Precise control of approximation quality in multicriteria optimization

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Abstract

Although many algorithms for multicriteria optimization provide good approximations, a precise control of their quality is challenging. In this paper we provide algorithmic tools to obtain exact approximation quality values for given approximations and develop a new method for multicriteria optimization guided by this quality. We show that the well-established ε-indicator measure is NP-hard to compute when the number of criteria is unlimited already in the case of simple underlying linear optimization problems. Despite this hardness result, we develop a practically efficient method to compute the ε-indicator for an arbitrary fixed number of criteria, applicable also to approximations consisting of polyhedra. Based on the corresponding characterization we develop an algorithm for the approximation of Pareto frontiers which computes the ϵ-indicator and improves its value in each iteration. Comparisons to other algorithms for multicriteria optimization on benchmarks with two to five criteria show that the performance of our method is competitive.

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1. Introduction

Multicriteria optimization allows to perform optimization with conflicting objectives and has a diverse range of application areas, e.g. engineering (Marler & Arora, 2004) or finance (Ponsich et al., 2013). To obtain approximations of the corresponding Pareto frontier, a large set of methods has been proposed (see Ruzika & Wiecek (2005) for a survey and Boland et al. (2015); Fattahi &

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Turkay (2018); Perini et al. (2020); Rasmi & Türkay (2019) for recent methods). As it is often computationally intractable to compute the exact Pareto frontier, these methods stop when some custom criterion is reached. Since these stopping rules differ from method to method, a comparison of the approximation quality obtained by different algorithms is difficult. Even when algorithms provide some minimal guarantees on the approximation quality, the actually obtained quality, which is essential for practical problems, can differ significantly. This is an issue in applications where an easily-interpretable approximation guarantee is needed. Although some theoretical measures of approximation quality have been considered in the literature, there are often no practically efficient computational methods available to compute these measures for given approximations. Available methods are mainly focused on the case that two explicitly given point sets are compared. We consider the computation of the approximation quality of a given approximation with respect to an unknown exact Pareto frontier which is just implicitly represented by a multicriteria problem. Based on this we are able to create an approximation method with precise approximation quality guarantee.

In this paper we develop an efficient method to compute the widely used quality measure known as ε-indicator (Zitzler et al., 2003). Although we show that the computation of this measure is NP-hard in general, we can develop algorithms which are efficient in practice. The NP-hardness result holds for unlimited number of criteria even if the the underlying optimization problem is a linear program and the approximation is linear.

However, efficient algorithms are still possible to obtain for fixed number of criteria, as our approach shows. We establish a characterization of the ε-indicator that leads to a formulation as an optimization problem. For this abstract optimization problem, we provide some concrete formulations that are usable in practice. Our approach works for an arbitrary number of criteria without restriction on the problem type.

Many algorithms for multicriteria optimization output approximations that are not given as sets of isolated points, but instead as set of segments or facets. Such algorithms are prevalent in multicriteria linear programming (Ehrgott, 2005) and more recently multicriteria mixed integer programming (Boland et al., 2015; Perini et al., 2020; Rasmi & Türkay, 2019). We design our algorithms to directly deal efficiently with approximations given in such a form.

We explicitly construct a characterization of the ε-indicator for the case of approximations

consisting of a finite set of points. This leads to a practically efficient method to compute the ε-indicator for any number of criteria. Based on this method we develop an algorithm for the approximation of Pareto frontiers which achieves an improvement on the ε-indicator in each iteration by adding a point with maximal distance to the approximation. Since the explicit value of the ε-indicator is known for the current approximation in each iteration, the approximation quality provided by this algorithm can be precisely controlled. This distinguishes this approach from other methods where the approximation quality can only be indirectly controlled. By adding only points to the approximation that improve the ε-indicator, our algorithm creates a succinct representation of the Pareto frontier which is easier to interpret by the user.

1.1. Literature Review

In the literature there exists a large range of approximation quality measures for multicriteria optimization (Laszczyk & Myszkowski, 2019; Zitzler et al., 2003). However only a few of them have been widely applied. In particular, although theoretical definitions exist, a computation of these measures in practice is often not addressed. One of these widely-used measures is given by the notion of the ε-indicator (related to ε-approximate Pareto-frontiers) which appears (implicitly) in theoretical studies of multicriteria optimization since the beginning of the field (Loridan, 1984) and is widely used with slight variations (Legriel et al., 2010; Papadimitriou & Yannakakis, 2000; Vassilvitskii & Yannakakis, 2005). This measure fulfills many suitable properties and is easily interpretable, which make it appealing in practice.

The computation of other approximation quality measures has been considered in the literature. Sayın (2000) defined the related measure of *coverage error* and developed an algorithm for its computation. Another quality measure is the *hypervolume*, i.e. the volume above the approximation boundary (Auger et al., 2012) whose computation is related to the Klee measure problem (Beume, 2009) for which several computational approaches have been developed (Bader & Zitzler, 2011). In the field of evolutionary multicriteria optimization a large variety of further measures is used which however only use the points of the approximation without taking into account the true Pareto frontier. These measures are for example related to the spread and uniformity of the points in the approximation.

Each of these approximation quality measures has its specific use and should be chosen depending on the application and the requirements by the users. An important factor is the simple inter-

pretability of the approximation quality measure independently of the algorithm. The ε-indicator fulfills this criterion.

Note that we compare the approximation error with respect to the true Pareto frontier which is not explicitly known. Computing the ε-indicator with respect to a given set of points can be easily done within a running time which is polynomial in the number of points (Branke et al., 2008). For constructing approximations, as subsets of fixed sets of points with small enough ε-indicator value, efficient algorithms are available (Bringmann et al., 2014).

Measuring the approximation quality is closely related to the goal of computing approximations to Pareto frontiers in multicriteria optimization. Algorithms for this task can be divided into two large categories: they output either finite sets of points or an infinite number of points parametrized with a finite description, for example by giving the endpoints of a line segment.

- Algorithms that output an approximation consisting of a finite set of points form the majority of multicriteria optimization methods. This approach can be applied to any type of multicriteria problem, including pure integer problems where this is the only possible type of method. Some examples for the mixed-integer linear case include Mavrotas & Diakoulaki (1998), the works by Kirlik & Sayın (2014); de Santis et al. (2020b); Stidsen et al. (2014) consider the pure integer case, while Cabrera-Guerrero et al. (2021) describe the general convex case. Our algorithm for approximating a Pareto frontier based on the ε-indicator discussed in Section 6 is also included in this category.

- Methods that implicitly output an infinite number of points by creating an appropriate parametrization are more tailored to their application areas. The classical example are methods for multiobjective linear programming which represent their solution as a set of extreme points. The approach has recently also been applied to mixed-integer problems with a significant continous part. Some algorithms of this type for the linear mixed-integer case are proposed by Boland et al. (2015); Perini et al. (2020); Rasmi & Türkay (2019). For the more general convex mixed-integer problems new methods are described by de Santis et al. (2020a) and Diessel (2020). Our method for computing the ε-indicator can be applied also to the approximations generated by this category of algorithms since their approximations can be written as a union of polyhedra.

Many of the algorithms noted above include parameters which influence the approximation quality,

however only indirectly. By returning an ε-indicator value in each iteration, our approximation algorithm provides a directly and precisely controllable quality.

1.2. Our results and organization of the paper

Multicriteria optimization problems and the ε-indicator are introduced in Section 2. The proof in Section 3 shows that computing the ε-indicator is NP-hard in the case of an unlimited number of criteria using instances of multicriteria linear programs with simple approximations. Section 4 describes our algorithmic approach to obtain methods which are practically efficient for a fixed number of criteria. In Subsection 4.1 we provide an alternative characterization of the ε-approximation quality measure that leads to a computational treatment as an optimization problem. We describe how this optimization problem can be formulated in the case that the approximation is given as a union of polyhedra. We provide two formulations: one based on a decomposition into parts of the boundary of the approximation (Subsection 4.3) and a full reformulation that uses one single MIP for the entire computation (Subsection 4.4). For the cases of two criteria and approximations with an arbitrary number of criteria consisting of a finite set of points we develop specialized formulations in Section 5.

Our method for approximating Pareto frontiers based on computing the ε-indicator is developed in Section 6. In Section 7 we provide numerical results on benchmark problems to compare the two methods for computing the ε-indicator. We also provide a comparison of our algorithm for the approximation of Pareto frontiers with other algorithms for multicriteria optimization with two to five criteria.

1.3. Notation

We use the symbol $\mathbf{1}_K \coloneqq (1,\dots,1)^T$ to denote the vector consisting of K ones. Throughout this paper we use the standard topology on the euclidean space $\mathbb{R}^K$. We denote the boundary of a set $B \subseteq \mathbb{R}^K$ by ∂B. The set of nonnegative vectors is denoted by

$$\mathbb{R}^n_{\geq 0} \coloneqq \{z \in \mathbb{R} \mid z_i \geq 0 \text{ for each } i = 1,\dots,n\}.$$

We use the relation "$\leq$" also for vectors. We write $a \leq b$ for two vectors $a, b \in \mathbb{R}^n$ if they fulfill component-wise $a_i \leq b_i$, $i = 1,\dots,n$. We denote the closed interval from $c \in \mathbb{R}$ to $d \in \mathbb{R}$ by $[c,d]$ and the corresponding open interval by $]c,d[$. The hyperrectangle induced by two vectors a and b fulfilling $a \leq b$ is written as

$$[a;b] \coloneqq [a_1,b_1] \times \cdots \times [a_n,b_n].$$

For sets $A, B \subseteq \mathbb{R}^n$ we denote their Minkowski sum by $A + B \coloneqq \{a + b \mid a \in A, b \in B\}$.

2. Approximation quality measures for multicriteria optimization

2.1. Introduction to Multicriteria optimization

A multicriteria optimization problem with K criteria has the form

$$\min (f_1(x), \ldots, f_K(x))^T$$

$$\text{s.t. } x \in X$$

where X is a set of feasible solutions in decision space with objective functions $f_i : X \to \mathbb{R}$, $i = 1 \ldots, K$. The set of feasible solutions X is usually implicitly given, e.g. by the constraints of a (mixed-integer) linear program or a combinatorial problem. Since it is in general not possible to minimize all objectives simultaneously, the goal is to find optimal trade-offs between the objective functions. We use the notation $f(x) = (f_1(x), \ldots, f_K(x))^T$ for the objective vector corresponding to a decision $x \in X$. The approximation is compared to the set of attainable objective vectors $Z \coloneqq f(X) \subseteq \mathbb{R}^K$ given by the image of the set of feasible decision vectors X. We call a given objective vector $z \in Z$ nondominated if there does not exist another vector $z' \in Z$ with $z'_i < z_i$ for some $i \in \{1, \ldots, n\}$ and $z' \leq z$ component-wise. The task of computing all optimal trade-offs can thus be described as finding all nondominated solutions. This set of nondominated solutions, also called Pareto frontier, is often infinitely large. Thus, algorithms for multicriteria optimization often return parametrized representations of the set, e.g. as the convex hull of extreme points in multicriteria linear programming. However even such representations are not always able to reflect the Pareto frontier exactly, as a number of extreme points can be required that is exponential in the number of underlying variables in the worst- and even average-case (Küfer, 1998).

Hence in practice only approximations to the nondominated set can be computed. Even algorithms that return the entire nondominated set in theory, when given unlimited resources, are used in practice with some stopping criterion. Thus, both theoretical approaches to analyze the approximation error as well as practical methods for its computation are essential. We assume that an approximation always overestimates the true Pareto frontier, i.e. the approximation A is always dominated by or equal to a point in the true Pareto frontier Z, guaranteeing the relation $A \subseteq Z + \mathbb{R}^K_{\geq 0}$. Additionally, we assume throughout the paper that the sets Z and A are closed in order to guarantee for the existence of minima and maxima. This is already the case for typical

formulations. If the original sets Z or A are not closed, their topological closures can be taken instead, without changing the approximation quality.

2.2. The ε-indicator

The notion of an ε-approximate Pareto frontier is widely used in the literature related to approximations in multicriteria optimization. In one of its various forms (Legriel et al., 2010) it can be defined as follows:

Definition 1 (ε-approximate Pareto frontier)**.** An approximation $A \subseteq Z + \mathbb{R}^K_{\geq 0}$ of the true set of attainable objective vectors $Z \subseteq \mathbb{R}^K$ is called ε-approximate Pareto frontier for Z if it fulfills: For every attainable objective vector $z \in Z$ there exists a point in the approximation $z' \in A$ such that $z' \leq z + \varepsilon \cdot \mathbf{1}_K$.

Remark 1. For other variants of the concept with a relative instead of an absolute deviation, see Papadimitriou & Yannakakis (2000). The relative measurement however requires further restriction on the considered set of objective vectors, i.e. strict positivity. Another related notion, however focused more on individual objective vectors, is the concept of ε-efficiency (White, 1986).

We now formally introduce the approximation quality measure considered in our paper: the *ε-indicator*. There exist various definitions that differ in the details, e.g. whether absolute or relative differences and a corresponding scaling are used (Laszczyk & Myszkowski, 2019; Zitzler et al., 2003). Our definition is based on ε-approximate Pareto frontiers to illustrate the connection between the two concepts.

Definition 2 (ε-indicator)**.** The ε-indicator value $\varepsilon_d(Z, A)$ of a closed, nonempty approximation $A \subseteq Z + \mathbb{R}^K_{\geq 0}$ of the closed set of attainable objective vectors $Z \subseteq \mathbb{R}^K$ is the smallest number $\varepsilon \geq 0$ such that A is an ε-approximate Pareto frontier for Z.

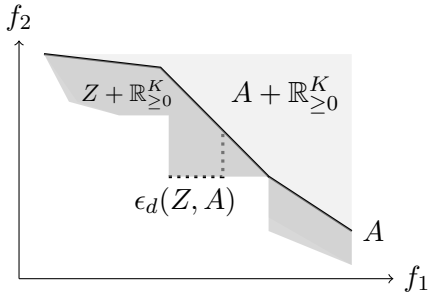


Figure 1: Illustration of the ε-indicator. The dotted line indicates the ε-indicator value between the true set of attainable objective vectors Z and the approximation A.

Note that the ε-indicator is well defined as a minimal number since the sets A and Z are assumed to be closed. Figure 1 illustrates the concept of the ε-indicator. By using a simple reformulation,

we can represent the ε-indicator value as the Hausdorff-distance in the ℓ_∞-metric between the two sets extended by the ordering cone $\mathbb{R}^K_{\geq 0}$.

Lemma 1 (Representation as Hausdorff-distance)**.** *The ε-indicator $\varepsilon_d(Z, A)$ of a closed, nonempty approximation $A \subseteq Z + \mathbb{R}^K_{\geq 0}$ of the closed set of attainable objective vectors $Z \subseteq \mathbb{R}^K$ can be represented as*

$$\varepsilon_d(Z, A) = \ell_\infty(Z, A + \mathbb{R}^K_{\geq 0}) \coloneqq \sup_{z \in Z} \inf_{z' \in A + \mathbb{R}^K_{\geq 0}} ||z - z||_\infty.$$

Proof. Assume that A is an ε-approximate Pareto frontier for Z. Let $z \in Z$ be some arbitrary attainable objective vector. Then there exists an approximation point $z' \in A$ such that $z' \leq z + \varepsilon \cdot \mathbf{1}_K$. Thus it holds $z + \varepsilon \cdot \mathbf{1}_K \in A + \mathbb{R}^K_{\geq 0}$. Since $||z + \varepsilon \cdot \mathbf{1}_K - z||_\infty = \varepsilon$ this implies $\ell_\infty(z, A + \mathbb{R}^K_{\geq 0}) \leq \varepsilon$. We hence have $\ell_\infty(Z, A + \mathbb{R}^K_{\geq 0}) \leq \varepsilon$.

Assume for the other direction that $\ell_\infty(Z, A + \mathbb{R}^K_{\geq 0}) \leq \varepsilon$. Let $z \in Z$ be again some arbitrary attainable objective vector. Then there exists a point $z^* \in A + \mathbb{R}^K_{\geq 0}$ with $||z - z^*||_\infty \leq \varepsilon$ since A is a closed set. Thus we can choose $z' \in A$ such that $z' \leq z^* \leq z + \varepsilon \cdot \mathbf{1}_K$. Hence A is an ε-approximate Pareto-frontier for Z. □

3. Complexity

We show that the ε-indicator is NP-hard to compute when the number of criteria is unlimited, already in the case of simple linear programs. For this, we use a reduction from the following feasibility problem of integer programming.

Definition 3 (0-1 Integer Linear Programming problem)**.** An instance of the problem asks to determine for a given matrix $B \in \mathbb{R}^{m \times n}$ and right-hand side $b \in \mathbb{R}^m$ if there exists a binary solution to the linear inequality system

$$\begin{aligned} By &\leq b, \\ y &\in \{0, 1\}^n. \end{aligned}$$

We reduce the NP-hard 0-1 Integer Linear Programming problem (Karp, 1972) to a computation of the ε-indicator for a given approximation of a simple linear multicriteria program. First we show that the 0-1 Integer Linear Programming problem can be restricted in the following way without losing NP-hardness.

Lemma 2. *The* 0-1 *Integer Linear Programming problem is* NP*-hard when restricted to instances such that there exists a solution $y^\circ \in]0, 1[^n$ of $By \leq b$.*

Proof. We show the claim by reducing an arbitrary 0-1 Integer Linear Programming problem to instances fulfilling this property. Suppose that there exists an index $j \in \{1, \ldots, n\}$ such that all solutions in the continuous relaxation

$$C \coloneqq \{y \in [0, 1]^n \mid By \leq b\}$$

fulfill $y_j \in \{0, 1\}$. This can be checked by minimizing $|y_i - 1/2|$ over the feasible set C for each $i \in \{1, \ldots, n\}$. A fractional value $y_i \in]0, 1[$ is feasible if and only if the optimal objective value

is less than 1/2. By introducing an additional variable, we can represent this task as a standard linear program, which allows an optimization in polynomial time.

If both cases $y_j = 0$ and $y_j = 1$ are feasible, then also the value $y_j = 1/2$ is feasible by convexity of the set of feasible solutions, contradicting the assumption. Thus, either only $y_j = 0$ or $y_j = 1$ can be reached with feasible solutions. We can distinguish between these two possibilities in polynomial time using a linear program (by maximizing or minimizing y_j over the set C). We can thus substitute the value y_j with the constant 0 or 1 in $By \leq b$, reducing the number of variables by 1. By repeating this process at most n times, a reduced set of the form $\{y \in [0,1]^{n'} \mid B'y \leq b'\}$ is reached which has a solution $y^\circ \in]0,1[^{n'}$. Due to the construction we have the equivalence that $\{y \in \{0,1\}^{n'} \mid B'y \leq b'\}$ has an integral solution if and only if the original problem $\{y \in \{0,1\}^n \mid By \leq b\}$ has one. Since the reduction can be done in polynomial time, this restricted class of 0-1 Integer Linear Programming problems is also NP-hard. □

Using this restricted form of the problem, we can give a reduction to the the determination of the ε-indicator value for a multicricriteria problem and a corresponding approximation.

Theorem 1 (NP-hardness of computing the ε-indicator)**.** *The computation of the ε-indicator for a given approximation A is NP-hard when the number of criteria is unbounded. In particular, it is NP-hard when the underlying optimization problem is a linear program.*

Proof. Consider an instance of the 0-1 Integer Linear Programming problem with coefficient matrix $B \in \mathbb{R}^{m \times n}$ and right-hand side $b \in \mathbb{R}^m$. According to Lemma 2 we can assume that there exists a solution $y^\circ \in]0,1[^n$ of $By^\circ \leq b$. In particular, we can choose $\delta \in]0, 1/2[$ such that the inclusion $y^\circ \in [\delta, 1-\delta]$ holds. We construct now a multicriteria problem with $2n$ criteria of the following form:

$$\begin{aligned} \min f(y) &= (f_1(y), f_2(y), \ldots, f_{2n}(y))^T \\ \text{s.t. } By &\leq b \\ 0 &\leq y \leq 1 \end{aligned} \tag{1}$$

where we choose the simple linear objective function vector

$$f(y) = (f_1(y), f_2(y), \ldots, f_{2n}(y))^T = (y_1, 1-y_1, y_2, 1-y_2, \ldots, y_n, 1-y_n)^T.$$

We denote the attainable set in objective space corresponding to (1) by $Z \subseteq \mathbb{R}^{2n}$. We use the simple approximation

$$A = \left\{ z' \in [0,1]^{2n} \;\middle|\; \sum_{i=1}^{2n} z'_i = 2n - \delta \right\}.$$

We first check that the approximation A is feasible, i.e. fulfills $A \subseteq Z + \mathbb{R}^{2n}_{\geq 0}$. Since $f_i(y^\circ) \leq 1 - \delta$ holds for each $i = 1, \ldots, 2n$ by choice of the fractional solution y° we have

$$(1-\delta)\mathbf{1}_{2n} \in f(y^\circ) + \mathbb{R}^{2n}_{\geq 0} \subseteq Z + \mathbb{R}^{2n}_{\geq 0}.$$

Every element of the approximation $y \in A$ has the property $y \geq (1-\delta)\mathbf{1}_{2n}$ since otherwise the sum $2n - \delta$ could not be reached with values in $[0,1]$. Thus, we fulfill the requirement

$$A \subseteq (1-\delta)\mathbf{1}_{2n} + \mathbb{R}^{2n}_{\geq 0} \subseteq Z + \mathbb{R}^{2n}_{\geq 0}.$$

To establish an equivalency, we show that the ε-indicator value $\varepsilon_d(Z, A)$ is at least $1 - \delta/n$ if and only if there is a feasible solution to the 0-1 Integer Linear Programming problem. For this, we show two statements:

1. If there exists a solution $y \in \{0,1\}^n$ with $By \leq b$, then the ε-indicator value is at least $\varepsilon_d(Z, A) \geq 1 - \delta/n$.
2. Otherwise, if there exists no such solution, the ε-indicator value is less than $\varepsilon_d(Z, A) < 1 - \delta/n$.

Proof of statement 1. Let $y \in \{0,1\}^n$ be an integral solution of $By \leq b$. Then, the objective vector

$$z \coloneqq f(y) = (y_1, 1-y_1, y_2, 1-y_2, \dots) \in Z$$

is reached while fulfilling $z \in \{0,1\}^{2n}$ and $\sum_{i=1}^{2n} z_i = n$.

We show now that the distance of this vector z to the approximation A is at least $\ell_\infty(z, A) \geq 1 - \delta/n$. This then implies the lower bound on the ε-indicator value $\varepsilon_d(Z, A) \geq 1 - \delta/n$. We have to show that for every approximation point $\tilde{z} \in A + \mathbb{R}^{2n}_{\geq 0}$ the distance to z is at least $||z - \tilde{z}||_\infty \geq 1 - \delta/n$. We decompose the vector $\tilde{z}$ as $\tilde{z} = z' + \Delta z$ into the approximation point $z' \in A$ and the corresponding shift $\Delta z \in \mathbb{R}^{2n}_{\geq 0}$. For every entry $i \in \{1, \dots, 2n\}$ with $z_i = 1$ the difference $z'_i - z_i \leq 0$ is nonpositive because by definition $z'_i \in [0,1]$. Since the point z has the properties $\sum_{i=1}^{2n} z_i = n$ and $z \in \{0,1\}^{2n}$ there are n of such entries with a nonpositive difference, leaving at most n indices j for which the difference $z'_j - z_j > 0$ is positive. Because by assumption the equations $\sum_{i=1}^{2n} z'_i = 2n - \delta$ and $\sum_{i=1}^{2n} z_i = n$ hold, we get the difference

$$\sum_{i=1}^{2n} \left(z'_i - z_i\right) = \sum_{i=1}^{2n} z'_i - \sum_{i=1}^{2n} z_i = 2n - \delta - n = n - \delta.$$

Since only at most n of the summands $z'_i - z_i$ are positive, there must be an index $k \in \{1, \dots, 2n\}$ fulfilling

$$z'_k - z_k \geq \frac{n - \delta}{n} = 1 - \frac{\delta}{n}.$$

We hence have

$$\tilde{z}_k - z_k = z'_k + (\Delta z)_k - z_k \geq z'_k - z_k \geq 1 - \delta/n$$

and thus $||\tilde{z} - z||_\infty \geq 1 - \delta/n$. This establishes the lower bound on the ε-indicator

$$\varepsilon_d(Z, A) = \ell_\infty(Z, A + \mathbb{R}^{2n}_{\geq 0}) \geq 1 - \delta/n$$

in the case that there exists a feasible solution to the 0-1 Integer Linear Programming instance.

Proof of statement 2. Assume now that there exists no solution $y \in \{0,1\}^n$ with $By \leq b$. Define the following distance measure d to the next binary solution for vectors $y \in [0,1]^n$:

$$d(y) = \sum_{i=1}^{n} \min\{y_i, 1 - y_i\}.$$

A vector $y \in [0,1]^n$ fulfills $d(y) = 0$ if and only if $y \in \{0,1\}^n$, otherwise it holds $d(y) > 0$. Since the relaxed set of solutions $C \coloneqq \{y \in [0,1]^n \mid By \leq b\}$ is a compact set and d is a continuous function, the infimum is attained in the expression $\inf_{y \in C} d(y)$, implying that its value κ' is strictly positive

$$\kappa' \coloneqq \inf_{y \in C} d(y) = \min_{y \in C} d(y) > 0$$

because $d(y) > 0$ holds for every $y \in C$ as we assume that no integral solutions exist. We choose now a value κ fulfilling $0 < \kappa \leq \kappa'$ such that

$$\kappa \leq 2\delta. \tag{2}$$

We assume w.l.o.g. that n is chosen large enough and δ and κ small enough such that the following two conditions hold:

$$\kappa/n \leq 1/2, \tag{3}$$

$$1 - \delta/n - \kappa/(2n^2 - 2n) \geq 1 - \delta/n - \kappa/(2n) \geq 1/2. \tag{4}$$

We now use this value κ to derive an upper bound on the ε-indicator $\varepsilon_d(Z, A)$. Let $z = f(y) \in Z$ be an arbitrary attainable objective vector with $y \in C$ being the corresponding solution. It holds

$d(y) \geq \kappa$. Thus, there must exist a component $j \in \{1, \dots, n\}$ with $\min\{y_j, 1 - y_j\} \geq \kappa/n$. We now construct an approximation point $z' \in A$ with sufficiently small ℓ_∞-distance to z. The components of this point are chosen as follows: For $i \in \{1, \dots, n\} \setminus \{j\}$, we set:

$$z'_{2i-1} = \begin{cases} 1 - \delta/n - \kappa/(2n^2 - 2n), & \text{if } y_i \leq 1/2, \\ 1, & \text{if } y_i > 1/2, \end{cases}$$

and

$$z'_{2i} = \begin{cases} 1, & \text{if } y_i \leq 1/2, \\ 1 - \delta/n - \kappa/(2n^2 - 2n), & \text{if } y_i > 1/2. \end{cases}$$

For the special components $2j - 1$ and $2j$ we set

$$z'_{2j-1} = \begin{cases} 1 - \delta/n + \kappa/(2n), & \text{if } y_j \leq 1/2, \\ 1, & \text{if } y_j > 1/2, \end{cases}$$

and

$$z'_{2j} = \begin{cases} 1, & \text{if } y_j \leq 1/2, \\ 1 - \delta/n + \kappa/(2n), & \text{if } y_j > 1/2. \end{cases}$$

We check that $z' \in A$ is a valid approximation point: By the choice of z'_{2i-1} and z'_{2i} it holds

$$z'_{2i-1} + z'_{2i} = 2 - \delta/n - \kappa/(2n^2 - 2n) \qquad \text{for every } i \in \{1, \dots, n\} \setminus \{j\}.$$

Additionally, it holds $z'_{2j-1} + z'_{2j} = 2 - \delta/n + \kappa/(2n)$. We hence get in sum

$$\sum_{i=1}^{2n} z'_i = (n-1)\left(2 - \delta/n - \kappa/(2n^2 - 2n)\right) + 2 - \delta/n + \kappa/(2n) = n(2 - \delta/n) = 2n - \delta.$$

The entries of z' fall in the range $z'_{2i-1}, z'_{2i} \in [0, 1]$ due to assumption (4). Similarly the special components are in the correct interval $z'_{2j-1}, z'_{2j} \in [0, 1]$ since $1 - \delta/n + \kappa/(2n) \in [0, 1]$ due to the bounds (2) and (4). Hence it also holds $z' \in [0, 1]^{2n}$ and we thus have verified that $z' \in A$ is a valid approximation point.

We now derive upper-bounds for the distance $||z - z'||_\infty$ to the objective vector. It is sufficient to obtain bounds for the absolute differences of the components $|z_{2i-1} - z'_{2i-1}|$ and $|z_{2i} - z'_{2i}|$ individually. For $i \in \{1, \dots, n\} \setminus \{j\}$ we arrive in the case $y_i \leq 1/2$ at the bound

$$\begin{aligned} \left|z_{2i-1} - z'_{2i-1}\right| &= \left|f_{2i-1}(y) - z'_{2i-1}\right| = \left|y_i - z'_{2i-1}\right| \\ &\overset{(4)}{\leq} \left|0 - (1 - \delta/n - \kappa/(2n^2 - 2n))\right| = 1 - \delta/n - \kappa/(2n^2 - 2n). \end{aligned}$$

In the case $y_i > 1/2$ we have

$$\left|z_{2i-1} - z'_{2i-1}\right| = \left|f_{2i-1}(y) - z'_{2i-1}\right| = \left|y_i - z'_{2i}\right| \leq |1/2 - 1| = 1/2.$$

We get equivalent bounds for the distances $|z_{2i} - z'_{2i}|$. If $y_i > 1/2$ holds, we have

$$\left|z_{2i} - z'_{2i}\right| = \left|f_{2i}(y) - z'_{2i}\right| = \left|1 - y_i - (1 - \delta/n - \kappa/(2n^2 - 2n))\right| \overset{(4)}{\leq} 1 - \delta/n - \kappa/(2n^2 - 2n).$$

In the case $y_i \leq 1/2$ we get the bound

$$\left|z_{2i} - z'_{2i}\right| = \left|f_{2i}(y) - z'_{2i}\right| = |1 - y_i - 1| = y_i \leq 1/2.$$

For the special index j we get the following bounds, using the property that $\min\{y_j, 1 - y_j\} \geq \kappa/n$. If the bound $y_j \leq 1/2$ holds, we have

$$\left|z_{2j-1} - z'_{2j-1}\right| = \left|f_{2j-1}(y) - z'_{2j-1}\right| = \left|y_j - z'_{2j-1}\right| \leq |\kappa/n - (1 - \delta/n + \kappa/(2n))| \overset{(4)}{\leq} 1 - \delta/n - \kappa/(2n).$$

In the case that $y_j > 1/2$ holds, we have

$$\left|z_{2j-1} - z'_{2j-1}\right| = \left|f_{2j-1}(y) - z'_{2j-1}\right| = \left|y_j - z'_{2j-1}\right| \leq |1/2 - 1| = 1/2.$$

We get equivalent bounds for the distances $|z_{2j}-z'_{2j}|$, using again the property $\min\{y_j, 1-y_j\} \geq \kappa/n$. If $y_j > 1/2$ is fulfilled it holds

$$\left|z_{2j} - z'_{2j}\right| = \left|f_{2j}(y) - z'_{2j}\right| = |(1-y_j) - (1-\delta/n+\kappa/(2n))| \overset{(4)}{\leq} 1-\delta/n-\kappa/(2n).$$

For the case $y_j \leq 1/2$ we get the inequality

$$\left|z_{2j} - z'_{2j}\right| = \left|f_{2j}(y) - z'_{2j}\right| = |1-y_j-1| = y_j \leq 1/2.$$

In summary, our considerations of the individual differences and assumption (4) yield the desired bound on the ℓ_∞-distance

$$||z - z'||_\infty < 1 - \delta/n.$$

It thus holds $\varepsilon_d(Z, A) = \ell_\infty(Z, A + \mathbb{R}^{2n}_{\geq 0}) < 1 - \delta/n$ in the case that no integral solution exists.

Statements 1 and 2 establish an equivalence between the ε-indicator value and the NP-hard 0-1 Integer Linear Programming problem. There exists an integral solution if and only if the ε-indicator value is at least $\varepsilon_d(Z, A) \geq 1 - \delta/n$. Otherwise it holds $\varepsilon_d(Z, A) < 1 - \delta/n$. Hence, computing the ε-indicator is NP-hard because the corresponding multicriteria problem and the approximation can be constructed in polynomial time. □

Although Theorem 1 shows that computing the ε-indicator is NP-hard, this difficulty only exists when the number of criteria is arbitrarily large. For a fixed number of criteria polynomial-time algorithms are possible also for the types of instances used in the NP-hardness proof, as our following methods show.

4. Algorithms to compute the ε-indicator

4.1. Reformulation to an optimization problem

We show that the ε-indicator can be represented as the solution of an optimization problem which enables an efficient computation. To establish this reformulation, we require two lemmas that characterize boundary points of the approximation.

Lemma 3 (Existence of boundary point)**.** *Let a point $z \in \mathbb{R}^K$ and a nonempty closed set $A \subseteq \mathbb{R}^K$ be given. Consider the distance*

$$d \coloneqq \ell_\infty\left(z, A + \mathbb{R}^K_{\geq 0}\right) \coloneqq \inf_{z' \in A + \mathbb{R}^K_{\geq 0}} ||z - z||_\infty > 0.$$

which we assume to be strictly positive. Then the point resulting from a shift by d in every dimension

$$z + d \cdot \mathbf{1}_K \in \partial\left(A + \mathbb{R}^K_{\geq 0}\right)$$

is contained in the boundary of the approximation.

Proof. We can characterize the value of d as a ℓ_∞-distance alternatively as

$$d = \inf\left\{||z - z'||_\infty \;\middle|\; z' \in \left(A + \mathbb{R}^K_{\geq 0}\right) \cap \left[z - \tilde{d} \cdot \mathbf{1}_K; z + \tilde{d} \cdot \mathbf{1}_K\right]\right\}$$

where $\tilde{d} \in \mathbb{R}_{\geq 0}$ is some upper bound for the distance $\tilde{d} \geq \ell_\infty(z, A + \mathbb{R}^K_{\geq 0})$. The ℓ_∞-norm is continuous and $(A + \mathbb{R}^K_{\geq 0}) \cap [z - \tilde{d} \cdot \mathbf{1}_K; z + \tilde{d} \cdot \mathbf{1}_K]$ is a compact set as A is closed. Hence, the infimum is

attained in the above expression, i.e. there exists a point $z' \in A + \mathbb{R}^K_{\geq 0}$ with $||z - z'||_\infty = d$. Thus the set $(A + \mathbb{R}^K_{\geq 0}) \cap [z; z + d \cdot \mathbf{1}_K] \neq \emptyset$ is nonempty.

By the subsequent argument the following inclusion holds

$$\left(A + \mathbb{R}^K_{\geq 0}\right) \cap [z; z + d \cdot \mathbf{1}_K] \subseteq \partial(A + \mathbb{R}^K_{\geq 0}).$$

We show first that the set $(A+\mathbb{R}^K_{\geq 0}) \cap [z; z+d \cdot \mathbf{1}_K]$ does not contain any interior point of $A+\mathbb{R}^K_{\geq 0}$. If the set would contain an interior point of $A + \mathbb{R}^K_{\geq 0}$, then a ball in the ℓ_∞-norm with some radius $\varepsilon > 0$ would be contained in the set $(A + \mathbb{R}^K_{\geq 0}) \cap [z; z + d \cdot \mathbf{1}_K]$ as well since $d > 0$ holds, which would imply that for $d' \coloneqq d - \varepsilon$ the intersection

$$\left(A + \mathbb{R}^K_{\geq 0}\right) \cap [z; z + d' \cdot \mathbf{1}_K] \neq \emptyset$$

would also be nonempty. This contradicts the choice of d as the infimum. Secondly, it holds

$$\left(A + \mathbb{R}^K_{\geq 0}\right) \cap [z; z + d \cdot \mathbf{1}_K] \subseteq A + \mathbb{R}^K_{\geq 0},$$

which combined with the first fact implies that the set can only contain boundary points of $A+\mathbb{R}^K_{\geq 0}$.

Since the set $(A + \mathbb{R}^K_{\geq 0}) \cap [z; z + d \cdot \mathbf{1}_K] \neq \emptyset$ is nonempty the point $z + d \cdot \mathbf{1}_K$ is contained in $A + \mathbb{R}^K_{\geq 0}$. With the above result we obtain the claim from the inclusion

$$z + d \cdot \mathbf{1}_K \in \left(A + \mathbb{R}^K_{\geq 0}\right) \cap [z; z + d \cdot \mathbf{1}_K] \subseteq \partial \left(A + \mathbb{R}^K_{\geq 0}\right). \qquad \square$$

Lemma 4 (Consequence of boundary point)**.** *Let $A \subseteq \mathbb{R}^K$ be a nonempty closed set. Let a point $z \in \mathbb{R}^K$ and number $d \in \mathbb{R}_{\geq 0}$ be given such that the corresponding shifted point*

$$z + d \cdot \mathbf{1}_K \in \partial(A + \mathbb{R}^K_{\geq 0})$$

is contained in the boundary of the approximation. Then the equality $\ell_\infty(z, A + \mathbb{R}^K_{\geq 0}) = d$ holds for the distance.

Proof. We show the equality using two inequalities relating the number d and the distance $\ell_\infty(z, A + \mathbb{R}^K_{\geq 0})$.

The inequality $\ell_\infty(z, A + \mathbb{R}^K_{\geq 0}) \leq d$ holds due to the inclusion

$$z + d \cdot \mathbf{1}_K \in \partial(A + \mathbb{R}^K_{\geq 0}) \subseteq A + \mathbb{R}^K_{\geq 0}$$

as $A + \mathbb{R}^K_{\geq 0}$ is closed. Thus we have the upper bound

$$\ell_\infty(z, A + \mathbb{R}^K_{\geq 0}) \leq \ell_\infty(z, z + d \cdot \mathbf{1}_K) = d.$$

We now show the lower bound $\ell_\infty(z, A + \mathbb{R}^K_{\geq 0}) \geq d$. Suppose for the sake of contradiction that $\ell_\infty(z, A + \mathbb{R}^K_{\geq 0}) < d$. Then there exists a value $0 < d' < d$ with

$$\left(A + \mathbb{R}^K_{\geq 0}\right) \cap [z - d' \cdot \mathbf{1}_K; z + d' \cdot \mathbf{1}_K] \neq \emptyset.$$

This implies that $z + d' \cdot \mathbf{1}_K \in A + \mathbb{R}^K_{\geq 0}$. Thus we have the inclusion

$$[z_1 + d', \infty[\times \cdots \times [z_K + d', \infty[\subseteq A + \mathbb{R}^K_{\geq 0}.$$

Hence, since the value $d > d'$ is larger, the point $z + d \cdot \mathbf{1}_K$ is an interior point of $A + \mathbb{R}^K_{\geq 0}$. This contradicts the assumption that $z+d\cdot\mathbf{1}_K \in \partial(A+\mathbb{R}^K_{\geq 0})$. Thus, also the inequality $\ell_\infty(z, A+\mathbb{R}^K_{\geq 0}) \geq d$ holds. $\square$

Application to computing the ε-indicator. Based on the representation of the ε-indicator as a Hausdorff distance in Lemma 1, we can use the two previous Lemmas 3 and 4 to formulate the computation of the ε-indicator as the optimization problem given in Theorem 2 whose construction is also illustrated in Figure 2.

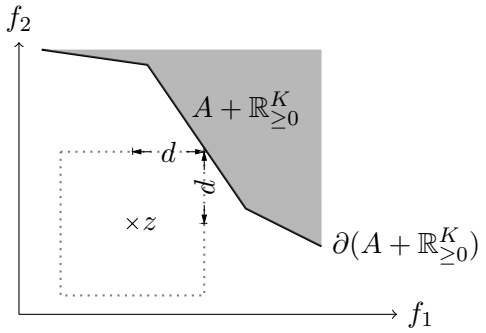


Figure 2: Illustration of the reformulation for computing the ε-indicator.

Theorem 2 (Reformulating ε-indicator)**.** *For a closed, nonempty approximation $A \subseteq \mathbb{R}^K$, we can express the ε-indicator $\varepsilon_d(Z, A)$ as the optimal objective value of the optimization problem*

$$\varepsilon_d(Z, A) = \max \left\{ d \geq 0 \mid \exists z \in Z : z + d \cdot \mathbf{1}_K \in \partial \left(A + \mathbb{R}^K_{\geq 0} \right) \right\}.$$

Proof. Consider a point $z \in \mathbb{R}^K$ and a closed, nonempty set $A \subseteq \mathbb{R}^K$. We distinguish the two cases $\ell_\infty(z, A + \mathbb{R}^K_{\geq 0}) > 0$ and $\ell_\infty(z, A + \mathbb{R}^K_{\geq 0}) = 0$.

In the case $\ell_\infty(z, A + \mathbb{R}^K_{\geq 0}) > 0$ the two Lemmas 3 and 4 above imply the equality

$$\ell_\infty \left(z, A + \mathbb{R}^K_{\geq 0} \right) = d$$

where d is the unique value $d \in \mathbb{R}$ such that

$$z + d \cdot \mathbf{1}_K \in \partial \left(A + \mathbb{R}^K_{\geq 0} \right).$$

In the form of an optimization problem, we can state this as

$$\ell_\infty \left(z, A + \mathbb{R}^K_{\geq 0} \right) = \max \left\{ d \mid z + d \cdot \mathbf{1}_K \in \partial \left(A + \mathbb{R}^K_{\geq 0} \right) \right\}$$

where the feasible set however only contains at most one element.

If $\ell_\infty(z, A + \mathbb{R}^K_{\geq 0}) = 0$, then there does not exist any $d > 0$ such that $z + d \cdot \mathbf{1}_K \in \partial(A + \mathbb{R}^K_{\geq 0})$ since this would contradict Lemma 4. Thus, the optimal objective value of the optimization problem $\max \left\{ d \mid z + d \cdot \mathbf{1}_K \in \partial \left(A + \mathbb{R}^K_{\geq 0} \right) \right\}$ is not larger than 0 or the problem is infeasible, resulting in an objective value of $-\infty$.

Thus we can conclude with the case distinction

$$\max \left\{ d \mid z + d \cdot \mathbf{1}_K \in \partial \left(A + \mathbb{R}^K_{\geq 0} \right) \right\} \begin{cases} = \ell_\infty(z, A + \mathbb{R}^K_{\geq 0}), & \text{if } \ell_\infty(z, A + \mathbb{R}^K_{\geq 0}) > 0, \\ \leq 0, & \text{if } \ell_\infty(z, A + \mathbb{R}^K_{\geq 0}) = 0, \end{cases}$$

which implies the relation

$$\ell_\infty(z, A + \mathbb{R}^K_{\geq 0}) = \max\{\max \left\{ d \mid z + d \cdot \mathbf{1}_K \in \partial \left(A + \mathbb{R}^K_{\geq 0} \right) \right\}, 0\}$$

since $\ell_\infty(z, A + \mathbb{R}^K_{\geq 0}) \geq 0$ is nonnegative.

Because the ε-indicator is defined as the maximal distance over all points $z \in Z$, we obtain the desired representation

$$\varepsilon_d(Z, A) = \max_{z \in Z} \ell_\infty \left(z, A + \mathbb{R}^K_{\geq 0} \right) = \max_{z \in Z} \max \left\{ \max \left\{ d \geq 0 \mid z + d \cdot \mathbf{1}_K \in \partial \left(A + \mathbb{R}^K_{\geq 0} \right) \right\}, 0 \right\}$$

$$= \max \left\{ d \geq 0 \mid \exists z \in Z : z + d \cdot \mathbf{1}_K \in \partial \left(A + \mathbb{R}^K_{\geq 0} \right) \right\}$$

where for the last equality we use the fact that there always exists a point $z \in Z$ and a value $d \geq 0$ with $z + d \cdot \mathbf{1}_K \in \partial \left(A + \mathbb{R}^K_{\geq 0} \right)$. For this, we only need to choose any point $y \in A \subseteq Z + \mathbb{R}^K_{\geq 0}$ which is not dominated by any other point in A and then take an appropriate value $z \in Z$ with $y \in z + \mathbb{R}^K_{\geq 0}$. Since z is nondominated by any point in A it is not contained in the interior of $A + \mathbb{R}^K_{\geq 0}$ which implies that there is a nonnegative value $d \geq 0$ for which $z + d \cdot \mathbf{1}_K \in \partial \left(A + \mathbb{R}^K_{\geq 0} \right)$ holds. □

4.2. Decomposing the computation

The above method reduces the computation of the ε-indicator to the solution of an optimization problem. However, we need to represent the boundary set $\partial(A + \mathbb{R}^K_{\geq 0})$ in the optimization problem. In order to turn this requirement into a linear programming formulation, linear inequalities have to be constructed. If the approximation A is given as a union of polyhedral objects (e.g. points, segments or simplices), we can represent the boundary $\partial(A + \mathbb{R}^K_{\geq 0})$ as a union of polyhedra.

This decomposition can be obtained as follows. Let $A = Q_1 \cup \cdots \cup Q_k$ be a decomposition of A into (not-necessarily disjoint) polyhedra $Q_1, \ldots, Q_k$. Then, each set $Q_i + \mathbb{R}^K_{\geq 0}$ is also a polyhedron. Thus we get the corresponding decomposition $A + \mathbb{R}^K_{\geq 0} = (Q_1 + \mathbb{R}^K_{\geq 0}) \cup \cdots \cup (Q_k + \mathbb{R}^K_{\geq 0})$ into a union of polyhedra. Each facet of each polyhedron can then be decomposed into simplices. For the resulting set of simplices, the lower envelope can be computed, yielding again a set of polyhedra that together cover $\partial(A+\mathbb{R}^K_{\geq 0})$. The computation of the lower envelope can be done in time $O(m^K)$ where m is the number of simplices, if we assume that the relative interiors of the simplices are pairwise disjoint (Edelsbrunner et al., 1989).

In the following section we assume that such a decomposition is given. Let $P_1, \ldots, P_m$ be polyhedra that cover the boundary, i.e. fulfill $P_1 \cup \cdots \cup P_m = \partial(A + \mathbb{R}^K_{\geq 0})$. For each polyhedron P_i we have given the matrix $B^{(i)}$ and vector $b^{(i)}$ for the representation

$$P_i = \left\{ x \in \mathbb{R}^K \;\middle|\; B^{(i)} x \leq b^{(i)} \right\}.$$

Figure 3 shows an approximation A together with the corresponding boundary $\partial(A + \mathbb{R}^2_{\geq 0})$ in the case of two criteria.

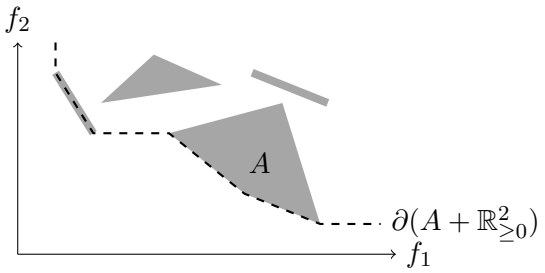


Figure 3: Illustration of an approximation A consisting of polyhedra (gray parts) with its corresponding boundary $\partial(A + \mathbb{R}^2_{\geq 0})$, represented as a dashed line.

4.3. Segment-wise method

In the segment-wise method we compute the ε-indicator $\varepsilon_d(Z, A)$ by solving separately for each part of the boundary $i = 1, \ldots, m$ the optimization problem

$$\begin{aligned} &\max\ d \\ \text{s.t.}\quad &B^{(i)}(z + d \cdot \mathbf{1}_K) \leq b^{(i)} \\ &z \in Z,\ d \in \mathbb{R}_{\geq 0} \end{aligned} \tag{5}$$

and then output the maximal obtained objective value. Due to Theorem 2 this results in the correct value for the ε-indicator $\varepsilon_d(Z, A)$. Note that the formulation (5) requires only one additional variable compared to the constraints and variables that are originating from the LP or MIP formulation for Z. Thus, the required time for the ε-indicator computation for each P_i is of the same order of magnitude as performing a linear optimization on Z. In particular, if the underlying problem is a continuous linear program, the value for a part P_i can be computed in polynomial time.

The disadvantage of this method is that we require one optimization for each of the m parts of the cover of the boundary $\partial(A + \mathbb{R}^K_{\geq 0})$. This number m of required parts can become exponentially large in n in the worst case. For the NP-hard instances constructed in the proof of Theorem 1 such a case occurs, resulting in an exponential running time although only polynomial time is needed per part.

4.4. Full reformulation

An alternative to performing an optimization for each facet describing the boundary $\partial(A + \mathbb{R}^K_{\geq 0})$ is to set up a single MIP that yields the desired ε-indicator value. For this full reformulation of the ε-indicator, we insert additional constraints and binary variables that together describe the boundary $\partial(A + \mathbb{R}^K_{\geq 0})$.

We use again the cover

$$P_1 \cup \cdots \cup P_m = \partial\left(A + \mathbb{R}^K_{\geq 0}\right)$$

of the boundary by polyhedra with representation $P_i = \{x \in \mathbb{R}^K \mid B^{(i)}x \leq b^{(i)}\}$ where $b^{(i)} \in \mathbb{R}^n$. Based on this cover we formulate a mixed-integer program which uses binary variables $a_i \in \{0, 1\}$ to ensure that the point is contained in one of these polyhedra, i.e. in the boundary of the approx-

imation. This results in the following mixed-integer program:

$$\max\ d \tag{6a}$$

$$\text{s.t.} \quad \sum_{i=1}^{m} a_i = 1 \tag{6b}$$

$$B^{(i)}(z + d \cdot \mathbf{1}_K) - M^{(i)} \cdot (1 - a_i) \leq b^{(i)} \qquad \text{for } i \in \{1, \ldots, m\} \tag{6c}$$

$$a_i \in \{0, 1\} \qquad \text{for } i \in \{1, \ldots, m\} \tag{6d}$$

$$z \in Z,\ d \in \mathbb{R}_{\geq 0} \tag{6e}$$

The constraint (6b) ensures that exactly one of the activation variables a_i is set to 1 and all others to 0. Thus at least one of the polyhedra has to be active.

Constraint (6c) ensures that the vector $z + d \cdot \mathbf{1}_K$ is contained in the polyhedron P_i whenever the polyhedron is set active by $a_i = 1$. Here the vector of big-M-constants $M^{(i)} \geq \mathbf{0}$ is chosen large enough such that every optimal solution fulfills this constraint when $a_i = 0$ is set. A method to choose $M^{(i)}$ is described below.

Together the constraints (6b) to (6d) imply that in a feasible solution, the vector $z + d \cdot \mathbf{1}_m$ is contained in one of the polyhedra $P_1, \ldots, P_m$ and thus in the boundary $\partial(A + \mathbb{R}^K_{\geq 0})$. Hence by Theorem 2 the optimal objective value is equal to the ε-indicator $\varepsilon_d(Z, A)$.

The vector of big-M-constants can be determined when the ideal point z^I and nadir point z^N are known. Since every nondominated point is contained in $[z^I, z^N]$ and the ε-indicator is bounded by $\varepsilon_d(Z, A) \leq \ell_\infty(z^I, A)$ for optimal solutions, we have the inclusion

$$z + d \cdot \mathbf{1}_K \in [z^I; z^N + \ell_\infty(z^I, A) \cdot \mathbf{1}_K].$$

It is thus sufficient to choose $M^{(i)}$ large enough such that for this range of values for $z + d \cdot \mathbf{1}_K$ it always holds $B^{(i)}(z + d \cdot \mathbf{1}_K) - M^{(i)} \leq b^{(i)}$. This can be guaranteed by setting the entries $M^{(i)}_k$ for $k \in \{1, \ldots, n\}$ to

$$M^{(i)}_k \coloneqq -b^{(i)}_k + \sum_{k \in \{1\ldots,n\} : B^{(i)}_{jk} \geq 0} B^{(i)}_{jk} \left(z^N_k + \ell_\infty(z^I, A)\right) + \sum_{k \in \{1\ldots,n\} : B^{(i)}_{jk} < 0} B^{(i)}_{jk} z^I_k.$$

5. Concrete Implementations for computing the ε-indicator

We illustrate the performance of our algorithm by running it on various approximate Pareto frontiers for bicriteria problems. In the following section the concrete formulations for this case are discussed.

5.1. Detailed formulation for the bicriteria case

For the case of $K = 2$ criteria we illustrate how the formulations of sections 4.3 and 4.4 can be implemented.

In the case that the approximation consists of line segments, rays and isolated points we can construct the boundary as follows: For each isolated point or endpoint of a segment $(z_1, z_2) \in A$ we create two rays extending from this point $\{z_1\} \times [z_2, \infty[$ and $[z_1, \infty[\times\{z_2\}$. This corresponds to adding all points which are dominated by (z_1, z_2) to the approximation. Then we compute the lower envelope of this set of segments and rays with respect to the first coordinate axis. The computation of the lower envelope can be done in the near-linear running time $O(m\alpha(m)\log m)$ where m is the number of segments and rays and α the inverse Ackermann function (Hart & Sharir, 1986).

Each resulting segment s contained in the boundary $\partial(A + \mathbb{R}^2_{\geq 0})$ can be represented in its parametric form as $s = \{q^{(s)} + tv^{(s)} \mid t \in [0, 1]\}$ where $q^{(s)}, v^{(s)} \in \mathbb{R}^2$. Based on this representation the segment s can be written as a polyhedron $s = \{x \in \mathbb{R}^2 \mid Bx \leq b\}$ with the system of linear inequalities

$$\left(x - q^{(s)}\right)^T \cdot \begin{pmatrix} v_2^{(s)} \\ -v_1^{(s)} \end{pmatrix} = 0, \tag{7a}$$

$$\left(x - q^{(s)}\right)^T \cdot v^{(s)} \geq 0, \tag{7b}$$

$$\left(x - q^{(s)}\right)^T \cdot v^{(s)} \leq \left|\left|v^{(s)}\right|\right|_2^2. \tag{7c}$$

See Figure 4 for an illustration. For a ray of the form $\{q^{(s)} + tv^{(s)} \mid t \in [0, \infty[\}$ a similar representation can be used with the last inequality (7c) dropped. This representation allows to use the segment-wise method (Section 4.3).

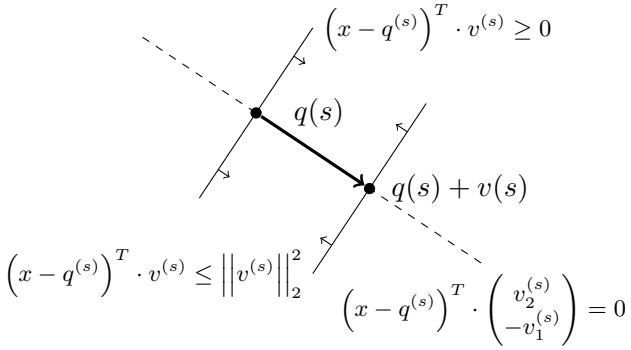


Figure 4: Illustration of the representation of segments via inequalities. The segment (thick line) is given as the intersection of the line (dashed) and two half-spaces (perpendicular).

We now give a corresponding concrete form of the optimization problem (6a)–(6e) for the full reformulation (introduced in Section 4.4) in the case of two criteria. Let S be the set of segments and R the set of rays of the boundary $\partial(A + \mathbb{R}^2_{\geq 0})$. The detailed form of the optimization problem is then given as follows.

$$\max \; d \tag{8a}$$

$$\text{s.t.} \qquad \left(z + d \cdot \mathbf{1}_2 - q^{(s)}\right)^T \begin{pmatrix} v_2^{(s)} \\ -v_1(s) \end{pmatrix} + M_s(1 - a_s) \geq 0 \qquad \text{for } s \in S \cup R \tag{8b}$$

$$\left(z + d \cdot \mathbf{1}_2 - q^{(s)}\right)^T \begin{pmatrix} v_2^{(s)} \\ -v_1(s) \end{pmatrix} - M_s(1 - a_s) \leq 0 \qquad \text{for } s \in S \cup R \tag{8c}$$

$$\left(z + d \cdot \mathbf{1}_2 - q^{(s)}\right)^T \cdot v^{(s)} + M_s(1 - a_s) \geq 0 \qquad \text{for } s \in S \cup R \tag{8d}$$

$$\left(z + d \cdot \mathbf{1}_2 - q^{(s)}\right)^T \cdot v^{(s)} - M_s(1 - a_s) \leq \left|\left|v^{(s)}\right|\right|_2^2 \qquad \text{for } s \in S \tag{8e}$$

$$\sum_{s \in S \cup R} a_s = 1 \tag{8f}$$

$$a_s \in \{0, 1\} \qquad \text{for } s \in S \cup R \tag{8g}$$

$$z \in Z, \; d \in \mathbb{R}_{\geq 0} \tag{8h}$$

The inequalities (8b)-(8e) represent the constraint (6c) of the general formulation applied to the segment of the approximation represented by the system (7a)-(7c). In particular, the inequalities (8b) and (8c) correspond to the line equality (7a) of the approximation segment.

The constant $M_s \in \mathbb{R}_{\geq 0}$ has to be chosen large enough such that every optimal point fulfills all the inequalities. Based on the discussion of the choice of M for the general case we provide here some simplification. We can bound the absolute value of the term involving $z + d \cdot \mathbf{1}_2 - q^{(s)}$ by

$$\left|\left(z + d \cdot \mathbf{1}_2 - q^{(s)}\right)^T \cdot v^{(s)}\right| \leq \left|\left|z + d \cdot \mathbf{1}_2 - q^{(s)}\right|\right|_\infty \cdot \left|\left|v^{(s)}\right|\right|_1$$

$$\leq \left(||z + d \cdot \mathbf{1}_2||_\infty + \left|\left|q^{(s)}\right|\right|_\infty\right) \left|\left|v^{(s)}\right|\right|_1 \leq \left(\max\left\{||z^I||_\infty, ||z^N||_\infty + \ell_\infty(z^I, A)\right\} + \left|\left|q^{(s)}\right|\right|_\infty\right) \left|\left|v^{(s)}\right|\right|_1$$

where we used the relation

$$z + d \cdot \mathbf{1}_K \in [z^I; z^N + \ell_\infty(z^I, A) \cdot \mathbf{1}_K]$$

known for the general case. An equivalent relation holds for the other terms involving $z + d \cdot \mathbf{1}_K - q^{(s)}$ as well, since the 1-norm of the other vector $\begin{pmatrix} v_2^{(s)} \\ -v_1(s) \end{pmatrix}$ used as a factor is equal to the 1-norm

of $v^{(s)}$. Thus, the choice

$$M_s := \left(\max\left\{||z^I||_\infty, ||z^N||_\infty + \ell_\infty(z^I, A)\right\} + \left|\left|q^{(s)}\right|\right|_\infty\right) \left|\left|v^{(s)}\right|\right|_1$$

is sufficient to not exclude any optimal solutions. Thus the value of M_s can be of the same order of magnitude as the other involved quantities, ensuring good LP-relaxations.

5.2. Formulation for the case of a finite approximation

For the important special case that the approximation A consists of a finite set of points, we can adapt our general approach to enable a more efficient computation of the ε-indicator. Approximations of this type are produced by the majority of algorithms for multicriteria optimization. Thus, an effective approach for computing the ε-indicator in this situation for problems with an arbitrary number of objectives has a large practical significance. The efficient computation for this special case will also form the basis of an algorithm for the approximation of Pareto frontiers discussed in section 6.

Our specialized method for finite approximations is based on Theorem 2 and a new characterization of the relevant set $\partial(A + \mathbb{R}^K_{\geq 0})$ with our concept of *corner-points*. This concept is related to orthogonal polyhedra which have been studied in computational geometry (Bournez et al., 1999) since $A + \mathbb{R}^K_{\geq 0}$ is an orthogonal polyhedron when A is finite.

In the following, we use the symbol $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ to denote the extended real number line. For two points $a, b \in \overline{\mathbb{R}}^K$ we say that a strictly dominates b if it holds $a_i < b_i$ for all $i \in \{1, \ldots, K\}$. We use the special vectors $v^{(1)}, \ldots, v^{(K)} \in \overline{\mathbb{R}}^K = (\mathbb{R} \cup \{-\infty, +\infty\})^K$ with entries

$$v^{(i)}_j = \begin{cases} +\infty, & \text{if } i = j, \\ -\infty, & \text{if } i \neq j. \end{cases}$$

Definition 4 (Corner point)**.** A corner point x of A is the componentwise maximum of K points in $A \cup \{v^{(1)}, \ldots, v^{(K)}\}$ such that x is not strictly dominated by any point in A. We denote the set of all corner points of A by $C(A)$.

In Figure 5 an example of the corner points corresponding to $K = 2$ criteria is given. Intuitively, they consist of the upper-right corners in the boundary of $\partial(A + \mathbb{R}^K_{\geq 0})$.

Lemma 5. *Let $y \in \partial(A + \mathbb{R}^K_{\geq 0})$ be an arbitrary boundary point. Then, there exists a corner point $x \in C(A)$ which fulfills $y \leq x$.*

Proof. We define iteratively points $x^{(0)}, \ldots, x^{(K)} \in \overline{\mathbb{R}}^K$ and $z^{(1)}, \ldots, z^{(K)} \in A \cup \{v^{(1)}, \ldots, v^{(k)}\}$. We set $x^{(0)} := y$. For $i \in \{1, \ldots, K\}$ we now perform in this order the following steps: We set

$$t_i := \sup\{t \in \mathbb{R} : x^{(i-1)} + t \cdot e_i \text{ is not strictly dominated by any point in } A\}.$$

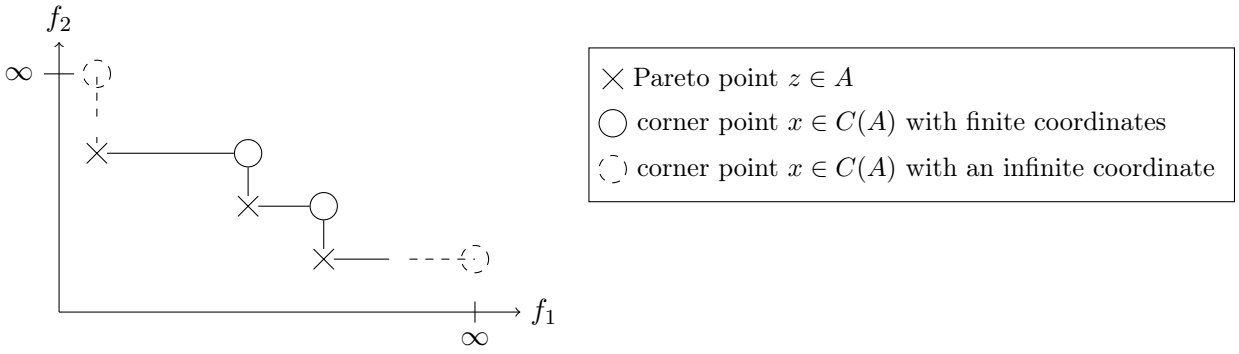


Figure 5: Illustration of the corner points in $K = 2$ dimensions for an approximation A consisting of three points, with two corner points with finite coordinates and two corner points having an infinite coordinate.

If $t_i = \infty$, we choose the special vector $z^{(i)} := v^{(i)}$. Otherwise, we have $t_i < \infty$. By the choice of the supremum there must then exist a point such that $x^{(i-1)} + t' \cdot e_i$ is strictly dominated by a point in A for every value of $t' > t_i$. Since A is finite there must exist a point $z^{(i)} \in A$ that strictly dominates $x^{(i-1)} + t' \cdot e_i$ for every value $t' > t$. We then choose this value $z^{(i)}$.

We set $x^{(i)} := \max\{y, z^{(1)}, \dots, z^{(i)}\}$. This can also be written as the recursion $x^{(i)} = \max\{x^{(i-1)}, z^{(i)}\}$.

We show by induction on $j \in \{0, \dots, K\}$ that $x^{(j)}$ is not strictly dominated by any point in A. Since $x^{(0)} = y \in \partial(A + \mathbb{R}^K_{\geq 0})$ is a point in the boundary it is not strictly dominated by any point in A. Otherwise, a strictly dominating point $z \in A$ would imply that an open ball around y is contained in $A + \mathbb{R}^K_{\geq 0}$ which contradicts the assumption that y is in the boundary of this set.

Suppose that $t_j < 0$. Then, it would hold that $x^{(j-1)}$ is strictly dominated, contradicting the induction hypothesis. Hence we can conclude that $t_j \geq 0$ is nonnegative. In the case that $t_j = \infty$, we have that $x^{(j-1)} + t \cdot e_j$ is not strictly dominated by a point in A for every value of $t \in \mathbb{R}$. By construction, it holds then $z^{(j)} = v^{(j)}$ which fulfills $v_i^{(j)} = -\infty \leq x_i^{(j-1)}$ for all $i \in \{1, \dots, K\} \setminus \{j\}$. Thus, the point $x^{(j)}$ is in all coordinates besides j identical to $x^{(j-1)}$. Since $x^{(j-1)} + t \cdot e_j$ is not strictly dominated by any point in A for all $t \in \mathbb{R}$, also $x^{(j)}$ is not strictly dominated by any point in A. If $t_j < \infty$, we choose $z^{(j)} \in A$ as the point which strictly dominates $x^{(j-1)} + t \cdot e_j$ for every value $t > t_j$. Since the set

$$\{t \in \mathbb{R} : x^{(i-1)} + t \cdot e_i \text{ is not strictly dominated by any point in } A\}$$

is closed, $x^{(j-1)} + t_j \cdot e_j$ is not strictly dominated by any point in A. The point $z^{(j)}$ fulfills $z^{(j)} \leq x^{(j-1)} + t_j \cdot e_j$ by construction since it otherwise could not strictly dominate $x^{(j-1)} + t \cdot e_j$. Thus, we have $x^{(j)} = \max\{x^{(j-1)}, z^{(j)}\} = x^{(j-1)} + t_j \cdot e_j$ which implies that $x^{(j)}$ is not strictly dominated by any point in A.

Since $t_j \geq 0$ it additionally holds $z_j^{(j)} \geq y_j$ because

$$z_j^{(j)} = x_j^{(j-1)} + t_j \geq y_j + t_j \geq y_j.$$

This implies $\max\{z^{(1)}, \dots, z^{(K)}\} \geq y$. Thus, we have

$$\max\{z^{(1)}, \dots, z^{(K)}\} = \max\{y, z^{(1)}, \dots, z^{(K)}\} = x^{(K)}.$$

Hence, $x^{(K)}$ is a corner point as the maximum of K points $z^{(1)}, \dots, z^{(K)} \in A \cup \{v^{(1)}, \dots, v^{(K)}\}$ which is not strictly dominated by any point in A.

□

Theorem 3. *For an approximation $A \subseteq Z + \mathbb{R}^K_{\geq 0}$ of the set of attainable objective vectors $Z \subseteq \mathbb{R}^K$ it holds*

$$\varepsilon_d(Z, A) = \max_{x \in C(A)} \max_{z \in Z} \min\{x_1 - z_1, \ldots, x_K - z_K\}.$$

Proof. We show the two inequalities "$\leq$" and "$\geq$" separately, starting with "$\leq$". It holds

$$\max_{x \in C(A)} \max_{z \in Z} \min\{x_1 - z_1, \ldots, x_K - z_K\} = \max_{z \in Z} \max\{d \in \mathbb{R} \mid \exists x \in C(A) : z + d \cdot \mathbf{1}_K \leq x\}$$

since setting $d = \min\{x_1 - z_1, \ldots, x_K - z_K\}$ is the maximal possible value for which $z + d \cdot \mathbf{1}_K \leq x$ is fulfilled.

By Lemma 5 for every point $y \in \partial(A + \mathbb{R}^K_{\geq 0})$ there exists a corner point $x \in C(A)$ with $x \geq y$. Thus, the following set inclusion follows:

$$\partial(A + \mathbb{R}^K_{\geq 0}) \subseteq \{u \in \mathbb{R}^K \mid \exists x \in C(A) : u \leq x\}.$$

Hence the condition $\exists x \in C(A) : z + d \cdot \mathbf{1}_K \leq x$ is implied by $z + d \cdot \mathbf{1}_K \in \partial(A + \mathbb{R}^K_{\geq 0})$. Thus, we obtain the claimed upper bound

$$\begin{aligned}
&\max_{z \in Z} \max\{d \in \mathbb{R} \mid \exists x \in C(A) : z + d \cdot \mathbf{1}_K \leq x\} \\
\geq &\max_{z \in Z} \max\{d \in \mathbb{R} \mid z + d \cdot \mathbf{1}_K \in \partial(A + \mathbb{R}^K_{\geq 0})\} \\
= &\max\{d \geq 0 \mid \exists z \in Z : z + d \cdot \mathbf{1}_K \in \partial(A + \mathbb{R}^K_{\geq 0})\} \\
= &\varepsilon_d(Z, A)
\end{aligned}$$

where we used Theorem 2 for the last equality.

We now show the other inequality. Let $x \in C(A)$ be a corner point of A and $z \in Z$ be chosen arbitrarily. We show the inequality

$$\varepsilon_d(Z, A) \geq \min\{x_1 - z_1, \ldots, x_K - z_K\}.$$

By definition, the corner point x is not strictly dominated by any point in A. Let any fixed number $d < \min\{x_1 - z_1, \ldots, x_K - z_K\}$ be given. Then the intersection $[z; z + d \cdot \mathbf{1}_K] \cap (A + \mathbb{R}^K_{\geq 0})$ is empty since any point u in the intersection would fulfill $u_i \leq z_i + d < z_i + (x_i - z_i) = x_i$ for all $i \in \{1, \ldots, K\}$ which would imply that a point in A strictly dominates the corner point x, contradicting the definition of a corner point. Hence, it holds

$$\ell_\infty(z, A + \mathbb{R}^K_{\geq 0}) \geq d$$

which implies $\varepsilon_d(Z, A) \geq d$. Since this holds for every value $d < \min\{x_1 - z_1, \ldots, x_K - z_K\}$ we obtain the desired lower bound

$$\varepsilon_d(Z, A) \geq \max_{x \in C(A)} \max_{z \in Z} \min\{x_1 - z_1, \ldots, x_K - z_K\}.$$

Note that the value of $\max_{z \in Z} \min\{x_1 - z_1, \ldots, x_K - z_K\}$ is never negative since we can always choose one of the points $z \in Z$ corresponding to the corner point x with $z \leq x$ which obtains a value of $\min\{x_1 - z_1, \ldots, x_K - z_K\} \geq 0$.

□

From Theorem 3 we obtain the following simple procedure for computing the ε-indicator of a given approximation $A \subseteq \mathbb{R}^K$: For each corner point $x \in C(A)$ solve the optimization problem

$$\max \ \min\{x_1 - z_1, \ldots, x_K - z_K\} \text{ s.t. } z \in Z. \tag{9}$$

Then, the ε-indicator is given as the maximal objective value among all the corner points. The optimization problem (9) can be written in the following form with linear constraints by introducing the auxilary variable t:

$$\begin{aligned} &\max\ t \\ &\text{s.t. } t \leq x_i - z_i \qquad \text{for all } i \in \{1, \dots, K\} \\ &\quad z \in Z,\ t \in \mathbb{R} \end{aligned} \tag{10}$$

We can compute all corner points by taking subsets of size K among the points in A and then creating their componentwise-maximum. We then filter out all maxima that are strictly dominated by a point in A. Note that the number of corner points is significantly lower than the $\binom{|A|}{K}$ possible subsets. For example with $K = 2$ criteria, the number of corner points is at most $|A| + 1$, i.e. grows linearly in $|A|$ compared to the quadratic growth of $\binom{|A|}{2} \in \Theta(|A|^2)$.

6. An algorithm for Pareto-frontier approximation based on the ε-indicator computation

We now present an algorithm which is based on computing the ε-indicator value for the current approximation in each iteration and adding a corresponding point with maximal distance to the approximation.

This approach detailed in Algorithm 1 has the following main benefits: After each iteration the ε-indicator value is exactly known. Thus, the stopping can be precisely controlled and the approximation guarantee of the current approximation is exactly known. Additionally, only points which provide significant improvements of the approximation quality are added to the Pareto frontier, keeping the number of points in the approximation small.

Since optimizing the subproblem (10) directly might lead to points z that are dominated we perform in Algorithm 1 a lexicographic optimization with (10) as the primary objective and the minimization of $\sum_{i=1}^{K} z_i$ as the secondary objective. This guarantees, that all found points z are Pareto-optimal.

To enable an efficient update of the corner points we use a special data structure in our implemention. We store all subsets of up to K points in A whose componentwise maximum is not strictly dominated by any point in A. An update after the insertion of a new point z^* into A can be performed by adding the componentwise maxima of the stored points with z^*. Afterwards, all points which are strictly dominated by z^* are filtered out.

Algorithm 1: Pareto frontier approximation based on ε-indicator

Input: Threshold ε_0 for the ε-indicator value
Output: Approximation $A \subseteq Z$ with $\varepsilon_d(Z, A) \leq \varepsilon_0$
$A \leftarrow \{$one arbitrary nondominated point $z \in Z\}$
repeat
  foreach *corner point* $x \in C(A)$ **do**
    Solve the optimization problem:

$$\begin{aligned} \text{lex-max } & \left(t, -\sum_{i=1}^{K} z_i\right) \\ \text{s.t. } & t \leq x_i - z_i \qquad \text{for all } i \in \{1, \ldots, K\} \\ & z \in Z,\ t \qquad\qquad\qquad\qquad\qquad\qquad \in \mathbb{R} \end{aligned}$$

    Store the pair (z, x) with obtained value $z \in Z$ and corresponding corner point x
  Set $(z^*, x^*) \coloneqq \arg\max_{(z,x)} \min\{x_1 - z_1, \ldots, x_K - z_K\}$ using the stored (z, x) pairs.
  $\varepsilon \leftarrow \min\{x_1^* - z_1^*, \ldots, x_K^* - z_K^*\}$
  $A \leftarrow A \cup \{z^*\}$
  Update the set of corner points $C(A)$.
until $\varepsilon \leq \varepsilon_0$

The following theorem establishes the convergence of the algorithm.

Theorem 4. *Assume that the ideal and nadir point of Z have finite coordinates. Then, for every bound $\varepsilon_0 > 0$ Algorithm 1 terminates in a finite number of steps with an approximation A that fulfills $\varepsilon_d(Z, A) \leq \varepsilon_0$.*

Proof. Since the algorithm only adds nondominated points to the approximation it holds $A \subseteq [z^I; z^N]$ and thus also all corner points $x \in C(A)$ are contained in the region $[z^I; z^N]$. Consider now an arbitrary iteration of the algorithm with the current approximation A. Let the corner point $x^* \in C(A)$ and the point $z^* \in Z$ be the choices which achieved the maximal value of $\min\{x_1^* - z_1^*, \ldots, x_K^* - z_K^*\}$. By definition the corner point x^* is not strictly dominated by any point in A. Hence the interior of $[z^*; x^*]$ is a subset of $(Z + \mathbb{R}^K_{\geq 0}) \setminus (A + \mathbb{R}^K_{\geq 0})$.

Consider the finite volume $V(A)$ of the set dominated by A in the relevant region given by

$$V(A) \coloneqq \operatorname{vol}\left((A + \mathbb{R}^K_{\geq 0}) \cap [z^I; z^N]\right) \leq \operatorname{vol}\left([z^I; z^N]\right) < \infty.$$

Let A' be the new approximation $A' = A \cup \{z^*\}$ in an iteration where the value of ε was larger than ε_0. Since $[z^*; x^*] \subseteq (A' + \mathbb{R}^K_{\geq 0}) \setminus (A + \mathbb{R}^K_{\geq 0})$, the volume increases in one iteration from A to A' by at least

$$V(A') - V(A) \geq \operatorname{vol}([z^*; x^*]) = (x_1^* - z_1^*) \cdots (x_K^* - z_K^*) \geq (\min\{x_1^* - z_1^*, \ldots, x_K^* - z_K^*\})^K \geq \varepsilon_0^K.$$

Thus, in each iteration where $\varepsilon_d(Z, A) \geq \varepsilon_0$ holds, the volume increases by at least ε_0^K. Since the volume is nonnegative and bounded from above by $\operatorname{vol}([z^I; z^N])$, after at most $\lceil \operatorname{vol}([z^I; z^N])/\varepsilon_0^K \rceil$ iterations the algorithm must reach a value of $\varepsilon \leq \varepsilon_0$ and will stop. By Theorem 3 this corresponds to an ε-indicator value of $\varepsilon_d(Z, A) \leq \varepsilon_0$. □

7. Numerical experiments

7.1. Configuration

For solving the mixed-integer programs formulated by our algorithm, we use Gurobi Version 8.1 (Gurobi Optimization, LLC, 2020) with default settings. The benchmarks were run on a personal laptop with an Intel i7-8665U CPU with 1.9 GHz and 16 GB RAM.

When solving the MIP of the full reformulation, better solution times can be achieved by giving a higher branching priority to the activation variables. By setting one activation variable to $a_i = 1$, all other activation variables are forced to 0. In this way, the branch-and-bound algorithm considers every part of the boundary. However some parts, where the ε-indicator is shown to be not maximal, can be pruned during the branching. Using this setting, the full reformulation outperforms the segment-wise approach.

7.2. Benchmark problem

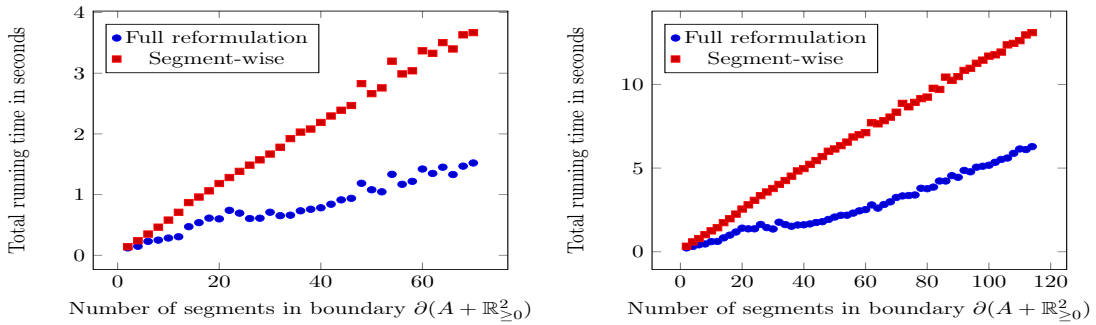
We use the benchmark problems by Mavrotas & Diakoulaki (1998) which are widely used in multicriteria mixed-integer optimization. The details for the generation are given in AppendixA. We randomly generated instances with two criteria according to this benchmark. For the benchmark we chose the following standard configuration: the number of constraints is equal to the number of variables and there are as many binary as continuous variables.

7.3. Evaluating dependence on number of segments

To evaluate the ability of our methods to compute the ε-indicator of given approximations we use the approximations generated by the Adaptive Patch Approximation Algorithm, corresponding to the problems of AppendixA. These approximations of bicriteria Pareto frontiers consist of line segments with slopes which provide a good complementary test set to the fundamentally different approximations consisting of a finite set of points generated by Algorithm 1. On each of this set of segments our methods for computing the ε-indicator are run. This allows us to analyze the dependence of the running time on the number of segments of the boundary $\partial(A + \mathbb{R}^m_{\geq 0})$.

In Figures 6a and 6b, the running times for a benchmark problem with 80 respectively 160 variables are illustrated. The running times are averages over 10 instances. The comparison shows that the full reformulation approach of Section 4.4 is in general faster than the segment-wise method of Section 4.3. For both algorithms the running time depends roughly linearly on the number of

segments in the boundary $\partial(A+\mathbb{R}^2_{\geq 0})$. The running time of the full reformulation fluctuates slightly due to effects in the used branch-and-bound method.



(a) 80 variables (b) 160 variables

Figure 6: Running times of the segment-wise approach and the full reformulation on instances with different numbers of variables.

7.4. Measuring quality of existing approximations

Our method allows to efficiently measure the quality of existing approximations to Pareto frontiers. In this way we can evaluate existing algorithms for multicriteria optimization problems. We illustrate this by computing the ε-indicator of the bicriteria approximations given as supplementary material by Boland et al. (2015) for benchmark instances which are also of the type described in AppendixA. These approximations consist of line segments, which make classical methods for computing the ε-indicator inapplicable. The objectives have been scaled such that the ideal point has coordinates $(0, 0)$ and the nadir point coordinates $(1, 1)$. In Table 1 the obtained ε-indicator values are shown. The results illustrate that the ε-indicator value is significantly larger than 0 even for algorithms that aim to compute the exact Pareto frontier.

size	**ε-indicator**	**full reformulation**	**segment-wise**	**Boland et al. (2015)**
20	0.007 055	0.21	0.51	0.60
40	0.002 941	1.01	2.99	2.76
80	0.000 656	6.76	12.00	29.08
160	0.000 242	25.64	32.23	274.54
320	0.000 321	81.97	137.77	3852.63

Table 1: Results on the approximations given by Boland et al. (2015) of bicriteria problems with varying sizes, given by the number of decision variables. The second column shows the obtained ε-indicator value. The other columns provide a comparison of the average running times in seconds for computing the ε-indicator and the original time needed for computing the approximations.

Table 1 also illustrates the running times for computing the ε-indicator by our methods and

compares them to the computation times for the original approximation provided by Boland et al. (2015). The running times of both the full reformulation and the segment-wise approach grow significantly slower with increasing problem size than the running time originally needed to compute the approximation. The comparison shows that our methods allow to evaluate the approximation quality order of magnitudes faster than the time needed for computing the evaluated approximation. Additionally, our algorithms scale well with increasing problem size.

7.5. Performance of the Pareto frontier approximation algorithm

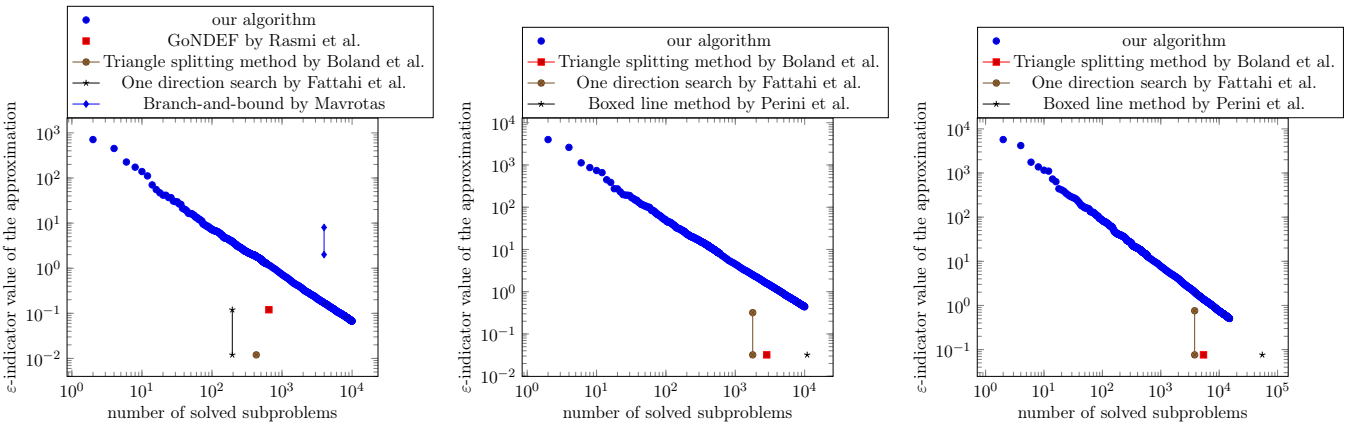
Algorithm 1 enables the approximation of Pareto frontiers with arbitrary numbers of criteria. All major problem classes can be solved with the algorithm, including mixed-integer problems with linear or nonlinear constraints, provided the scalarized subproblems are solvable. We discuss and compare results for linear mixed-integer problems since this class provides a significant challenge while enough algorithms are available for comparison in the literature.

In particular, we compare our algorithm to two alternative methods that are capable to solve mixed-integer linear programming problems with an arbitrary number of criteria: GoNDEF by Rasmi & Türkay (2019, Table 6) and the branch-and-bound method by Mavrotas & Diakoulaki (2005, Table 1)

Additionally, we perform a comparison to three algorithms that are restricted to bicriteria problems: the Boxed Line Method by Perini et al. (2020), the One Direction Search Method by Fattahi & Turkay (2018, Table 1), and the Triangle Splitting Method by Boland et al. (2015, Table 3).

Figure 7 illustrates the convergence of our algorithm on bicriteria benchmark problems measured by the achieved ε-indicator values. We compare the number of single-criteria MIPs solved as subproblems by our algorithm to the subproblems solved by other algorithms. The number of iterations of our algorithm was limited appropriately, with more iterations approximations with even better quality would be obtained. The approximation quality achieved by the other algorithms is difficult to quantify precisely, we thus use estimates based on the described parameter settings where possible. In cases where no accurate estimate can be made, we give a range for the possibly obtained ε-indicator values as a vertical segment in the plot.

With varying problem sizes our algorithm achieves a stable convergence. The comparison on these bicriteria problems with specialized algorithms for this problem type shows that our algorithm



(a) 40 variables (b) 160 variables (c) 320 variables

Figure 7: Dependence of the achieved ϵ-indicator value on the number of single-criteria MIPs solved for a bicriteria benchmark problem by Mavrotas. We used three different sizes of the benchmark problems with 40 (7a), 160 (7b) and 320 variables (7c).

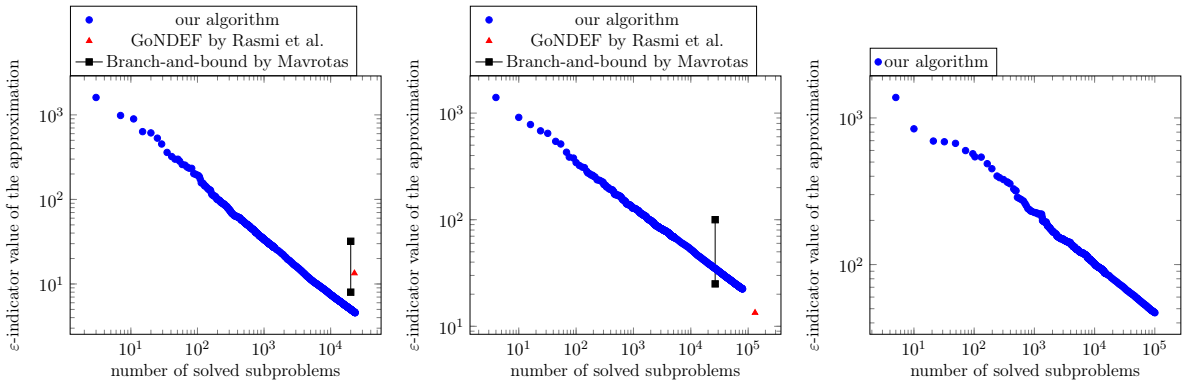
requires a larger number of subproblems to solve.

Figure 8 illustrates the convergence for multicriteria problems with three up to five criteria. We compare the results to the measurement obtained by Rasmi & Türkay (2019) with their GoNDEF algorithm and the method of Mavrotas & Diakoulaki (2005). Note that Rasmi & Türkay (2019) did not provide explicit values for the ε-indicator but their accuracy settings allow for an implicit estimate. The approximation quality of Mavrotas & Diakoulaki (2005) is estimated based on the number of nondominated points found. Our algorithm requires a similar number of subproblems to solve as the GoNDEF algorithm. The results on the benchmark problem with five criteria show that our approach is scalable to large number of criteria. Together with the precise guarantee of the achieved ε-indicator value provided by the algorithm, our method enables the precisely-controlled approximation of Pareto frontiers for every number of criteria.

In Figure 9 the number of MIPs required to solve up to different numbers of iterations are shown. Additionally, we plot the number of corner points that are existing in the approximation after each iteration. These results show that these quantities grow with an approximately linear rate in the number of iterations, with a slope depending on the number of criteria.

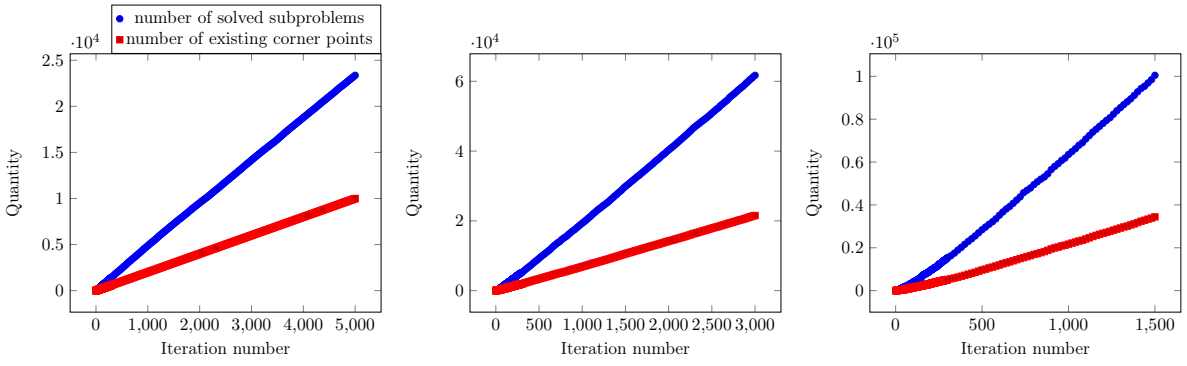
8. Conclusion and outlook

Although our result in Theorem 1 showed that computing the ε-indicator is NP-hard in the case of unlimited numbers of criteria, our developed methods are able to compute the ε-indicator



(a) $K = 3$ criteria (b) $K = 4$ criteria (c) $K = 5$ criteria

Figure 8: Dependence of the achieved ϵ-indicator value on the number of single-criteria MIPs solved (our algorithm and GoNDEF), respectively the number of nodes in the branch-and-bound-tree (for Mavrotas & Diakoulaki (2005)). We use a benchmark problems by Mavrotas with 40 variables and three to five criteria.



(a) $K = 3$ criteria (b) $K = 4$ criteria (c) $K = 5$ criteria

Figure 9: Dependence of the number of solved subproblems and the number of existing corner points on the number of iterations. The values shown were measured with instances of the benchmark problem by Mavrotas with 40 variables and varying numbers of criteria.

efficiently when the number of criteria is fixed. The numerical evaluation illustrates that with these techniques it is possible to compute the approximation quality of a Pareto frontier significantly faster than the time needed for the construction of the approximation.

This gives the opportunity to objectively compare the quality of various approximation methods for multicriteria optimization. In particular it allows to analyze the convergence behavior of algorithms with respect to this quality measure. This can form the basis of a systematic study of approximation methods and their performance on different problem classes.

We adapted our characterization of the ε-indicator to the special case of a finite approximation which lead to an algorithm for the approximation of Pareto frontiers. Since our algorithm returns with each approximation also a precise value of its corresponding ε-indicator, users are provided with an exact and easily interpretable guarantee of the approximation quality. A comparison to other state-of-the art algorithms on benchmarks with different numbers of criteria established the practical efficiency of this algorithm. In summary, we obtained an efficient algorithm for the precisely-controlled approximation of Pareto frontiers.

The observed effectiveness of our multicriteria optimization algorithm leads to some further theoretical questions regarding its analysis which should be adressed in future work:

- What is the typical growth rate of the number of corner points depending on the number of points in the approximation and the number of criteria?
- How quickly does the ε-indicator value decrease during the iterations of the algorithm for typical multicriteria problems?
- With which rate does the ε-indicator value decrease when running other approximation algorithms for multicriteria optimization problems?

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AppendixA. Benchmark problem by Mavrotas

The following classs of multicriteria linear mixed-integer problems proposed by (Mavrotas & Diakoulaki, 1998) is a widely used benchmark problem. Our notation is based on the description by (Rasmi & Türkay, 2019). Additionally, we change the optimization direction to minimization. The problem has the following form, where x and y are the decision variables and the number of criteria $K \in \mathbb{N}$ can be freely chosen:

$$\begin{aligned}
\min \quad & \left(\sum_{i=1}^{n} c_i^{(1)} x_i + \sum_{j=1}^{q} f_j^{(1)} y_j, \ldots, \sum_{i=1}^{n} c_i^{(K)} x_i + \sum_{j=1}^{q} f_j^{(K)} y_j \right) \\
\text{s. t.} \quad & \sum_{i=1}^{n} a_{ij} x_i + a'_j y_j \leq b_j && \text{for all } j = 1 \ldots, q \\
& \sum_{i=1}^{n} a_{ij} x_i \leq b_j && \text{for all } j = q+1, \ldots, m-1 \\
& \sum_{j=1}^{q} y_j \leq \frac{q}{3} \\
& x_i \in \mathbb{R}_{\geq 0} && \text{for all } i = 1, \ldots, n \\
& y_j \in \{0, 1\} && \text{for all } j = 1, \ldots, q
\end{aligned}$$

The problem consists of m constraints, n continuous variables and q binary variables. We choose the corresponding coefficients randomly independently from the following distributions:

- $c_i^{(1)}, \ldots, c_i^{(K)} \sim \mathcal{U}([-10, 10])$, $i = 1, \ldots, n$,
- $f_j^{(1)}, \ldots, f_j^{(K)} \sim \mathcal{U}([-200, 200])$, $j = 1, \ldots, q$,
- $b_j \sim \mathcal{U}([50, 100])$, $j = 1, \ldots, m-1$,
- $a_{ij}, a'_j \sim \mathcal{U}([-1, 20]) \circ B(0.25)$,

where $\mathcal{U}$ denotes the uniform distribution on a set and $\mathcal{U}([-1, 20]) \circ B(0.25)$ represents the distribution with values having a value of 0 with a probability of 0.75 and otherwise uniform values in $[-1, 20]$.