choice of θ_S if p is set to the value of p^* in the statement of the lemma. Note that $\theta_S(p)$ is well-defined since it is the inverse of a strictly increasing function (cf. Lemma 1), and it is continuous and strictly increasing for similar reasons. Moreover, the image of $\theta_S(p)$ over $p \in [1, \infty)$ is $[\theta_S(1), \bar{r}(S))$ with $0 < \theta_S(1) < \bar{r}(S)$.

Next, for $S' \in \mathcal{S} \setminus \{S\}$ including at least one most expensive product from \mathcal{N} , we define the function

$$\epsilon_{S'}(p) = \mathbb{P} \left[\tilde{R}(S', u) \geq \theta_S(p) \right] = 1 - F_{S'}(\theta_S(p)),$$

where $F_{S'}$ is the cumulative distribution function common to all $\tilde{R}(S', u)$, $u \in \mathcal{U}$. We will equate $\epsilon_{S'}(p)$ with $|\mathcal{U}| \sqrt{\frac{\delta}{q \cdot |\mathcal{S}|}}$ to compute a lower bound of q such that $\mathbb{P} \left[\tilde{R}(S', u) \geq \theta_S(p) \right] \geq |\mathcal{U}| \sqrt{\frac{\delta}{q \cdot |\mathcal{S}|}}$. Since the image of $\theta_S(p)$ over $p \in [1, \infty)$ is $[\theta_S(1), \bar{r}(S)) \subset [0, \bar{r}(S')]$ by the fact that S' contains a most expensive product from \mathcal{N} , and both $\theta_S(p)$ and $F_{S'}$ are strictly increasing and continuous, $\epsilon_{S'}(p)$ is strictly decreasing and continuous over $p \in [1, \infty)$. Moreover, we have $\epsilon_{S'}(1) > 0$ since $\theta_S(1) < \bar{r}(S) \leq \bar{r}(S')$.

Finally, we denote the minimum q such that $\mathbb{P} \left[\tilde{R}(S', u) \geq \theta_S(p) \right] \geq |\mathcal{U}| \sqrt{\frac{\delta}{q \cdot |\mathcal{S}|}}$ holds by

$$q_{S'}(p) = \frac{\delta}{|\mathcal{S}| \cdot \epsilon_{S'}(p)^{|\mathcal{U}|}}.$$

Since $\epsilon_{S'}(p)$ is strictly decreasing and continuous as well as strictly positive over $p \in [1, \infty)$, $q_{S'}(p)$ is strictly increasing, continuous and finite over $p \in [1, \infty)$.

In summary, Step 1 has shown that for any $p \in [1, \infty)$, the functions $\theta_S(p)$ and $q_{S'}(p)$ provide a value of θ_S and the minimum value of q such that for all $q \geq q_{S'}(p)$, we have

$$\mathbb{P} \left[\tilde{R}(S, u) \geq \theta_S \right] \leq |\mathcal{U}| \sqrt{\frac{\delta}{p \cdot |\mathcal{S}|}} \quad \text{and} \quad \mathbb{P} \left[\tilde{R}(S', u) \geq \theta_S \right] \geq |\mathcal{U}| \sqrt{\frac{\delta}{q \cdot |\mathcal{S}|}}.$$

As for the second step, we show that the interval $[q_{S'}(p), \infty)$ contains the conjugate of some $p \in [1, \infty)$. To this end, define $q'(p) = p/(p-1)$ if $p > 1$; $+\infty$ otherwise as the conjugate of p . By construction, q' is continuous and strictly decreasing over $p \in [1, \infty)$. Since $q_{S'}(1) < \infty$ but $q'(1) = \infty$, the continuity of both functions implies the existence of $\eta > 0$ such that $q'(p) \geq q_{S'}(p)$ for all $p \in [1, 1 + \eta]$. Hence, for all $p \in (1, 1 + \eta]$, the choice $q = q'(p)$ and $\theta_S = \theta_S(p)$ implies that

$$\mathbb{P} \left[\tilde{R}(S, u) \geq \theta_S \right] \leq |\mathcal{U}| \sqrt{\frac{\delta}{p \cdot |\mathcal{S}|}} \quad \text{and} \quad \mathbb{P} \left[\tilde{R}(S', u) \geq \theta_S \right] \geq |\mathcal{U}| \sqrt{\frac{\delta}{q \cdot |\mathcal{S}|}}.$$

In view of the third step, finally, Assumption 1 (ii) ensures that for every $S \in \mathcal{S}$ there is a set $\mathcal{S}_S \subseteq \mathcal{S} \setminus \{S\}$ of assortments, $|\mathcal{S}_S| \geq |\mathcal{S}|^\alpha - 1$, such that each $S' \in \mathcal{S}_S$ contains a most expensive product. For each $S \in \mathcal{S}$ and $S' \in \mathcal{S}_S$, denote by $\eta_{S,S'} > 0$ the value of η determined above, and define $\eta_S = \min\{\eta_{S,S'} : S' \in \mathcal{S}_S\} > 0$. For any $p \in (1, 1 + \eta_S]$, the choice $q = q'(p)$ and $\theta_S = \theta_S(p)$ then implies that

$$\mathbb{P} \left[\tilde{R}(S, u) \geq \theta_S \right] \leq |\mathcal{U}| \sqrt{\frac{\delta}{p \cdot |\mathcal{S}|}} \quad \text{and} \quad \mathbb{P} \left[\tilde{R}(S', u) \geq \theta_S \right] \geq |\mathcal{U}| \sqrt{\frac{\delta}{q \cdot |\mathcal{S}|}} \quad \forall S' \in \mathcal{S}_S.$$

For $\eta = \min_{S \in \mathcal{S}} \eta_S > 0$ and any $p^* \in (1, 1 + \eta)$, finally, we can find a conjugate q^* and $\{\theta_S\}_{S \in \mathcal{S}}$ so that the statements from the lemma hold. \square

LEMMA 3. Fix a random robust assortment optimization problem $(\mathcal{N}, \mathcal{S}, \tilde{\mathcal{C}}, \mathcal{U}, \mathbf{r})$ satisfying Assumption 1. For any $\delta \in (0, 1)$, there is a conjugate pair (p^*, q^*) and $\theta \in \mathbb{R}$ satisfying $|\mathcal{S}|^{-1} \sqrt{\frac{\delta}{q^* \cdot |\mathcal{S}|}} \leq \mathbb{P} \left[\tilde{R}(S, u) \leq \theta \right] \leq |\mathcal{S}|^{-1} \sqrt{\frac{\delta}{p^* \cdot |\mathcal{S}|}}$ for all $S \in \mathcal{S}$ and $u \in \mathcal{U}$.

Proof of Lemma 3. For any $S \in \mathcal{S}$ and $p \in [1, \infty)$, we define the function $\theta_S(p) = F_S^{-1} \left(\sqrt[|S|-1]{\frac{\delta}{p \cdot |S|}} \right)$, where F_S^{-1} is the inverse cumulative distribution function common to all random variables $\tilde{R}(S, u)$, $u \in \mathcal{U}$. Similar arguments as in the proof of Lemma 2 show that θ_S is continuous, strictly decreasing as well as strictly positive over $p \in [1, \infty)$, and these properties are inherited to the function $\theta(p) = \min\{\theta_S(p) : S \in \mathcal{S}\}$. We next define the function $\epsilon_S(p) = \mathbb{P} \left[\tilde{R}(S, u) \leq \theta(p) \right] = F_S(\theta(p))$, where F_S is the cumulative distribution function common to all random variables $\tilde{R}(S, u)$, $u \in \mathcal{U}$. Again, similar arguments as in the proof of Lemma 2 show that ϵ_S is continuous, strictly decreasing as well as strictly positive over $p \in [1, \infty)$, and these properties are inherited to the function $\epsilon(p) = \min\{\epsilon_S(p) : S \in \mathcal{S}\}$. Note that $\mathbb{P} \left[\tilde{R}(S, u) \leq \theta(p) \right] \leq \sqrt[|S|-1]{\frac{\delta}{p \cdot |S|}}$ for all $S \in \mathcal{S}$, $u \in \mathcal{U}$ and $p \in [1, \infty)$ by definition of $\theta(p)$. Similarly, we have

$$\mathbb{P} \left[\tilde{R}(S, u) \leq \theta(p) \right] \geq \sqrt[|S|-1]{\frac{\delta}{q \cdot |S|}} \quad \forall p \in [1, \infty), \forall q \geq q(p) = \frac{\delta}{|S| \cdot \epsilon(p)^{|S|-1}},$$

for all $S \in \mathcal{S}$ and $u \in \mathcal{U}$ by construction of $\epsilon(p)$, and our earlier discussion implies that $q(p)$ is finite for all $p \in [1, \infty)$. The same arguments as in the proof of Lemma 2 then show that there is $\eta > 0$ such that for all $p \in (1, 1 + \eta]$, the conjugate of p is contained in the interval $[q(p), \infty)$. Thus, for any $p^* \in (1, 1 + \eta]$, the corresponding conjugate $q^* = p^*/(p^* - 1)$ and $\theta = \theta(p^*)$ satisfy

$$\mathbb{P} \left[\tilde{R}(S, u) \leq \theta \right] \leq \sqrt[|S|-1]{\frac{\delta}{p^* \cdot |S|}} \quad \text{and} \quad \mathbb{P} \left[\tilde{R}(S, u) \leq \theta \right] \geq \sqrt[|S|-1]{\frac{\delta}{q^* \cdot |S|}} \quad \forall S \in \mathcal{S}$$

as desired, which completes the proof. \square

Proof of Theorem 2. We first note that

$$\begin{aligned} & \mathbb{P} \left[\text{problem } (\mathcal{N}, \mathcal{S}, \tilde{\mathcal{C}}, \mathcal{U}, \mathbf{r}) \text{ is randomization-proof} \right] \\ &= \mathbb{P} \left[\left(\text{problem } (\mathcal{N}, \mathcal{S}, \tilde{\mathcal{C}}, \mathcal{U}, \mathbf{r}) \text{ is randomization-proof} \right) \wedge \bigcup_{S \in \mathcal{S}} (S \text{ is an optimal assortment}) \right] \\ &= \mathbb{P} \left[\bigcup_{S \in \mathcal{S}} \left(\text{problem } (\mathcal{N}, \mathcal{S}, \tilde{\mathcal{C}}, \mathcal{U}, \mathbf{r}) \text{ is randomization-proof} \wedge S \text{ is an optimal assortment} \right) \right] \\ &\leq \sum_{S \in \mathcal{S}} \mathbb{P} \left[\left(\text{problem } (\mathcal{N}, \mathcal{S}, \tilde{\mathcal{C}}, \mathcal{U}, \mathbf{r}) \text{ is randomization-proof} \wedge S \text{ is an optimal assortment} \right) \right] \\ &= \sum_{S \in \mathcal{S}} \mathbb{P} \left[\tilde{R}^*(S) \geq \max_{S' \in \mathcal{S} \setminus \{S\}} \tilde{R}(S', \tilde{u}(S)) \right], \end{aligned}$$

where the inequality is due to Bonferroni's inequality, $\tilde{u}(S)$ is any minimizer of $\tilde{R}(S, u)$ over $u \in \mathcal{U}$, and the last equality is due to Corollary 2 and Lemma 1. Fix any assortment $S \in \mathcal{S}$. The independence of the choice vectors $\tilde{\mathcal{C}}$ then allows us to conclude that

$$\mathbb{P} \left[\tilde{R}^*(S) \geq \max_{S' \in \mathcal{S} \setminus \{S\}} \tilde{R}(S', \tilde{u}(S)) \right] = \mathbb{P} \left[\min_{u \in \mathcal{U}} \tilde{R}(S, u) \geq \max_{S' \in \mathcal{S} \setminus \{S\}} \tilde{R}(S') \right],$$

where for all $S' \in \mathcal{S}$, $\tilde{R}(S')$ follows the same distribution $\mathbb{P}_{S'}$ as all $\tilde{R}(S', u)$, $u \in \mathcal{U}$.

We next 'break the connection' between the two random variables on the left-hand side and right-hand side of the above expression by introducing a suitably chosen scalar θ_S :

$$\mathbb{P} \left[\min_{u \in \mathcal{U}} \tilde{R}(S, u) \geq \max_{S' \in \mathcal{S} \setminus \{S\}} \tilde{R}(S') \right] = 1 - \mathbb{P} \left[\min_{u \in \mathcal{U}} \tilde{R}(S, u) < \max_{S' \in \mathcal{S} \setminus \{S\}} \tilde{R}(S') \right]$$

$$\begin{aligned}
&\leq 1 - \mathbb{P} \left[\min_{u \in \mathcal{U}} \tilde{R}(S, u) \leq \theta_S < \max_{S' \in \mathcal{S} \setminus \{S\}} \tilde{R}(S') \right] \\
&= 1 - \mathbb{P} \left[\min_{u \in \mathcal{U}} \tilde{R}(S, u) \leq \theta_S \right] \cdot \mathbb{P} \left[\theta_S < \max_{S' \in \mathcal{S} \setminus \{S\}} \tilde{R}(S') \right] \\
&= 1 - \mathbb{P} \left[\min_{u \in \mathcal{U}} \tilde{R}(S, u) \leq \theta_S \right] \cdot \left(1 - \mathbb{P} \left[\max_{S' \in \mathcal{S} \setminus \{S\}} \tilde{R}(S') \leq \theta_S \right] \right)
\end{aligned}$$

We further observe that

$$\mathbb{P} \left[\min_{u \in \mathcal{U}} \tilde{R}(S, u) \leq \theta_S \right] = 1 - \mathbb{P} \left[\min_{u \in \mathcal{U}} \tilde{R}(S, u) > \theta_S \right] = 1 - \mathbb{P} \left[\max_{u \in \mathcal{U}} -\tilde{R}(S, u) \leq -\theta_S \right],$$

where the use of a weak inequality in the last expression is justified since \mathbb{P}_S is atomless by [Lemma 1](#). The overall bound on the randomization-proofness probability thus amounts to

$$\begin{aligned}
&\sum_{S \in \mathcal{S}} \left[1 - \mathbb{P} \left[\min_{u \in \mathcal{U}} \tilde{R}(S, u) \leq \theta_S \right] \cdot \left(1 - \mathbb{P} \left[\max_{S' \in \mathcal{S} \setminus \{S\}} \tilde{R}(S') \leq \theta_S \right] \right) \right] \\
&= \sum_{S \in \mathcal{S}} \left[1 - \left(1 - \mathbb{P} \left[\max_{u \in \mathcal{U}} -\tilde{R}(S, u) \leq -\theta_S \right] \right) \cdot \left(1 - \mathbb{P} \left[\max_{S' \in \mathcal{S} \setminus \{S\}} \tilde{R}(S') \leq \theta_S \right] \right) \right] \\
&= \sum_{S \in \mathcal{S}} \mathbb{P} \left[\max_{S' \in \mathcal{S} \setminus \{S\}} \tilde{R}(S') \leq \theta_S \right] + \mathbb{P} \left[\max_{u \in \mathcal{U}} -\tilde{R}(S, u) \leq -\theta_S \right] \\
&\quad - \mathbb{P} \left[\max_{u \in \mathcal{U}} -\tilde{R}(S, u) \leq -\theta_S \right] \cdot \mathbb{P} \left[\max_{S' \in \mathcal{S} \setminus \{S\}} \tilde{R}(S') \leq \theta_S \right] \\
&\leq \sum_{S \in \mathcal{S}} \mathbb{P} \left[\max_{S' \in \mathcal{S} \setminus \{S\}} \tilde{R}(S') \leq \theta_S \right] + \mathbb{P} \left[\max_{u \in \mathcal{U}} -\tilde{R}(S, u) \leq -\theta_S \right] \\
&= \sum_{S \in \mathcal{S}} \mathbb{P} \left[\max_{S' \in \mathcal{S} \setminus \{S\}} \tilde{R}(S') \leq \theta_S \right] + \mathbb{P} \left[\min_{u \in \mathcal{U}} \tilde{R}(S, u) \geq \theta_S \right].
\end{aligned}$$

We first study the case where the number $|\mathcal{S}|$ of admissible assortments is large. To this end, fix any

$$N \geq N_1 = \frac{\delta}{q} \cdot \left(\left[\frac{|\mathcal{U}|}{|\mathcal{U}| \cdot \alpha - 1} + \left(\left[\frac{|\mathcal{U}|}{|\mathcal{U}| \cdot \alpha - 1} \right]^2 + \left[\frac{\delta}{q} \right]^\alpha \right)^{0.5} \right] \cdot \left(\frac{q}{\delta} \right)^\alpha \right)^{\frac{2|\mathcal{U}|}{|\mathcal{U}| \cdot \alpha - 1}},$$

where α is any value strictly greater than $1/|\mathcal{U}|$ and q is any value satisfying the conditions described in the proof of [Lemma 2](#), as well as (p, q) and $\{\theta_S\}_{S \in \mathcal{S}}$ as described in the proof of [Lemma 2](#). We then have

$$\mathbb{P} \left[\min_{u \in \mathcal{U}} \tilde{R}(S, u) \geq \theta_S \right] = \prod_{u \in \mathcal{U}} \mathbb{P} \left[\tilde{R}(S, u) \geq \theta_S \right] = \mathbb{P} \left[\tilde{R}(S) \geq \theta_S \right]^{|\mathcal{U}|} \leq \frac{\delta}{p \cdot |\mathcal{S}|},$$

where the two equalities hold since $\tilde{R}(S, u), u \in \mathcal{U}$, are i.i.d. random variables, and the inequality is due to [Lemma 2](#) (i). Similarly, [Lemma 2](#) (ii) implies the existence of a set $\mathcal{S}_S \subseteq \mathcal{S} \setminus \{S\}$, $|\mathcal{S}_S| \geq |\mathcal{S}|^\alpha - 1$ for some $\alpha > \frac{1}{|\mathcal{U}|}$, such that

$$\begin{aligned}
&\mathbb{P} \left[\tilde{R}(S') \leq \theta_S \right] \leq 1 - \sqrt[|\mathcal{U}|]{\frac{\delta}{q \cdot |\mathcal{S}|}} \quad \forall S' \in \mathcal{S}_S \\
\implies &\mathbb{P} \left[\max_{S' \in \mathcal{S} \setminus \{S\}} \tilde{R}(S') \leq \theta_S \right] \leq \mathbb{P} \left[\max_{S' \in \mathcal{S}_S} \tilde{R}(S') \leq \theta_S \right] \leq \left(1 - \sqrt[|\mathcal{U}|]{\frac{\delta}{q \cdot |\mathcal{S}|}} \right)^{|\mathcal{S}|^\alpha - 1}.
\end{aligned}$$

Define $x = [\delta/(q \cdot |\mathcal{S}|)]^{-\frac{1}{|\mathcal{U}|}}$ so that $|\mathcal{S}| = (\delta/q) \cdot x^{|\mathcal{U}|}$. Clearly, $x > 1$ by the fact that $|\mathcal{S}| \geq 2$, $\delta \in (0, 1)$, $q \geq 1$ and $|\mathcal{U}| \geq 2$. We can then bound the above probability by

$$\left(1 - \sqrt[|\mathcal{U}|]{\frac{\delta}{q \cdot |\mathcal{S}|}} \right)^{|\mathcal{S}|^\alpha - 1} = \left(1 - \frac{1}{x} \right)^{\left(x^{|\mathcal{U}| \cdot \frac{\delta}{q}} \right)^\alpha - 1} = \left(1 - \frac{1}{x} \right)^{x \cdot \left(x^{(|\mathcal{U}| \cdot \alpha - 1)} \cdot \left(\frac{\delta}{q} \right)^\alpha - \frac{1}{x} \right)}$$

$$= \left[\left(1 - \frac{1}{x}\right)^x \right]^{x^{(|\mathcal{U}| \cdot \alpha - 1) \cdot \left(\frac{\delta}{q}\right)^\alpha - \frac{1}{x}}} \leq \left(\frac{1}{e}\right)^{x^{(|\mathcal{U}| \cdot \alpha - 1) \cdot \left(\frac{\delta}{q}\right)^\alpha - \frac{1}{x}}} \leq \left(\frac{1}{e}\right)^{x^{(|\mathcal{U}| \cdot \alpha - 1) \cdot \left(\frac{\delta}{q}\right)^\alpha - 1}},$$

where the penultimate inequality holds since $(1 - 1/x)^x < 1/e$ for $x > 1$ and $x^{(|\mathcal{U}| \cdot \alpha - 1) \cdot (\delta/q)^\alpha - 1/x} = (|\mathcal{S}|^\alpha - 1)/x > 0$. Likewise, the last inequality holds since $1/e < 1$ and $1/x < 1$. For $|\mathcal{S}| \geq N$, we then have that

$$\begin{aligned} & \left(\frac{1}{e}\right)^{x^{(|\mathcal{U}| \cdot \alpha - 1) \cdot \left(\frac{\delta}{q}\right)^\alpha - 1}} \leq \frac{\delta}{q \cdot |\mathcal{S}|} \quad \forall |\mathcal{S}| > N \\ \Leftrightarrow & \left(\frac{1}{e}\right)^{x^{(|\mathcal{U}| \cdot \alpha - 1) \cdot \left(\frac{\delta}{q}\right)^\alpha - 1}} \leq x^{-|\mathcal{U}|} \quad \forall x > \left(\frac{q \cdot N}{\delta}\right)^{1/|\mathcal{U}|} \\ \Leftrightarrow & -x^{(|\mathcal{U}| \cdot \alpha - 1) \cdot \left(\frac{\delta}{q}\right)^\alpha} + 1 + |\mathcal{U}| \cdot \log x \leq 0 \quad \forall x > \left(\frac{q \cdot N}{\delta}\right)^{1/|\mathcal{U}|}. \end{aligned} \quad (16)$$

This implies that $\mathbb{P}\left[\max_{S' \in \mathcal{S} \setminus \{S\}} \tilde{R}(S') \leq \theta_S\right] \leq \frac{\delta}{q \cdot |\mathcal{S}|}$ since tedious but otherwise straightforward algebraic manipulations show that the condition in (16) indeed holds for all $x > (q \cdot N/\delta)^{1/|\mathcal{U}|}$. In conclusion, we have

$$\sum_{S \in \mathcal{S}} \mathbb{P}\left[\min_{u \in \mathcal{U}} \tilde{R}(S, u) \geq \theta_S\right] + \mathbb{P}\left(\max_{S' \in \mathcal{S} \setminus \{S\}} \tilde{R}(S') \leq \theta_S\right) \leq \sum_{S \in \mathcal{S}} \left(\frac{\delta}{p \cdot |\mathcal{S}|} + \frac{\delta}{q \cdot |\mathcal{S}|}\right) = \delta$$

by the fact that (p, q) is a conjugate pair, which concludes the first part of the proof.

We now study the case where the cardinality $|\mathcal{U}|$ of the uncertainty set is large. To this end, fix any

$$N \geq N_2 = \log_{\left(1 - \frac{1}{\sqrt{q^* \cdot |\mathcal{S}|}}\right)} \left(\frac{\delta}{q^* \cdot |\mathcal{S}|}\right)$$

and select (p^*, q^*) as well as θ as described in the proof of Lemma 3. We then have

$$\mathbb{P}\left[\max_{S \in \mathcal{S} \setminus \{S\}} \tilde{R}(S) \leq \theta\right] = \prod_{S \in \mathcal{S} \setminus \{S\}} \mathbb{P}\left[\tilde{R}(S) \leq \theta\right] \leq \frac{\delta}{p^* \cdot |\mathcal{S}|},$$

where the equality holds since $\tilde{R}(S), S \in \mathcal{S}$, are independent random variables, as well as

$$\mathbb{P}\left[\min_{u \in \mathcal{U}} \tilde{R}(S, u) \geq \theta\right] = \prod_{u \in \mathcal{U}} \left(1 - \mathbb{P}\left[\tilde{R}(S, u) \leq \theta\right]\right) \leq \left(1 - \frac{1}{\sqrt{q^* \cdot |\mathcal{S}|}}\right)^{|\mathcal{U}|} \quad \forall S \in \mathcal{S},$$

where the reversion of the weak inequality in the second expression is justified since $\tilde{\mathcal{C}}$ and thus \tilde{R} is atomless. Note that the above quantity indeed does not exceed $\delta/(q^* \cdot |\mathcal{S}|)$ whenever $|\mathcal{U}| \geq N_2$.

In summary, the statement of the theorem holds for any $N \geq \max\{N_1, N_2\}$. \square

Appendix C: Auxiliary Results and Proofs for Section 4

Proof of Corollary 3. This result follows directly from the strong duality property of the robust assortment optimization problem under the MNL model (cf. Désir et al. 2021) and from Corollary 1. \square

For our analysis of the cardinality-constrained MNL model, we consider for any number of products $n \geq 2$ and any cardinality restriction $|S| \leq C$, $C \in \{1, \dots, n-1\}$ the instance where the price of each product i , $i \in \{1, \dots, n\}$, is $r_i = 1$, while the uncertainty set is $\mathcal{V} = \{(1, \mathbf{v}) : \sum_{i \in \mathcal{N}} v_i = n - C, \mathbf{v} \in \{0, 1\}^n\}$. In the remainder of this section, we refer to this model as the *reference model*.

The proof of Theorem 3 relies on the following auxiliary results, which we state and prove first.

LEMMA 4. *Under the reference model, the worst-case expected revenues of any feasible solution to the deterministic robust assortment optimization problem are 0.*

Proof of Lemma 4. Fix any feasible solution $S \in \mathcal{S}$, $|S| \leq C$, to the deterministic robust assortment optimization problem. Under any valuation scenario $\mathbf{v} = (1, \mathbf{v}') \in \mathcal{V}$ with $v'_i = 0$ if $i \in S$, the expected revenues of S vanish. The statement now follows from the fact that the worst-case expected revenues are always non-negative. \square

LEMMA 5. *Under the reference model, the randomized assortment strategy $\mathbf{p}^* \in \Delta(\mathcal{S})$, where $p_S^* = 1/|\{S \in \mathcal{S} : |S| = C\}|$ if $|S| = C$ and $p_S^* = 0$ otherwise, strictly outperforms any other feasible randomized assortment strategy \mathbf{p} that only places positive probability $p_S > 0$ on assortments S of full cardinality $|S| = C$.*

Proof of Lemma 5. Assume to the contrary that there is a feasible randomized assortment strategy $\mathbf{p} \neq \mathbf{p}^*$ that (i) places positive probability $p_S > 0$ only on assortments S of cardinality $|S| = C$ and that (ii) has worst-case revenues larger than or equal to those of \mathbf{p}^* . Since $\mathbf{p} \neq \mathbf{p}^*$, there must be two components i and j of \mathbf{p} —without loss of generality $i = 1$ and $j = 2$ —such that $p_i > p_j$ and the associated assortments S_i and S_j satisfy $|S_i| = |S_j| = C$. Let Π denote the group of all permutations $\pi : \mathbb{R}^{2^n} \rightarrow \mathbb{R}^{2^n}$ that permute those components of a randomization strategy in $\Delta(\mathcal{S})$ that correspond to assortments S of cardinality $|S| = C$. Moreover, assume that $\pi_1 \in \Pi$ is the identity and that $\pi_2 \in \Pi$ reverses the order of the first two components but leaves all other components unchanged.

Consider now the randomization strategy \mathbf{p}' defined through

$$\begin{aligned} \mathbf{p}' &= \frac{1}{|\Pi \setminus \{\pi_1, \pi_2\}|} \sum_{\pi \in \Pi \setminus \{\pi_1, \pi_2\}} \pi(\mathbf{p}) + \left(\frac{1}{|\Pi|} + \frac{\epsilon}{p_1 - p_2} \right) \cdot \pi_1(\mathbf{p}) + \left(\frac{1}{|\Pi|} - \frac{\epsilon}{p_1 - p_2} \right) \cdot \pi_2(\mathbf{p}) \\ &= \mathbf{p}^* + \epsilon(+1, -1, 0, \dots, 0)^\top. \end{aligned}$$

For $\epsilon > 0$ sufficiently small, \mathbf{p}' is a convex combination of the permutations $\pi(\mathbf{p})$, $\pi \in \Pi$, and as such $\mathbf{p}' \in \Delta(\mathcal{S})$. Moreover, we have $p'_S > 0$ only for assortments $S \in \mathcal{S}$ with $|S| = C$. By the symmetry of the problem instance, any permutation $\pi(\mathbf{p})$, $\pi \in \Pi$, has the same worst-case expected revenues as \mathbf{p} . The concavity of the worst-case expected revenues then implies that \mathbf{p}' has worst-case expected revenues that are at least as large as those of \mathbf{p} and thus—by our earlier assumption—of \mathbf{p}^* . We next show that this leads to a contradiction, which will complete the proof.

To compare the worst-case expected revenues of \mathbf{p}' and \mathbf{p}^* , we note that

$$\begin{aligned}
& \min_{\mathbf{v} \in \mathcal{V}} \sum_{S \in \mathcal{S}} p'_S \cdot R(S, \mathbf{v}) - \min_{\mathbf{v} \in \mathcal{V}} \sum_{S \in \mathcal{S}} p^*_S \cdot R(S, \mathbf{v}) \\
&= \min_{\mathbf{v} \in \mathcal{V}} \sum_{S \in \mathcal{S}} [p'_S - p^*_S] \cdot R(S, \mathbf{v}) \\
&= \min_{\mathbf{v} \in \mathcal{V}} \epsilon \cdot [R(S_1, \mathbf{v}) - R(S_2, \mathbf{v})] \\
&= \min_{\mathbf{v} \in \mathcal{V}} \epsilon \cdot \left[\frac{\sum_{i \in S_1} v_i}{v_0 + \sum_{i \in S_1} v_i} - \frac{\sum_{i \in S_2} v_i}{v_0 + \sum_{i \in S_2} v_i} \right] \\
&= \min_{\mathbf{v} \in \mathcal{V}} \epsilon \cdot \left[\frac{\sum_{i \in S_1 \cap S_2} v_i + \sum_{i \in S_1 \setminus S_2} v_i}{v_0 + \sum_{i \in S_1 \cap S_2} v_i + \sum_{i \in S_1 \setminus S_2} v_i} - \frac{\sum_{i \in S_1 \cap S_2} v_i + \sum_{i \in S_2 \setminus S_1} v_i}{v_0 + \sum_{i \in S_1 \cap S_2} v_i + \sum_{i \in S_2 \setminus S_1} v_i} \right]. \tag{17}
\end{aligned}$$

Here, the first identity holds since the symmetry of \mathbf{p}^* implies that any valuation scenario $\mathbf{v} \in \mathcal{V}$ is a worst-case valuation for \mathbf{p}^* . The second identity assumes that S_1 and S_2 are the assortments corresponding to the first two components of \mathbf{p}' and \mathbf{p}^* , and it follows from the construction of \mathbf{p}' . By choosing $\mathbf{v} \in \mathcal{V}$ such that as many components $i \in S_1 \setminus S_2$ of \mathbf{v} as possible are 0 and as many components $i \in S_2 \setminus S_1$ of \mathbf{v} as possible are 1, we can see that the expression (17) must be strictly negative. Since this implies that the worst-case revenues of \mathbf{p}' are strictly lower than those of \mathbf{p}^* , we obtain the desired contradiction. \square

LEMMA 6. *Under the reference model, any optimal randomization strategy $\mathbf{p} \in \Delta(\mathcal{S})$ satisfies $p_S = 0$ for all $S \in \mathcal{S}$ with $|S| \neq C$.*

Proof of Lemma 6. Note first that any optimal randomization strategy \mathbf{p} has to be feasible and hence place zero probability on any assortment $S \subseteq \mathcal{N}$ violating the cardinality constraint $|S| \leq C$. Assume to the contrary that there is an optimal randomization strategy $\mathbf{p} \in \Delta(\mathcal{S})$ with $p_S > 0$ for some $S \in \mathcal{S}$ satisfying $|S| < C$. We show that \mathbf{p} can be transformed into another optimal randomization strategy $\mathbf{p}' \in \Delta(\mathcal{S})$ satisfying $p'_S = 0$ for all $S \in \mathcal{S}$ with $|S| < C$ and $\mathbf{p}' \neq \mathbf{p}^*$, where \mathbf{p}^* is defined in Lemma 5. Since this contradicts the statement of Lemma 5, we obtain the desired contradiction.

We construct \mathbf{p}' as follows. For any assortment $S \in \mathcal{S}$ with $|S| \leq C$, let $\theta(S) \subseteq \mathcal{N}$ be an assortment satisfying $S \subseteq \theta(S)$ and $|\theta(S)| = C$; for all other assortments S , we define $\theta(S) = S$. We set $p'_T = \sum_{S \in \mathcal{S}: \theta(S)=T} p_S$ for all $T \subseteq \mathcal{N}$ with $|T| = C$, and $p'_T = 0$ otherwise. For later use we note that there are multiple ways to define the mapping θ , and that multiple different probability vectors \mathbf{p}' can be obtained through different θ . We have $\mathbf{p}' \in \Delta(\mathcal{S})$, and the worst-case expected revenues of $\mathbf{p}' \in \Delta(\mathcal{S})$ are at least as large as those of \mathbf{p} since for any valuation scenario $\mathbf{v} \in \mathcal{V}$, we have

$$\begin{aligned}
\sum_{S \in \mathcal{S}} p'_S \cdot R(S, \mathbf{v}) &= \sum_{\substack{S \in \mathcal{S}: \\ |S|=C}} p'_S \cdot R(S, \mathbf{v}) = \sum_{\substack{S \in \mathcal{S}: \\ |S|=C}} p_S \cdot R(S, \mathbf{v}) + \sum_{\substack{S \in \mathcal{S}: \\ |S|<C}} p_S \cdot R(\theta(S), \mathbf{v}) \\
&\geq \sum_{\substack{S \in \mathcal{S}: \\ |S|=C}} p_S \cdot R(S, \mathbf{v}) + \sum_{\substack{S \in \mathcal{S}: \\ |S|<C}} p_S \cdot R(S, \mathbf{v}) = \sum_{S \in \mathcal{S}} p_S \cdot R(S, \mathbf{v}).
\end{aligned}$$

Here, the inequality holds since for every $\mathbf{v} \in \mathcal{V}$, we have

$$R(S, \mathbf{v}) = \frac{\sum_{i \in S} v_i}{v_0 + \sum_{i \in S} v_i} \leq \frac{\sum_{i \in S} v_i + \sum_{i \in \theta(S) \setminus S} v_i}{v_0 + \sum_{i \in S} v_i + \sum_{i \in \theta(S) \setminus S} v_i} = R(\theta(S), \mathbf{v}), \tag{18}$$

where the inequality follows from the fact that $a/b \leq (a+c)/(b+c)$ whenever $0 \leq a < b$ and $c \geq 0$.

If $\mathbf{p}' \neq \mathbf{p}^*$, where \mathbf{p}^* is defined in [Lemma 5](#), then we have the desired contradiction to [Lemma 5](#). Otherwise, if $\mathbf{p}' = \mathbf{p}^*$, then we can exercise our aforementioned flexibility in the choice of θ to derive another optimal randomization strategy $\mathbf{p}'' \neq \mathbf{p}'$ that results in the desired contradiction. \square

Proof of [Theorem 3](#). The first statement follows immediately from [Lemmas 4](#) and [5](#) and the fact that for any valuation scenario $\mathbf{v} \in \mathcal{V}$, the optimal randomized strategy \mathbf{p}^* places strictly positive probability on assortments S that include products $i \in \mathcal{N}$ with valuation $v_i = 1$. The second statement, on the other hand, follows immediately from [Lemmas 5](#) and [6](#). \square

Proof of [Corollary 4](#). The proof of this corollary follows directly from the first part of [Theorem 3](#). \square

Proof of [Corollary 5](#). We consider our reference model for an even number of products n and set $C = n/2$. (An analogous argument can be made for the case where n is odd.) In this case, the unique optimal randomization strategy \mathbf{p}^* defined in [Lemma 5](#) places strictly positive probability on

$$\binom{n}{C} = \frac{n!}{(n-C)!C!} = \frac{n!}{\frac{n}{2}! \frac{n}{2}!} \approx \frac{\sqrt{2\pi n} \left(\frac{n}{e}\right)^n}{\left(\sqrt{2\pi \frac{n}{2}} \left(\frac{n}{2e}\right)^{\frac{n}{2}}\right)^2} = \sqrt{\frac{2}{\pi}} n^{-\frac{1}{2}} 2^n$$

different assortments, where the approximation is due to Stirling's formula. \square

Appendix D: Proofs for Section 5

Proof of Observation 1. As per Definition 2, we need to verify that the initial state distribution q is a probability distribution, that for all transition kernels $p \in \mathcal{P}$ and all $x \in \mathcal{X}$ and $a \in \mathcal{A}$, $p(\cdot|x, a)$ is a probability distribution, and that the discount factor γ is contained in the open interval $(0, 1)$.

Since λ is a probability distribution over \mathcal{N}_0 , so is q . The discount factor γ is strictly less than 1 since $\rho_{i0} > 0$ for all $\rho \in \mathcal{U}$ and $i \in \mathcal{N}$. At the same time, γ is strictly positive since $\rho_{i0} < 1$ for at least one $\rho \in \mathcal{U}$ and $i \in \mathcal{N}$. Next, fix any transition kernel $p \in \mathcal{P}$ and state $x \in \mathcal{X}$. For the action $a = \top$, there is precisely one state $x' \in \mathcal{X}$ for which $p(x'|x, a) = 1$, while $p(x''|x, a) = 0$ for all other states $x'' \in \mathcal{X}$. For the action $a = \perp$, we note that for any state $x' \neq 0$, we have

$$0 \leq \rho_{xx'}/\gamma = p(x'|x, a) \leq \sum_{x'' \in \mathcal{N}} p(x''|x, a) = \sum_{x'' \in \mathcal{N}} \rho_{xx''}/\gamma = (1 - \rho_{x0})/\gamma \leq 1,$$

where the first two inequalities hold since $\rho_{xx''} \geq 0$ for all $x'' \neq 0$ and $\gamma > 0$ by construction, the last equality holds since $(\rho_{x0}, \dots, \rho_{xn})$ is a probability distribution over \mathcal{N}_0 , and the last inequality follows from the construction of γ . We thus have $p(x'|x, a) \in [0, 1]$ for all $x' \neq 0$ as well as $\sum_{x'' \in \mathcal{N}} p(x''|x, a) \in [0, 1]$. It then follows directly from our construction in (6) that $p(0|x, a) \in [0, 1]$ as well as $\sum_{x'' \in \mathcal{X}} p(x''|x, a) = 1$. \square

Proof of Theorem 4. For any assortment $S \in \mathcal{S}$, the worst-case expected revenues $R^*(S)$ are given by equation (13) of Désir et al. (2021):

$$\begin{aligned} & \text{minimize} && \sum_{i \in \mathcal{N}_0} \lambda_i \cdot g_i \\ & \text{subject to} && g_i = r_i && \forall i \in S \\ & && g_i = \sum_{j \in \mathcal{N}} \rho_{ij} \cdot g_j && \forall i \in \mathcal{N} \setminus S \\ & && \rho \in \mathcal{U}, \mathbf{g} \in \mathbb{R}_+^{n+1}. \end{aligned}$$

Similarly, for a policy $\pi_S \in \Pi$, the worst-case expected total discounted reward is given by equation (28) of Nilim and El Ghaoui (2005):

$$\begin{aligned} & \text{minimize} && \sum_{x \in \mathcal{X}} q_x \cdot v_x \\ & \text{subject to} && v_x \geq (1 - \gamma) \cdot r_x + \gamma \cdot v_x && \forall x \in \mathcal{X} : \pi_S(x) = \top \\ & && v_x \geq 0 + \gamma \cdot \sum_{x' \in \mathcal{X}} p(x'|x, \perp) \cdot v_{x'} && \forall x \in \mathcal{X} : \pi_S(x) = \perp \\ & && p \in \mathcal{P}, \mathbf{v} \in \mathbb{R}_+^{n+1}. \end{aligned}$$

Lemma 2 of Nilim and El Ghaoui (2005) implies that there is an optimal solution to this problem that satisfies all inequalities as equalities. We can thus replace all inequalities with equalities, and substituting p with its definition yields

$$\begin{aligned} & \text{minimize} && \sum_{x \in \mathcal{X}} q_x \cdot v_x \\ & \text{subject to} && v_x = r_x && \forall x \in \mathcal{X} : \pi_S(x) = \top \\ & && v_x = \sum_{x' \in \mathcal{X}} \rho_{xx'} \cdot v_{x'} && \forall x \in \mathcal{X} : \pi_S(x) = \perp \\ & && \rho \in \mathcal{U}, \mathbf{v} \in \mathbb{R}_+^{n+1}. \end{aligned}$$

The statement now follows from the fact that $\mathbf{q} = \lambda$ and $\mathcal{X} = \mathcal{N}_0$. \square

Proof of Theorem 5. We first show that a product-wise substitution set in the MC model implies that the ambiguity set of the associated robust MDP in Definition 3 is (x, a) -rectangular. We can then use established results from the robust MDP literature to prove the statement of the theorem.

A robust MDP has an (x, a) -rectangular ambiguity set whenever

$$\mathcal{P} = \{p : \mathcal{X} \times \mathcal{A} \times \mathcal{X} \rightarrow \mathbb{R}_+ : \forall x \in \mathcal{X}, a \in \mathcal{A} \exists p^{xa} \in \mathcal{P} \text{ such that } p(\cdot|x, a) = p^{xa}(\cdot|x, a)\},$$

see Iyengar (2005), Nilim and El Ghaoui (2005) and Wiesemann et al. (2013). In other words, for any selection of conditional distributions p^{xa} , $x \in \mathcal{X}$ and $a \in \mathcal{A}$, we need to show that the composite distribution p satisfying $p(\cdot|x, a) = p^{xa}(\cdot|x, a)$ for all $x \in \mathcal{X}$ and $a \in \mathcal{A}$ is also contained in the ambiguity set \mathcal{P} . Fix any such selection of conditional distributions p^{xa} , $x \in \mathcal{X}$ and $a \in \mathcal{A}$, which by Definition 3 must correspond to a selection of substitution matrices $\rho^{xa} \in \mathcal{U}$, $x \in \mathcal{X}$ and $a \in \mathcal{A}$. Consider the substitution matrix ρ whose i -th row coincides with the i -th row of $\rho^{i\perp}$, $i = 0, \dots, n$. Our definition of product-wise substitution sets guarantees that $\rho \in \mathcal{U}$. Moreover, since $p(\cdot|x, \top)$ does not depend on the substitution matrix, the transition kernel p associated with ρ satisfies $p(\cdot|x, a) = p^{xa}(\cdot|x, a)$ for all $x \in \mathcal{X}$ and $a \in \mathcal{A}$ as desired. We thus conclude that the ambiguity set of the associated robust MDP in Definition 3 is (x, a) -rectangular.

Theorem 4 of Nilim and El Ghaoui (2005) shows that strong duality holds for robust MDPs with (x, a) -rectangular ambiguity sets, that is,

$$\max_{\pi \in \Pi} \min_{p \in \mathcal{P}} \mathbb{E} \left[\sum_{t=0}^{\infty} \gamma^t \cdot r(x_t, a_t) \mid x_0 \sim q \right] = \min_{p \in \mathcal{P}} \max_{\pi \in \Pi} \mathbb{E} \left[\sum_{t=0}^{\infty} \gamma^t \cdot r(x_t, a_t) \mid x_0 \sim q \right].$$

Theorem 4 of our paper implies that

$$\max_{S \in \mathcal{S}} R^*(S) = \max_{\pi \in \Pi} \min_{p \in \mathcal{P}} \mathbb{E} \left[\sum_{t=0}^{\infty} \gamma^t \cdot r(x_t, a_t) \mid x_0 \sim q \right].$$

Moreover, the same theorem implies that

$$\min_{p \in \mathcal{P}} \max_{\pi \in \Pi} \mathbb{E} \left[\sum_{t=0}^{\infty} \gamma^t \cdot r(x_t, a_t) \mid x_0 \sim q \right] = \min_{\rho \in \mathcal{U}} \max_{S \in \mathcal{S}} R(S, \rho)$$

since Definition 3 implies the existence of a one-to-one correspondence between transition kernels $p \in \mathcal{P}$ and substitution matrices $\rho \in \mathcal{U}$, and Theorem 4 of our paper can be applied individually to each singleton uncertainty set $\{\rho\}$ constructed from any substitution matrix $\rho \in \mathcal{U}$. We have thus established strong duality for our robust assortment optimization problem, that is,

$$\max_{S \in \mathcal{S}} \min_{\rho \in \mathcal{U}} R(S, \rho) = \min_{\rho \in \mathcal{U}} \max_{S \in \mathcal{S}} R(S, \rho).$$

The right-hand side of this equation bounds from above the worst-case expected revenues achievable by any (deterministic or randomized) assortment strategy. Since this bound is attained by the optimal deterministic assortment on the left-hand side of this equation, we cannot improve by adopting a randomized assortment strategy. \square

Proof of Theorem 6. For any $n \geq 3$ and $C \leq n - 2$, we construct the following instance of the cardinality-constrained robust MC problem. The product-wise prices are $r_1 = 0$ and $r_i = 1$, $i \in \{2, \dots, n\}$, the initial

choice probabilities satisfy $\lambda_1 = 1$ and $\lambda_i = 0$, $i \neq 1$, and the uncertainty set \mathcal{U} contains all substitution matrices $\boldsymbol{\rho}$ satisfying $\boldsymbol{\rho}_0 = (1, 0, \dots, 0)$, $\boldsymbol{\rho}_1 = (0, \boldsymbol{\rho}')$ where $\boldsymbol{\rho}'$ is any element of the probability simplex in \mathbb{R}^n , as well as $\boldsymbol{\rho}_i$ being any element of the probability simplex in \mathbb{R}^{n+1} , $i \in \{2, \dots, n\}$. Note that by construction, this uncertainty set exhibits product-wise substitution sets.

The worst-case expected revenues of every deterministic assortment $S \in \mathcal{S}$ satisfying $|S| \leq C$ are 0. This is clearly the case if $1 \in S$. If $1 \notin S$, fix any product $i \in \mathcal{N} \setminus (S \cup \{1\})$, which exists since $C \leq n - 2$. For any substitution matrix $\boldsymbol{\rho}^i$ satisfying $\rho_{1i}^i = \rho_{i0}^i = 1$, we have $R(S, \boldsymbol{\rho}^i) = 0$.

We now consider any randomized assortment strategy that places a strictly positive probability on every assortment $S \in \mathcal{S} \setminus \{1\}$ satisfying $|S| \leq C$ and that places zero probability on all other assortments. Fix any substitution matrix $\boldsymbol{\rho} \in \mathcal{U}$. Since $\rho_{1i} > 0$ for at least one $i \in \{2, \dots, n\}$ and since this product i will be offered with strictly positive probability, the randomized assortment strategy must generate strictly positive expected revenues under the substitution matrix $\boldsymbol{\rho}$. Since $\boldsymbol{\rho}$ was arbitrary, the worst-case expected revenues of the randomized assortment strategy must be strictly positive as well. \square

Proof of Corollary 6. The proof of this corollary follows directly from [Theorem 6](#). \square

Proof of Proposition 1. For $n = 2$, we can offer three different assortments: $\{1\}$, $\{2\}$ and $\{1, 2\}$. Fix any worst-case substitution matrices $\boldsymbol{\rho}^1 \in \mathcal{U}$ and $\boldsymbol{\rho}^2 \in \mathcal{U}$ for the two assortments $\{1\}$ and $\{2\}$, respectively (they may not be unique). The worst-case expected revenues for the three assortments amount to

$$R(\{1\}, \boldsymbol{\rho}^1) = \lambda_1 \cdot r_1 + \lambda_2 \cdot \rho_{21}^1 \cdot r_1, \quad R(\{2\}, \boldsymbol{\rho}^2) = \lambda_1 \cdot \rho_{12}^2 \cdot r_2 + \lambda_2 \cdot r_2 \quad \text{and} \quad R(\{1, 2\}, \boldsymbol{\rho}) = \lambda_1 \cdot r_1 + \lambda_2 \cdot r_2;$$

note in particular that the expected revenues of the assortment $\{1, 2\}$ do not depend on the realized substitution matrix $\boldsymbol{\rho} \in \mathcal{U}$. We show that the problem is randomization-proof by considering two cases: (i) $R(\{1, 2\}, \boldsymbol{\rho}) \leq \max\{R(\{1\}, \boldsymbol{\rho}^1), R(\{2\}, \boldsymbol{\rho}^2)\}$ and (ii) $R(\{1, 2\}, \boldsymbol{\rho}) > \max\{R(\{1\}, \boldsymbol{\rho}^1), R(\{2\}, \boldsymbol{\rho}^2)\}$.

Case (i). We assume that $R(\{1, 2\}, \boldsymbol{\rho}) \leq R(\{1\}, \boldsymbol{\rho}^1)$; the case where $R(\{1, 2\}, \boldsymbol{\rho}) \leq R(\{2\}, \boldsymbol{\rho}^2)$ is symmetric. Since $R(\{1, 2\}, \boldsymbol{\rho}) \leq R(\{1\}, \boldsymbol{\rho}^1)$, we have $\lambda_1 \cdot r_1 + \lambda_2 \cdot r_2 \leq \lambda_1 \cdot r_1 + \lambda_2 \cdot \rho_{21}^1 \cdot r_1$ and thus $\lambda_2 = 0$ or $r_2 \leq \rho_{21}^1 \cdot r_1$. Since only $R(\{2\}, \boldsymbol{\rho})$ depends on $\boldsymbol{\rho}$ if $\lambda_2 = 0$, it is easy to see that the problem is randomization-proof. Assume now that $r_2 \leq \rho_{21}^1 \cdot r_1$. For any substitution matrix $\boldsymbol{\rho} \in \mathcal{U}$, we then have

$$R(\{2\}, \boldsymbol{\rho}) = \lambda_1 \cdot \rho_{12} \cdot r_2 + \lambda_2 \cdot r_2 \leq \lambda_1 \cdot \rho_{12} \cdot \rho_{21}^1 \cdot r_1 + \lambda_2 \cdot \rho_{21}^1 \cdot r_1 \leq \lambda_1 \cdot r_1 + \lambda_2 \cdot \rho_{21}^1 \cdot r_1 = R(\{1\}, \boldsymbol{\rho}^1), \quad (19)$$

where the first inequality holds since $r_2 \leq \rho_{21}^1 \cdot r_1$, while the second inequality is due to the fact that $\rho_{12}, \rho_{21}^1 \leq 1$. Now consider an arbitrary randomized assortment strategy $\alpha \cdot \{1\} + \beta \cdot \{2\} + \gamma \cdot \{1, 2\}$ with $\alpha, \beta, \gamma \geq 0$ and $\alpha + \beta + \gamma = 1$, and let $\boldsymbol{\rho}^r$ be a worst-case substitution matrix for this strategy. We then have that

$$\alpha \cdot R(\{1\}, \boldsymbol{\rho}^r) + \beta \cdot R(\{2\}, \boldsymbol{\rho}^r) + \gamma \cdot R(\{1, 2\}, \boldsymbol{\rho}^r) \leq \alpha \cdot R(\{1\}, \boldsymbol{\rho}^1) + \beta \cdot R(\{1\}, \boldsymbol{\rho}^1) + \gamma \cdot R(\{1\}, \boldsymbol{\rho}^1) = R(\{1\}, \boldsymbol{\rho}^1),$$

where the first inequality holds since $R(\{2\}, \boldsymbol{\rho}^r) \leq R(\{1\}, \boldsymbol{\rho}^1)$ by (19) and $R(\{1, 2\}, \boldsymbol{\rho}^r) \leq R(\{1\}, \boldsymbol{\rho}^1)$ by assumption. We thus conclude that the problem is randomization-proof.

Case (ii). We prove this case by contradiction. We first show that if a problem instance is randomization-receptive, then it is optimal to randomize between the singleton assortments $\{1\}$ and $\{2\}$. We then derive

a necessary and sufficient condition for the randomization between $\{1\}$ and $\{2\}$ to be beneficial. Finally, we show that this condition contradicts our assumption that $R(\{1, 2\}, \boldsymbol{\rho}) > \max\{R(\{1\}, \boldsymbol{\rho}^1), R(\{2\}, \boldsymbol{\rho}^2)\}$.

Assume that the problem is randomization-receptive, and let $\alpha \cdot \{1\} + \beta \cdot \{2\} + \gamma \cdot \{1, 2\}$ with $\alpha, \beta \geq 0, \gamma > 0$ and $\alpha + \beta + \gamma = 1$ be an optimal randomization strategy. Denote by $\boldsymbol{\rho}^r$ one of its worst-case substitution matrices. We then have $\alpha \cdot R(\{1\}, \boldsymbol{\rho}^r) + \beta \cdot R(\{2\}, \boldsymbol{\rho}^r) + \gamma \cdot R(\{1, 2\}, \boldsymbol{\rho}^r) > R(\{1, 2\}, \boldsymbol{\rho}^r)$, and hence $(1 - \gamma) \cdot R(\{1, 2\}, \boldsymbol{\rho}^r) < \alpha \cdot R(\{1\}, \boldsymbol{\rho}^r) + \beta \cdot R(\{2\}, \boldsymbol{\rho}^r)$. This implies that

$$\begin{aligned} \alpha \cdot R(\{1\}, \boldsymbol{\rho}^r) + \beta \cdot R(\{2\}, \boldsymbol{\rho}^r) + \gamma \cdot R(\{1, 2\}, \boldsymbol{\rho}^r) &< \left[1 + \frac{\gamma}{1 - \gamma}\right] \cdot (\alpha \cdot R(\{1\}, \boldsymbol{\rho}^r) + \beta \cdot R(\{2\}, \boldsymbol{\rho}^r)) \\ &= \frac{\alpha}{\alpha + \beta} \cdot R(\{1\}, \boldsymbol{\rho}^r) + \frac{\beta}{\alpha + \beta} \cdot R(\{2\}, \boldsymbol{\rho}^r), \end{aligned}$$

where the inequality uses the fact that $(1 - \gamma) \cdot R(\{1, 2\}, \boldsymbol{\rho}^r) < \alpha \cdot R(\{1\}, \boldsymbol{\rho}^r) + \beta \cdot R(\{2\}, \boldsymbol{\rho}^r)$, while the equality holds since $\gamma = 1 - \alpha - \beta$. Since $R(\{1, 2\}, \boldsymbol{\rho})$ does not depend on $\boldsymbol{\rho}$, we have $\boldsymbol{\rho}^r \in \arg \min_{\boldsymbol{\rho} \in \mathcal{U}} \frac{\alpha}{\alpha + \beta} \cdot R(\{1\}, \boldsymbol{\rho}) + \frac{\beta}{\alpha + \beta} \cdot R(\{2\}, \boldsymbol{\rho})$. It is therefore optimal to set $\gamma = 0$ and randomize only between the assortments $\{1\}$ and $\{2\}$.

The problem of randomizing between the assortments $\{1\}$ and $\{2\}$ can be written as

$$\begin{aligned} &\text{maximize}_{\beta \in [0, 1]} \left[\min_{\boldsymbol{\rho} \in \mathcal{U}} \{\beta \cdot R(\{1\}, \boldsymbol{\rho}) + (1 - \beta) \cdot R(\{2\}, \boldsymbol{\rho})\} \right] \\ &= \text{maximize}_{\beta \in [0, 1]} \left[\min_{\boldsymbol{\rho} \in \mathcal{U}} \{\lambda_1 \cdot r_1 + \lambda_2 \cdot r_2 + \beta \cdot \lambda_2 (\rho_{21} \cdot r_1 - r_2) + (1 - \beta) \cdot \lambda_1 \cdot (\rho_{12} \cdot r_2 - r_1)\} \right]. \end{aligned}$$

For the problem to be randomization-receptive, it thus needs to satisfy

$$\begin{aligned} &\text{maximize}_{\beta \in [0, 1]} \left[\min_{\boldsymbol{\rho} \in \mathcal{U}} \{\lambda_1 \cdot r_1 + \lambda_2 \cdot r_2 + \beta \cdot \lambda_2 \cdot (\rho_{21} \cdot r_1 - r_2) + (1 - \beta) \cdot \lambda_1 \cdot (\rho_{12} \cdot r_2 - r_1)\} \right] \\ &\quad > \lambda_1 \cdot \rho_{12} \cdot r_2 + \lambda_2 \cdot r_2 \\ \iff &\exists \beta \in (0, 1) \text{ such that } \beta \cdot \lambda_2 \cdot (\rho_{21} \cdot r_1 - r_2) + (1 - \beta) \cdot \lambda_1 \cdot (\rho_{12} \cdot r_2 - r_1) > 0 \quad \forall \boldsymbol{\rho} \in \mathcal{U}, \end{aligned} \quad (20)$$

where we use our case assumption that the assortment $\{1, 2\}$ generates the highest worst-case expected revenues. We now show that (20) contradicts the assumption that $R(\{1, 2\}, \boldsymbol{\rho}) > \max\{R(\{1\}, \boldsymbol{\rho}^1), R(\{2\}, \boldsymbol{\rho}^2)\}$.

By our case assumption, we have $\lambda_1 \cdot r_1 + \lambda_2 \cdot r_2 > \lambda_1 \cdot r_1 + \lambda_2 \cdot \rho_{21}^1 \cdot r_1$ and $\lambda_1 \cdot r_1 + \lambda_2 \cdot r_2 > \lambda_2 \cdot r_2 + \lambda_1 \cdot \rho_{12}^2 \cdot r_2$. These inequalities imply that $\lambda_1, \lambda_2 > 0$ as well as $r_2 > \rho_{21}^1 \cdot r_1$ and $r_1 > \rho_{12}^2 \cdot r_2$. For notational convenience, we introduce the constants a, b and c defined via:

$$r_2 = a \cdot r_1 \text{ for some } a \in (\rho_{21}^1, 1/\rho_{12}^2), \quad \lambda_2 = b \cdot \lambda_1 \text{ for some } b \in \mathbb{R}_{++}, \quad 1 - \beta = c \cdot \beta \text{ for some } c \in \mathbb{R}_{++}.$$

Using this notation, we can re-express the inequality in (20) as

$$\begin{aligned} &\beta \cdot \lambda_2 \cdot (\rho_{21} \cdot r_1 - r_2) + (1 - \beta) \cdot \lambda_1 \cdot (\rho_{12} \cdot r_2 - r_1) > 0 \\ \iff &\beta \cdot b \cdot \lambda_1 \cdot (\rho_{21} \cdot r_1 - a \cdot r_1) + c \cdot \beta \cdot \lambda_1 \cdot (\rho_{12} \cdot a \cdot r_1 - r_1) > 0 \\ \iff &b \cdot \rho_{21} - b \cdot a > c - c \cdot a \cdot \rho_{12}. \end{aligned}$$

Here, the first equivalence results from the definitions of a, b and c , whereas the second equivalence follows from dividing the expressions by $\beta \lambda_1 r_1 > 0$. Since this inequality has to hold for all $\boldsymbol{\rho} \in \mathcal{U}$, it has to hold in particular for $\boldsymbol{\rho}^1$ and $\boldsymbol{\rho}^2$, which respectively result in the conditions

$$\begin{aligned} b \cdot \rho_{21}^1 - b \cdot a > c - c \cdot a \cdot \rho_{12}^1 &\implies b \cdot \rho_{21}^1 - b \cdot a > c - c \cdot a \\ &\implies b \cdot (\rho_{21}^1 - a) > c \cdot (1 - a) &\implies a > 1 \\ \text{and } b \cdot \rho_{21}^2 - b \cdot a > c - c \cdot a \cdot \rho_{12}^2 &\implies b - b \cdot a > c - c \cdot a \cdot \rho_{12}^2 \\ &\implies b \cdot (1 - a) > c \cdot (1 - a \cdot \rho_{12}^2) &\implies a < 1. \end{aligned}$$

Here, the respective first implications hold since $\rho_{12}^1, \rho_{21}^2 \in [0, 1]$, and the last ones follow from the fact that $a \in (\rho_{21}^1, 1/\rho_{12}^2)$ and $b, c > 0$. This is a contradiction, and we thus conclude that the problem must be randomization-proof under case (ii) as well. \square

Proof of Observation 2. We extend the randomization-receptive instance from Example 4 to $n > 3$ products as follows. The product-wise prices are $\mathbf{r} = (0, 4.66, 1.00, 10.00, 1, \dots, 1)$, the initial choice probabilities are $\boldsymbol{\lambda} = (0, 0.37, 0.62, 0.01, 0, \dots, 0)$, and the uncertainty set is

$$\mathcal{U} = \left\{ \left(\begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0.01 & 0 & 0.05 & 0.94 & 0 & \dots & 0 \\ 0.26 & 0.69 & 0 & 0.05 & 0 & \dots & 0 \\ 0.90 & 0.05 & 0.05 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & 0 & 0 & \dots & 0 \end{array} \right), \left(\begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0.90 & 0 & 0.05 & 0.05 & 0 & \dots & 0 \\ 0.01 & 0.44 & 0 & 0.55 & 0 & \dots & 0 \\ 0.01 & 0.94 & 0.05 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & 0 & 0 & \dots & 0 \end{array} \right) \right\}.$$

Since the products $4, \dots, n$ are never purchased and hence do not affect the revenues, the reasoning of Example 4 immediately carries over to this problem. \square

Proof of Observation 3. We show that $R_{\text{det}}^*(\mathcal{U}) \geq \sum_{i \in \mathcal{N}} \lambda_i \cdot \min\{r_i : i \in \mathcal{N}\}$ and $R_{\text{rand}}^*(\mathcal{U}) \leq \sum_{i \in \mathcal{N}} \lambda_i \cdot \max\{r_i : i \in \mathcal{N}\}$, which together imply the statement. To see that $R_{\text{det}}^*(\mathcal{U}) \geq \sum_{i \in \mathcal{N}} \lambda_i \cdot \min\{r_i : i \in \mathcal{N}\}$, consider the deterministic assortment $S = \mathcal{N}$. The worst-case expected revenues of this assortment amount to $\sum_{i \in \mathcal{N}} \lambda_i \cdot r_i \geq \sum_{i \in \mathcal{N}} \lambda_i \cdot \min\{r_i : i \in \mathcal{N}\}$, and hence $R_{\text{det}}^*(\mathcal{U}) \geq \sum_{i \in \mathcal{N}} \lambda_i \cdot \min\{r_i : i \in \mathcal{N}\}$. To see that $R_{\text{rand}}^*(\mathcal{U}) \leq \sum_{i \in \mathcal{N}} \lambda_i \cdot \max\{r_i : i \in \mathcal{N}\}$, on the other hand, note that any customer will exercise the no-purchase option with a probability of at least $\lambda_0 = 1 - \sum_{i \in \mathcal{N}} \lambda_i$. Thus, even if a randomized assortment strategy would achieve expected revenues of $\max\{r_i : i \in \mathcal{N}\}$ whenever a customer purchases any product, its worst-case expected revenues are bounded above by $\sum_{i \in \mathcal{N}} \lambda_i \cdot \max\{r_i : i \in \mathcal{N}\}$. \square

Proof of Corollary 7. The proof of this corollary follows directly from Observation 3. \square

Proof of Proposition 2. For general substitution sets, the result directly follows from Corollary 5 since the MNL model is a special case of the MC model. The result for product-wise substitution sets requires a separate proof, however, since the MC models corresponding to MNL models may exhibit general (as opposed to product-wise) substitution sets.

Consider an instance of the randomized robust assortment optimization problem under an MC model with product-wise substitution sets, $n \geq 2$ products with $r_i = 1$ and $\lambda_i = 1/n$ for all $i \in \mathcal{N}$, as well as a cardinality constraint $|S| \leq C$ with $C \in \{1, \dots, n-2\}$. We consider the product-wise substitution sets

$$\mathcal{U}_i = \left\{ \frac{1}{n-C} \cdot (\mathbf{e}_0 + \sum_{k \in \mathcal{K}} \mathbf{e}_k) : \mathcal{K} \subseteq \mathcal{N} \setminus \{i\}, |\mathcal{K}| = n - C - 1 \right\} \quad \forall i \in \mathcal{N},$$

where by a slight abuse of notation the indices of $\mathbf{e}_i \in \mathbb{R}^{n+1}$ start from 0 so as to match the product labels. The uncertainty set \mathcal{U}_0 associated with product 0 is the singleton set $\{(1, 0, \dots, 0)\}$.

Under this setting, we can follow the same steps as in the proof of Lemma 5 to show that the randomized assortment strategy $\mathbf{p}^* \in \Delta(\mathcal{S})$, $p_S^* = 1/|\{S \in \mathcal{S} : |S| = C\}|$ if $|S| = C$ and $p_S^* = 0$ otherwise, strictly outperforms any other feasible randomized assortment strategy \mathbf{p} that only places positive probability $p_S > 0$ on assortments S with $|S| = C$. The only aspect that needs to be verified is equation (17), which holds since

$$\begin{aligned}
& \min_{\rho \in \mathcal{U}} \epsilon \cdot (R(S_1, \rho) - R(S_2, \rho)) \\
= & \min_{\rho \in \mathcal{U}} \epsilon \cdot \left[\sum_{i \in \mathcal{N}} \lambda_i \cdot g_i^{S_1} - \sum_{i \in \mathcal{N}} \lambda_i \cdot g_i^{S_2} \right] \quad \text{with } g_i^{S_1} = \begin{cases} r_i & \text{if } i \in S_1 \\ \sum_{j \in \mathcal{N}} \rho_{ij} \cdot g_j^{S_1} & \text{if } i \in \mathcal{N} \setminus S_1 \end{cases} \\
& \quad \text{and } g_i^{S_2} = \begin{cases} r_i & \text{if } i \in S_2 \\ \sum_{j \in \mathcal{N}} \rho_{ij} \cdot g_j^{S_2} & \text{if } i \in \mathcal{N} \setminus S_2 \end{cases} \\
= & \min_{\rho \in \mathcal{U}} \epsilon \cdot \left[\sum_{i \in \mathcal{N} \setminus S_1} \lambda_i \cdot g_i^{S_1} - \sum_{i \in \mathcal{N} \setminus S_2} \lambda_i \cdot g_i^{S_2} \right] < 0.
\end{aligned}$$

Here, the first identity is due to the definition of $R(S_1, \rho)$ and $R(S_2, \rho)$, while the second equality holds since both S_1 and S_2 contain C products and the prices r_i and initial probabilities λ_i of all products are the same. The inequality, finally, is due to the fact that we can choose $\rho_i \in \mathcal{U}_i$, $i \in \mathcal{N}$, whose associated choice of \mathcal{K} satisfies $\mathcal{K} \cap S_2 \neq \emptyset$ whenever $i \in S_1 \setminus S_2$ and $\mathcal{K} \cap S_1 = \emptyset$ otherwise. This choice of ρ ensures that (i) when S_2 is offered, customers demanding products in $\mathcal{N} \setminus S_2$ subsequently consider products in S_2 and hence generate revenues, whereas (ii) when S_1 is offered, customers demanding products in $\mathcal{N} \setminus S_1$ will not consider any products in S_1 , and hence no extra revenues accrue. This explains why the difference is strictly negative.

Having established the analogue of [Lemma 5](#) for our setting, we can now apply the steps in the proof of [Lemma 6](#) to show that any optimal randomization strategy $\mathbf{p} \in \Delta(\mathcal{S})$ satisfies $p_S = 0$ for all $S \in \mathcal{S}$ with $|S| \neq C$. To this end, the only aspect that needs to be verified is

$$R(S, \rho) = \sum_{i \in \mathcal{N}} \lambda_i \cdot g_i^S \leq \sum_{i \in \mathcal{N}} \lambda_i \cdot g_i^{\theta(S)} = R(\theta(S), \rho),$$

where $\theta(S)$ is any assortment of cardinality C that contains S . The inequality holds since for all $i \in \theta(S) \setminus S$, we have $g_i^{\theta(S)} = 1 \geq g_i^S$, which in turn implies that $g_i^{\theta(S)} \geq g_i^S$ for all $i \in \mathcal{N} \setminus \theta(S)$. A rigorous proof of this claim is tedious but straightforward. The result then follows the same strategy as the proof of [Corollary 5](#).

□

Appendix E: Auxiliary Results and Proofs for Section 6

Proof of Observation 4. We extend the randomization-receptive instance from Example 5 to $n \geq 3$ products as follows. The product-wise prices are $r_1 = 1$, $r_2 = 2$ and $r_i = 1$ for $i \geq 3$. The preference rankings are

$$1 \rightarrow 2 \rightarrow 0 \rightarrow 3 \rightarrow 4 \rightarrow \dots \rightarrow n, \quad 2 \rightarrow 1 \rightarrow 0 \rightarrow 3 \rightarrow 4 \rightarrow \dots \rightarrow n, \quad 1 \rightarrow 0 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow \dots \rightarrow n.$$

We assume that the ambiguity set Γ contains the two occurrence probability scenarios $\lambda^1 = (1, 0, 0)$ and $\lambda^2 = (0, 0.5, 0.5)$. The same computations as in Example 5 show that this problem is randomization-receptive.

□

Proof of Observation 5. The bound for a generic number of products n can be derived using similar arguments as in the proof of Observation 3; we therefore only prove the bound for $n = 2$ products. To avoid trivial cases, we assume that $r_1 \neq r_2$ and $r_1, r_2 > 0$. Without loss of generality, we further assume that $r_1 > r_2$. In this case, including product 1 into an assortment weakly increases the expected revenues under any occurrence probability scenario, and therefore we can restrict ourselves to deterministic and randomized assortments that include product 1. We then consider the preference rankings

$$\begin{aligned} 1: & \quad 1 \rightarrow 0 && \text{with probability } \lambda_1 \\ 2: & \quad 2 \rightarrow 1 \rightarrow 0 && \text{with probability } \lambda_2 \\ 3: & \quad 2 \rightarrow 0 && \text{with probability } \lambda_3, \end{aligned}$$

where $\lambda_1, \lambda_2, \lambda_3 \geq 0$ and $\lambda_1 + \lambda_2 + \lambda_3 = 1$. Note that we do not consider any preference rankings that contain the no-purchase option 0 as highest ranked choice; this is without loss of generality as such a preference ranking affects every (deterministic and randomized) assortment in the same way. We now compute the expected revenues of the deterministic and randomized assortments of interest:

$$\begin{aligned} \{1\}: & \quad (\lambda_1 + \lambda_2) \cdot r_1 = (1 - \lambda_3) \cdot r_1 \\ \{1, 2\}: & \quad \lambda_1 \cdot r_1 + (\lambda_2 + \lambda_3) \cdot r_2 = r_2 + \lambda_1 \cdot (r_1 - r_2) \\ \alpha \cdot \{1\} + \beta \cdot \{1, 2\}: & \quad \alpha \cdot (1 - \lambda_3) \cdot r_1 + \beta \cdot [r_2 + \lambda_1 \cdot (r_1 - r_2)] \end{aligned}$$

Fix an optimal randomization strategy (α^*, β^*) as well as the worst-case distributions (not necessarily unique) $\lambda^1 = (\lambda_1^1, \lambda_2^1, \lambda_3^1) \in \mathcal{U}$, $\lambda^{12} = (\lambda_1^{12}, \lambda_2^{12}, \lambda_3^{12}) \in \mathcal{U}$ and $\lambda^r = (\lambda_1^r, \lambda_2^r, \lambda_3^r) \in \mathcal{U}$ for the deterministic assortments $\{1\}$ and $\{1, 2\}$ as well as the randomized assortment $\alpha^* \cdot \{1\} + \beta^* \cdot \{1, 2\}$, respectively.

The benefit of randomization can now be computed as

$$R = \frac{\alpha^* \cdot (1 - \lambda_3^r) \cdot r_1 + \beta^* \cdot [r_2 + \lambda_1^r \cdot (r_1 - r_2)]}{\max \{(1 - \lambda_3^1) \cdot r_1, r_2 + \lambda_1^{12} \cdot (r_1 - r_2)\}}.$$

Assume first that $(1 - \lambda_3^1) \cdot r_1 \geq r_2 + \lambda_1^{12} \cdot (r_1 - r_2)$ in the denominator of R . We then have

$$\begin{aligned} R &= \frac{\alpha^* \cdot (1 - \lambda_3^r) \cdot r_1 + \beta^* \cdot [r_2 + \lambda_1^r \cdot (r_1 - r_2)]}{(1 - \lambda_3^1) \cdot r_1} \leq \frac{\alpha^* \cdot (1 - \lambda_3^1) \cdot r_1 + \beta^* \cdot [r_2 + \lambda_1^1 \cdot (r_1 - r_2)]}{(1 - \lambda_3^1) \cdot r_1} \\ &= 1 + \frac{-\beta^* \cdot (1 - \lambda_3^r) \cdot r_1 + \beta^* \cdot [r_2 + \lambda_1^r \cdot (r_1 - r_2)]}{(1 - \lambda_3^1) \cdot r_1} = 1 + \beta^* \cdot \frac{-(1 - \lambda_3^r) \cdot r_1 + r_2 + \lambda_1^r \cdot (r_1 - r_2)}{(1 - \lambda_3^1) \cdot r_1} \\ &\leq 1 + \beta^* \cdot \frac{-(1 - \lambda_3^r) \cdot r_1 + r_2 + (1 - \lambda_3^r) \cdot (r_1 - r_2)}{(1 - \lambda_3^1) \cdot r_1} = 1 + \beta^* \cdot \frac{\lambda_3^r \cdot r_2}{(1 - \lambda_3^1) \cdot r_1} \leq 1 + \beta^* \leq 2, \end{aligned}$$

where the first equality is due to $(1 - \lambda_3^1) \cdot r_1 \geq r_2 + \lambda_1^{12} \cdot (r_1 - r_2)$, the first inequality holds since the occurrence probabilities λ^r minimize the expected revenues of the randomized assortment, the second equality follows from $\alpha^* + \beta^* = 1$, the second inequality holds since $\lambda_1^1, \lambda_2^1, \lambda_3^1 \geq 0$ and $\lambda_1^1 + \lambda_2^1 + \lambda_3^1 = 1$, and the third inequality follows from the fact that $(1 - \lambda_3^1) \cdot r_1 \geq r_2 + \lambda_1^{12} \cdot (r_1 - r_2) \geq r_2$ since $r_1 > r_2$ by assumption.

Assume next that $(1 - \lambda_3^1) \cdot r_1 < r_2 + \lambda_1^{12} \cdot (r_1 - r_2)$ in the denominator of R . We then have

$$\begin{aligned} R &= \frac{\alpha^* \cdot (1 - \lambda_3^1) \cdot r_1 + \beta^* \cdot [r_2 + \lambda_1^1 \cdot (r_1 - r_2)]}{r_2 + \lambda_1^{12} \cdot (r_1 - r_2)} \leq \frac{\alpha^* \cdot (1 - \lambda_3^1) \cdot r_1 + \beta^* \cdot [r_2 + \lambda_1^1 \cdot (r_1 - r_2)]}{r_2 + \lambda_1^{12} \cdot (r_1 - r_2)} \\ &= \frac{\alpha^* \cdot (1 - \lambda_3^1) \cdot r_1 + \beta^* \cdot [r_2 + \lambda_1^1 \cdot (r_1 - r_2)]}{(1 - \lambda_3^1) \cdot r_1} \cdot \frac{(1 - \lambda_3^1) \cdot r_1}{r_2 + \lambda_1^{12} \cdot (r_1 - r_2)} \\ &\leq \left(1 + \beta^* \cdot \frac{\lambda_3^1 \cdot r_2}{(1 - \lambda_3^1) \cdot r_1}\right) \cdot \frac{(1 - \lambda_3^1) \cdot r_1}{r_2 + \lambda_1^{12} \cdot (r_1 - r_2)} = \frac{(1 - \lambda_3^1) \cdot r_1}{r_2 + \lambda_1^{12} \cdot (r_1 - r_2)} + \beta^* \cdot \frac{\lambda_3^1 \cdot r_2}{r_2 + \lambda_1^{12} \cdot (r_1 - r_2)} \leq 2, \end{aligned}$$

where the calculations follow the same ideas as in the previous case. Note that our derivations assume that $(1 - \lambda_3^1) \cdot r_1 \neq 0$. Indeed, if $(1 - \lambda_3^1) \cdot r_1 = 0$, then $\lambda_3^1 = 1$ since $r_1 > 0$ by assumption. In that case, however, the problem is readily verified to be randomization-proof. \square

EXAMPLE 6. Consider the unconstrained robust preference ranking problem with two products, $r_1 = M$ (with M being a large positive number) and $r_2 = 1$ as well as the preference rankings

$$\begin{array}{lll} 1: & 1 \rightarrow 0 & \text{with probability } \lambda_1 \\ 2: & 2 \rightarrow 1 \rightarrow 0 & \text{with probability } \lambda_2 \\ 3: & 2 \rightarrow 0 & \text{with probability } \lambda_3. \end{array}$$

We assume that the ambiguity set \mathcal{U} contains the two occurrence probability scenarios $\boldsymbol{\lambda}^1 = (0, 1, 0)$ and $\boldsymbol{\lambda}^2 = (1/M, 0, 1 - 1/M)$. Following the same reasoning as in the proof of [Observation 5](#), we can without loss of generality restrict ourselves to the consideration of the following assortments with the associated worst-case expected revenues:

$$\begin{array}{ll} \{1\}: & \min \{M, 1\} = 1 \\ \{1, 2\}: & \min \{1, 2 - 1/M\} = 1 \\ \alpha \cdot \{1\} + \beta \cdot \{1, 2\}: & \min \{\alpha \cdot M + \beta, 1 + \beta \cdot (1 - 1/M)\} \end{array}$$

Clearly, $R_{\det}^*(\mathcal{V}) = 1$ in this instance. The worst-case expected revenues of the randomized assortment strategy $(\alpha^*, \beta^*) = \left(\frac{1-1/M}{M-1/M}, \frac{M-1}{M-1/M}\right)$, on the other hand, evaluate to

$$\min \left\{ \frac{1-1/M}{M-1/M} \cdot M + \frac{M-1}{M-1/M}, 1 + \frac{M-1}{M-1/M} \cdot (1-1/M) \right\} = \frac{2M-2}{M-1/M},$$

and the last expression evidently converges to 2 as $M \rightarrow \infty$.

Proof of Corollary 8. The proof this result follows directly from [Observation 5](#). \square

Proof of Theorem 7. In view of the first statement, fix any optimal randomization strategy \mathbf{p}^* for the robust preference ranking problem. If at most $K + 1$ components of \mathbf{p}^* are positive, then there is nothing to prove. Otherwise, note that the worst-case revenues of \mathbf{p}^* evaluate to

$$\min_{\boldsymbol{\lambda} \in \Lambda} \sum_{S \in \mathcal{S}} p_S^* \cdot R(S, \boldsymbol{\lambda}) = \min_{\boldsymbol{\lambda} \in \Lambda} \sum_{S \in \mathcal{S}} \sum_{k \in \mathcal{K}} p_S^* \cdot \lambda_k \cdot R_k(S) = \min_{\boldsymbol{\lambda} \in \Lambda} \boldsymbol{\lambda}^\top \left[\sum_{S \in \mathcal{S}} p_S^* \cdot \mathbf{R}(S) \right], \quad (21)$$

where $\mathbf{R}(S) = (R_1(S), \dots, R_K(S))^\top \in \mathbb{R}^K$. Here, the summation $\sum_{S \in \mathcal{S}} p_S^* \cdot \mathbf{R}(S)$ can be interpreted as a convex combination of the 2^n expected revenues vectors $\mathbf{R}(S) \in \mathbb{R}^K$, $S \in \mathcal{S}$. Carathéodory's theorem then implies that there are alternative weights q_S^* , $S \in \mathcal{S}$, such that at most $K + 1$ components of \mathbf{q}^* are positive and that satisfy

$$\sum_{S \in \mathcal{S}} q_S^* \cdot \mathbf{R}(S) = \sum_{S \in \mathcal{S}} p_S^* \cdot \mathbf{R}(S).$$

Substituting the expression on the left-hand side of this identity into equation (21), we conclude that

$$\min_{\lambda \in \Lambda} \sum_{S \in \mathcal{S}} q_S^* \cdot R(S, \lambda) = \min_{\lambda \in \Lambda} \sum_{S \in \mathcal{S}} p_S^* \cdot R(S, \lambda),$$

which implies that the parsimonious randomization strategy \mathbf{q}^* attains the same objective value as the optimal randomization strategy \mathbf{p}^* .

In view of the second statement, assume that the uncertainty set is of the form

$$\Lambda = \mathbf{F}\Xi \quad \text{with} \quad \Xi = \{\boldsymbol{\xi} \in \mathbb{R}_+^m : \mathbf{A}\boldsymbol{\xi} \leq \mathbf{b}\} \subseteq \mathbb{R}_+^m.$$

Fix any optimal randomization strategy \mathbf{p}^* for the randomized robust preference ranking problem. If at most $m + 1$ components of \mathbf{p}^* are positive, then there is nothing to prove. Otherwise, similar to (21), the worst-case revenues of \mathbf{p}^* evaluate to

$$\min_{\lambda \in \Lambda} \sum_{S \in \mathcal{S}} p_S^* \cdot R(S, \lambda) = \min_{\boldsymbol{\xi} \in \Xi} (\mathbf{F}\boldsymbol{\xi})^\top \left[\sum_{S \in \mathcal{S}} p_S^* \cdot \mathbf{R}(S) \right] = \min_{\boldsymbol{\xi} \in \Xi} \boldsymbol{\xi}^\top \left[\sum_{S \in \mathcal{S}} p_S^* \cdot \mathbf{F}^\top \mathbf{R}(S) \right].$$

Here, the summation $\sum_{S \in \mathcal{S}} p_S^* \cdot \mathbf{F}^\top \mathbf{R}(S)$ can be interpreted as a convex combination of the 2^n vectors $\mathbf{F}^\top \mathbf{R}(S) \in \mathbb{R}^m$, $S \in \mathcal{S}$. Applying Carathéodory's theorem again shows that there are alternative weights q_S^* , $S \in \mathcal{S}$, with at most $m + 1$ components positive such that \mathbf{q}^* attains the same objective value as the optimal randomization strategy \mathbf{p}^* . This concludes the proof. \square

Proof of Theorem 8. Consider the cardinality-constrained preference ranking problem with product-wise prices $r_1 = \dots = r_n = 1$ and the n preference rankings

$$1 \rightarrow 0 \rightarrow \dots, \quad 2 \rightarrow 0 \rightarrow \dots, \quad \dots, \quad n \rightarrow 0 \rightarrow \dots$$

with an associated ambiguity set \mathcal{U} that contains all probability distributions $\boldsymbol{\lambda} \in \mathbb{R}^n$ contained in the n -dimensional probability simplex. Any deterministic assortment $S \in \mathcal{S}$ satisfies $|S| \leq C$ and thus generates zero revenues under the Dirac distribution that places all probability mass on a preference ranking containing a product $i \in \mathcal{N} \setminus S$ as highest ranked product. Consider now any randomized assortment strategy that places a strictly positive probability on every assortment $S \in \mathcal{S}$. Since every product is offered with a strictly positive probability, this strategy generates strictly positive expected revenues under any occurrence probability scenario $\boldsymbol{\lambda} \in \mathcal{U}$. We thus conclude that the worst-case expected revenues of this strategy are also strictly positive. \square

Proof of Corollary 9. This corollary follows directly from Theorem 8. \square

Proof of Corollary 10. The proof of this corollary follows the same steps as the proof of Theorem 7 and is hence omitted. \square

Appendix F: Supplementary Numerical Results for Section 7

Tables 12 and 13 provide supplementary information on the runtimes as well as the performance of the different problem formulations for the MC choice model.

n	K	deterministic		randomized	
		exact runtime	exact objective	runtime	heuristic objective runtime
20	5	0.22s	0.91%	1.02s	0.90% 0.50s
20	8	0.73s	1.29%	7.32s	1.28% 5.96s
20	10	2.22s	1.35%	31.32s	1.33% 29.29s
20	12	8.17s	1.48%	165.12s	1.47% 150.10s
30	5	0.60s	0.65%	2.14s	0.65% 1.35s
30	8	1.65s	0.81%	12.18s	0.81% 9.78s
30	10	4.29s	0.96%	55.10s	0.95% 47.74s
30	12	14.82s	1.05%	247.61s	1.05% 230.06s
40	5	1.07s	0.45%	3.37s	0.44% 3.21s
40	8	2.79s	0.60%	18.19s	0.59% 16.09s
40	10	6.09s	0.63%	75.93s	0.62% 71.59s
40	12	20.08s	0.73%	331.86s	0.72% 303.40s
50	5	1.77s	0.37%	4.68s	0.36% 6.54s
50	8	5.65s	0.44%	29.14s	0.44% 30.08s
50	10	11.10s	0.47%	126.39s	0.47% 118.87s
50	12	29.71s	0.55%	543.70s	0.55% 508.67s
60	5	3.11s	0.29%	7.21s	0.28% 13.84s
60	8	7.86s	0.32%	39.07s	0.32% 47.93s
60	10	19.32s	0.37%	180.43s	0.37% 176.34s
60	12	41.94s	0.41%	657.93s	0.41% 642.21s
70	5	3.68s	0.22%	8.55s	0.21% 25.21s
70	8	10.26s	0.26%	48.36s	0.26% 70.98s
70	10	25.73s	0.32%	203.54s	0.32% 227.00s
70	12	54.15s	0.32%	792.60s	0.32% 755.90s
80	5	6.56s	0.20%	11.66s	0.19% 43.60s
80	8	15.46s	0.23%	58.25s	0.23% 96.88s
80	10	35.85s	0.27%	254.84s	0.26% 300.97s
80	12	75.24s	0.27%	1,118.74s	0.27% 1,165.78s
90	5	8.66s	0.17%	15.07s	0.16% 83.25s
90	8	27.11s	0.17%	81.19s	0.17% 158.98s
90	10	57.35s	0.22%	366.83s	0.22% 451.94s
90	12	89.78s	0.22%	1,109.80s	0.22% 1,105.29s
100	5	17.66s	0.17%	23.08s	0.17% 142.46s
100	8	36.31s	0.17%	111.31s	0.17% 260.03s
100	10	68.60s	0.18%	404.69s	0.17% 538.53s
100	12	107.03s	0.21%	1,244.29s	0.21% 1,412.64s

Table 12 Deterministic vs. randomized solution of the unconstrained MC problem under general substitution sets. All percentages are reported relative to the worst-case expected revenues of the optimal deterministic robust assortment.

n	K	deterministic		randomized		
		exact runtime	objective	exact runtime	objective	heuristic runtime
20	5	1.26s	4.51%	6.23s	4.45%	1.30s
20	8	5.08s	7.53%	35.01s	7.48%	18.47s
20	10	14.11s	8.70%	144.44s	8.65%	100.67s
20	12	43.62s	9.20%	556.59s	9.16%	474.12s
30	5	6.04s	6.35%	35.69s	6.29%	4.06s
30	8	35.29s	9.93%	200.57s	9.86%	49.72s
30	10	98.11s	12.01%	565.06s	11.92%	299.34s
30	12	233.42s	13.48%	2,112.93s	13.41%	1,531.79s
40	5	18.21s	7.85%	131.25s	7.76%	11.94s
40	8	174.41s	12.52%	1,042.42s	12.41%	137.71s
40	10	498.01s	15.20%	2,898.64s	15.10%	684.67s
40	12	1,404.03s	16.94%	8,288.72s	16.83%	3,852.12s
50	5	44.81s	9.17%	458.82s	9.06%	33.16s
50	8	572.67s	14.82%	3,955.41s	14.70%	282.57s
50	10	2,260.97s	17.45%	9,981.55s	17.33%	1,481.38s
50	12	3,608.87s	17.80%	17,054.40s	17.70%	6,706.03s
60	5	80.90s	9.99%	1,073.07s	9.90%	61.95s
60	8	1,441.70s	15.90%	9,103.87s	15.78%	519.43s
60	10	4,789.94s	18.22%	21,139.40s	18.11%	2,592.26s
60	12	6,440.95s	18.27%	27,527.50s	18.17%	10,547.90s
70	5	141.82s	11.40%	2,093.76s	11.30%	132.29s
70	8	2,784.57s	16.60%	19,390.60s	16.47%	928.64s
70	10	6,113.36s	17.82%	32,640.60s	17.68%	3,538.27s
70	12	6,237.76s	16.93%	33,286.90s	16.85%	13,432.00s
80	5	220.94s	11.92%	3,198.93s	11.80%	222.90s
80	8	4,132.22s	17.56%	30,642.75s	17.43%	1,431.79s
80	10	6,352.38s	16.79%	34,369.30s	16.69%	4,526.13s
80	12	10,617.26s	17.70%	28,164.65s	17.65%	15,879.15s
90	5	367.60s	12.94%	5,059.18s	12.83%	399.70s
90	8	4,020.61s	17.15%	33,973.50s	17.03%	1,960.26s
90	10	4,929.50s	16.19%	35,324.40s	16.11%	6,670.96s
90	12	n/a	n/a	n/a	n/a	n/a
100	5	468.03s	13.46%	6,736.09s	13.31%	596.70s
100	8	4,474.56s	16.63%	39,533.50s	16.52%	2,587.09s
100	10	4,174.16s	15.24%	47,062.60s	15.19%	8,783.42s
100	12	n/a	n/a	n/a	n/a	n/a

Table 13 Deterministic vs. randomized solution of the constrained MC problem under general substitution sets and a cardinality constraint $|S| \leq C = 5$. All percentages are reported relative to the worst-case expected revenues of the optimal deterministic robust assortment