RISK-AVERSE MULTISTAGE STOCHASTIC PROGRAMS WITH EXPECTED CONDITIONAL RISK MEASURES*

MARYAM KHATAMI†, THUENER SILVA‡, BERNARDO K. PAGNONCELLI §, AND LEWIS NTAIMO ¶

Abstract. We study decomposition algorithms for risk-averse multistage stochastic programs with expected conditional risk measures (ECRMs). ECRMs are attractive because they are time-consistent, which means that a plan made today will not be changed in the future if the problem is re-solved given a realization of the random variables. We show that solving risk-averse problems based on ECRMs is as complex as solving risk-neutral ones. We consider ECRMs for both quantile and deviation mean-risk measures, deriving the Bellman equations in each case. We illustrate our results with extensive numerical computations for problems from two applications: hydrothermal scheduling and portfolio selection. The results show that the ECRM approach provides higher expected costs in the early stages to hedge against cost spikes in later stages for the hydrothermal scheduling problem. For the portfolio selection problem, the new approach gives well-diversified portfolios over time. Overall, the ECRM approach provides superior performance over the risk-neutral approach under extreme scenario conditions.

Key words. Multistage stochastic programming, decomposition algorithms, expected conditional risk measures

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1. Introduction.

The presence of uncertainty in real-life problems has been widely recognized, and extensive research has been developed to incorporate such randomness into the decision-making process. Customer demand in production planning and supply chain management, rainfall in hydrothermal scheduling, and the return on assets in financial problems are only a few examples of random elements that should not be left out of a realistic model. In the operations research literature, there are several approaches to deal with such problems such as simulation-optimization [16] and robust optimization [5]. In this work, we focus on stochastic programming (SP), a framework whose defining characteristic is the assumption of full knowledge of the distribution of the random parameters [6, 34].

SP is a very popular approach to cope with uncertainty within an optimization model, and its tools have been applied to a wide variety of practical problems [39]. The classic two-stage SP problem with recourse, which was established with the early works [10, 38], has been used in recent decades for applications in transportation planning [3], health care [11] and energy management [25], among others.

In classical two-stage stochastic programming (TSSP), the decision-maker aims at minimizing expected costs by selecting a first-stage decision based on the knowledge of the distribution of the parameters. Recourse actions taken in the second stage aim at minimizing the expected cost incurred by the first-stage decision as a result of deviations from the goals or targets. Minimizing costs on average is not the only possible goal: in finance the decision-maker might want to avoid periods with extreme losses, or in health care management more stable solutions that take into account the patients' welfare may be desirable. In those situations, it is natural to include risk

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†Department of Operations and Decision Technologies, Indiana University, Bloomington, IN.

‡Department of Industrial Engineering, Pontifical Catholic University of Rio de Janeiro, Brazil.

§Escuela de Negocios, Universidad Adolfo Ibáñez, Santiago, Chile.

¶Department of Industrial and Systems Engineering, Texas A&M University, College Station, TX (ntaimo@tamu.edu).

aversion in the problem with the goal of having less variability in the solution. While there is still active research in the area, risk-averse TSSP is well understood (see for example [22] and [34]), and the resulting problems can be efficiently solved in practice [24], [36]. Several different risk measures are available in the literature, and tractable formulations exist to incorporate them into TSSP [1], [9], [32].

Some problems are inherently dynamic, in the sense that sequential decisions need to be implemented to properly capture the essential elements of the model. For example, in hydrothermal scheduling problems it is probably not optimal to use all the water available in the reservoirs in the first period because having to rely solely on thermal plants on subsequent stages would be too expensive. For those situations, multistage stochastic programming (MSSP) is an adequate framework since at every stage the decision-maker has to optimize present plus expected future costs, and the goal is to minimize costs across the entire horizon under consideration.

MSSP problems are challenging to solve. One of the most popular methods is the stochastic dual dynamic programming (SDDP) algorithm [28], which was developed to solve risk-neutral, hydrothermal scheduling problems where the goal is to minimize expected operation costs. The algorithm has been widely adopted by academia and industry, and it is one of the most efficient techniques to deal with large-scale problems with hundreds of stages.

Unlike the TSSP case, risk-averse extensions to MSSP are much more challenging. The majority of the papers in the literature of risk averse MSSP uses a nested formulation of the problem and replaces the expected value by some risk measure. The works [19], [30] and [35] use a nested formulation to include risk aversion into MSSP, and consider a convex combination of Conditional Value-at-risk (CVaR) and expected value as the risk measure.[1] Those three papers use some form of SDDP to solve the resulting problem.

Their approach, albeit valid, is problematic for two reasons. First, in a nested formulation it is very hard to interpret what is being measured, that is, how exactly risk is being captured by a sequence of nested CVaRs. Second, and most important for our work, the application of SDDP is not as simple as the risk neutral (expected value) case. In particular, an upper bound for the optimal value is not readily available. In [30] an inner approximation for evaluating the upper bound is proposed, but the construction is computationally expensive. In [19] a method that depends on both the risk measure and on specific characteristics of the problem is derived. Finally, in [35] the authors use a stopping criteria based on the stabilization of the lower bound.

Other approaches include [26], where the authors apply mean-risk multistage stochastic integer programs to optimize the flu shot design for seasonal influenza. Risk measures used to formulate this problem include the absolute semideviation and the expected shortage. A branch-and-price algorithm is developed to solve the models. Finally, in [4] the authors use a minimax approach and propose a global, end-of-horizon risk measure that can be implemented in practice.

A different approach, which was originally proposed in [29], involves using the conditional expectation of a risk measure at every stage. In the original paper the authors consider only the CVaR, and in [18] the measure is called $m-$CVaR. In [17] the authors extend the original definition by allowing risk measures that are translation-invariant and monotone, and called them Expected Conditional Risk Measures (ECRMs). ECRMs are attractive because they are *time-consistent*, that is, a plan made today will not be changed in the future if the problem is re-solved given a

[1] The work [35] considers additionally a mean-upper semideviation risk measure.

realization of the random elements within the formulation. Moreover, ECRMs have a particular nested structure which is amenable to decomposition algorithms. In [17] a pension fund problem for the CVaR case is formulated and solved, but no decomposition algorithm is proposed to solve the problem because the problem has a deterministic equivalent formulation that is tractable and is solved to optimality.

Our first contribution to the theory for stochastic programming is the introduction of the risk-neutral convertible (RNC) property. In words, a risk-averse formulation has this property if it can be converted to a risk-neutral MSSP. The concept is related—but different—to the polyhedral risk measures proposed in [14], which are defined as optimal values of RNC problems. We will show that ECRMs satisfy this property, which makes it possible to use standard solution methods such as the SDDP algorithm exactly as it is done in the risk neutral case to solve the problem.

The second contribution is providing explicit Bellman equations for the ECRMs that we propose in this work, which are based on the expected excess, quantile deviation, conditional value-at-risk, and absolute semideviation. In each case, we are able to obtain the corresponding risk-averse MSSP formulation in a recursive form, and since they are RNC such measures can be used within the SDDP algorithm.

The final contribution is a report on extensive numerical experiments of ECRMs in a large-scale hydrothermal scheduling problem with 84 stages, and a portfolio selection problem with 16 stages and five industry sectors. We show the effects of different risk measures in the solution of each of those problems, and discuss some managerial insights that can be extracted from the results.

The rest of the paper is organized as follows: We cover the preliminaries including an overview of deviation and quantile risk measures and definition of coherence in section 2. We derive Bellman equations for MSSPs with ECRM in section 3 based on the risk measures discussed in section 2. In section 4 we introduce two applications, a hydrothermal scheduling and a portfolio selection problem, and report on the computational results of implementing MSSPs with ECRM on the two applications in section 5. We end the paper with some concluding remarks in section 6.

2. Preliminaries.

In this section, we discuss two important classes of risk measures and provide examples of each class. For simplicity, we present static versions of the risk measures, and discuss them in the multistage context later. In all examples presented in this section, the risk measures are functions from L^1 to the real numbers, where

$$L^1 := \{Y \mid Y \text{ is a random variable with } \mathbb{E}[Y] < \infty\}.$$

There are several different ways of categorizing risk measures, either by their theoretical properties (polyhedral, distortion, coherent) or by the type of application one is interested in (deviation measures, quantile measures, drawdown measures, etc.). In this discussion, we focus on deviation risk measures and quantile risk measures.

2.1. Deviation risk measures.

Deviation risk measures capture the average deviation of outcomes with respect to a certain threshold. Our first example is the expected excess (EE), which reflects the expectation of the excess over a given target. As before, $Y \in L^1$ and let $\zeta \in \mathbb{R}$ be a target defined by the decision-maker. Then, EE is defined as

$$\mathrm{EE}_\zeta(Y) = \mathbb{E}[(Y - \zeta)_+].$$

Similar to EE, the absolute semideviation (ASD) reflects the expectation over a target. However, for ASD the target is the average of the random variable under consideration:

$$\mathrm{ASD}(Y) = \mathbb{E}[(Y - \mathbb{E}(Y))_+].$$

2.2. Quantile Risk Measures. Quantile risk measures capture the average deviation of outcomes with respect to a specified quantile. In this work we focus on two quantile measures, Conditional Value-at-Risk (CVaR) and Quantile Deviation (QDEV). Given $\alpha \in (0,1)$, and a random variable $Y \in L^1$, CVaR is defined as:

$$\mathrm{CVaR}_\alpha(Y) = \operatorname*{Min}_{\eta\in\mathbb{R}} \left\{ \eta + \frac{1}{1-\alpha}\mathbb{E}(Y-\eta)_+ \right\}, \tag{2.1}$$

where $(Y-\eta)_+ = \max\{Y-\eta, 0\}$. CVaR reflects the conditional expectation of the $(1-\alpha)100\%$ worst outcomes for a given level $\alpha \in (0,1)$. An optimal solution η^* in problem (2.1) is the α Value-at-Risk of the random variable Y [32].

Given $\alpha \in (0,1)$ and $\epsilon_1, \epsilon_2 > 0$ such that $\alpha = \epsilon_2/(\epsilon_1+\epsilon_2)$ and a random variable $Y \in L^1$, the quantile deviation (QDEV) is defined as

$$\mathrm{QDEV}_{\epsilon_1,\epsilon_2}(Y) = \operatorname*{Min}_{\eta} \mathbb{E}(\epsilon_1[\eta - Y]_+ + \epsilon_2[Y-\eta]_+).$$

Unlike CVaR, QDEV is a two-sided quantile risk measure which can capture expected deviation of a random variable both above and below the η level.

When we move to an optimization setting, it is useful to consider the mean-risk versions of those measures. In general mean-risk problems can be written as

$$\operatorname*{Min}_{x\in X} \quad \mathbb{E}[f(x,\tilde{\omega})] + \lambda\mathbb{D}[f(x,\tilde{\omega})], \tag{2.2}$$

where $f(\cdot,\tilde{\omega})$ is assumed to be convex, and $f(x,\cdot)$ has finite expectation. The risk measure $\mathbb{D}$ in formulation (2.2) can be replaced by any of the risk measures we presented, transforming the objective into a combination of expected value and risk.

2.3. Coherent Risk Measures. The properties of coherence were originally proposed by [2] in an attempt to make explicit desirable characteristics a risk measure should have. The properties are also guided by tractability, since it is important in practice to be able to solve a problem with a given risk measure. For completeness, we state here the axioms of coherence:

- Axiom 1. *Translation invariance:*
  If $a \in \mathbb{R}$ and $Y \in L^1$, then $\rho(Y+a) = a + \rho(Y)$.
- Axiom 2. *Positive homogeneity:*
  If $c \geq 0$ and $Y \in L^1$, then $\rho(cY) = c\rho(Y)$.
- Axiom 3. *Monotonicity:*
  If $Y_1 \leq Y_2$, with $Y_1, Y_2 \in L^1$, then $\rho(Y_1) \leq \rho(Y_2)$.
- Axiom 4. *Convexity:*
  If $\lambda \in (0,1)$ and $Y_1, Y_2 \in L^1$, then $\rho(\lambda Y_1 + (1-\lambda)Y_2) \leq \lambda\rho(Y_1) + (1-\lambda)\rho(Y_2)$.

Tables 1 and 2 summarize the properties that quantile and deviation risk measures satisfy, both as stand-alone measures and as mean-risk measures, as described in (2.2). To differentiate the pure risk measure from its mean counterpart, we add a letter 'M' before the abbreviation to indicate mean-risk. For example, mean expected excess is denoted MEE and mean ASD as MASD.

TABLE 1
Properties of Risk Measures.

Risk measure	Translation invariance	Positive homogeneity	Monotonicity	Convexity
EE	✗	✗	✓	✓
ASD	✗	✓	✗	✓
QDEV	✗	✓	✗	✓
CVaR	✓	✓	✓	✓

TABLE 2
Properties of Mean Risk Measures.

Mean-risk measure	Translation invariance	Positive homogeneity	Monotonicity	Convexity
MEE	✗	✗	✓	✓
MASD	✓	✓	✓	✓
MQD	✓	✓	✗	✓
MCVaR	✓	✓	✓	✓

3. Multistage Stochastic Programming with ECRM. Following the notation in [17], let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots \subset \mathcal{F}_T$ be sub sigma-algebras of $\mathcal{F}$, with $\mathcal{F}_1 = \{\emptyset, \Omega\}$, and $\mathcal{F}_T = \mathcal{F}$. Spaces $\mathcal{Z}_t$ are composed of $\mathcal{F}_t$-measurable functions from Ω to $\mathbb{R}$, and $\mathcal{Z} = \mathcal{Z}_1 \times \ldots \times \mathcal{Z}_T$. A multiperiod risk function $\mathbb{F}$ is simply a function from $\mathcal{Z}$ to $\mathbb{R}$. Our focus in this work is to study multiperiod risk measures in the context of MSSP. Our object of study is the ECRM, which is defined as follows:

$$\begin{aligned}(3.1)\qquad \mathbb{F}(Z_1, \cdots, Z_T) = & Z_1 + \rho_2(Z_2) + \mathbb{E}_{\xi_2}[\rho_3^{\xi_{[2]}}(Z_3)] + \mathbb{E}_{\xi_{[3]}}[\rho_4^{\xi_{[3]}}(Z_4)] + \cdots + \\ & \mathbb{E}_{\xi_{[T-1]}}[\rho_T^{\xi_{[T-1]}}(Z_T)],\end{aligned}$$

where $Z_t \in \mathcal{Z}_T$ and the subscript in $\mathbb{E}$ indicates that the expectation is with respect to the corresponding variables, and dependence on the whole history of the process up to time t is denoted by $\xi_{[t]}$. Risk measure $\rho_t^{\xi_{[t-1]}}(Z_t)$ is a one-period conditional risk measure in stage t, given the history of the process up to time $t-1$. Unlike the nested approach where the risk measure in every stage t captures uncertainty from stage t up to the last stage, evaluation of the risk in ECRM is stage-based.

Using the law of total expectation, equation (3.1) is equivalent to

$$\begin{aligned}(3.2)\qquad \mathbb{F}(Z_1, \cdots, Z_T) = & Z_1 + \rho_2(Z_2) + \mathbb{E}_{\xi_2}[\rho_3^{\xi_{[2]}}(Z_3) + \mathbb{E}_{\xi_3}^{\xi_{[2]}}[\rho_4^{\xi_{[3]}}(Z_4) + \cdots + \\ & \mathbb{E}_{\xi_{T-1}}^{\xi_{[T-2]}}[\rho_T^{\xi_{[T-1]}}(Z_T)]]],\end{aligned}$$

where the superscript in the expectation denotes that it is conditional on some random variable. In the next subsections, we use equation (3.2) as a basis to derive dynamic programming formulations for risk-averse MSSP problems with ECRMs.

3.1. Bellman Formulations for ECRMs. This subsection presents the explicit Bellman formulations for ECRMs when the inner risk measure in (3.1) is one of the risk measures presented in Section 2. In all cases we use the mean-risk version of those measures because they are more general than the measures themselves, and as shown in Table 2, they have more desirable properties in most cases.

3.1.1. ECRM with Mean Expected Excess. Using equation (3.2), the optimization problem formulation for ECRM with mean expected excess $\mathbb{E}$-MEE is written as follows:

$$\begin{aligned} \text{Min } & g_1(x_1) + \text{MEE}(g_2(x_2,\xi_2)) + \mathbb{E}_{\xi_2}[\text{MEE}^{\xi_{[2]}}(g_3(x_3,\xi_3)) + \mathbb{E}_{\xi_3}^{\xi_{[2]}}[\text{MEE}^{\xi_{[3]}}(g_4(x_4,\xi_4)) \\ & + \cdots + \mathbb{E}_{\xi_{T-1}}^{\xi_{[T-2]}}[\text{MEE}^{\xi_{[T-1]}}(g_T(x_T,\xi_T))|\xi_{[T-2]}]\cdots|\xi_{[2]}]] \\ \text{s.t. } & x_t \in \mathcal{X}_t(x_{[t-1]},\xi_{[t]}), \quad t = 1,\cdots,T, \end{aligned} \tag{3.3}$$

where $\mathcal{X}_t(x_{[t-1]},\xi_{[t]})$ represents the feasible set at every stage $t = 1,\ldots,T$. Using $\text{MEE}(Z) = \mathbb{E}(Z) + \lambda\mathbb{E}[(Z-\zeta)_+] = \mathbb{E}[Z + \lambda(Z-\zeta)_+]$, formulation (3.3) is rewritten as follows:

$$\begin{aligned} \text{Min } & g_1(x_1) + \mathbb{E}_{\xi_2}[g_2(x_2,\xi_2) + \lambda_2(g_2(x_2,\xi_2) - \zeta_2)_+] + \\ & \mathbb{E}_{\xi_2}[\mathbb{E}_{\xi_3}^{\xi_{[2]}}[g_3(x_3,\xi_3) + \lambda_3(g_3(x_3,\xi_3) - \zeta_3)_+] \\ & \quad + \mathbb{E}_{\xi_3}^{\xi_{[2]}}[\mathbb{E}_{\xi_4}^{\xi_{[3]}}[g_4(x_4,\xi_4) + \lambda_4(g_4(x_4,\xi_4) - \zeta_4)_+] + \cdots \\ & \quad + \mathbb{E}_{\xi_{T-1}}^{\xi_{[T-2]}}[\mathbb{E}_{\xi_T}^{\xi_{[T-1]}}[g_T(x_T,\xi_T) + \lambda_T(g_T(x_T,\xi_T) - \zeta_T)_+]]\cdots]] \\ \text{s.t. } & x_t \in \mathcal{X}_t(x_{[t-1]},\xi_{[t]}), \quad t = 1,\cdots,T. \end{aligned} \tag{3.4}$$

Note that each stage t has its own target ζ_t, which represents the decision-maker's target for the immediate cost.

Let us now write Bellman equations for the problem. The formulation can be written as

$$\begin{aligned} \underset{x_1}{\text{Min}} \ & g_1(x_1) + \mathbb{E}_{\xi_2}[Q_2(x_1,\xi_2)] \\ \text{s.t. } & x_1 \in \mathcal{X}_1, \end{aligned} \tag{3.5}$$

where for $t \in \{2,\cdots,T-1\}$ we have

$$\begin{aligned} Q_t(x_{t-1},\xi_{[t]}) = \underset{x_t}{\text{Min}} \ & g_t(x_t,\xi_t) + \lambda_t(g_t(x_t,\xi_t) - \zeta_t)_+ + \mathbb{E}_{\xi_{t+1}}[Q_{t+1}(x_t,\xi_{t+1})|\xi_{[t]}] \\ \text{s.t. } & x_t \in \mathcal{X}_t(x_{[t-1]},\xi_{[t]}), \end{aligned} \tag{3.6}$$

and for $t = T$ the problem can be written as

$$\begin{aligned} Q_T(x_{T-1},\xi_T) = \underset{x_T}{\text{Min}} \ & g_T(x_T,\xi_T) + \lambda_T(g_T(x_T,\xi_T) - \zeta_T)_+ \\ \text{s.t. } & x_T \in \mathcal{X}_T(x_{[T-1]},\xi_{[T]}). \end{aligned} \tag{3.7}$$

To linearize the deviation function $(g_t(x_t,\tilde{\xi}_t) - \zeta_t)_+$, we use auxiliary variables $u_t, t \in \{2,\cdots,T-1\}$:

$$\begin{aligned} Q_t(x_{t-1},\xi_{[t]}) = \underset{x_t,u_t}{\text{Min}} \ & g_t(x_t,\xi_t) + \lambda_t u_t + \mathbb{E}_{\xi_{t+1}}[Q_{t+1}(x_t,\xi_{t+1})|\xi_{[t]}] \\ \text{s.t. } & x_t \in \mathcal{X}_t(x_{[t-1]},\xi_{[t]}) \\ & u_t \geq g_t(x_t,\xi_t) - \zeta_t, \\ & u_t \geq 0. \end{aligned}$$

A similar substitution is done for the last stage.

3.1.2. ECRM with Mean Quantile Deviation. We now present the multistage formulation for ECRM with mean quantile deviation $\mathbb{E}$-MQD:

$$
\begin{aligned}
&\underset{x_1,\cdots,x_T}{\text{Min}}\ g_1(x_1) + \text{MQD}(g_2(x_2,\xi_2)) + \mathbb{E}_{\xi_2}[\text{MQD}^{\xi_{[2]}}(g_3(x_3,\xi_3))\\
&\qquad + \mathbb{E}_{\xi_3}^{\xi_{[2]}}[\text{MQD}^{\xi_{[3]}}(g_4(x_4,\xi_4))\\
&\qquad + \cdots + \mathbb{E}_{\xi_{T-1}}^{\xi_{[T-2]}}[\text{MQD}^{\xi_{[T-1]}}(g_T(x_T,\xi_T))|\xi_{[T-2]}]\cdots|\xi_{[2]}]]\\
&\text{s.t. } x_t \in \mathcal{X}_t(x_{[t-1]},\xi_{[t]}), \quad t = 1,\cdots,T.
\end{aligned}
\tag{3.8}
$$

Given $\alpha \in (0,1)$ and $\epsilon_1, \epsilon_2 \geq 0$ such that $\alpha = \epsilon_2/(\epsilon_1+\epsilon_2)$, the mean quantile deviation can be written as

$$
\begin{aligned}
\text{MQD}(Z) &= \mathbb{E}(Z) + \lambda \underset{\eta\in\mathbb{R}}{\text{Min}}\{\mathbb{E}[\epsilon_1(\eta - Z)_+ + \epsilon_2(Z-\eta)_+]\}\\
&= \underset{\eta\in\mathbb{R}}{\text{Min}}\{\mathbb{E}[Z + \lambda\epsilon_1(\eta - Z)_+ + \lambda\zeta_2(Z-\eta)_+]\}\\
&= \underset{\eta\in\mathbb{R}}{\text{Min}}\{\lambda\epsilon_1\eta + (1-\lambda\epsilon_1)\mathbb{E}[Z] + \lambda(\epsilon_1+\epsilon_2)\mathbb{E}[(Z-\eta)_+]\}\\
&= \underset{\eta\in\mathbb{R}}{\text{Min}}\{\lambda\epsilon_1\eta + \mathbb{E}[(1-\lambda\epsilon_1)Z + \lambda(\epsilon_1+\zeta_2)(Z-\eta)_+]\}.
\end{aligned}
$$

Substituting the last expression in problem (3.8) gives

$$
\begin{aligned}
&\underset{x_1}{\text{Min}}\ g_1(x_1) + \underset{\eta_2,x_2}{\text{Min}}\{\lambda_2\epsilon_1\eta_2 + \mathbb{E}_{\xi_2}[(1-\lambda_2\epsilon_1)g_2(x_2,\xi_2)\\
&\quad + \lambda_2(\epsilon_1+\epsilon_2)(g_2(x_2,\xi_2)-\eta_2)_+]\}\\
&\quad + \mathbb{E}_{\xi_2}[\underset{\eta_3,x_3}{\text{Min}}\{\lambda_3\epsilon_1\eta_3 + \mathbb{E}_{\xi_3}^{\xi_{[2]}}[(1-\lambda_3\epsilon_1)g_3(x_3,\xi_3) + \lambda_3(\epsilon_1+\epsilon_2)(g_3(x_3,\xi_3)-\eta_3)_+]\}\\
&\quad + \mathbb{E}_{\xi_3}^{\xi_{[2]}}[\underset{\eta_4,x_4}{\text{Min}}\{\lambda_4\epsilon_1\eta_4 + \mathbb{E}_{\xi_4}^{\xi_{[3]}}[(1-\lambda_4\epsilon_1)g_4(x_4,\xi_4) + \lambda_4(\epsilon_1+\epsilon_2)(g_4(x_4,\xi_4)-\eta_4)_+]\}\\
&\quad + \cdots + \mathbb{E}_{\xi_{T-1}}^{\xi_{[T-2]}}[\underset{\eta_T,x_T}{\text{Min}}\{\lambda_T\epsilon_1\eta_T + \mathbb{E}_{\xi_T}^{\xi_{[T-1]}}[(1-\lambda_T\epsilon_1)g_T(x_T,\xi_T)\\
&\quad + \lambda_T(\epsilon_1+\epsilon_2)(g_T(x_T,\xi_T)-\eta_T)_+]\}]\cdots]]\\
&\text{s.t. } x_t \in \mathcal{X}_t(x_{[t-1]},\xi_{[t]}), \quad t = 1,\ldots,T.
\end{aligned}
$$

Let us now write Bellman equations for the problem. The formulation can be written as

$$
\begin{aligned}
&\underset{x_1,\eta_2}{\text{Min}}\ g_1(x_1) + \lambda_2\epsilon_1\eta_2 + \mathbb{E}_{\xi_2}[Q_2(x_1,\xi_2,\eta_2)]\\
&\qquad \text{s.t. } x_1 \in \mathcal{X}_1,
\end{aligned}
\tag{3.9}
$$

where for $t \in \{2,\cdots,T-1\}$ we have

$$
\begin{aligned}
Q_t(x_{t-1},\xi_{[t]},\eta_t) = &\underset{x_t,\eta_{t+1}}{\text{Min}}\ (1-\lambda_t\epsilon_1)g_t(x_t,\xi_t) + \lambda_t(\epsilon_1+\epsilon_2)(g_t(x_t,\xi_t)-\eta_t)_+\\
&+ \lambda_{t+1}\epsilon_1\eta_{t+1} + \mathbb{E}_{\xi_{t+1}}^{\xi_{[t]}}[Q_{t+1}(x_t,\xi_{t+1},\eta_{t+1})]\\
&\text{s.t. } \eta_{t+1}\in\mathbb{R},\ x_t \in \mathcal{X}_t(x_{[t-1]},\xi_{[t]}),
\end{aligned}
\tag{3.10}
$$

and for $t = T$ the problem can be written as

$$
\begin{aligned}
Q_T(x_{T-1},\xi_T,\eta_T) = &\underset{x_T}{\text{Min}}\ (1-\lambda_T\epsilon_1)g_T(x_T,\xi_T) + \lambda_T(\epsilon_1+\epsilon_2)(g_T(x_T,\xi_T)-\eta_T)_+\\
&\text{s.t. } x_T \in \mathcal{X}_t(x_{[T-1]},\xi_{[T]}).
\end{aligned}
\tag{3.11}
$$

To linearize the deviation function $(g_t(x_t,\xi_t)-\eta_t)_+$, we use auxiliary variables $u_t, t \in \{2,\cdots,T-1\}$:

$$
\begin{aligned}
&Q_t(x_{t-1},\xi_{[t]},\eta_t) = \underset{x_t,\eta_{t+1},u_t}{\text{Min}} \ (1-\lambda_t\epsilon_1)g_t(x_t,\xi_t) + \lambda_t(\epsilon_1+\epsilon_2)u_t + \lambda_{t+1}\epsilon_1\eta_{t+1} + \\
&\mathbb{E}^{\xi_{[t]}}_{\xi_{t+1}}[Q_{t+1}(x_t,\xi_{t+1},\eta_{t+1})] \\
&\ \text{s.t. } u_t \geq g_t(x_t,\xi_t) - \eta_t, \\
&\qquad \eta_{t+1} \in \mathbb{R}, u_t \geq 0, \ x_t \in \mathcal{X}_t(x_{[t-1]},\xi_{[t]}).
\end{aligned}
$$

A similar substitution is done for the last stage.

3.1.3. ECRM with Mean Conditional Value-at-Risk. In [17] the authors present ECRM for CVaR. We now extend the derivation for the mean-CVaR:

$$
\begin{aligned}
&\underset{x_1,\cdots,x_T}{\text{Min}} \ g_1(x_1) + \text{MCVaR}(g_2(x_2,\xi_2)) + \mathbb{E}_{\xi_2}[\text{MCVaR}^{\xi_{[2]}}(g_3(x_3,\xi_3)) + \\
&\mathbb{E}^{\xi_{[2]}}_{\xi_3}[\text{MCVaR}^{\xi_{[3]}}(g_4(x_4,\xi_4)) + \cdots + \mathbb{E}^{\xi_{[T-2]}}_{\xi_{T-1}}[\text{MCVaR}^{\xi_{[T-1]}}(g_T(x_T,\xi_T))|\xi_{[T-2]}]\cdots|\xi_{[2]}]] \\
&\ \text{s.t. } x_t \in \mathcal{X}_t(x_{[t-1]},\xi_{[t]}), \quad t=1,\ldots,T.
\end{aligned}
\tag{3.12}
$$

Before writing the Bellman equations we express the mean-CVaR in the following form:

$$
\begin{aligned}
&\text{MCVaR}_\alpha(Z) = (1-\lambda)\mathbb{E}(Z) + \lambda \underset{\eta\in R}{\text{Min}}\{\eta + \frac{1}{(1-\alpha)}\mathbb{E}[(Z-\eta)_+]\} = \\
&\underset{\eta\in R}{\text{Min}}\{\lambda\eta + \mathbb{E}[(1-\lambda)Z + \frac{\lambda}{(1-\alpha)}(Z-\eta)_+]\}.
\end{aligned}
$$

The optimization problem formulation (3.12) is equivalent to the following formulation:

$$
\begin{aligned}
&\underset{x_1}{\text{Min}} \ g_1(x_1) + \underset{\eta_2,x_2}{\text{Min}}\{\lambda_2\eta_2 + \mathbb{E}_{\xi_2}[(1-\lambda_2)g_2(x_2,\xi_2) + \frac{\lambda_2}{(1-\alpha)}(g_2(x_2,\xi_2)-\eta_2)_+]\} \\
&\quad + \mathbb{E}_{\xi_2}[\underset{\eta_3,x_3}{\text{Min}}\{\lambda_3\eta_3 + \mathbb{E}^{\xi_{[2]}}_{\xi_3}[(1-\lambda_3)g_3(x_3,\xi_3) + \frac{\lambda_3}{(1-\alpha)}(g_3(x_3,\xi_3)-\eta_3)_+]\} \\
&\quad + \mathbb{E}^{\xi_{[2]}}_{\xi_3}[\underset{\eta_4,x_4}{\text{Min}}\{\lambda_4\eta_4 + \mathbb{E}^{\xi_{[3]}}_{\xi_4}[(1-\lambda_4)g_4(x_4,\xi_4) + \frac{\lambda_4}{(1-\alpha)}(g_4(x_4,\xi_4)-\eta_4)_+]\} + \cdots \\
&\quad + \mathbb{E}^{\xi_{[T-2]}}_{\xi_{T-1}}[\underset{\eta_T,x_T}{\text{Min}}\{\lambda_T\eta_T + \mathbb{E}^{\xi_{[T-1]}}_{\xi_T}[(1-\lambda_T)g_T(x_T,\xi_T) \\
&\quad + \frac{\lambda_T}{(1-\alpha)}(g_T(x_T,\xi_T)-\eta_T)_+]\}]\cdots]] \\
&\ \text{s.t. } x_t \in \mathcal{X}_t(x_{[t-1]},\xi_{[t]}), \quad t=1,\ldots,T.
\end{aligned}
$$

Let us now write Bellman equations for the problem. The formulation can be written as

$$
\begin{aligned}
&\underset{x_1,\eta_2}{\text{Min}} \ g_1(x_1) + \lambda_2\eta_2 + \mathbb{E}_{\xi_2}[Q_2(x_1,\xi_2,\eta_2)] \\
&\qquad \text{s.t. } x_1 \in \mathcal{X}_1,
\end{aligned}
\tag{3.13}
$$

where for $t \in \{2, \cdots, T-1\}$ we have

(3.14)

$$
\begin{aligned}
Q_t(x_{t-1}, \xi_{[t]}, \eta_t) = \underset{x_t, \eta_{t+1}}{\operatorname{Min}} \ & \lambda_{t+1}\eta_{t+1} + (1-\lambda_t) g_t(x_t, \tilde{\xi}_t) + \frac{\lambda_t}{(1-\alpha)} (g_t(x_t, \tilde{\xi}_t) - \eta_t)_+ \\
& + \mathbb{E}^{\xi_{[t]}}_{\xi_{t+1}} [Q_{t+1}(x_t, \xi_{t+1}, \eta_{t+1})] \\
& \text{s.t. } \eta_{t+1} \in \mathbb{R}, \ x_t \in \mathcal{X}_t(x_{[t-1]}, \xi_{[t]}),
\end{aligned}
$$

and for $t = T$ the problem can be written as

$$
\begin{aligned}
Q_T(x_{T-1}, \tilde{\xi}_T, \eta_T) = & \underset{x_T}{\operatorname{Min}} \ (1-\lambda_T) g_T(x_T, \tilde{\xi}_T) + \frac{\lambda_T}{(1-\alpha)} (g_T(x_T, \tilde{\xi}_T) - \eta_T)_+ \\
& \text{s.t. } x_T \in \mathcal{X}_t(x_{[T-1]}, \xi_{[T]}).
\end{aligned} \tag{3.15}
$$

To linearize the deviation function $(g_t(x_t, \tilde{\xi}_t) - \eta_t)_+$, we use auxiliary variables $u_t, t \in \{2, \cdots, T-1\}$:

$$
\begin{aligned}
Q_t(x_{t-1}, \xi_{[t]}, \eta_t) = \underset{x_t, \eta_{t+1}}{\operatorname{Min}} \ & \lambda_{t+1}\eta_{t+1} + (1-\lambda_t) g_t(x_t, \tilde{\xi}_t) + \frac{\lambda_t}{(1-\alpha)} u_t \\
& + \mathbb{E}^{\xi_{[t]}}_{\xi_{t+1}} [Q_{t+1}(x_t, \xi_{t+1}, \eta_{t+1})] \\
\text{s.t.} \ & u_t \geq g_t(x_t, \tilde{\xi}_t) - \eta_t, \\
& u_t \geq 0, \\
& \eta_{t+1} \in \mathbb{R}, \ x_t \in \mathcal{X}_t(x_{[t-1]}, \xi_{[t]}).
\end{aligned} \tag{3.16}
$$

A similar reformulation is done for the last stage.

Remark: As Bellman equation (3.14) shows, η_t is known in stage t since it is an argument for function Q_t. Therefore, it is a decision variable in stage $t-1$ and thus a parameter in stage t.

3.1.4. ECRM with Mean Absolute Semideviation.

The optimization problem formulation for the ECRM with mean absolute semideviaton $\mathbb{E}$-MASD can be written as follows:

$$
\begin{aligned}
\operatorname{Min} \ & g_1(x_1) + \operatorname{MASD}(g_2(x_2, \xi_2)) + \mathbb{E}_{\xi_2}[\operatorname{MASD}^{\xi_{[2]}}(g_3(x_3, \xi_3)) \\
& + \mathbb{E}^{\xi_{[2]}}_{\xi_3}[\operatorname{MASD}^{\xi_{[3]}}(g_4(x_4, \xi_4)) \\
& + \cdots + \mathbb{E}^{\xi_{[T-2]}}_{\xi_{T-1}}[\operatorname{MASD}^{\xi_{[T-1]}}(g_T(x_T, \xi_T)) | \xi_{[T-2]}] \cdots | \xi_{[2]}]] \\
\text{s.t. } & x_t \in \mathcal{X}_t(x_{[t-1]}, \xi_{[t]}), \quad t = 1, \dots, T.
\end{aligned} \tag{3.17}
$$

We can rewrite MASD as follows:

$$
\begin{aligned}
\operatorname{MASD}(Z) &= \mathbb{E}(Z) + \lambda \mathbb{E}[(Z - \mathbb{E}(Z))_+] = (1-\lambda)\mathbb{E}(Z) + \lambda \mathbb{E}[\max\{Z, \mathbb{E}(Z)\}] \\
&= \mathbb{E}[(1-\lambda)Z + \lambda \max\{Z, \mathbb{E}(Z)\}].
\end{aligned} \tag{3.18}
$$

Using (3.18), formulation (3.17) can be rewritten as follows:

$$
\begin{aligned}
\text{Min }\ & g_1(x_1) + \mathbb{E}_{\xi_2}[(1-\lambda_2)g_2(x_2,\xi_2) + \lambda_2 \max\{g_2(x_2,\xi_2), \mathbb{E}_{\xi_2}[g_2(x_2,\xi_2)]\}] \\
& \quad + \mathbb{E}_{\xi_2}[\mathbb{E}_{\xi_3}^{\xi_{[2]}}[(1-\lambda_3)g_3(x_3,\xi_3) + \lambda_3 \max\{g_3(x_3,\xi_3), \mathbb{E}_{\xi_3}[g_3(x_3,\xi_3)]\}] \\
& \quad + \mathbb{E}_{\xi_3}^{\xi_{[2]}}[\mathbb{E}_{\xi_4}^{\xi_{[3]}}[(1-\lambda_4)g_4(x_4,\xi_4) + \lambda_4 \max\{g_4(x_4,\xi_4), \mathbb{E}_{\xi_4}[g_4(x_4,\xi_4)]\}] + \cdots \\
& \quad + \mathbb{E}_{\xi_{T-1}}^{\xi_{[T-2]}}[\mathbb{E}_{\xi_T}^{\xi_{[T-1]}}[(1-\lambda_T)g_T(x_T,\xi_T) \\
& \quad + \lambda_T \max\{g_T(x_T,\xi_T), \mathbb{E}_{\xi_T}[g_T(x_T,\xi_T)]\}]] \cdots]] \\
\text{s.t. }\ & x_t \in \mathcal{X}_t(x_{[t-1]}, \xi_{[t]}), \quad t = 1, \dots, T.
\end{aligned}
$$

The Bellman equations for $\mathbb{E}$-MASD are much more involved. The reason is because the problem does not have block angular structure, as pointed out by [9]. In practice it means that the problem cannot be decomposed by scenarios because there is a linking constraint based on the average of all realizations for a given node. In their work [9], the authors propose a decomposition scheme for the two-stage MASD problem which relies on an explicit formula for the second stage problem in order to construct a subgradient for expected second stage costs. In the multistage setting, due to the implict expression for the future cost function, closed-formulas are not available and the proposed methodology cannot be applied directly.

The vast majority of multistage models can be decomposed by scenarios at each node, which in the literature is referred to as hazard-decision (HD) framework. The concept was originally proposed in [27] in the context of hydrothermal scheduling problems, and the SDDP algorithm was originally designed to handle HD formulations only. In this framework, at every stage the current uncertainty is revealed and the decision-maker has to choose a course of action taking into account current and future costs.

Whenever there is a constraint that couples the possible scenario realizations one is forced to use the decision-hazard (DH) framework, which explicitly considers those realizations simultaneously. In this framework, the decision has to be taken without knowing neither the stage uncertainty nor the future realizations of the process. An example is [37], where the authors consider CVaR in the constraints and are forced to use a DH formulation to solve the problem since all scenarios are needed to compute the CVaR at every stage. Formal definitions of the two frameworks can be found in [7], where the author shows that DH formulations can be written as HD formulations by extending the representation of each node in a policy graph.

For the first stage we have

$$
\begin{aligned}
\min_{x_1}\ & g_1(x_1) + \mathbb{E}_{\xi_2}[Q_2(x_1)] \\
\text{s.t. }\ & x_1 \in \mathcal{X}_1.
\end{aligned} \tag{3.19}
$$

We define N_t as the number of outcomes of the random variable ξ_t and Ω_t as the set that contains all possible histories of the process up to time t. By denoting ξ_t^i as the i-th possible realization of random variable ξ_t, we have for $t \in \{2, \cdots, T-1\}$

$$Q_t(x_{t-1}) = \underset{x_t, \nu_t^i}{\text{Min}} \ (1-\lambda_t)\frac{1}{N_t}\sum_{i=1}^{N_t} g_t(x_t, \xi_t^i) + \lambda_t \frac{1}{N_t}\sum_{i=1}^{N_t}\nu_t^i + \mathbb{E}_{\xi_{t+1}}^{\xi_{[t]}}[Q_{t+1}(x_t)] \tag{3.20}$$

$$\text{s.t.} \quad x_t \in \mathcal{X}_t(x_{t-1}, \xi_{[t]}),\ \xi_{[t]} \in \Omega_t, \tag{3.21}$$

$$\nu_t^i \geq g_t(x_t, \xi_t^i) \quad \forall\ i = 1, \dots, N_t, \tag{3.22}$$

$$\nu_t^i \geq \frac{1}{N_t}\sum_{i=1}^{N_t} g_t(x_t, \xi_t^i), \quad \forall\ i = 1, \dots, N_t. \tag{3.23}$$

Constraints (3.21) have to hold for each process history. Constraints (3.22) and (3.23) linearize the max function in (3.18), and the auxiliary variable ν_t^i stores deviations from the mean at each scenario. Constraint (3.23) is somewhat similar to the CVaR constraint in [37] because it links all N_t realizations at the current stage.

For the last stage we have

$$\begin{aligned}
Q_T(x_{T-1}) = \underset{x_T, \nu_T^i}{\text{Min}} \ & (1-\lambda_T)\frac{1}{N_T}\sum_{i=1}^{N_T} g_T(x_T, \xi_T^i) + \lambda_T \frac{1}{N_T}\sum_{i=1}^{N_T}\nu_T^i \\
\text{s.t.} \quad & x_T \in \mathcal{X}_T(x_{T-1}, \xi_{[T]}),\ \xi_{[T]} \in \Omega_T, \\
& \nu_T^i \geq g_T(x_T, \xi_T^i) \quad \forall\ i = 1, \dots, N_T, \\
& \nu_T^i \geq \frac{1}{N_T}\sum_{i=1}^{N_T} g_T(x_T, \xi_T^i), \quad \forall\ i = 1, \dots, N_T.
\end{aligned}$$

Next, we address the risk-neutral convertible property of MSSPs with ECRM.

3.2. Risk-Neutral Convertible Property of MSSPs with ECRM.

The Bellman equations presented in previous subsections are similar to Bellman equations for risk-neutral MSSPs. Such similarity has important algorithmic consequences since risk-neutral MSSPs can be efficiently solved by decomposition schemes without significant changes. We now define a property that lays down the necessary structure a MSSP must have to be amenable to decomposition algorithms. Since DH formulations can be converted to HD formulations, the definition is based on HD MSSPs.

DEFINITION 3.1. *Consider a risk averse formulation of an MSSP problem. The formulation is said to be risk-neutral convertible (RNC) if it can be written in the following form:*

$$\begin{aligned}
\underset{x_1,\dots,x_T}{\mathit{Min}} \ & \mathbb{E}_{\xi_1,\dots,\xi_T}\left[g_1(x_1) + g_2(x_2, \xi_2) + \dots, g_T(x_T, \xi_t)\right] \\
\mathit{s.t.}\ & x_t \in \mathcal{X}_t(x_{[t-1]}, \xi_{[t]}), \quad t = 1, \dots, T.
\end{aligned}$$

Importantly, the constraints defined by $\mathcal{X}_t$ have to be explicit functions, e.g., in the linear case we must have

$$A_t x_t \leq \xi_t - \sum_{m=1}^{t-1} B_m x_m$$

for matrices A_t and $B_m, m = 1, \dots, T-1$. Equivalently, a problem is RNC if it can be written in recursive form:

$$\begin{aligned}
\underset{x_1}{\mathit{Min}} \ & g(x_1) + \mathbb{E}_{\xi_1}\left[Q_2(x_1, \xi_1)\right] \\
\mathit{s.t.}\ & x_1 \in \mathcal{X}_1,
\end{aligned}$$

where

$$Q_t(x_{t-1}, \xi_{[t]}) = \underset{x_t}{Min}\ g(x_t, \xi_t) + \mathbb{E}_{\xi_{t+1}}\left[Q_{t+1}(x_{t+1}, \xi_{[t]})\right] \tag{3.24}$$

$$s.t.\ x_t \in \mathcal{X}_t(x_{[t-1]}, \xi_{[t]}). \tag{3.25}$$

The explicit Bellman equations presented in the previous subsection prove that $\mathbb{E}$-MEE, $\mathbb{E}$-MQD, $\mathbb{E}$-MCVaR, and $\mathbb{E}$-MASD have RNC formulations. It is worth noting that the nested formulation proposed in [19] does *not* have this property because the value function is present in the constraints. More importantly, being RNC, risk-averse problems with ECRMs can be solved using the SDDP as if the problem were risk neutral. Moreover, convergence properties derived for the risk-neutral case (e.g. [31, 33]) can be directly applied to problems with ECRMs, and upper bounds are automatically available. To have a self-contained paper we provide a description of the SDDP algorithm in the accompanying supplement to this paper.

4. Applications.

We consider two different applications for the ECRM approach. The first application is a *hydrothermal scheduling problem*. This problem involves the planning and operation of an energy systems to minimize total costs under uncertainty. Typically, inflows are uncertain during the planning horizon, and demand follows a deterministic pattern. The goal is to determine a configuration of the electrical network, with its generating units, transmission lines, bars and demand in each bar, and to find an operation schedule of each component in the system to meet the load demand of each bar at the lowest possible cost. The second application is a *portfolio selection problem*, where the decision-maker has to determine how to invest their funds among a given set of assets over a planning horizon. The future returns for each asset are unknown, and the goal is to maximize expected returns while controlling for the risk of the investments. Next, we provide further details of each application and state the ECRM formulation for each problem, and for different risk measures.

4.1. Hydrothermal Scheduling Problem.

Long-term hydrothermal scheduling problems were the main motivation behind the development of SDDP, and those problems still make extensive use of the algorithm to obtain solutions to realistic instances, with 100 or more stages. The planning and operation of electrical systems aim at minimizing the expected value of the total cost of operation, subject to satisfying the demand in each bar of the system. The solution gives an optimal configuration of the electrical network, with its generating units, transmission lines, bars and demand in each bar, finding an operation schedule at the lowest possible cost.

We use the following notation to define the hydrothermal scheduling problem:

v_t	Storage level.
$\mathbf{y}_t$	Water spillage (s_t) and water release (u_t).
$\mathbf{g}_t$	Thermal generation.
$\mathbf{f}_t$	Power flow.
$\mathbf{d}_t$	Demand in each node.
$a_{t,i}$	Random inflow outcomes for each scenario ($i \in \Omega_t$).
$p_{t,i}$	Probability associated with random inflow outcomes for each scenario ($i \in \Omega_t$).
A	Assigns thermoelectric generation units to buses.
B	Assigns hydro generation units to buses.
C	Incidence matrix.
H	Water balance of each reservoir.
τ	Discount factor for net present value (NPV).
$\mathbf{l}_t$	Amount of load shedding.
$\mathbf{c}_l$	Load shedding cost.

The risk-neutral monthly hydrothermal scheduling problem can be stated as follows:

$$
\begin{aligned}
Q_t(v_{t-1}, a_{t,i}) = \underset{\{\mathbf{y_t}, \mathbf{g_t}, \mathbf{f_t}, v_t\} \in \mathcal{X}_t}{\text{Min}} \quad & \mathbf{c}^\top \mathbf{g}_t + \mathbf{c}_l^\top \mathbf{l}_t + \tau \sum_{i \in \Omega_t} p_{t,i} Q_{t+1}(v_t, a_{t+1,i}), \\
\text{s.t.} \quad & A\mathbf{g_t} + B\mathbf{y_t} + C\mathbf{f_t} = \mathbf{d_t} - \mathbf{l}_t, \\
& v_t + H\mathbf{y_t} = v_{t-1} + a_{t,i}, \\
& \mathbf{f_t} \in \mathcal{F}_t, \mathbf{y_t} \in \mathcal{Y}_t, \mathbf{g_t} \in \mathcal{G}_t, v_t \in \mathcal{V}_t.
\end{aligned} \tag{4.1}
$$

The objective function involves minimizing the current generation and expected future cost, with a discount factor τ. The first constraint in (4.1) models the system power flow to meet the demand in each node, while the second constraint balances storage levels with the system state and inflows. The last set of constraints are the decision variables' lower and upper bounds. We point out that problem (4.1) has complete recourse. To avoid repetition and to improve the flow of the exposition, the mathematical models for different ECRM risk measures are presented in the supplement to this paper.

4.2. Portfolio Selection Problem. We consider a portfolio selection problem with N risky assets plus a risk-free one, which will be cash. At each time period t the decision-maker has to allocate the current wealth among the $N + 1$ investment possibilities, and a cost is incurred for each transaction. The quantity $r_{i,t}$ represents the excess return of asset i with respect to cash in time period t.

A fundamental assumption of the SDDP is that the underlying process has to be stage-wise independent. Since in most applications such an assumption is too restrictive, it is necessary to find ways around it to apply the algorithm in practice. For hydrothermal models it has been observed that autoregressive (AR) process are an excellent fit for inflows. As shown in [21], by increasing the state space it is possible to construct a stagewise independent model using AR processes.

In finance the task is more challenging. First of all, there is no consensus to which process better describes the dynamics of returns. Second, in case one process, or a family of processes is identified, it is unclear if it will be possible to include it in SDDP. It is widely accepted that financial returns series have nonlinear temporal dependence [8, 23], and incorporating a model with those characteristics within SDDP would be a formidable challenge.

An alternative is to use Markov chains. Instead of choosing a process, one can define states of the economy at every time period, such as *bear*, *bull* and *average*, and within each state generate data according to a factor model. The Markov chain approach has been used in two recent publications [20, 37], and the resulting models are slightly more involved but still amenable to SDDP. To formalize the portfolio selection problem, we will use the following notation:

N	Number of risky assets.
$\mathcal{A}$	Set of risky assets, $\mathcal{A} = \{1, \cdots, N\}$.
$\mathcal{K}$	Set of market states, indexed j, k.
$\mathbf{c}$	Transaction costs, in percentage.
$\mathbf{u}_t$	Amount of money allocated in each risky asset at stage t.
$u_{i,t}$	Amount of money allocated in risky asset i at stage t.
a_t	Amount of money allocated in risk-free asset at stage t.
$\mathbf{d}_t$	Amount (in money) sold of each risky asset at stage t.
$\mathbf{b}_t$	Amount (in money) bought of each risky asset at stage t.
$\mathbf{r}_t^j(s)$	Returns for each risky asset at stage t at market state j and scenario s.
$P_{k\|j}$	Transitional probability from state j to state k.
$p_{s\|k}$	Probability of scenario s on state k.

The portfolio selection problem is usually formulated as a maximization problem, but to be consistent with the generic formulations we have presented, we will cast it as a minimization problem. We begin with a monthly risk-neutral asset allocation problem (hazard-decision) which can be stated for a given j and t as follows:

$$\begin{aligned}
(4.2) \qquad Q_t^j(\mathbf{u}_{t-1}, \mathbf{r}_t^j(s)) = -\mathbf{u}_{t-1}^\top \mathbf{r}_t^j(s) + \underset{\mathbf{u_t}, \mathbf{b_t}, \mathbf{d_t}}{\text{Min}} \quad & \mathbf{c}^\top(\mathbf{b}_t + \mathbf{d}_t) + \sum_{k \in \mathcal{K}} \mathcal{Q}_{t+1}^k(\mathbf{u}_t) P_{k|j} \\
\text{s.t.} \quad & a_t + (\mathbf{1} + \mathbf{c})^\top \mathbf{b}_t - (\mathbf{1} - \mathbf{c})^\top \mathbf{d}_t = a_{t-1}, \\
& u_{i,t} - b_{i,t} + d_{i,t} = (1 + r_{i,t}^j(s)) u_{i,t-1}, \quad \forall i \in \mathcal{A}, \\
& \mathbf{u}_t, \mathbf{b}_t, \mathbf{d}_t \geq 0,
\end{aligned}$$

with

$$(4.3) \qquad \mathcal{Q}_{t+1}^k(\mathbf{u}_t) = \sum_{s \in \mathcal{S}_k} Q_{t+1}^k\big(\mathbf{u}_t, \mathbf{r}_{t+1}^k(s)\big) p_{s|k}.$$

The objective function involves minimizing current transaction costs and the future expected costs, while maximizing expected returns from the risky assets. The returns from funds invested in a risky-asset at time period $t-1$ are realized at the end of that time period, and the positions in each of the assets are a state variable represented by $\mathbf{u}_{t-1}$. The first constraint discounts the transaction costs from the risk-free asset, that has zero return. The second constraint registers the dollar amounts of assets bought and sold while rebalancing the portfolio. In this model we do not allow short selling, so every decision variable has to be nonnegative.

In the last stage T, there is no optimization problem to be solved:

$$Q_T^j(\mathbf{u}_{T-1}, \mathbf{r}_T^j(s)) = -\mathbf{u}_{T-1}^\top \mathbf{r}_T^j(s).$$

We point out that this is a complete recourse problem, and the lowest possible return is -100%. Thus, given any $\mathbf{u_t}$ and $\mathbf{r_{t+1}}(s)$, it is possible to make an allocation that satisfies all the constraints of problem (4.2). Risk-averse formulations for this problem are presented in supplement to this paper.

5. Computational Study. To understand how the proposed ECRMs behave in practice and how different risk measures will impact decisions, an extensive computational study was performed for our two applications. Both applications were constructed using real datasets used in several other studies. The models were implemented in Julia language 1.0.3 using SDDP.jl [12] and JuMP[13] frameworks to construct and solve SDDP decomposition. We used both Gurobi 9.0.2 and CPLEX 12.9.0.0 to solve the linear programming problems, and all experiments were conducted on a single Core i7-5820K 3.3GHz 64GB of memory with Ubuntu 18.04.

5.1. Hydrothermal Scheduling Problem. Our hydrothermal scheduling instance has data from the Brazilian power system, and was discussed in [7]. Generation resources in this instance include four aggregated hydro plants in North, South, Northeast, and Southeast regions, and 95 thermal plants spread throughout these regions. The instance consists of five lines connecting the four regions. Demand in each node is deterministic, but varies for different stages. Figure 1 gives an illustration of the topology of the instance.

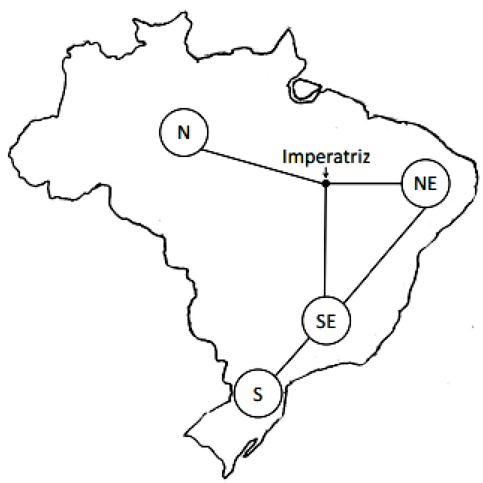

Fig. 1. *An aggregated Brazilian interconnected power system [7].*

The instance is solved for 84 stages (months). The amount of rainfall per month in each region is a random variable with 25 observations, resulting in 25^{84} scenarios. The random process is stage-wise independent, which is required by the SDDP algorithm. Data on constraints on hydro-electric generation, reservoir levels, and the initial amount of water in each reservoir is summarized in Table 3, and Table 4 shows the number of thermal plants in each region.

TABLE 3
Data for the hydro plants.

	H1	H2	H3	H4
Initial reservoir	119428.8	11535.1	29548.2	6649.4
Max. reservoir	200717.6	19617.2	51806.1	12679.9
Max. hydro generation	45829.1	13381.8	9780.9	7740.2

TABLE 4
The data for thermal plants.

	N	S	NE	SE
# of Thermal Plants	0	17	36	42

We formulated the problem using MSSPs with $\mathbb{E}$-MEE, $\mathbb{E}$-MCVaR, and $\mathbb{E}$-MQD and performed extensive numerical experiments in each case. $\mathbb{E}$-MASD is more involved and we would not be able to solve large-scale instances as efficiently as we did for other ECRMs. The results show that $\mathbb{E}$-MEE is the risk measure that better captures risk aversion in this problem. The solutions obtained for $\mathbb{E}$-MCVaR and $\mathbb{E}$-MQD were very similar to risk-neutral one for a wide range of parameter choices, showing that they did not capture risk-aversion well for this problem. We investigated the instance in detail and we observed that the quantiles for more extreme scenarios (high and low inflow) were cancelling out after averaging them with $\mathbb{E}$-MCVaR and $\mathbb{E}$-MQD . It is possible that such behavior would disappear for distributions with heavier tails, but since we are working with a real-world instance we did not want to alter the original data. Therefore, we report detailed computational results for $\mathbb{E}$-MEE. We use the same target ζ for all stages, which is equal to the average of the 90th-percentile of costs at every stage obtained from the risk-neutral policy.

The MSSP with $\mathbb{E}$-MEE was solved using the SDDP algorithm limiting the number of iteration in 10000. In Figure 2 we report lower and upper bounds for one particular experiment. Such result is representative of all the different variations we solved for $\mathbb{E}$-MEE, and we remind the reader that an attractive feature of ECRMs is that upper bounds can be directly obtained from SDDP.

After solving the problem we evaluated the performance of the optimal policy for 7500 randomly selected scenarios, and presented the results for different parameter choices. In all plots we show the 50%, 90%, 95%, and 99% quantiles of some performance measures (e.g. cost) for the optimal policy obtained from the risk-neutral and the MSSP with $\mathbb{E}$-MEE over the same 7500 random scenario paths. We assume different levels of risk aversion by varying the value of $\lambda \in \{0.5, 5, 50, 100\}$.

In the upper-left panel of Figure 3 we see that the risk-neutral policy has the lowest average cost until around stage 60, while $\mathbb{E}$-MEE with $\lambda = 100$ has the highest. Such behavior is intuitive as the risk-averse model tends to save water in the reservoirs for drier months (or scenarios) in the future, and has to rely on expensive thermal plants in initial stages. The risk-neutral model starts using mostly the free hydro plants, and at some point ends up with low reservoir levels, which makes using thermal plants inevitable. A similar behavior occurs for the $\mathbb{E}$-MEE with lower values of λ.

When we move to the quantiles, we observe that having lower average costs hides cost spikes that can have negative consequences in the system's operation. Figure 3 shows that the risk-neutral policy exhibits increasingly higher spikes for the 90%, 95% and 99% quantiles, especially in later stages. It is also apparent that for higher

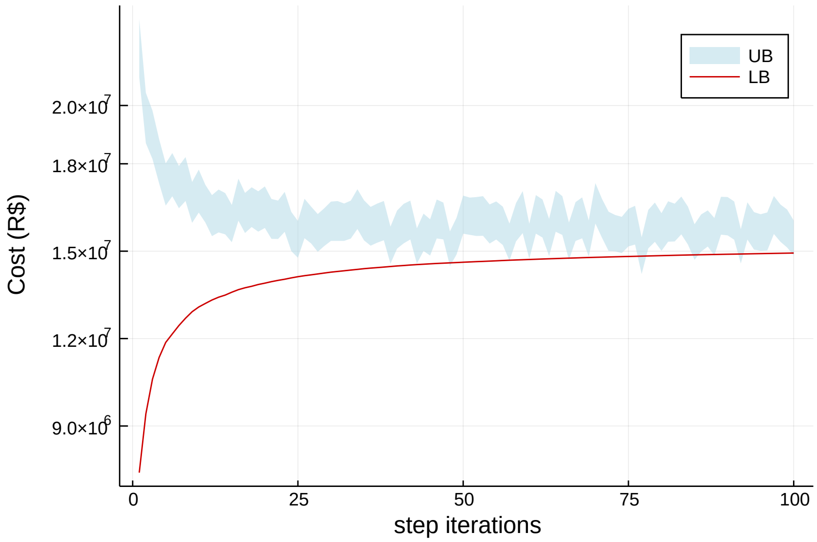

Fig. 2. *Upper and lower bounds for EMEE $\lambda = 0.1$ (step = 100).*

values of λ the spikes are very small or nonexistent, an advantage obtained by having higher average costs for the early stages. We believe the insights are useful from a decision-maker perspective: minimizing expected costs is certainly not the only goal, and if the consequences of having extreme costs are severe (e.g. blackouts) it may very well be worth to pay a premium in the first stages to avoid the possibility of price spikes.

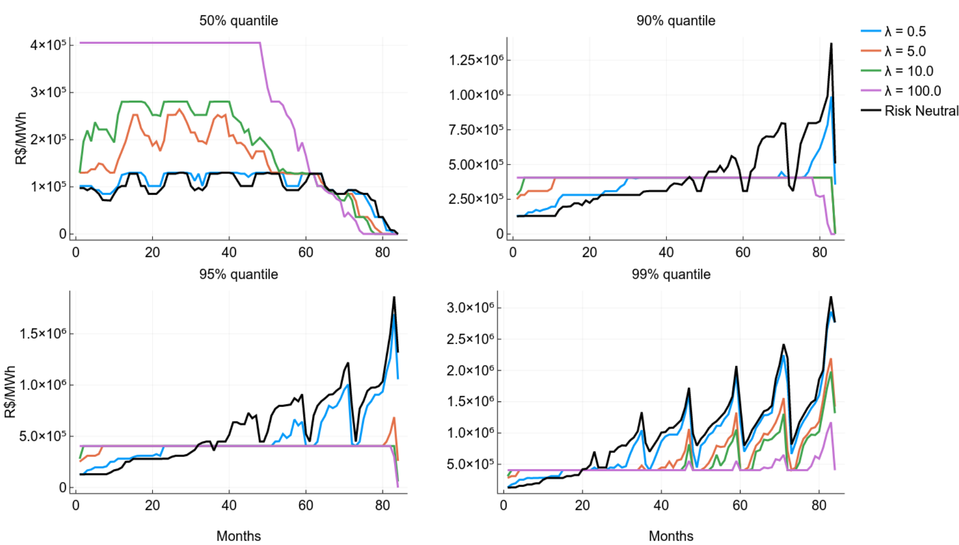

Fig. 3. *Stage-wise cost of MSP with $\mathbb{E}$-MEE for different λ values vs. risk-neutral.*

Figure 4 represents average and quantiles of the reservoir levels in Southeast region of Brazil. We see that the $\mathbb{E}$-MEE with higher values of λ store more water in the reservoir, and that the risk-neutral policy has the lowest reservoir levels across all stages, and for all quantiles.

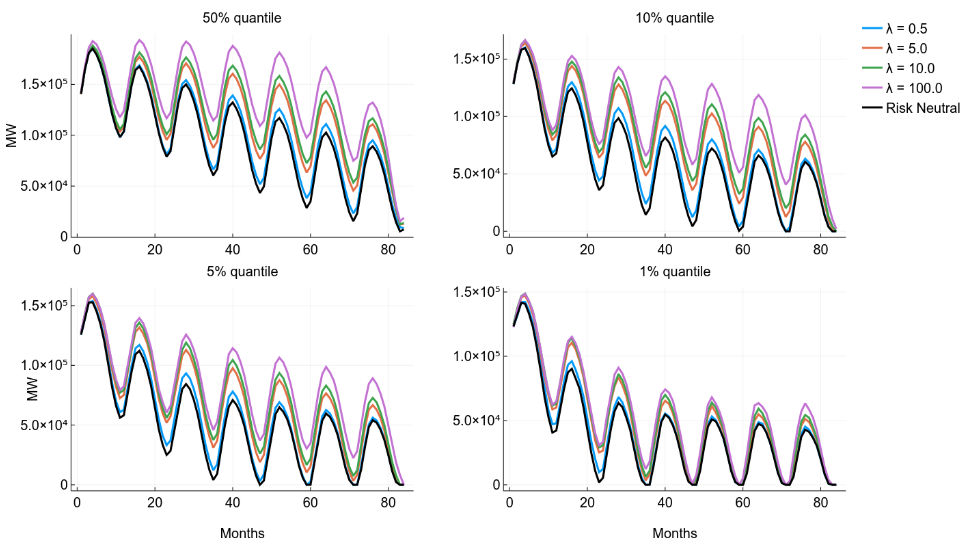

Fig. 4. *Stage-wise reservoir level of MSP with $\mathbb{E}$-MEE vs. risk-neutral in the Southeast.*

The reservoir levels associated with different policies follow the same pattern in the South and Northeast regions. A different pattern occurs in the Northern region as there are no thermal plants in this area, leaving the hydro plant as the only available source of electricity generation in the region. Figures for these regions are included in the supplement to this paper.

5.2. Portfolio Selection Problem. As mentioned in Section 4.2, the key element in our portfolio selection problem is how to model asset's returns. We use data from five industry sectors taken from Kenneth R. French website[2]. The data comprises 50 years of monthly returns, and the descriptive statistics are summarized in Table 5.

Portfolio	Mean	St. dev.
Cnsmr	0.0099	0.0456
Manuf	0.0090	0.0447
HiTec	0.0095	0.0559
Hlth	0.0101	0.0489
Other	0.0088	0.0534

Table 5

Mean and standard deviation for the five-industry data obtained from Kenneth R. French website, from 1970 to March 2020.

In our computational study we use a Markov chain that is constructed using a Hidden Markov Model (HMM). The HMM is used to estimate the transition probabili-

[2]urlhttp://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data library.html

ties of the Markov chain and the distribution of returns of each state. The estimation procedure considers three states in the Markov chain, and the transition matrix is shown in Table 6.

TABLE 6
Matrix of states and transition probabilities.

From\To	State 1	State 2	State 3
State 1	0.29	0.43	0.28
State 2	0.06	0.35	0.59
State 3	0.13	0.38	0.49

We consider the five-factor model described in [15] to obtain the expected returns under each state, as shown in in Table 7.

TABLE 7
Expected returns for each investment, under each state.

State	Cnsmr	Manuf	HiTec	Hlth	Other
1	-0.0654	-0.0651	-0.0779	-0.0589	-0.0850
2	0.0482	0.0443	0.0532	0.0470	0.0534
3	-0.0019	-0.0008	-0.0045	-0.0026	-0.0038

Similar to HSP, for all risk measures, we limit the number of iteration; however, in this case, the SDDP algorithm was limited to 5000 iterations. After solving the problem, we evaluated the optimal policy's performance for 1000 randomly selected scenarios and presented the results for different parameter choices.

Remark: We present formulation (4.2)-(4.3) as a minimization problem to follow the convention adopted in the whole paper. However, for a portfolio selection problem it is not natural to say that negative values are better than positive ones. For the presentation of results we inverted the sign of the objective function values, and in all figures higher values are better. For $\mathbb{E}$-MCVaR, $\mathbb{E}$-MEE, and $\mathbb{E}$-MQD we report a series of three plots for each risk measure: the first shows the average losses for the 50th, 10th, 5th and 1st percentiles for different values of λ. The second shows which percentage of the budget was allocated to the risk-free and and risky assets. Finally, the third plot shows an area graph with the average allocation at each stage for one specific choice of the parameters.

5.2.1. $\mathbb{E}$-MCVAR. In Figure 5, we observe that for the 50th quantile the risk neutral policy obtains higher average returns across all stages. As the value of λ increases, average returns decrease in an orderly fashion, aligned with our intuition that higher levels of risk aversion offer some protection at the expense of returns.

As we move through the quantiles we see a smooth transition in terms of losses: in the 1% quantile all policies exhibit a similar behavior in terms of average losses. As we reach the 10% quantile we observe that the risk neutral policy can have much more severe losses than the risk-averse ones.

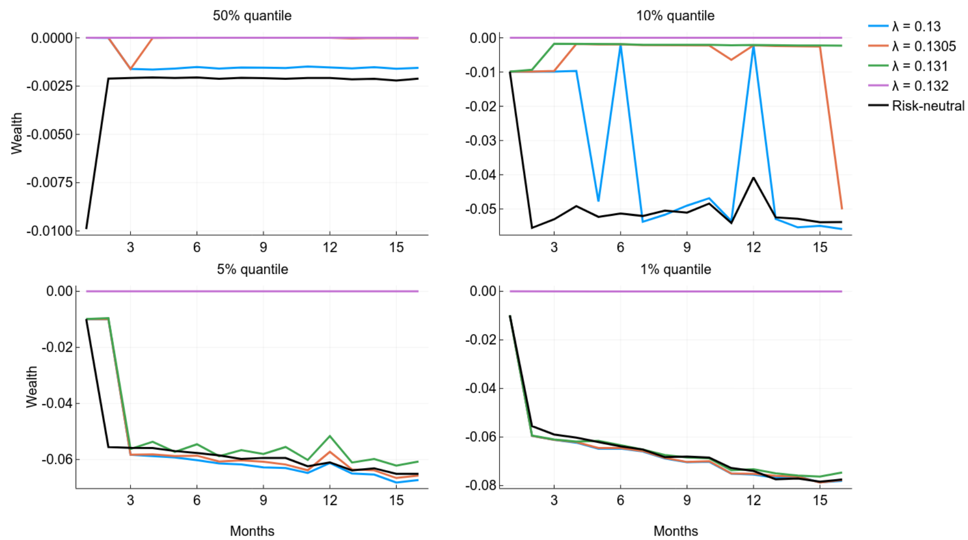

FIG. 5. *Stage-wise average returns with 1000 simulations, comparing* $\mathbb{E}$*-MCVaR(*$\alpha = 0.9$*) and risk-neutral.*

As we move to Figure 6 we can identify which decisions led to the average returns observed in Figure 5. The risk-neutral policy never allocates the budget to the risk-free option, investing all on the risky assets at every stage. As λ grows the risk-averse policy allocates around 20% to risk-free assets. When $\lambda = 0.131$ this value doubles, reaching 40%. As λ grows the policy becomes overly conservative and invest everything in the risk-free asset.

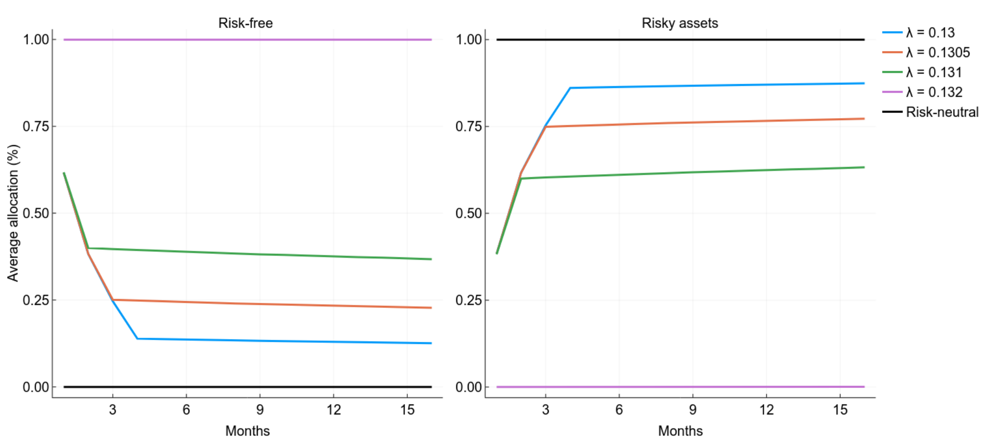

FIG. 6. *Stage-wise average allocation with 1000 simulations for* $\mathbb{E}$*-MCVaR (*$\alpha = 0.9$*).*

We end this section showing in detail the solution for one specific risk-averse case. When $\lambda = 0.131$ and $\alpha = 0.9$, the average allocation across stages is given by Figure 7. A few comments are in order. First, it is interesting to notice the balance between risk and return: even though the majority of the portfolio is invested in cash (starting above 50% and ending below 40%) a significant portion is invested in risky assets. Consumer goods (Cnsmr) has close to 25% of investments, and it is a somewhat safe alternative: it has the lowest standard deviation, it offers attractive returns under state 2 (as shown in Table 4), and lower or comparable losses in the other states with respect to other alternatives. A similar case can be made for health companies (Hlth). Manufacturing (Manuf) is an intermediate choice, and the average exposure was slightly below 15% across all stages. High technology companies (Hitec) and the class Other receives little and no investment, respectively: their potential losses are high, and for this choice of λ the expected returns were not attractive enough to justify allocating a significant portion of the budget in those assets. In summary, the portfolio ends up highly diversified, which is a fundamental property for using the solution in practice.

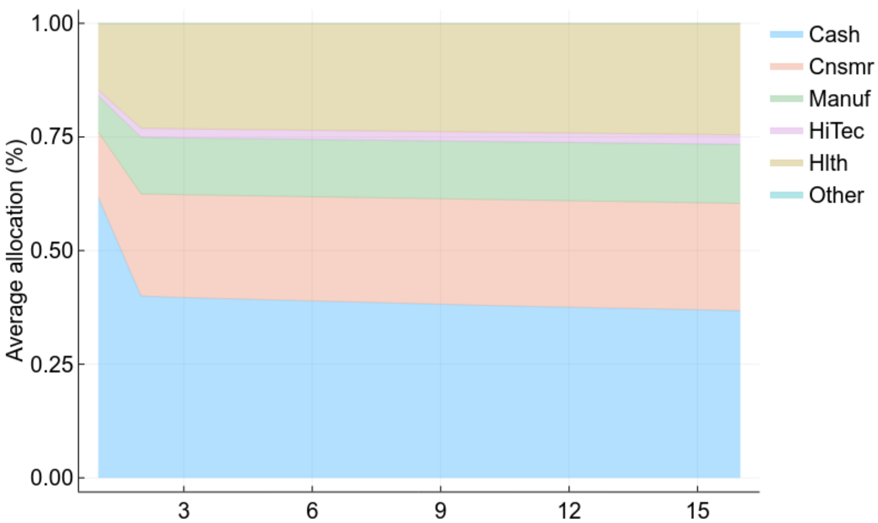

FIG. 7. *Stage-wise average allocation with 1000 simulations, for* $\mathbb{E}$*-MCVaR* ($\alpha = 0.9$ *and* $\lambda = 0.131$*).*

5.2.2. $\mathbb{E}$-MEE. For the $\mathbb{E}$-MEE we consider the same threshold ζ used for the Hydrothermal scheduling problem. In Figure 8 we observe a similar pattern to the $\mathbb{E}$-MCVaR case: higher returns for the 50% quantile, and high potential losses (around 5%) for the other quantiles. We highlight two important differences here: first, the differences between the RN and risk-averse policies is more pronounced, and occurs at all quantiles showed. Second, and perhaps most important, the results are less sensitive to the choice of λ, and the differences in the solutions can be observed for values of λ between zero and 1,000.

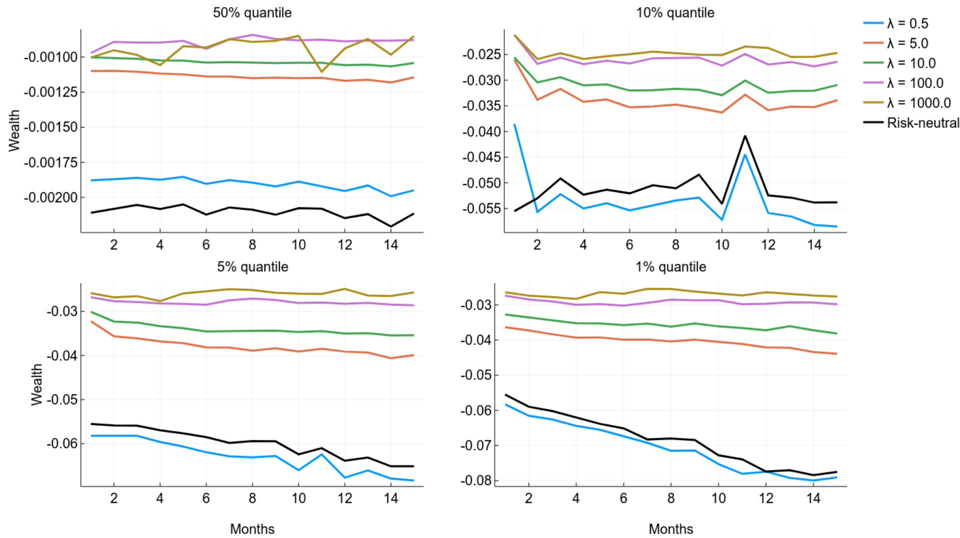

FIG. 8. *Stage-wise average returns with 1000 simulations, comparing $\mathbb{E}$-MEE and risk-neutral.*

Figure 9 shows that even for very high levels of risk aversion the solution still has around 40% invested in risky assets. The reason is that for $\mathbb{E}$-MEE we have a threshold, and the model assumes losses are limited. In practice, this means that the rest of cash can be allocated more freely among the risky assets aiming at higher returns.

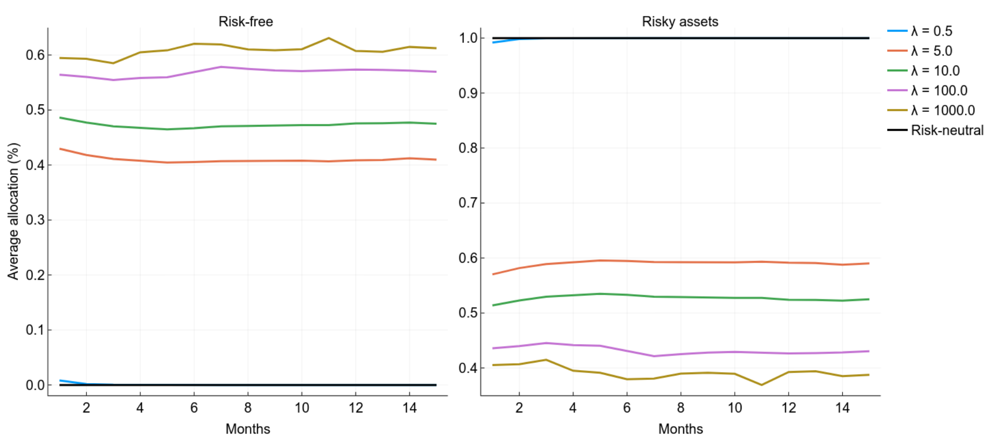

FIG. 9. *Stage-wise average allocation with 1000 simulations, for $\mathbb{E}$-MEE.*

We show in Figure 10 the average allocation for $\lambda = 100$. More than half of the portfolio is allocated to cash since a value of 100 gives significant weight to the risk component of $\mathbb{E}$-MEE. It is interesting to note how evenly spread are the remaining 40% among the risky assets. The solution is extremely balanced and diversified, using all five risky asset investment options.

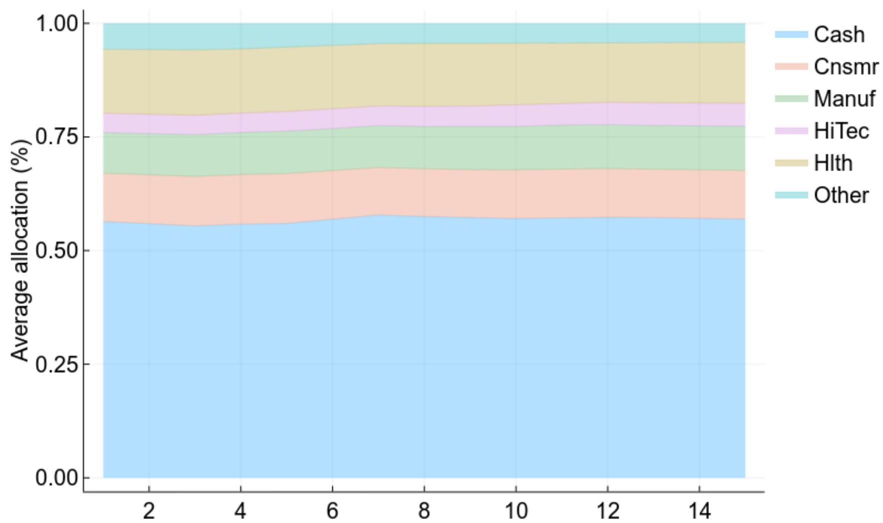

FIG. 10. *Stage-wise average allocation with for 1000 simulations, for* $\mathbb{E}$*-MEE with* $\lambda = 100$.

We conclude our numerical illustrations with the results for $\mathbb{E}$-MQD. Figure 11 is similar to Figure 5, which is not surprising since both are quantile deviation measures. Differences between the risk-neutral and risk averse policies are more significant for the 5th and 10th quantiles.

We can observe that Figures 12 and 13 are similar to what observed for other ECRMs. For the more conservative values of λ the allocation in cash is around 40%, and there is diversification in the allocation of risky assets among the five remaining alternatives.

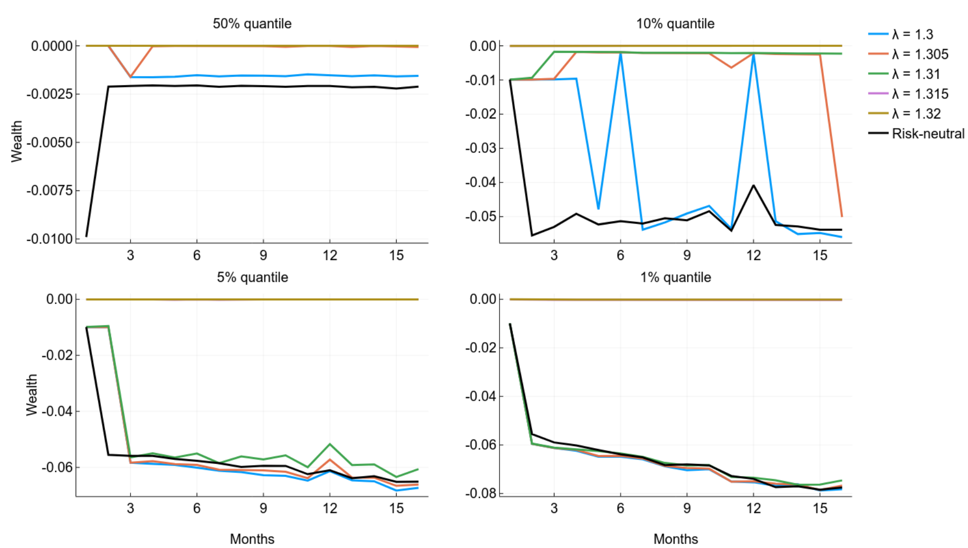

FIG. 11. *Stage-wise average returns for 1000 simulations, comparing* $\mathbb{E}$*-MQD* ($\zeta_1 = 0.1$ *and* $\zeta_2 = 0.9$) *and risk-neutral.*

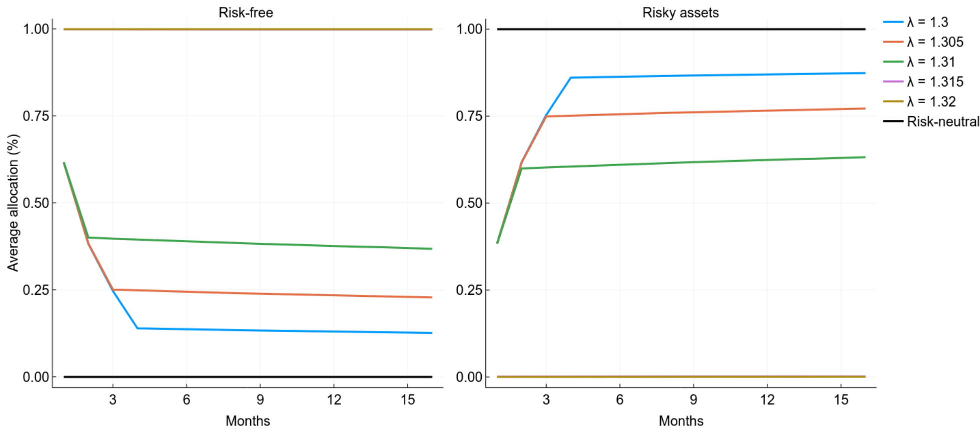

FIG. 12. *Stage-wise average allocation with 1000 simulations, for* $\mathbb{E}$*-MQD* ($\zeta_1 = 0.1$ *and* $\zeta_2 = 0.9$*).*

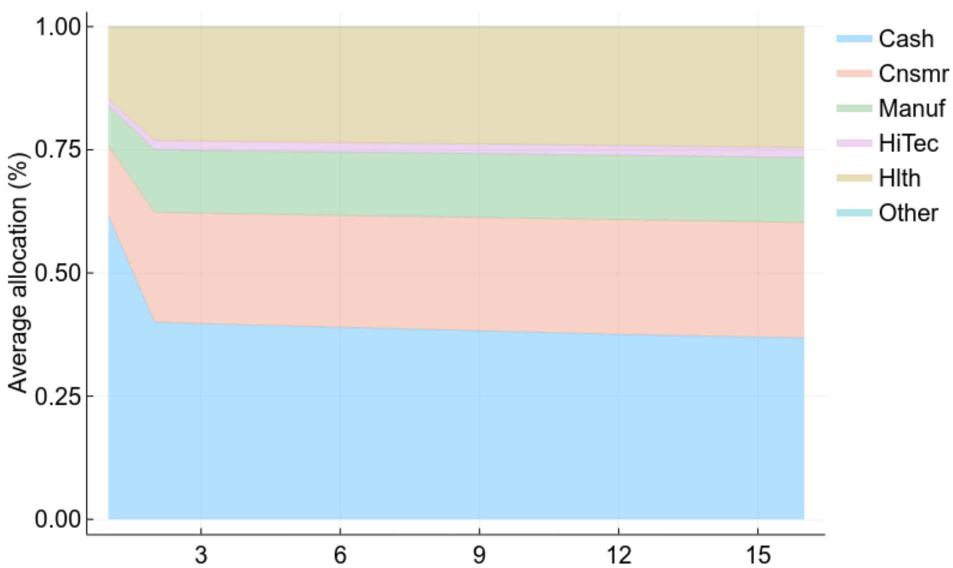

FIG. 13. *Stage-wise average allocation with 1000 simulations, for* $\mathbb{E}$*-MQD* ($\zeta_1 = 0.1$*,* $\zeta_2 = 0.9$ *and* $\lambda = 1.31$*).*

6. Conclusion.

In this work we studied decomposition algorithms for risk-averse MSSP with expected conditional risk measures (ECRMs). ECRMs are time-consistent, and we show that solving risk-averse problems in the ECRM framework is just as complex as solving risk-neutral ones. We consider ECRMs for both deviation and quantile mean-risk measures and derive Bellman equations for each case. We implemented the SDDP algorithm for risk-averse problems with ECRMs and illustrate our approach with an extensive numerical study of problems arising in two applications: hydrothermal scheduling and portfolio selection. For the first problem the $\mathbb{E}$-MEE had the best performance, generating polices significantly different than the risk neutral case. In particular, we observed that higher average costs in the first stages avoided potential costs spikes at later stages. In other words, the $\mathbb{E}$-MEE policy saved water and for that reason coped better with scenarios where expensive thermal plants were needed.

For the second application $\mathbb{E}$-MCVaR, $\mathbb{E}$-MEE and $\mathbb{E}$-MQD offered meaningful risk control. In all risk-averse cases the investment policy had lower returns for the 50th quantile than those of the risk neutral policy, but controlled losses better for more extreme quantiles. For higher values of λ we observed that between 40% and 60% of the portfolio was invested in cash. Interestingly, the remaining portion was well diversified among the risky options, which is fundamental property if a model is used in practice.

Future work includes developing ECRMs for convex risk measures such as the entropic risk measure, and comparing our results with other MSSP risk frameworks such as the nested CVaR, and multistage distributionally robust models.

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