

# Optimization with Least Constraint Violation

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## Abstract

Study about theory and algorithms for constrained optimization usually assumes that the feasible region of the optimization problem is nonempty. However, there are many important practical optimization problems whose feasible regions are not known to be nonempty or not, and optimizers of the objective function with the least constraint violation prefer to be found. A natural way for dealing with these problems is to extend the constrained optimization problem as the one optimizing the objective function over the set of points with the least constraint violation. Firstly, the minimization problem with least constraint violation is proved to be an Lipschitz equality constrained optimization problem when the original problem is a convex optimization problem with possible inconsistent conic constraints, and it can be reformulated as an MPEC problem. Secondly, for nonlinear programming problems with possible inconsistent constraints, various types of stationary points are presented for the MPCC problem which is equivalent to the minimization problem with least constraint violation, and an elegant necessary optimality condition, named as L-stationary condition, is established from the classical optimality theory of Lipschitz continuous optimization. Finally, the smoothing Fischer-Burmeister function method for nonlinear programming case is constructed for solving the problem minimizing the objective function with the least constraint violation. It is demonstrated that, when the positive smoothing parameter approaches to zero, any point in the outer limit of the KKT-point mapping is an L-stationary point of the equivalent MPCC problem.

**Key words:** least constraint violation, L-stationary point, MPEC, MPCC, smoothing function.

## 1 Introduction

For studying an ordinary nonlinear optimization problem, a basic assumption is that feasible region of the optimization problem is nonempty. Many important theoretical issues are well studied for an optimization problem under this assumption. For example, optimality theory and sensitivity analysis are two main theoretical topics. Optimality theory consists of necessary optimality conditions and sufficient optimality conditions. Sensitivity analysis studies continuity properties of the optimal value and the solution mapping when the optimization is perturbed. For nonlinear programming, for a local minimizer, the first-order necessary optimality conditions and the second-order optimality conditions can be developed under certain constraint qualifications, and the second-order sufficient optimality conditions imply the second-order growth condition, see for instance the famous textbook [10]. For nonlinear programming, a series of stability results were obtained by Robinson, see [11], [12] and [13]. Bonnans and Shapiro [1] established the optimality theory and the stability theory for general optimization problems, including problems whose decision variables are infinite dimensional, nonlinear semidefinite programming problems and other conic optimization problems.

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However, when the feasible set is empty or the constraints are inconsistent, infeasibility detection is an important issue for algorithmic design. Many numerical algorithms have been proposed to find infeasible stationary points; namely, stationary points for minimizing certain infeasibility measure. Byrd, Curtis and Nocedal [2] presented a set of conditions to guarantee the superlinear convergence of their SQP algorithm to an infeasible stationary point. Burke, Curtis and Wang [3] considered the general program with equality and inequality constraints, and proved that their SQP method has strong global convergence and rapid convergence to the KKT point, and has superlinear/quadratic convergence to an infeasible stationary point. Recently, Dai, Liu and Sun [5] proposed a primal-dual interior-point method, which can be superlinearly or quadratically convergent to the Karush-Kuhn-Tucker point if the original problem is feasible, and can be superlinearly or quadratically convergent to the infeasible stationary point when the problem is infeasible.

These algorithms can find a stationary point of the infeasibility measure, which have nothing to do with the objective function of the problem. In practice, there are many important problems that we need to find minimizers of the objective function over the points with the least constraint violation. A natural way to deal with such problems is to extend the constrained optimization problem as the one that optimizes the objective function over the set of points with least constraint violation. When the the feasible region is nonempty, the set of points with least constraint violation coincides with the feasible region of the constrained optimization problem and hence the extended constrained optimization problem coincides with the original problem.

Now we give a formal definition of infeasibility measure of the optimization problem. Suppose that the problem is of the following form

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & c(x) \in K, \end{aligned} \tag{1.1}$$

where  $f : \mathcal{X} \rightarrow \mathfrak{R}$ ,  $c : \mathcal{X} \rightarrow \mathcal{Y}$ ,  $K \subset \mathcal{Y}$ ,  $\mathcal{X}$  and  $\mathcal{Y}$  are finite dimensional Hilbert spaces.

**Definition 1.1** A function  $\theta : \mathcal{X} \rightarrow \mathfrak{R}$  is said to be an infeasibility measure of constraint  $c(x) \in K$  if there exists an increasing continuous function  $\varrho : \mathfrak{R}_+ \rightarrow \mathfrak{R}_+$  with  $\varrho(0) = 0$  such that

$$\theta(x) = \varrho(\text{dist}(c(x), K)),$$

where

$$\text{dist}(c(x), K) = \inf\{\|y - c(x)\| : y \in K\}$$

is the distance from  $c(x)$  to  $K$  under the norm  $\|\cdot\|$  in  $\mathcal{Y}$ .

Under the infeasibility measure defined above, we introduce the mathematical model of minimizing the objective  $f(x)$  over the set of points with least infeasibility measure.

**Definition 1.2** For an infeasibility measure  $\theta(x)$  of the constraint  $c(x) \in K$ , the mathematical model of minimizing the objective  $f(x)$  over the set of points with least constraint violation associated with  $\theta$ , is defined by

$$\begin{cases} \min & f(x) \\ \text{s.t.} & x \in \text{Argmin}_z \theta(z). \end{cases} \tag{1.2}$$

Obviously, if the feasible region  $c^{-1}(K)$  is nonempty, then  $\min_z \theta(z) = 0$ ,  $\text{Argmin}_z \theta(z) = c^{-1}(K)$ , and Problem (1.2) is just the original problem (1.1). Thus Problem (1.2) can be regarded as an extension of the original problem (1.1). As the minimization of the constraint violation is considered absolutely prior to the optimization of the objective function, we call the optimum (optimizer) of Problem (1.2) as the *optimum (optimizer) of Problem (1.1) with least constraint violation*, or simply *constrained optimum (optimizer) of Problem (1.1)*.

We do not identify the notion of  $\text{Argmin}_z \theta(z)$  in Definition 1.2. If  $\theta(z)$  is convex, it is obvious that the solution set is just the set of global minimizers. However, if  $\theta(z)$  is non-convex (this happens when  $\{x : c(x) \in K\}$  do not represent a convex set if feasible),  $\text{Argmin}_z \theta(z)$  may be understood as a set of local minimizers or even the set of stationary points. Infeasibility detection is well known difficult in the nonconvex optimization case. Indeed, for a nonconvex problem, infeasibility detection has many of the difficulties inhered in global

optimization since even if an algorithm identifies an infeasible point where constraint violations are locally minimized, there may exist feasible points in other regions of the space of decision variables.

Now we give a simple example to explain the above concepts. Consider the simple quadratic programming problem

$$\begin{aligned} \min \quad & x_1^2 + x_2^2 \\ \text{s.t.} \quad & x_1 + x_2 - 1 \leq 0, \\ & -x_1 - x_2 + 2 \leq 0. \end{aligned} \tag{1.3}$$

It is easy to find that the feasible region is empty. We consider the minimization problem over the set of points with least constraint violation. We can regard the least violation for the constraints as the optimal value of the following problem

$$\begin{aligned} \min \quad & \frac{1}{2}(y_1^2 + y_2^2) \\ \text{s.t.} \quad & x_1 + x_2 - 1 + y_1 \leq 0, \\ & -x_1 - x_2 + 2 + y_2 \leq 0. \end{aligned} \tag{1.4}$$

Then the set of points with the least violation is given by

$$S = \{x : (x, y) \text{ solves Problem (1.4)}\}.$$

It is not difficult to obtain

$$S = \{(x_1, x_2) : x_1 + x_2 - 3/2 = 0\}.$$

Therefore the minimum point of the objective over the violation set is  $(3/4, 3/4)$ . It is not difficult to verify that  $\varrho(t) = \frac{1}{2}t^2$  for  $t \geq 0$  and

$$\theta(x) = \frac{1}{2} \left\{ [x_1 + x_2 - 1]_+^2 + [-x_1 - x_2 + 2]_+^2 \right\}$$

is an infeasibility measure in Definition 1.1 and

$$\text{Argmin}_x \theta(x) = S.$$

A popular method for the minimization with least violation constraint is the penalty method. Define the penalty function as

$$P_c(x) = x_1^2 + x_2^2 + c \left\{ [x_1 + x_2 - 1]_+^2 + [-x_1 - x_2 + 2]_+^2 \right\},$$

where  $[t]_+ = \max\{0, t\}$  for  $t \in \mathfrak{R}$ . Obviously  $P_c$  is a smooth convex function with

$$\nabla P_c(x_1, x_2) = \begin{bmatrix} 2x_1 + 2c \{ [x_1 + x_2 - 1]_+ - [-x_1 - x_2 + 2]_+ \} \\ 2x_2 + 2c \{ [x_1 + x_2 - 1]_+ - [-x_1 - x_2 + 2]_+ \} \end{bmatrix}.$$

By solving  $\nabla P_c(x) = 0$ , we obtain the minimizer of  $P_c(x)$  is

$$x^*(c) = \left( \frac{3c}{1+4c}, \frac{3c}{1+4c} \right)^T. \tag{1.5}$$

Thus we have

$$\lim_{c \rightarrow \infty} x^*(c) = \left( \frac{3}{4}, \frac{3}{4} \right)^T;$$

namely, the limit of the minimizer of  $P_c$  approaches the optimal solution. However, for finite  $c > 0$ , the minimizer of  $P_c$  never coincides with the optimal solution. Therefore, it is significant to find a method, different from the penalty method, for solving the minimization optimization problem over the set of points with least violation constraint.

The rest of this paper is organized as follows. In Section 2, we formulate the minimization problem with the smallest constraint violation for a general conic optimization problem as a Lipschitz continuous equality constrained optimization problem. For a convex conic problem, we prove that  $\theta(x)$  is convex and the minimization problem over the set of the least constraint violation can be reformulated as an MPEC problem. In Section 3, for nonlinear programming with possible inconsistent constraints, we present various types of stationary points for the MPCC problem associated with the minimization problem over the set of the least constraint violation. Especially, an elegant necessary optimality condition, named as L-stationary condition, is established from the classical optimality theory of Lipschitz continuous optimization. In Section 4, we propose the smoothing Fischer-Burmeister function method for solving the minimization problem over the set of the least constraint violation for nonlinear programming. It is demonstrated that, when the positive smoothing parameter approaches to 0, any point in the outer limit of the KKT-point mapping is an L-stationary point of our problem. Some discussions are made in the last section.

## 2 Conic Optimization Problem with Least Constraint Voilation

In this section, we consider the general mathematical programming problem of the form

$$(P) \quad \begin{aligned} \min_x \quad & f(x) \\ \text{s.t.} \quad & g(x) \in K, \end{aligned} \quad (2.1)$$

where  $\mathcal{X}$  and  $\mathcal{Y}$  are finite dimensional Hilbert spaces,  $f : \mathcal{X} \rightarrow \Re$  and  $g : \mathcal{X} \rightarrow \mathcal{Y}$  are continuously differentiable mappings, and  $K \subset \mathcal{Y}$  is a closed convex cone.

For  $\varrho(t) = \frac{1}{2}t^2$  for  $t \geq 0$ , the least violation for the constraint is defined as the optimal value of the following problem

$$\begin{aligned} \min \quad & \frac{1}{2}\|y\|^2 \\ \text{s.t.} \quad & g(x) + y \in K. \end{aligned} \quad (2.2)$$

The set of points with the least violation is given by

$$S = \{x : (x, y) \text{ solves Problem (2.2)}\}.$$

Our problem is to minimize  $f$  over  $S$ ; namely,

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & (x, y) \text{ solves} \\ & \left\{ \begin{array}{l} \min_{w,z} \quad \frac{1}{2}\|z\|^2 \\ \text{s.t.} \quad g(w) + z \in K. \end{array} \right. \end{aligned} \quad (2.3)$$

Denote the lower problem of Problem (2.3) by  $P_L$ ; namely,

$$(P_L) \quad \begin{aligned} \min_{w,z} \quad & \frac{1}{2}\|z\|^2 \\ \text{s.t.} \quad & g(w) + z \in K. \end{aligned} \quad (2.4)$$

Associated with the above  $\varrho(t)$ , we have that

$$\theta(w) = \min_z \left\{ \frac{1}{2}\|z\|^2 : g(w) + z \in K \right\}. \quad (2.5)$$

It is easy to verify that  $\theta(w)$  is an infeasibility measure for Problem (2.1). Obviously, we have

$$\begin{aligned}\theta(w) &= \min_z \left\{ \frac{1}{2} \|z\|^2 : g(w) + z \in K \right\} \\ &= \min_{z'} \left\{ \frac{1}{2} \|z' - g(w)\|^2 : z' \in K \right\} \\ &= \frac{1}{2} \|g(w) - \Pi_K(g(w))\|^2.\end{aligned}$$

Then the optimal value of Problem (P<sub>L</sub>) can be expressed as

$$\text{Argmin}(\text{P}_L) = \text{Argmin}_x \theta(x) = \text{Argmin}_x \frac{1}{2} \|g(x) - \Pi_K(g(x))\|^2 = \text{Argmin}_x \frac{1}{2} \|\Pi_{K^\circ}(g(x))\|^2,$$

where  $K^\circ$  means the polar cone of  $K$ . Therefore Problem (2.3) can equivalently be expressed as

$$\begin{aligned}\min \quad & f(x) \\ \text{s.t.} \quad & x \in \text{argmin}_w \frac{1}{2} \|\Pi_{K^\circ}(g(w))\|^2.\end{aligned}\tag{2.6}$$

Now we discuss when the infeasibility measure  $\theta$  is a convex function.

**Definition 2.1** Let  $g$  be a continuous mapping. We say that the set-valued mapping  $\mathcal{F}_g : x \rightarrow K - g(x)$  is graph-convex if

$$\text{gph } \mathcal{F}_g = \{(x, y) : g(x) + y \in K\} \subset \mathcal{X} \times \mathcal{Y}$$

is a convex set.

**Proposition 2.1** Let  $g$  be a continuous mapping and  $\mathcal{F}_g : x \rightarrow K - g(x)$  be graph-convex. Then the function

$$\theta(x) = \frac{1}{2} \|\Pi_{K^\circ}(g(x))\|^2$$

is convex.

**Proof.** From the definition of  $\theta(x)$  in (2.5), we have

$$\theta(x) = \min_z \left\{ \frac{1}{2} \|z\|^2 : g(x) + z \in K \right\} = \min_z \left\{ \frac{1}{2} \|z\|^2 : (x, z) \in \text{gph } \mathcal{F}_g \right\}.$$

For any  $x^i \in \mathcal{X}$ ,  $i = 1, 2$ , there exist  $z^1$  and  $z^2$  such that

$$\text{Argmin}(x^i) = \frac{1}{2} \|z^i\|^2, \quad (x^i, z^i) \in \text{gph } \mathcal{F}_g, i = 1, 2.$$

For any  $t \in [0, 1]$ , one has that from the convexity of  $\text{gph } \mathcal{F}_g$  that

$$\left( (1-t)x^1 + tx^2, (1-t)z^1 + tz^2 \right) \in \text{gph } \mathcal{F}_g.$$

Thus

$$\begin{aligned}\theta((1-t)x^1 + tx^2) &= \min_z \left\{ \frac{1}{2} \|z\|^2 : ((1-t)x^1 + tx^2, z) \in \text{gph } \mathcal{F}_g \right\} \\ &\leq \frac{1}{2} \|(1-t)z^1 + tz^2\|^2 \\ &\leq (1-t) \frac{1}{2} \|z^1\|^2 + t \frac{1}{2} \|z^2\|^2 \\ &= (1-t)\theta(x^1) + t\theta(x^2),\end{aligned}$$

which implies that  $\theta$  is a convex function. □

It follows from the literature that  $\theta(x)$  is differentiable if  $K$  is a closed convex cone and if  $\mathcal{X} = \mathfrak{R}^n$ ,

$$\nabla\theta(x) = \text{D}g(x)^* \Pi_{K^\circ}(g(x)). \quad (2.7)$$

Therefore, for the case when  $\mathcal{X} = \mathfrak{R}^n$ ,  $g : \mathfrak{R}^n \rightarrow Y$  is continuously differentiable,  $K \subset Y$  is a closed convex cone, and  $\mathcal{F}_g$  is graph convex, Problem (2.6) is a convex optimization problem and is reduced to

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & \text{D}g(x)^* \Pi_{K^\circ}(g(x)) = 0. \end{aligned} \quad (2.8)$$

Although Problem (2.8) is a convex optimization problem, we can not handle the constraints easily because they are nonsmooth equalities. We have to transform the constraints to smoothing constraints and then construct numerical algorithms.

If  $\mathcal{F}_g$  defined by Definition 2.1 is graph-convex, then Problem (P<sub>L</sub>) is a convex optimization problem and thus  $(x, y)$  solves (P<sub>L</sub>) if and only if there exists  $\lambda \in Y^*$  such that

$$\begin{aligned} \text{D}g(x)^* \lambda &= 0, \\ y + \lambda &= 0, \\ \lambda &\in N_K(g(x) + y). \end{aligned}$$

Defining  $z = g(x) + y$ , the above system can be rewritten as

$$F(x, y, z) = 0, \quad (y, z) \in \Omega, \quad (2.9)$$

where

$$F(x, y, z) = \begin{bmatrix} \text{D}g(x)^* y \\ g(x) + y - z \end{bmatrix} \text{ and } \Omega = \{(y, z) : K^* \in y \perp z \in K\}. \quad (2.10)$$

Therefore, Problem (2.6) is equivalently expressed as

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & F(x, y, z) = 0, \\ & (y, z) \in \Omega. \end{aligned} \quad (2.11)$$

### 3 Convex Nonlinear Programming with Least Constraint Violation

For simplicity we consider the following convex nonlinear programming problem

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & h(x) = 0, \\ & g(x) \geq 0, \end{aligned} \quad (3.1)$$

where  $f : \mathfrak{R}^n \rightarrow \mathfrak{R}$ ,  $h : \mathfrak{R}^n \rightarrow \mathfrak{R}^q$  and  $g : \mathfrak{R}^n \rightarrow \mathfrak{R}^p$ . This is the simple case where  $K$  is a closed polyhedral cone:

$$K = \{0_q\} \times \mathfrak{R}_+^p \quad (3.2)$$

and

$$g(x) = \begin{bmatrix} Ax - b \\ c(x) \end{bmatrix},$$

where  $A \in \mathfrak{R}^{q \times n}$ ,  $b \in \mathfrak{R}^q$  and  $c : \mathfrak{R}^n \rightarrow \mathfrak{R}^p$  with  $c_i(x)$  is concave and twice continuously differentiable for each  $i = 1, \dots, p$ . In this case,

$$F(x, y, z) = \begin{bmatrix} A^T y^E + \mathcal{J}c(x)^T y^I \\ Ax - b + y^E - z^E \\ c(x) + y^I - z^I \end{bmatrix},$$

where  $y = (y^E, y^I)$ ,  $z = (z^E, z^I)$  and  $y^E = (y_1, \dots, y_q)^T$ ,  $z^E = (z_1, \dots, z_q)^T$ ,  $y^I = (y_{q+1}, \dots, y_{q+p})^T$  and  $z^I = (z_{q+1}, \dots, z_{q+p})^T$ . The Jacobian of  $F$  at  $(x, y, z)$  is of the form

$$\mathcal{J}F(x, y, z) = \begin{bmatrix} \sum_{j=1}^p y_j^I \nabla^2 c_j(x) & A^T & \mathcal{J}c(x)^T & 0 & 0 \\ A & I_q & 0 & -I_q & 0 \\ \mathcal{J}c(x) & 0 & I_p & 0 & -I_p \end{bmatrix}. \quad (3.3)$$

For  $K = \{0_q\} \times \mathfrak{R}_-^p$ , one has  $K^* = \mathfrak{R}^q \times \mathfrak{R}_+^p$  and

$$\Omega = \{(y, x) : y^E \in \mathfrak{R}^q, z^E = 0, 0 \leq y^I \perp z^I \geq 0\}.$$

We use  $\Phi$  to denote the feasible set of Problem (2.11); namely,

$$\Phi = \{(x, y, z) \in \mathfrak{R}^n \times \Omega : F(x, y, z) = 0\}.$$

Let

$$\Theta = \{(a, b) \in \mathfrak{R}^p \times \mathfrak{R}^p : 0 \leq a \perp b \geq 0\}.$$

Then  $\Phi$  is simplified as

$$\Phi = \{(x, y, z) : F(x, y, z) = 0, z^E = 0, (y^I, z^I) \in \Theta\} \quad (3.4)$$

and Problem (2.11) is simplified as an MPCC problem

$$\min f(x) \quad \text{s.t. } (x, y, z) \in \Phi. \quad (3.5)$$

In the following, we derive the tangent cone, the regular normal cone and the normal cone of  $\Phi$  at  $(x, y, z) \in \Phi$ , which are useful in developing S-stationary conditions and M-stationary conditions for Problem (3.5).

The tangent cone of  $\Phi$  at  $(x, y, z)$  denoted by  $T_\Phi(x, y, z)$ , the regular normal cone of  $\Phi$  at  $(x, y, z)$  denoted by  $\widehat{N}_\Phi(x, y, z)$  and the normal cone of  $\Phi$  at  $(x, y, z)$  denoted by  $N_\Phi(x, y, z)$ , are defined respectively by

$$\begin{aligned} T_\Phi(x, y, z) &= \left\{ (d_x, d_y, d_z) : \begin{array}{l} \exists t_k \searrow 0, \exists (d_x^k, d_y^k, d_z^k) \rightarrow (d_x, d_y, d_z) \\ \text{satisfying } (x, y, z) + t_k (d_x^k, d_y^k, d_z^k) \in \Phi \end{array} \right\}; \\ \widehat{N}_\Phi(x, y, z) &= \left\{ (v_x, v_y, v_z) : \begin{array}{l} \langle (v_x, v_y, v_z), (x', y', z') - (x, y, z) \rangle \\ \leq o(\|(x', y', z') - (x, y, z)\|), (x', y', z') \in \Phi \end{array} \right\}; \\ N_\Phi(x, y, z) &= \left\{ (v_x, v_y, v_z) : \begin{array}{l} \exists (x^k, y^k, z^k) \xrightarrow{\Phi} (x, y, z), \exists (v_x^k, v_y^k, v_z^k) \rightarrow (v_x, v_y, v_z) \\ \text{satisfying } (v_x^k, v_y^k, v_z^k) \in \widehat{N}_\Phi(x^k, y^k, z^k) \end{array} \right\}. \end{aligned}$$

Let  $\omega = \{(\zeta_1, \zeta_2) \in \mathfrak{R}_+^2 : \zeta_1 \zeta_2 = 0\}$ . For  $\Theta$  with complementarity constraints, we have the following lemma about the variational geometry of  $\Theta$  at a point  $(\bar{a}, \bar{b}) \in \Theta$ .

**Lemma 3.1** *For  $(\bar{a}, \bar{b}) \in \Theta$ , the tangent cone, the regular normal cone and normal cone of  $\Theta$  at  $(\bar{a}, \bar{b})$  are calculated by*

$$T_\Theta(\bar{a}, \bar{b}) = \bigotimes_{i=1}^p T_\omega(\bar{a}_i, \bar{b}_i), \widehat{N}_\Theta(\bar{a}, \bar{b}) = \bigotimes_{i=1}^p \widehat{N}_\omega(\bar{a}_i, \bar{b}_i) \text{ and } N_\Theta(\bar{a}, \bar{b}) = \bigotimes_{i=1}^p N_\omega(\bar{a}_i, \bar{b}_i),$$

where

$$\begin{aligned}
\bigotimes_{i=1}^p T_\omega(\bar{a}_i, \bar{b}_i) &= \{(u, v) \mid (u_i, v_i) \in T_\omega(\bar{a}_i, \bar{b}_i), i = 1, \dots, p\}, \\
\bigotimes_{i=1}^p \widehat{N}_\omega(\bar{a}_i, \bar{b}_i) &= \{(u, v) \mid (u_i, v_i) \in \widehat{N}_\omega(\bar{a}_i, \bar{b}_i), i = 1, \dots, p\}, \\
\bigotimes_{i=1}^p N_\omega(\bar{a}_i, \bar{b}_i) &= \{(u, v) \mid (u_i, v_i) \in N_\omega(\bar{a}_i, \bar{b}_i), i = 1, \dots, p\}, \\
T_\omega(\bar{a}_i, \bar{b}_i) &= \begin{cases} \mathfrak{R} \times \{0\}, & \text{if } a_i > 0, b_i = 0; \\ \{0\} \times \mathfrak{R}, & \text{if } a_i = 0, b_i > 0; \\ \omega, & \text{if } a_i = 0, b_i = 0, \end{cases} \\
\widehat{N}_\omega(\bar{a}_i, \bar{b}_i) &= \begin{cases} \{0\} \times \mathfrak{R}, & \text{if } a_i > 0, b_i = 0; \\ \mathfrak{R} \times \{0\}, & \text{if } a_i = 0, b_i > 0; \\ \mathfrak{R}_- \times \mathfrak{R}_-, & \text{if } a_i = 0, b_i = 0, \end{cases} \\
N_\omega(\bar{a}_i, \bar{b}_i) &= \begin{cases} \{0\} \times \mathfrak{R}, & \text{if } a_i > 0, b_i = 0; \\ \mathfrak{R} \times \{0\}, & \text{if } a_i = 0, b_i > 0; \\ (\mathfrak{R} \times \{0\}) \cup (\{0\} \times \mathfrak{R}) \cup (\mathfrak{R}_- \times \mathfrak{R}_-), & \text{if } a_i = 0, b_i = 0. \end{cases}
\end{aligned}$$

For deriving the tangent cone, the regular normal cone and the normal cone of  $\Phi$  at  $(x, y, z) \in \Phi$ , we need the following assumption.

**Assumption 1** The Jacobian  $\mathcal{J}F(x, y, z)$  given by (3.3) is of full row rank.

**Proposition 3.1** Assume Assumption 1 is satisfied. Then

$$T_\Phi(x, y, z) = \left\{ d \in \mathfrak{R}^n \times \mathfrak{R}^{q+p} \times \mathfrak{R}^{q+p} : \begin{cases} \sum_{j=1}^p y_j^l \nabla^2 c_j(x) d_x + A^T d_{y^E} + \mathcal{J}c(x)^T d_{y^l} = 0 \\ Ad_x + d_{y^E} - d_{z^E} = 0 \\ \mathcal{J}c(x) d_x + d_{y^l} - d_{z^l} = 0 \\ d_{z^E} = 0 \\ (d_{y^l}, d_{z^l}) \in T_\Theta(y^l, z^l) \end{cases} \right\}, \quad (3.6)$$

$$\widehat{N}_\Phi(x, y, z) = \left\{ \left( \begin{array}{l} \sum_{j=1}^p y_j^l \nabla^2 c_j(x) \eta_1 + A^T \eta_2 + \mathcal{J}c(x)^T \eta_3 \\ A \eta_1 + \eta_2 \\ \mathcal{J}c(x) \eta_1 + \eta_3 + \xi_a \\ -\eta_2 + \eta_4 \\ -\eta_3 + \xi_b \end{array} \right) : \begin{array}{l} (\eta_1, \eta_2, \eta_3, \eta_4) \in \mathfrak{R}^{n+q+p+q} \\ (\xi_a, \xi_b) \in \widehat{N}_\Theta(y^l, z^l) \end{array} \right\} \quad (3.7)$$

and

$$N_\Phi(x, y, z) = \left\{ \left( \begin{array}{l} \sum_{j=1}^p y_j^l \nabla^2 c_j(x) \eta_1 + A^T \eta_2 + \mathcal{J}c(x)^T \eta_3 \\ A \eta_1 + \eta_2 \\ \mathcal{J}c(x) \eta_1 + \eta_3 + \xi_a \\ -\eta_2 + \eta_4 \\ -\eta_3 + \xi_b \end{array} \right) : \begin{array}{l} (\eta_1, \eta_2, \eta_3, \eta_4) \in \mathfrak{R}^{n+q+p+q} \\ (\xi_a, \xi_b) \in N_\Theta(y^l, z^l) \end{array} \right\} \quad (3.8)$$



**Proof.** Since the Jacobian  $\mathcal{J}F(x, y, z)$  given by (3.3) is of full row rank, we now prove the following equality (see 6.7 Exercise of [14] for similar result):

$$T_\Phi(x, y, z) = \{d : d_{z^E} = 0, (d_{y^I}, d_{z^I}) \in T_\Theta(y^I, z^I) : \mathcal{J}F(x, y, z)d = 0\}. \quad (3.9)$$

It is obvious that the set in the left hand-side is contained in the right hand-side and hence we only need to prove the opposite inclusion. For any  $d = (d_x, d_y, d_z)$  satisfying  $d_{z^E} = 0, (d_{y^I}, d_{z^I}) \in T_\Theta(y^I, z^I), \mathcal{J}F(x, y, z)d = 0$ , one has that there exist  $d^k = (d_x^k, d_y^k, d_z^k) \rightarrow d$  and  $t_k \searrow 0$  such that

$$(x, y^E, z^E, (y^I, z^I)) + t_k(d_x^k, d_{y^E}^k, d_{z^E}^k, (d_{y^I}^k, d_{z^I}^k)) \in \mathfrak{X}^n \times \mathfrak{X}^q \times \{0_q\} \times \Theta.$$

It follows from Lemma 3.1 that  $[d_{y^I}]_i [d_{z^I}]_i = 0$  for  $i = 1, \dots, p$ . Let

$$\alpha = \{i : y_i^I > 0, z_i^I = 0\}, \beta = \{i : y_i^I = z_i^I = 0\}, \gamma = \{i : y_i^I = 0, z_i^I > 0\}$$

and

$$\beta_a = \{i \in \beta : [d_{y^I}]_i > 0, [d_{z^I}]_i = 0\}, \beta_b = \{i \in \beta : [d_{y^I}]_i = [d_{z^I}]_i = 0\}, \beta_c = \{i \in \beta : [d_{y^I}]_i = 0, [d_{z^I}]_i > 0\}.$$

Let

$$\Gamma_d = \left\{ \begin{array}{l} (y_{\alpha \cup \beta_a}^I, z_{\alpha \cup \beta_a}^I) \in \mathfrak{X}_+^{|\alpha| + |\beta_a|} \times \{0_{|\alpha| + |\beta_a|}\} \\ (y^I, z^I) \in \mathfrak{X}^p \times \mathfrak{X}^p : \begin{array}{l} (y_{\beta_c \cup \gamma}^I, z_{\beta_c \cup \gamma}^I) \in \{0_{|\beta_c| + |\gamma|}\} \times \mathfrak{X}_+^{|\beta_c| + |\gamma|} \\ (y_{\beta_b}^I, z_{\beta_b}^I) = (0_{|\beta_b|}, 0_{|\beta_b|}) \end{array} \end{array} \right\}.$$

Then  $\Gamma_d$  is a convex set and  $\Gamma_d \subset \Theta$ . Since the Jacobian  $\mathcal{J}F(x, y, z)$  given by (3.3) is of full row rank, it follows from Theorem 2.87 of [1] that there exist a neighborhood  $\mathcal{V}$  of  $(x, y, z)$  and a positive constant  $\kappa$  such that

$$\begin{aligned} & \text{dist}((x', [y^E]', [z^E]', ([y^I]', [z^I]')), [\mathfrak{X}^n \times \mathfrak{X}^q \times \{0_q\} \times \Gamma_d] \cap F^{-1}(0)) \\ & \leq \kappa \|F(x', y', z'), \Pi_{\mathfrak{X}^n \times \mathfrak{X}^q \times \{0_q\} \times \Gamma_d}((x', [y^E]', [z^E]', ([y^I]', [z^I]')))\|, (x', y', z') \in \mathcal{V}. \end{aligned}$$

Noticing that for  $(x^k, y^k, z^k) = (x, y, z) + t_k d^k, [z^E]^k = 0$ , and

$$\begin{aligned} & ([y^I]_{\alpha \cup \beta_a}^k, [z^I]_{\alpha \cup \beta_a}^k) \in \mathfrak{X}_+^{|\alpha| + |\beta_a|} \times \{0_{|\alpha| + |\beta_a|}\}, \\ & ([y^I]_{\beta_c \cup \gamma}^k, [z^I]_{\beta_c \cup \gamma}^k) \in \{0_{|\beta_c| + |\gamma|}\} \times \mathfrak{X}_+^{|\beta_c| + |\gamma|}, \\ & ([y^I]_{\beta_b}^k, [z^I]_{\beta_b}^k) = (t_k [d_{y^I}^k]_{\beta_b}, t_k [d_{z^I}^k]_{\beta_b}) = o(t_k), \end{aligned}$$

we have that

$$\begin{aligned} & \text{dist}((x, y, z) + t_k d^k, \Phi) \\ & = \text{dist}((x, y, z) + t_k d^k, [\mathfrak{X}^n \times \mathfrak{X}^q \times \{0_q\} \times \Theta] \cap F^{-1}(0)) \\ & = \text{dist}((x, y, z) + t_k d^k, [\mathfrak{X}^n \times \mathfrak{X}^q \times \{0_q\} \times \Gamma_d] \cap F^{-1}(0)) \\ & \leq \kappa \left[ \|(t_k [d_{y^I}^k]_{\beta_b}, t_k [d_{z^I}^k]_{\beta_b})\| + \|F(x^k, y^k, z^k)\| \right] \\ & = \kappa \left\| F(x, y, z) + t_k \mathcal{J}F(x, y, z)d^k + \int_0^1 [\mathcal{J}F((x, y, z) + st_k d^k) - \mathcal{J}F(x, y, z)] d(st_k d^k) \right\| \\ & \quad + \kappa \|(t_k [d_{y^I}^k]_{\beta_b}, t_k [d_{z^I}^k]_{\beta_b})\| \\ & = o(t_k), \end{aligned}$$

which implies that  $d \in T_\Phi(x, y, z)$ . Therefore we obtain equality (3.9).

Combining with (3.3) and (3.9), we obtain (3.6).

Since  $\mathcal{J}F(x, y, z)$  is of full row rank, formula (3.7) comes from the equality

$$\widehat{N}_\Phi(x, y, z) = \mathcal{J}F(x, y, z)^*(\mathfrak{X}^n \times \mathfrak{X}^{q+p} \times \mathfrak{X}^{q+p}) + \widehat{N}_{\mathfrak{R}^n \times \Omega}(x, y, z)$$

and

$$\widehat{N}_{\mathfrak{R}^n \times \Omega}(x, y, z) = \{0_n\} \times \{(v_y, v_z) : v_{y^E} = 0_q, v_{z^E} \in \mathfrak{X}^q, (v_{y^I}, v_{z^I}) \in \widehat{N}_\Theta(y^I, z^I)\}.$$

Formula (3.8) can be established in the same way. As  $\mathcal{J}F(x, y, z)$  is of full row rank, one has

$$N_\Phi(x, y, z) = \mathcal{J}F(x, y, z)^*(\mathfrak{X}^n \times \mathfrak{X}^{q+p} \times \mathfrak{X}^{q+p}) + N_{\mathfrak{R}^n \times \Omega}(x, y, z)$$

and

$$N_{\mathfrak{R}^n \times \Omega}(x, y, z) = \{0_n\} \times \{(v_y, v_z) : v_{y^E} = 0_q, v_{z^E} \in \mathfrak{X}^q, (v_{y^I}, v_{z^I}) \in N_\Theta(y^I, z^I)\}.$$

The proof is completed.  $\square$

From the above lemma, we can easily develop the necessary optimality conditions for a local minimizer of Problem (2.11). For this purpose, we define

$$\alpha = \{i : [y^I]_i^* = 0 < [z^I]_i^*\}, \quad \beta = \{i : [y^I]_i^* = 0 = [z^I]_i^*\}, \quad \gamma = \{i : [y^I]_i^* > 0 = [z^I]_i^*\}.$$

**Theorem 3.1 (S-stationary point)** *Let  $(x^*, y^*, z^*)$  be a local minimizer of Problem (2.11). Let Assumption 1 be satisfied at  $(x^*, y^*, z^*)$ . Then there exist  $\eta_1^* \in \mathfrak{X}^n$ ,  $[\xi_a]_\beta^* \in \mathfrak{X}_-^{|\beta|}$  and  $[\xi_b]_\beta^* \in \mathfrak{X}_-^{|\beta|}$  such that*

$$\begin{aligned} \nabla f(x^*) + \left[ \sum_{j=1}^p [y^I]_j^* \nabla^2 c_j(x^*) - A^T A - \mathcal{J}c_\gamma(x^*)^T \mathcal{J}c_\gamma(x^*) \right] \eta_1^* + \mathcal{J}c_\beta(x^*)^T [\xi_b]_\beta^* &= 0, \\ \mathcal{J}c_\beta(x^*) \eta_1^* + [\xi_a]_\beta^* + [\xi_b]_\beta^* &= 0. \end{aligned} \quad (3.10)$$

**Proof.** From [14], we obtain the inclusion

$$0 \in \nabla_{x,y,z} f(x)|_{x=x^*} + \widehat{N}_\Phi(x^*, y^*, z^*),$$

where  $\widehat{N}_\Phi(x^*, y^*, z^*)$  is from Lemma 3.1. Namely, there exist  $(\eta_1^*, \eta_2^*, \eta_3^*, \eta_4^*) \in \mathfrak{X}^{n+q+p+q}$  and  $([\xi_a]^*, [\xi_b]^*) \in \widehat{N}_\Theta([y^I]^*, [z^I]^*)$  such that

$$\begin{pmatrix} \nabla f(x^*) + \sum_{j=1}^p [y^I]_j^* \nabla^2 c_j(x^*) \eta_1^* + A^T \eta_2^* + \mathcal{J}c(x^*)^T \eta_3^* \\ A \eta_1^* + \eta_2^* \\ \mathcal{J}c(x^*) \eta_1^* + \eta_3^* + [\xi_a]^* \\ -\eta_2^* + \eta_4^* \\ -\eta_3^* + [\xi_b]^* \end{pmatrix} = 0.$$

This set of equalities is simplified as

$$\begin{pmatrix} \nabla f(x^*) + \sum_{j=1}^p [y^I]_j^* \nabla^2 c_j(x^*) \eta_1^* + -A^T A \eta_1^* + \mathcal{J}c(x^*)^T [\xi_b]^* \\ \mathcal{J}c(x^*) \eta_1^* + [\xi_a]^* + [\xi_b]^* \end{pmatrix} = 0. \quad (3.11)$$

Since  $([\xi_a]^*, [\xi_b]^*) \in \widehat{N}_\Theta([y^I]^*, [z^I]^*)$ , one has from Lemma 3.1 that  $\mathcal{J}c(x^*) \eta_1^* + [\xi_a]^* + [\xi_b]^* = 0$  becomes

$$\begin{aligned} \mathcal{J}c_\alpha(x^*) \eta_1^* + [\xi_a]_\alpha^* &= 0, \quad [\xi_b]_\alpha^* = 0, \\ \mathcal{J}c_\gamma(x^*) \eta_1^* + [\xi_b]_\gamma^* &= 0, \quad [\xi_a]_\gamma^* = 0, \\ \mathcal{J}c_\beta(x^*) \eta_1^* + [\xi_a]_\beta^* + [\xi_b]_\beta^* &= 0, \quad [\xi_a]_\beta^* \leq 0, \quad [\xi_b]_\beta^* \leq 0. \end{aligned}$$

Thus (3.11) is simplified as (3.10).  $\square$

**Theorem 3.2 (M-stationary point)** Let  $(x^*, y^*, z^*)$  be a local minimizer of Problem (2.11). Let Assumption 1 be satisfied at  $(x^*, y^*, z^*)$ . Then there exist  $\eta_1^* \in \mathfrak{R}^n$ ,  $[\xi_a]_\beta^* \in \mathfrak{R}^{|\beta|}$  and  $[\xi_b]_\beta^* \in \mathfrak{R}^{|\beta|}$  satisfying

$$\{[\xi_a]_i^* [\xi_b]_i^* = 0\} \text{ or } \{[\xi_a]_i^* \leq 0 \text{ and } [\xi_b]_i^* \leq 0\} \quad \forall i \in \beta, \quad (3.12)$$

such that

$$\begin{aligned} \nabla f(x^*) + \left[ \sum_{j=1}^p [y^j]_j^* \nabla^2 c_j(x^*) - A^T A - \mathcal{J}c_\gamma(x^*)^T \mathcal{J}c_\gamma(x^*) \right] \eta_1^* + \mathcal{J}c_\beta(x^*)^T [\xi_b]_\beta^* &= 0, \\ \mathcal{J}c_\beta(x^*) \eta_1^* + [\xi_a]_\beta^* + [\xi_b]_\beta^* &= 0. \end{aligned} \quad (3.13)$$

Let

$$H(x, y, z^l) = \begin{bmatrix} A^T y^E + \mathcal{J}c(x)^T y^l \\ Ax - b + y^E \\ c(x) + y^l - z^l \\ \min\{y^l, z^l\} \end{bmatrix}, \quad (3.14)$$

Then Problem (2.11) is expressed as

$$\begin{aligned} \min_{x, y, z^l} \quad & f(x) \\ \text{s.t.} \quad & H(x, y, z^l) = 0. \end{aligned} \quad (3.15)$$

Noticing that  $F$  is a Lipschitz continuous mapping, Problem (3.15) is a Lipschitz continuous optimization problem. So we may use the optimality conditions for Lipschitz continuous optimization developed in Clarke (1983). This leads to the so-called C-stationary point. We say that the point  $(x^*, y^*, z^*)$  is a C-stationary point if there exist  $\eta_1^* \in \mathfrak{R}^n$ ,  $[\xi_a]_\beta^* \in \mathfrak{R}^{|\beta|}$  and  $[\xi_b]_\beta^* \in \mathfrak{R}^{|\beta|}$  satisfying

$$[\xi_a]_i^* [\xi_b]_i^* \geq 0 \quad \forall i \in \beta, \quad (3.16)$$

such that

$$\begin{aligned} \nabla f(x^*) + \left[ \sum_{j=1}^p [y^j]_j^* \nabla^2 c_j(x^*) - A^T A - \mathcal{J}c_\gamma(x^*)^T \mathcal{J}c_\gamma(x^*) \right] \eta_1^* + \mathcal{J}c_\beta(x^*)^T [\xi_b]_\beta^* &= 0, \\ \mathcal{J}c_\beta(x^*) \eta_1^* + [\xi_a]_\beta^* + [\xi_b]_\beta^* &= 0. \end{aligned} \quad (3.17)$$

It follows from Theorem 3.1 that under Assumption 1, the point  $(x^*, y^*, z^*)$  is a strong stationary point of Problem (2.11). Thus,  $(x^*, y^*, z^*)$  is an M-stationary point and also a C-stationary point of Problem (2.11).

C-stationary conditions are preferred to describe the necessary conditions for MPCC problems. In the following, we will see that for Problem (3.15), we may obtain a better result than conditions in (3.17) by using the optimality conditions for Lipschitz continuous optimization given by [16].

**Proposition 3.2 (Fritz-John stationary point)** Let  $(x^*, y^*, z^*)$  be a local minimizer of Problem (3.15). Then there exist  $\eta_0^* \in \mathfrak{R}_+$ ,  $\eta_1^* \in \mathfrak{R}^n$  and  $[v_b]_\beta \in \mathfrak{R}^{|\beta|}$  satisfying

$$[v_b]_i \in [0, 1], \quad i \in \beta \quad (3.18)$$

such that

$$\eta_0^* \nabla f(x^*) + \left[ \sum_{j=1}^p [y^j]_j^* \nabla^2 c_j(x^*) - A^T A - \mathcal{J}c_\gamma(x^*)^T \mathcal{J}c_\gamma(x^*) - \mathcal{J}c_\beta(x^*)^T \text{Diag}([v_b]_\beta) \mathcal{J}c_\beta(x^*) \right] \eta_1^* = 0, \quad (3.19)$$

where  $\text{Diag}(v) = \text{Diag}(v_1, \dots, v_m)$  for  $v \in \mathfrak{R}^m$ .

**Proof.** The generalized Lagrangian of (3.15) is

$$\begin{aligned} L^s(x, y^E, y^I, z^I, \eta_0, \eta_1, \eta_2, \eta_3, \xi) \\ = \eta_0 f(x) + \langle \eta_1, A^T y^E + \mathcal{J}c(x)^T y^I \rangle + \langle \eta_2, Ax - b + y^E \rangle + \langle \eta_3, c(x) + y^I - z^I \rangle + \langle \xi, \min\{y^I, z^I\} \rangle. \end{aligned}$$

It follows from the necessary optimality conditions for Lipschitz continuous optimization in [16] that there exist nonzero vectors  $(\eta_0^*, \eta_1^*, \eta_2^*, \eta_3^*, \xi^*)$  with  $\eta_0^* \geq 0$  such that

$$0 \in \partial_c L(x^*, [y^E]^*, [y^I]^*, [z^I]^*, \eta_0^*, \eta_1^*, \eta_2^*, \eta_3^*, \xi^*),$$

where  $\partial_c$  is the Clarke generalized Jacobian. Noting that

$$\partial_c \min\{[y^I]^*, [z^I]^*\} = [\text{Diag}(v_a) \quad \text{Diag}(v_b)], \quad (3.20)$$

where  $v_a \in \mathfrak{R}^p$  and  $v_b \in \mathfrak{R}^p$  satisfy

$$\begin{aligned} [v_a]_i &= 1, \quad [v_b]_i = 0, & \text{if } i \in \alpha; \\ [v_a]_i &= 0, \quad [v_b]_i = 1, & \text{if } i \in \gamma; \\ [v_a]_i &= t, \quad [v_b]_i = 1 - t, & \text{for some } t \in [0, 1] \text{ if } i \in \beta. \end{aligned} \quad (3.21)$$

Then we get from  $0 \in \partial_c L(x^*, [y^E]^*, [y^I]^*, [z^I]^*, \eta_0^*, \eta_1^*, \eta_2^*, \eta_3^*, \xi^*)$  that there exist  $v_a \in \mathfrak{R}^p$  and  $v_b \in \mathfrak{R}^p$  satisfying (3.21) such that

$$\begin{pmatrix} \eta_0^* \nabla f(x^*) + \sum_{j=1}^p [y^I]^*_j \nabla^2 c_j(x^*) \eta_1^* + A^T \eta_2^* + \mathcal{J}c(x^*)^T \eta_3^* \\ A \eta_1^* + \eta_2^* \\ \mathcal{J}c(x^*) \eta_1^* + \eta_3^* + \text{Diag}(v_a) \xi^* \\ -\eta_3^* + \text{Diag}(v_b) \xi^* \end{pmatrix} = 0.$$

This set of equalities can be simplified as

$$\begin{pmatrix} \eta_0^* \nabla f(x^*) + \sum_{j=1}^p [y^I]^*_j \nabla^2 c_j(x^*) \eta_1^* - A^T A \eta_1^* + \mathcal{J}c(x^*)^T \text{Diag}(v_b) \xi^* \\ \mathcal{J}c(x^*) \eta_1^* + \text{Diag}(v_b) \xi^* + \text{Diag}(v_a) \xi^* \end{pmatrix} = 0. \quad (3.22)$$

In view of (3.21), we have

$$\text{Diag}(v_b) + \text{Diag}(v_a) = I_p.$$

So we get from (3.22) that  $\xi^* = -\mathcal{J}c(x^*) \eta_1^*$ . Substituting this expression back to the first equation in (3.22), we obtain

$$\eta_0^* \nabla f(x^*) + \sum_{j=1}^p [y^I]^*_j \nabla^2 c_j(x^*) \eta_1^* - A^T A \eta_1^* - \mathcal{J}c(x^*)^T \text{Diag}(v_b) \mathcal{J}c(x^*) = 0.$$

Using (3.21) again, we obtain (3.19) where  $[v_b]_\beta$  satisfies (3.24).  $\square$

From Proposition 3.2, we obtain an elegant set of necessary optimality conditions as follows.

**Theorem 3.3** *Let  $(x^*, y^*, z^*)$  be a local minimizer of Problem (3.15). Suppose that the matrix*

$$\left[ \sum_{i=1}^p [y^I]^*_i \nabla^2 c_i(x^*) - A^T A - \mathcal{J}c_\gamma(x^*)^T \mathcal{J}c_\gamma(x^*) \right] \quad (3.23)$$

is negatively definite. Then there exist  $\eta_1^* \in \mathfrak{R}^n$  and  $[v_b]_\beta \in \mathfrak{R}^{|\beta|}$  satisfying

$$[v_b]_i \in [0, 1], \quad i \in \beta \quad (3.24)$$

such that

$$\nabla f(x^*) + \left[ \sum_{i=1}^p [y^I]_j^* \nabla^2 c_j(x^*) - A^T A - \mathcal{J}c_\gamma(x^*)^T \mathcal{J}c_\gamma(x^*) - \mathcal{J}c_\beta(x^*)^T \text{Diag}([v_b]_\beta) \mathcal{J}c_\beta(x^*) \right] \eta_1^* = 0. \quad (3.25)$$

**Definition 3.1** We say that  $x^*$  is a *L-stationary point* for Problem (3.15) if there exists  $\eta_1^*$  such that (3.25) is satisfied and (3.25) is called *L-stationary condition*.

Let us introduce the following notation

$$\mathcal{S}^* = \left\{ (x^*, \lambda^*) \in \mathfrak{R}^{2n} : \begin{array}{l} \nabla f(x^*) + \left[ \sum_{i=1}^p [y^I]_j^* \nabla^2 c_j(x^*) - A^T A - \mathcal{J}c_\gamma(x^*)^T \mathcal{J}c_\gamma(x^*) \right. \\ \left. - \mathcal{J}c_\beta(x^*)^T \text{Diag}([v_b]_\beta) \mathcal{J}c_\beta(x^*) \right] \lambda^* = 0 \end{array} \right\}. \quad (3.26)$$

In the next section, we will propose a smoothing function method to generate a sequence of  $\{(x^k, \lambda^k)\}$ , whose any cluster point is an element of  $\mathcal{S}^*$ .

**Remark 3.1** The condition that matrix (3.23) is negatively definite is not strict because it holds if either  $A^T A$  is positively definite or  $\nabla^2 c_i(x^*)$  is negatively definite and  $[y^I]_i^* > 0$  for some index  $i$ .

**Remark 3.2** It follows from Theorem 3.1 that under Assumption 1, the point  $(x^*, y^*, z^*)$  is a strong stationary point of Problem (2.11). Thus,  $(x^*, y^*, z^*)$  is an *M-stationary point*, also a *C-stationary point* of Problem (2.11). However, we do not know the relations between Theorem 3.1 and Theorem 3.3 because they adopt different constraint qualifications and the results are also different.

**Remark 3.3** If there are no inequality constraints, we may check that (3.25) or (3.10) is a set of sufficient optimality conditions for  $x^*$  being an optimal solution to Problem (2.8) when  $f$  is a convex function.

## 4 The Smoothing Fischer-Burmeister Function Method

In this section, we only consider the case when the constraints in Problem (3.1) are inconsistent. In this case, the optimization problem with least constraint violation is equivalent to Problem (3.5) if  $h$  is an affine mapping and  $g_i$  is a smooth concave function for  $i = 1, \dots, p$ . We will present a smoothing function method to solve Problem (3.5). Let

$$G(x, y, z^I) = \begin{bmatrix} A^T y^E + \mathcal{J}c(x)^T y^I \\ Ax - b + y^E \\ c(x) + y^I - z^I \end{bmatrix}, \quad (4.1)$$

Then Problem (3.5) is an MPCC problem of the following form

$$\begin{array}{ll} \min_{x, y, z^I} & f(x) \\ \text{s.t.} & G(x, y, z^I) = 0, \\ & (y^I, z^I) \in \Theta. \end{array} \quad (4.2)$$

It is well known that, for such a problem, it is not suitable to treat it as a traditional NLP problem because, as explained in [9, Example 3.1.1 and Example 3.1.2], even the basic constraint qualification (namely, the tangent cone is equal to the linearized cone at an optimal solution) does not hold. To overcome this difficulty,

various relaxation approaches have been proposed dealing with the complementarity constraints. Facchinei et al. (1999) [6] and Fukushima and Pang (1999) [7] used  $\psi_\varepsilon(a, b) = 0$  to approximate the complementarity relation:  $0 \leq a, 0 \leq b, ab = 0$ , where  $\psi_\varepsilon(a, b)$  is the smoothing Fischer-Burmeister function

$$\psi_\varepsilon(a, b) = a + b - \sqrt{a^2 + b^2 + 2\varepsilon^2}. \quad (4.3)$$

Other relaxations of the complementarity relation can be found in for example Scholtes [15], which uses

$$a \geq 0, b \geq 0, ab \leq \varepsilon,$$

and Lin and Fukushima [8], which uses

$$(a + \varepsilon)(b + \varepsilon) \geq \varepsilon^2 \text{ and } ab \leq \varepsilon^2.$$

In this section, we shall use  $\psi_\varepsilon(a, b) = 0$  to approximate the complementarity relation, where  $\phi_\varepsilon(a, b)$  is the smoothing Fischer-Burmeister function defined by (4.3).

Define

$$\Psi_\varepsilon(y^l, z^l) = \begin{bmatrix} \psi_\varepsilon(y_1^l, z_1^l) \\ \vdots \\ \psi_\varepsilon(y_p^l, z_p^l) \end{bmatrix} \quad (4.4)$$

and

$$\Theta(\varepsilon) := \{(y^l, z^l) \in \mathbb{R}^p \times \mathbb{R}^p : \Psi_\varepsilon(y^l, z^l) = 0\}. \quad (4.5)$$

Then if  $(y^l, z^l) \in \Theta(\varepsilon)$ , we have

$$y^l > 0, z^l > 0 \text{ and } y_i^l z_i^l = \varepsilon^2, i = 1, \dots, p.$$

Obviously,  $\psi_0(a, b) = 0$  if and only if  $0 \leq a, 0 \leq b, ab = 0$ . Therefore  $\Theta(0) = \Theta$ .

For any  $(y^l, z^l) \in \mathbb{R}^{2p}$ , we have

$$\mathcal{J}_{y^l, z^l} \Psi_\varepsilon(y^l, z^l) = \begin{bmatrix} \mathcal{J}_{y^l} \Psi_\varepsilon(y^l, z^l) & \mathcal{J}_{z^l} \Psi_\varepsilon(y^l, z^l) \end{bmatrix},$$

where

$$\mathcal{J}_{y^l} \Psi_\varepsilon(y^l, z^l) = \begin{bmatrix} 1 - \frac{[y^l]_1}{\sqrt{[y^l]_1^2 + [z^l]_1^2 + 2\varepsilon^2}} & & \\ & \ddots & \\ & & 1 - \frac{[y^l]_p}{\sqrt{[y^l]_p^2 + [z^l]_p^2 + 2\varepsilon^2}} \end{bmatrix}$$

and

$$\mathcal{J}_{z^l} \Psi_\varepsilon(y^l, z^l) = \begin{bmatrix} 1 - \frac{[z^l]_1}{\sqrt{[y^l]_1^2 + [z^l]_1^2 + 2\varepsilon^2}} & & \\ & \ddots & \\ & & 1 - \frac{[z^l]_p}{\sqrt{[y^l]_p^2 + [z^l]_p^2 + 2\varepsilon^2}} \end{bmatrix}.$$

Let  $(y^l, z^l) \in \Theta_\varepsilon$ . Then for  $i = 1, \dots, p$ ,

$$[y^l]_i + [z^l]_i - \sqrt{[y^l]_i^2 + [z^l]_i^2 + 2\varepsilon^2} = 0,$$

we have  $[y^l]_i > 0$ ,  $[z^l]_i > 0$  and  $[y^l]_i[z^l]_i = \varepsilon^2$ . Thus

$$\begin{aligned} 1 - \frac{[y^l]_i}{\sqrt{[y^l]_i^2 + [z^l]_i^2 + 2\varepsilon^2}} &= 1 - \frac{[y^l]_i}{\sqrt{[y^l]_i^2 + [z^l]_i^2 + 2[y^l]_i[z^l]_i}} \\ &= 1 - \frac{[y^l]_i}{[y^l]_i + [z^l]_i} \\ &= \frac{[z^l]_i}{[y^l]_i + [z^l]_i}, \end{aligned}$$

and in turn we obtain

$$1 - \frac{[y^l]_i}{\sqrt{[y^l]_i^2 + [z^l]_i^2 + 2\varepsilon^2}} = \frac{[z^l]_i}{[y^l]_i + [z^l]_i}, \quad 1 - \frac{[z^l]_i}{\sqrt{[y^l]_i^2 + [z^l]_i^2 + 2\varepsilon^2}} = \frac{[y^l]_i}{[y^l]_i + [z^l]_i}. \quad (4.6)$$

Obviously, for any  $\varepsilon > 0$ , both  $\mathcal{J}_y^l \Psi_\varepsilon(y^l, z)$  and  $\mathcal{J}_z \Psi_\varepsilon(y^l, z)$  are nonsingular matrices. We can easily obtain the following conclusion.

**Lemma 4.1** *Let  $\varepsilon > 0$ . Then for any  $(y^l, z^l) \in \Theta(\varepsilon)$ , the linear independence constraint qualification (LICQ) holds and the tangent cone of  $\Theta(\varepsilon)$  at  $(y^l, z^l)$  is*

$$T_{\Theta(\varepsilon)}(y^l, z^l) = \{(\Delta y^l, \Delta z^l) \in \mathfrak{R}^{2m} : \mathcal{J}_{y^l, z^l} \Psi_\varepsilon(y^l, z^l)(\Delta y^l, \Delta z^l) = 0\}, \quad (4.7)$$

and the normal cone of  $\Theta(\varepsilon)$  at  $(y^l, z^l)$  is

$$N_{\Theta(\varepsilon)}(y^l, z^l) = \widehat{N}_{\Theta(\varepsilon)}(y^l, z^l) = \mathcal{J}_{y^l, z^l} \Psi_\varepsilon(y^l, z^l)^T \mathfrak{R}^p. \quad (4.8)$$

We use the following problem, denoted by  $P_\varepsilon$ , to approximate Problem (4.2):

$$\begin{aligned} \min_{x, y, z^l} \quad & f(x) \\ \text{s.t.} \quad & G(x, y, z^l) = 0, \\ & (y^l, z^l) \in \Theta(\varepsilon), \end{aligned} \quad (4.9)$$

where  $\Theta(\varepsilon)$  is defined by (4.5). Furthermore, we use  $\Phi(\varepsilon)$  to denote the feasible set for Problem (4.9); namely,

$$\Phi(\varepsilon) = \{(x, y^E, y^l, z^l) \in \mathfrak{R}^n \times \mathfrak{R}^q \times \Theta(\varepsilon) : G(x, y, z^l) = 0\}. \quad (4.10)$$

Define

$$F_\varepsilon(x, y, z^l) = \begin{bmatrix} G(x, y, z^l) \\ \Psi_\varepsilon(y^l, z^l) \end{bmatrix}. \quad (4.11)$$

Then  $\Phi(\varepsilon)$  is expressed as

$$\Phi(\varepsilon) = \{(x, y^E, y^l, z^l) \in \mathfrak{R}^n \times \mathfrak{R}^q \times \mathfrak{R}^p \times \mathfrak{R}^p : F_\varepsilon(x, y, z^l) = 0\}.$$

By some calculations, we obtain

$$\mathcal{J}F_\varepsilon(\vartheta, \mu, y^l, z^l) = \begin{bmatrix} \sum_{i=1}^p [y^l]_i \nabla^2 c_i(x) & A^T & \mathcal{J}c(x)^T & 0 \\ A & I & 0 & 0 \\ \mathcal{J}c(x) & 0 & I & -I \\ 0 & 0 & \mathcal{J}_{y^l} \Psi_\varepsilon(y^l, z^l) & \mathcal{J}_{z^l} \Psi_\varepsilon(y^l, z^l) \end{bmatrix}. \quad (4.12)$$

Similarly to the proof of Proposition 3.1, we can establish the following result.

**Proposition 4.1** For  $(x, y, z^l) \in \Phi(\varepsilon)$ , if

$$\sum_{i=1}^p [y^l]_i \nabla^2 c_i(x) - AA^T - \mathcal{J}c(x)^T \mathcal{J}_{z^l} \Psi_\varepsilon(y^l, z^l) \mathcal{J}c(x) \quad (4.13)$$

is nonsingular, then  $\mathcal{J}F_\varepsilon(x, y, z^l)$  is full of row rank. In this case,

$$T_{\Phi(\varepsilon)}(x, y, z^l) = \{d \in \mathfrak{X}^n \times \mathfrak{X}^{q+p} \times \mathfrak{X}^p : \mathcal{J}F_\varepsilon(x, y, z^l)d = 0\} \quad (4.14)$$

and

$$N_{\Phi(\varepsilon)}(x, y, z^l) = \widehat{N}_{\Phi_\varepsilon}(x, y, z^l) = \mathcal{J}F_\varepsilon(\vartheta, \mu, y, z^l)^T \mathfrak{X}^{n+q+p+p}. \quad (4.15)$$

**Proof.** Let us check that  $\mathcal{J}F_\varepsilon(x, y, z^l)^T$  is of full rank in column. For  $\xi_1 \in \mathfrak{X}^n$ ,  $\xi_2 \in \mathfrak{X}^q$ ,  $\xi_3 \in \mathfrak{X}^p$  and  $\xi_4 \in \mathfrak{X}^p$ , consider

$$\mathcal{J}F_\varepsilon(x, y, z^l)^T \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix} = 0.$$

It is equivalent to

$$\begin{bmatrix} \sum_{i=1}^p [y^l]_i \nabla^2 c_i(x) \xi_1 + A^T \xi_2 + \mathcal{J}c(x)^T \xi_3 \\ A \xi_1 + \xi_2 \\ \mathcal{J}c(x) \xi_1 + \xi_3 + \mathcal{J}_{y^l} \Psi_\varepsilon(y^l, z^l) \xi_4 \\ -\xi_3 + \mathcal{J}_{z^l} \Psi_\varepsilon(y^l, z^l) \xi_4 \end{bmatrix} = 0. \quad (4.16)$$

Noting that

$$\mathcal{J}_{y^l} \Psi_\varepsilon(y^l, z^l) + \mathcal{J}_{z^l} \Psi_\varepsilon(y^l, z^l) = I_p,$$

one has from (4.16) that

$$\begin{aligned} \xi_2 &= -A \xi_1 \\ \xi_4 &= \mathcal{J}c(x) \xi_1 \\ \xi_3 &= \mathcal{J}_{z^l} \Psi_\varepsilon(y^l, z^l) \mathcal{J}c(x) \xi_1 \end{aligned} \quad (4.17)$$

and

$$\left[ \sum_{i=1}^p [y^l]_i \nabla^2 c_i(x) - AA^T - \mathcal{J}c(x)^T \mathcal{J}_{z^l} \Psi_\varepsilon(y^l, z^l) \mathcal{J}c(x) \right] \xi_1 = 0. \quad (4.18)$$

From the assumption that the matrix of 4.13 is nonsingular, we obtain from (4.18) that  $\xi_1 = 0$  and in turn from (4.17) that  $\xi_2 = 0$ ,  $\xi_3 = 0$  and  $\xi_4 = 0$ . Thus  $\mathcal{J}F_\varepsilon(x, y, z^l)$  is full of row rank and hence (4.14) and (4.15) follow from Chapter 6 of [14].  $\square$

**Lemma 4.2** For  $\Theta(\varepsilon)$  defined by (4.5), we have

$$\lim_{\varepsilon \searrow 0} \Theta(\varepsilon) = \Theta(0). \quad (4.19)$$

**Proof.** For any  $(y^l, z^l) \in \limsup_{\varepsilon \searrow 0} \Theta(\varepsilon)$ , there exist  $\varepsilon_k \searrow 0$  and  $([y^l]^k, [z^l]^k) \in \Theta(\varepsilon_k)$  such that  $([y^l]^k, [z^l]^k) \rightarrow (y^l, z^l)$ . The inclusion  $([y^l]^k, [z^l]^k) \in \Theta(\varepsilon_k)$  implies that

$$[y^l]^k + [z^l]^k - \sqrt{([y^l]^k)^2 + ([z^l]^k)^2 + 2\varepsilon_k^2} = 0.$$

Then, letting  $k \rightarrow \infty$ , we have

$$y^l + z^l - \sqrt{[y^l]^2 + [z^l]^2} = 0;$$



namely,  $\psi_0(y^l, z^l) = 0$  and  $(y^l, z^l) \in \Theta(0)$ . Therefore we have

$$\limsup_{\varepsilon \searrow 0} \Theta(\varepsilon) \subset \Theta(0).$$

For any  $(y^l, z^l) \in \Theta(0)$ , let

$$I_+ = \{i : [y^l]_i > 0\}, J_+ = \{i : [z^l]_i > 0\}, I_0 = \{1, \dots, m\} \setminus (I_+ \cup J_+).$$

For any  $\varepsilon > 0$  defined  $(y^l(\varepsilon), z(\varepsilon))$  by

$$([y^l]_i(\varepsilon), [z^l]_i(\varepsilon)) = \begin{cases} ([y^l]_i, \varepsilon^2/[y^l]_i), & \text{if } i \in I_+; \\ (\varepsilon^2/[z^l]_i, [z^l]_i), & \text{if } i \in J_+; \\ (\varepsilon, \varepsilon) & \text{if } i \in I_0, \end{cases} \quad (4.20)$$

Then  $\psi_\varepsilon([y^l]_i(\varepsilon), [z^l]_i(\varepsilon)) = 0$  for  $i = 1, \dots, m$ . Thus  $\Psi_\varepsilon(y^l(\varepsilon), z^l(\varepsilon)) = 0$  or, equivalently,  $(y^l(\varepsilon), z^l(\varepsilon)) \in \Theta(\varepsilon)$ . Obviously,  $(y^l(\varepsilon), z^l(\varepsilon)) \rightarrow (y^l, z^l)$ . This implies that

$$\liminf_{\varepsilon \searrow 0} \Theta(\varepsilon) \supset \Theta(0).$$

Therefore  $\Theta(\varepsilon) \rightarrow \Theta(0)$  as  $\varepsilon \searrow 0$ . □

**Corollary 4.1** *Let  $\Phi(\varepsilon)$  be defined by (4.10). Then*

$$\Phi(\varepsilon) \rightarrow \Phi \text{ as } \varepsilon \searrow 0.$$

**Proof.** The result can be obtained by noting that  $\Phi(\varepsilon)$  and  $\Phi$  can be expressed as

$$\Phi(\varepsilon) = \{(x, y^E, y^l, z^l) \in \mathfrak{X}^n \times \mathfrak{X}^q \times \mathfrak{X}^p \times \mathfrak{X}^p : G(x, y^E, y^l, z^l) = 0\} \cap \mathfrak{X}^n \times \mathfrak{X}^q \times \Theta(\varepsilon)$$

and

$$\Phi = \{(x, y^E, y^l, z^l) \in \mathfrak{X}^n \times \mathfrak{X}^q \times \mathfrak{X}^p \times \mathfrak{X}^p : G(x, y^E, y^l, z^l) = 0\} \cap \mathfrak{X}^n \times \mathfrak{X}^q \times \Theta,$$

respectively. □

Now denote the optimal value and the (global) solution set of Problem  $P_\varepsilon$  by  $\kappa(\varepsilon)$  and  $S(\varepsilon)$ , respectively; namely,

$$\kappa(\varepsilon) := \inf\{f(x) \mid (x, y, z^l) \in \Phi(\varepsilon)\},$$

$$S(\varepsilon) := \text{Argmin}\{f(x) \mid (x, y, z^l) \in \Phi(\varepsilon)\}.$$

**Theorem 4.1** *Let  $f$  be level-bounded; namely, the level set of  $f$  is bounded. Let  $P_\varepsilon$  is defined by (4.9), and  $\kappa(\varepsilon)$  and  $S(\varepsilon)$  be its optimal value and solution set, respectively. Then the function  $\kappa(\varepsilon)$  is continuous at 0 with respect to  $\mathfrak{X}_+$  and the set-valued mapping  $S(\varepsilon)$  is outer semi-continuous at 0 with respect to  $\mathfrak{X}_+$ .*

**Proof.** As  $f$  is level-bounded, we have  $\kappa(\varepsilon)$  is finite and  $S(\varepsilon) \neq \emptyset$  for any  $\varepsilon \geq 0$ . Let

$$\widehat{f}_\varepsilon(x, y, z^l) = f(x) + \delta_{\Phi(\varepsilon)}(x, y, z^l),$$

where  $\delta_{\Phi(\varepsilon)}$  is the indicator function of  $\Phi(\varepsilon)$ . From Lemma 4.2,  $\Phi(\varepsilon) \rightarrow \Theta(0)$  as  $\varepsilon \searrow 0$ ,  $\widehat{f}_\varepsilon$  epi-converges to  $\widehat{f}_0$ . The level-boundedness of  $\widehat{f}_\varepsilon$  is easily verified for  $\varepsilon \geq 0$ . Therefore, we have from Theorem 7.41 of Rockafellar and Wets (1998) that the function  $\kappa(\varepsilon)$  is continuous at 0 with respect to  $\mathfrak{X}_+$  and the set-valued mapping  $S(\varepsilon)$  is outer semi-continuous at 0 with respect to  $\mathfrak{X}_+$ . The proof is completed. □

If  $(x, y, z^l) \in \Phi(\varepsilon)$  is a local minimizer of  $P_\varepsilon$  and  $\mathcal{J}F_\varepsilon(x, y, z^l)$  is of full row rank, then there exists a vector  $\xi \in \mathfrak{R}^{n+q+2p}$  such that

$$\nabla_{x,y,z^l} f(x) + \mathcal{J}F_\varepsilon(x, y, z^l)^T \xi = 0,$$

which is reduced to

$$\nabla_x f(x) + \left[ \sum_{i=1}^p [y^l]_i \nabla^2 c_i(x) - AA^T - \mathcal{J}c(x)^T \mathcal{J}_{z^l} \Psi_\varepsilon(y^l, z^l) \mathcal{J}c(x) \right] \xi_1 = 0.$$

This leads to the following definition.

**Definition 4.1** We say  $(x, y, z^l) \in \Phi(\varepsilon)$  is a stationary point of  $P_\varepsilon$  if there exists a vector  $\lambda \in \mathfrak{R}^n$

$$\nabla_x f(x) + \left[ \sum_{i=1}^p [y^l]_i \nabla^2 c_i(x) - AA^T - \mathcal{J}c(x)^T \mathcal{J}_{z^l} \Psi_\varepsilon(y^l, z^l) \mathcal{J}c(x) \right] \lambda = 0. \quad (4.21)$$

The following theorem is about the convergence of the stationary points for  $P_\varepsilon$ , which shows that a cluster point of stationary points for  $P_\varepsilon$  is related to the condition (4.24) when  $\varepsilon \searrow 0$ .

**Theorem 4.2** Let  $(x(\varepsilon), y(\varepsilon), z^l(\varepsilon)) \in \mathfrak{R}^{n+q+2p}$  be a stationary point for  $P_\varepsilon$  for  $\varepsilon > 0$ , with multiplier  $\lambda(\varepsilon) \in \mathfrak{R}^n$ . Then for any

$$(x^*, y^*, [z^l]^*, \lambda^*) \in \limsup_{\varepsilon \searrow 0} \{(x(\varepsilon), y(\varepsilon), z^l(\varepsilon), \lambda(\varepsilon))\},$$

one has that  $(x^*, \lambda^*) \in \mathcal{S}^*$ , where  $\mathcal{S}^*$  is defined by (3.26).

**Proof.** Let  $(x^*, y^*, [z^l]^*, \lambda^*) \in \limsup_{\varepsilon \searrow 0} \{(x(\varepsilon), y(\varepsilon), z^l(\varepsilon), \lambda(\varepsilon))\}$ . Then there exists a sequence  $\varepsilon_k \searrow 0$  and  $(x^k, y^k, [z^l]^k, \lambda^k) = (x(\varepsilon_k), y(\varepsilon_k), [z^l](\varepsilon_k), \lambda(\varepsilon_k))$  such that  $(x^k, y^k, [z^l]^k, \lambda^k) \rightarrow (x^*, y^*, [z^l]^*, \lambda^*)$  with

$$\nabla_x f(x^k) + \left[ \sum_{i=1}^p [y^l]_i^k \nabla^2 c_i(x^k) - AA^T - \mathcal{J}c(x^k)^T \mathcal{J}_{z^l} \Psi_\varepsilon([y^l]^k, [z^l]^k) \mathcal{J}c(x^k) \right] \lambda^k = 0. \quad (4.22)$$

It follows from Lemma 4.2 that  $([y^l]^*, [z^l]^*) \in \Theta$ . Define

$$\alpha = \{i : [y^l]_i^* = 0 < [z^l]_i^*\}, \quad \beta = \{i : [y^l]_i^* = 0 = [z^l]_i^*\}, \quad \gamma = \{i : [y^l]_i^* > 0 = [z^l]_i^*\}.$$

Noting that

$$\mathcal{J}_{z^l} \Psi_\varepsilon([y^l]^k, [z^l]^k) = \begin{bmatrix} \frac{[y^l]_1^k}{[z^l]_1^k + [y^l]_1^k} & & & \\ & \ddots & & \\ & & & \frac{[y^l]_p^k}{[z^l]_p^k + [y^l]_p^k} \end{bmatrix},$$

we have

$$\frac{[y^l]_i^k}{[z^l]_i^k + [y^l]_i^k} \rightarrow \begin{cases} 0, & i \in \alpha; \\ 1, & i \in \gamma. \end{cases}$$

For  $i \in \beta$ , since  $\frac{[y^l]_i^k}{[z^l]_i^k + [y^l]_i^k} \in (0, 1)$ , it has an cluster point  $\eta_i \in [0, 1]$ . Thus there exists  $\{k_m : m \in \mathbf{N}\}$  such that

$$\frac{[y^l]_i^{k_m}}{[z^l]_i^{k_m} + [y^l]_i^{k_m}} \rightarrow \begin{cases} 0, & i \in \alpha; \\ \eta_i, & i \in \beta; \\ 1, & i \in \gamma. \end{cases} \quad (4.23)$$

Taking the limit for  $k = k_m$ ,  $m \rightarrow \infty$  in (4.22), we obtain

$$\nabla f(x^*) + \left[ \sum_{j=1}^p [y^I]_j^* \nabla^2 c_j(x^*) - A^T A - \mathcal{J}c_\gamma(x^*)^T \mathcal{J}c_\gamma(x^*) - \mathcal{J}c_\beta(x^*)^T \text{Diag}(\eta_\beta) \mathcal{J}c_\beta(x^*) \right] \lambda^* = 0 \quad (4.24)$$

with  $\eta_i \in [0, 1]$  for  $i \in \beta$ . Thus  $(x^*, y^*, [z^I]^*, \lambda^*)$  satisfied (4.24) and  $(x^*, \lambda^*) \in \mathcal{S}^*$ . The proof is completed.  $\square$

**Theorem 4.3** Let  $(x(\varepsilon), y(\varepsilon), z^I(\varepsilon)) \in \mathfrak{X}^{n+q+2p}$  be a local minimizer of  $P_\varepsilon$  for  $\varepsilon > 0$ . Let

$$(x^*, y^*, [z^I]^*) \in \limsup_{\varepsilon \searrow 0} \left\{ (x(\varepsilon), y(\varepsilon), z^I(\varepsilon)) \right\}.$$

If the matrix

$$\left[ \sum_{j=1}^p [y^I]_j^* \nabla^2 c_j(x^*) - A^T A - \mathcal{J}c_\gamma(x^*)^T \mathcal{J}c_\gamma(x^*) \right] \quad (4.25)$$

is negatively definite, then there exists a vector  $\lambda^* \in \mathfrak{X}^n$  such that  $(x^*, \lambda^*) \in \mathcal{S}^*$ .

**Proof.** For  $(x^*, y^*, [z^I]^*) \in \limsup_{\varepsilon \searrow 0} \left\{ (x(\varepsilon), y(\varepsilon), z^I(\varepsilon)) \right\}$ , there exists a sequence  $\varepsilon_k \searrow 0$  and  $(x^k, y^k, [z^I]^k) = (x(\varepsilon_k), y(\varepsilon_k), [z^I](\varepsilon_k))$  such that  $(x^k, y^k, [z^I]^k) \rightarrow (x^*, y^*, [z^I]^*)$ . Since the matrix in (4.25) is negatively definite, the matrix

$$\left[ \sum_{i=1}^p [y^I]_i^k \nabla^2 c_i(x^k) - AA^T - \mathcal{J}c(x^k)^T \mathcal{J}_{z^I} \Psi_\varepsilon([y^I]^k, [z^I]^k) \mathcal{J}c(x^k) \right] \quad (4.26)$$

is negatively definite for  $k$  large enough. Then there exists a unique vector  $\lambda^k \in \mathfrak{X}^n$  such that

$$\nabla f(x^k) + \left[ \sum_{i=1}^p [y^I]_i^k \nabla^2 c_i(x^k) - AA^T - \mathcal{J}c(x^k)^T \mathcal{J}_{z^I} \Psi_\varepsilon([y^I]^k, [z^I]^k) \mathcal{J}c(x^k) \right] \lambda^k = 0.$$

Then

$$\lambda^k = - \left[ \sum_{i=1}^p [y^I]_i^k \nabla^2 c_i(x^k) - AA^T - \mathcal{J}c(x^k)^T \mathcal{J}_{z^I} \Psi_\varepsilon([y^I]^k, [z^I]^k) \mathcal{J}c(x^k) \right]^{-1} \nabla f(x^k)$$

and  $\lambda^k$  has a cluster point  $\lambda^*$  such that there exists  $[v_b]_\beta \in \mathfrak{X}^{|\beta|}$  satisfying

$$[v_b]_i \in [0, 1], \quad i \in \beta$$

and

$$\nabla f(x^*) + \left[ \sum_{j=1}^p [y^I]_j^* \nabla^2 c_j(x^*) - A^T A - \mathcal{J}c_\gamma(x^*)^T \mathcal{J}c_\gamma(x^*) - \mathcal{J}c_\beta(x^*)^T \text{Diag}([v_b]_\beta) \mathcal{J}c_\beta(x^*) \right] \lambda^* = 0.$$

The proof is completed.  $\square$

From the above theorem, we see that the smoothing Fischer-Burmeister function method works for the convex nonlinear programming with inconsistent constraints. Specifically, when the positive smoothing parameter of the method approaches to zero, any point in the outer limit of the KKT-point mapping is an L-stationary point of the equivalent MPCC problem.

## 5 Discussions

This paper established the optimization model with least constraint violation to model constrained optimization problems with possible inconsistent constraints. If the constraints are consistent, the model is reduced

to the original problem. When the constraints in a conic optimization problem are possibly inconsistent, the model is reformulated as an MPEC problem. For the nonlinear programming problem with possibly inconsistent convex constraints, several stationary points for the equivalent MPCC problem were given. Importantly, the so-called L-stationary point is proposed, from the optimality theory for Lipschitz continuous optimization. The smoothing Fischer-Burmeister function method is constructed to solve the equivalent MPCC problem and any accumulation point of the sequence generated by the smoothing function method is an L-stationary point.

There are many topics left to investigate for optimization with least constraint violation. When we do not know whether the feasible region is nonempty or not, the optimization problem with least violation is always feasible, this is its advantage. However, if the constrained problem is feasible, then the model involves the infeasibility measure  $\theta(x)$ , which is usually smooth but not twice differentiable even functions of the original problem are all twice differentiable, this brings computational difficulties. The smoothing Fischer-Burmeister function method only deals with the case when the constraints are inconsistent for convex nonlinear programming, it has nothing to do with the original problem when it is feasible. Is it possible for us to propose a unified algorithm, which can solve the optimization problem with least violation no matter when the original problem is either infeasible or feasible? Another question is as follows. The smoothing function algorithm can only cope with the nonlinear programming problem with least constraint violation. Can we construct algorithms to deal with other conic optimization problems, for example nonlinear semidefinite optimization problem?

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