

New exact approaches for the combined cell layout problem and extensions of the multi-bay facility layout problem

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In this paper we consider the Combined Cell Layout Problem (CCLP), the Multi-Bay Facility Layout Problem (MBFLP) and several generalizations of the MBFLP, which have wide applications, e.g., in factory planning, heavy manufacturing, semiconductor fabrication and arranging rooms in hospitals. Given a set of cells of type single-row or directed-circular and a set of one-dimensional departments with pairwise transport weights between them, the CCLP asks for an assignment of the departments to the cells such that departments in the same cell do not overlap and such that the sum of the weighted center-to-center distances is minimized. Distances between departments in the same cell are measured according to the layout type of the cell and otherwise their distance equals the sum of the distances to the associated (un-)loading stations of the cells plus possible space between the cells. We solve the CCLP exactly by enumerating over all assignments of the departments to the cells and solving several CCLP with fixed-cell assignment. We show how to reduce the number of distinguishable cell assignments significantly by merging two cells of type single-row. This leads to new well-performing exact approaches for the CCLP, the MBFLP and its generalizations where arising subproblems are solved via (new) mixed-integer linear programming models. In a computational study we compare the computation times and the optimal values of various facility layout problems in order to support the decision maker to choose a layout.

Key words. Facility Planning and Design, Combined Cell Layout Problem, Multi-Bay Layout Problem, Pier-Type Layouts, Exact Approaches

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1 Introduction

The aim of facility layout problems is to find an optimal non-overlapping arrangement of departments inside a plant according to a given objective function, e. g., minimizing material-handling costs. It constitutes an important problem for manufacturing industries as up to 50 % of manufacturing costs are due to moving parts between different facilities, and thus a good arrangement of facilities might reduce up to 30 % of material-handling costs [53]. In contrast, a poor layout can add up to 36 % of material-handling costs [16]. Especially in terms of Industry 4.0 and Smart Manufacturing minimizing material-handling costs, and hence finding an optimal layout of the departments within the facilities, plays an enormous role [2, 46].

In this work we consider various facility layout problems and propose new well-performing exact approaches for solving them. We are given the following setting for all layout problems considered: A set of $n \in \mathbb{N}$ one-dimensional departments $[n] := \{1, \dots, n\}$, with lengths ℓ_i , $i \in [n]$, and pairwise weights w_{ij} , $i, j \in [n]$, $i \neq j$, and a set of cells \mathcal{C} . The task is to minimize the sum of the weighted center-to-center distances of the departments such that departments in the same cell do not overlap. The considered problems differ in two aspects – the structure of the layout, i. e., the number of cells and their (un-) loading positions – and the distance calculation. We describe the differences in the following.

At first, we consider problems consisting of one cell and we deal with the Single-Row Facility Layout Problem (SRFLP) and the Directed Circular Facility Layout Problem (DCFLP). In the SRFLP one looks for an arrangement of the departments in one row such that the weighted sum of the horizontal center-to-center distances between the departments is minimized. Like for all other layout problems considered in this work, the arrangement of machines within flexible manufacturing systems is a perfect application example [29]. Further, applications can be found in the alignment of departments in office buildings, hospitals or in supermarkets [52], the assignment of files to disk cylinders in computer storage, and the design of warehouse layouts [42, 48]. In the DCFLP the task is to find an arrangement of the departments along a circle such that the weighted sum of the center-to-center distances measured in clockwise direction is minimized. According to [1, 44], the DCFLP has several practical advantages over the SRFLP, e. g., relative low initial investment costs because of their space-saving design and high material handling flexibility. The DCFLP arises by, e. g., determining a space-free alignment around a cyclic conveyor system or the cyclic motion path of an industrial robot. Both, the SRFLP and the DCFLP are widely studied, see, e. g., [5, 15, 36, 37, 38, 40].

The focus of this work lies on facility layout problems consisting of several cells where we concentrate on problems with up to four cells in the computational experiments. At first, we consider the Combined Cell Layout Problem (CCLP), which is a generalization of the SRFLP and the DCFLP. We are given a set of cells $\mathcal{C} := \{1, \dots, m\}$, $m \in [n]$, $m \geq 2$, each with an (un-) loading station whose position is denoted by p_{E_k} , $k \in \mathcal{C}$ (in this paper we always assume that loading and unloading station of a cell are the same). The function $t: \mathcal{C} \rightarrow \{\text{SRFLP}, \text{DCFLP}\}$ specifies the associated layout type of each cell. For $t(k) = \text{SRFLP}$, $k \in \mathcal{C}$, the position p_{E_k} of the loading station is fixed at the left or right border of cell k and for $t(k) = \text{DCFLP}$, $k \in \mathcal{C}$, the loading station can be placed on an arbitrary position along the circle. The inner-cell distances depend on the type of the cell. The inter-cell distance between cell $k \in \mathcal{C}$ and cell $o \in \mathcal{C}$, $k < o$, is denoted by u_{ko} and the distance between departments in different cells equals the sum of the distances of the departments to the respective loading station in the same cell plus the corresponding inter-cell distance u_{ko} . We also write CCLP (m_1, m_2) , $m = m_1 + m_2$, where $m_1 \in \mathbb{Z}_{\geq 0}$ denotes the number of cells of type SRFLP and $m_2 \in \mathbb{Z}_{\geq 0}$ denotes the number of cells of type DCFLP.

Several variants of the CCLP have been studied in the literature in more detail. If the assignment of the departments to the cells is fixed, the problem is called Fixed-Cell Combined Cell Layout Problem (FC-CCLP)¹. If $t(k) = \text{SRFLP}$ for all $k \in \mathcal{C}$, then this problem is denoted as Multi-Bay

¹In [15, 32] this problem is denoted by Combined Cell Layout Problem instead of Fixed-Cell Combined Cell

Facility Layout Problem (MBFLP), see, e. g., [26, 41, 47]. The cells are arranged in a non-overlapping and in a parallel way, and hence the inter-cell distance is set to $u_{ko} = w_{path}(o - k)$, $k, o \in \mathcal{C}$, $k < o$, $w_{path} \in \mathbb{R}_{\geq 0}$. The MBFLP is in particular interesting in practice in heavy manufacturing, e. g., steel production and bridge crane manufacturing, and semiconductor fabrication [45, 47, 54], as well as arranging the rooms of the patients in hospitals where often only one side of a corridor has windows [48]. Many real-world factory layouts implicitly use these layout structures, see, e. g., [18, 45]. The layout problem similar to the MBFLP but with $u_{ko} = w_{path}$ for some $w_{path} \in \mathbb{R}_{\geq 0}$ is denoted by Pier-Type Material Flow Pattern (PMFP), see [21]. The PMFP has applications in the design of cross docking warehouses [21]. The size of the factory, and thus the costs of the initial investment increases with a large number of cells. Hence, we focus on the MBFLP and the PMFP with $m = 3$ and $m = 4$ which we call (3-BFLP), (4-BFLP), (3-PMFP) and (4-PMFP), respectively. For instance, in a hospital each floor corresponds to a cell and the initial investment costs for building a hospital usually increase with the number of floors. The departments are given as one-dimensional objects. Hence, we assume implicitly that the departments have the same height and we assume, w. l. o. g., that the height equals one. In the MBFLP with $m \geq 3$ the vertical distance between a department in cell i and a department in cell j , $i, j \in [m]$, $i < j$, is at least the height of row i plus the height of the cells between cell i and cell j . The vertical distances can be included by enlarging w_{path} , thus, in the following, we assume that the vertical distance is included in w_{path} if $w_{path} > 0$.

The T-Row Facility Layout Problem (TRFLP) was introduced in [23] and consists of two cells of type SRFLP where the position of the loading station in cell 2 is fixed at the border and in cell 1 the position p_{E_1} can be chosen arbitrarily. It is shown in [23] that the TRFLP is an extension of the 3-BFLP, and hence has the same applications. For further applications we refer to [23].

We continue this line of research and introduce the X-Row Facility Layout Problem (XRFLP). Recall that we assume that the height of the departments equals one. Given four non-overlapping cells of type SRFLP which form an X and let $d_{ip_{E_k}}$ denote the distance of the center of $i \in [n]$ to p_{E_k} if i is assigned to cell $k \in [4]$. Let C_2 (C_3) denote the set of departments assigned to cell 2 (cell 3), then one has to ensure that the departments in cell 2 and in cell 3 do not overlap, i. e., either $d_{ip_{E_2}} \geq 1 + \frac{\ell_i}{2}$, $i \in C_2$, or $d_{ip_{E_3}} \geq 1 + \frac{\ell_i}{2}$, $i \in C_3$, has to be satisfied. For an illustration we refer to Figure 1d. The XRFLP is an extension of the 4-BFLP because one has to ensure additionally that departments in cell 2 and cell 3 do not overlap. In factory planning it is realistic to take the width of the path between cells into account because the products have to be transported between the departments by a forklift or an automatic guided vehicle and the transportation systems usually travel in a rectangular manner. Considering the XRFLP, let w_{path}^1 (w_{path}^2) denote the width of the path between cell 1 and cell 3 (cell 2 and cell 4). Measuring the distances between cell 2 and cell 3 we do not cross a path, and hence we set the inter-cell distance to zero. In contrast, going from cell 1 to cell 4 we cross both paths, i. e., $u_{14} = w_{path}^1 + w_{path}^2$. For the remaining inter-cell distances we obtain $u_{12} = u_{13} = w_{path}^1$ and $u_{24} = u_{34} = w_{path}^2$.

We illustrate the distance calculation of the problems considered above by the following example.

Example 1. *We are given an instance with $n = 5$ departments with lengths $\ell_2 = 1$, $\ell_1 = \ell_3 = \ell_4 = 2$, $\ell_5 = 3$ and non-zero weights $w_{15} = w_{24} = w_{31} = 1$, $w_{21} = w_{23} = w_{52} = 2$, $w_{42} = 3$. The inter-cell distances w_{path}^1 , w_{path}^2 for the XRFLP are set to zero as well as $u_{12} = 0$ for the CCLP (1,1) and the CCLP (0,2). Optimal layouts of different facility layout problems are depicted in Figure 1. Note that in the X-row layout department 2 is in cell 3.*

a) *An optimal single-row layout is illustrated in Figure 1a with objective value*

$$2 \cdot 1.5 + 1 \cdot 5 + 1 \cdot 2.5 + 2 \cdot 3.5 + (3 + 1) \cdot 1.5 + 2 \cdot 4 = 31.5;$$

Layout Problem and additionally one department in each cell is fixed opposite the (un-) loading station.

b) an optimal directed-circular layout is illustrated in Figure 1b with objective value

$$1 \cdot 2.5 + 2 \cdot 3.5 + 2 \cdot 1.5 + 1 \cdot 8.5 + 1 \cdot 2 + 3 \cdot 1.5 + 2 \cdot 4 = 35.5;$$

c) an optimal T-row layout is illustrated in Figure 1c with objective value

$$2 \cdot 0.5 + 1 \cdot 2 + 1 \cdot 2.5 + 2 \cdot 2.5 + (3 + 1) \cdot 1.5 + 2 \cdot 3 = 22.5;$$

d) an optimal X-row layout is illustrated in Figure 1d with objective value

$$2 \cdot 1.5 + 1 \cdot 3 + 1 \cdot 2.5 + 2 \cdot 2.5 + (3 + 1) \cdot 1.5 + 2 \cdot 2 = 23.5;$$

e) an optimal combined cell layout with two circular layout cells is illustrated in Figure 1e with objective value

$$2 \cdot 1.5 + 1 \cdot (2 + 1.5) + 1 \cdot 2.5 + 2 \cdot (0 + 2) + (3 + 1) \cdot 0 + 2 \cdot 2 = 17;$$

f) an optimal combined cell layout with one single-row and one circular layout cell is illustrated in Figure 1f with objective value

$$2 \cdot 1.5 + 1 \cdot (3 + 1.5) + 1 \cdot 2.5 + 2 \cdot (0 + 3) + (3 + 1) \cdot (0 + 1) + 2 \cdot 2 = 24.$$

In the following example we compare the distance calculation of the 3-BFLP and the 4-BFLP with the distance calculation of the 3-PMFP and the 4-PMFP.

Example 2. We are given an instance with $n = 5$ departments with lengths $\ell_1 = \ell_5 = 6$, $\ell_3 = \ell_4 = 5$, $\ell_2 = 2$ and non-zero weights $w_{12} = 2$, $w_{13} = w_{14} = w_{15} = w_{34} = w_{35} = 1$, $w_{path} = 1$. Optimal 3-Bay, 4-Bay, 3-Pier-Type and 4-Pier-Type layouts are illustrated in Figure 2.

a) An optimal 3-Bay layout is illustrated in Figure 2a with an objective value of

$$2 \cdot 4 + 6.5 + 11.5 + 7 + 5 + 7.5 = 45.5;$$

b) An optimal 4-Bay layout is illustrated in Figure 2b with an objective value of

$$2 \cdot 4 + 6.5 + 7.5 + 7 + 6 + 7.5 = 42.5;$$

c) An optimal 3-Pier-Type layout is illustrated in Figure 2c with an objective value of

$$2 \cdot 4 + 6.5 + 11.5 + 7 + 5 + 6.5 = 44.5;$$

d) An optimal 4-Pier-Type layout is illustrated in Figure 2d with an objective value of

$$2 \cdot 4 + 6.5 + 6.5 + 7 + 6 + 6.5 = 40.5.$$

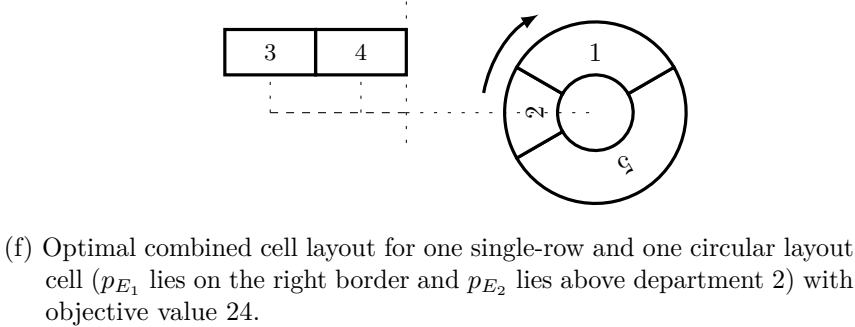
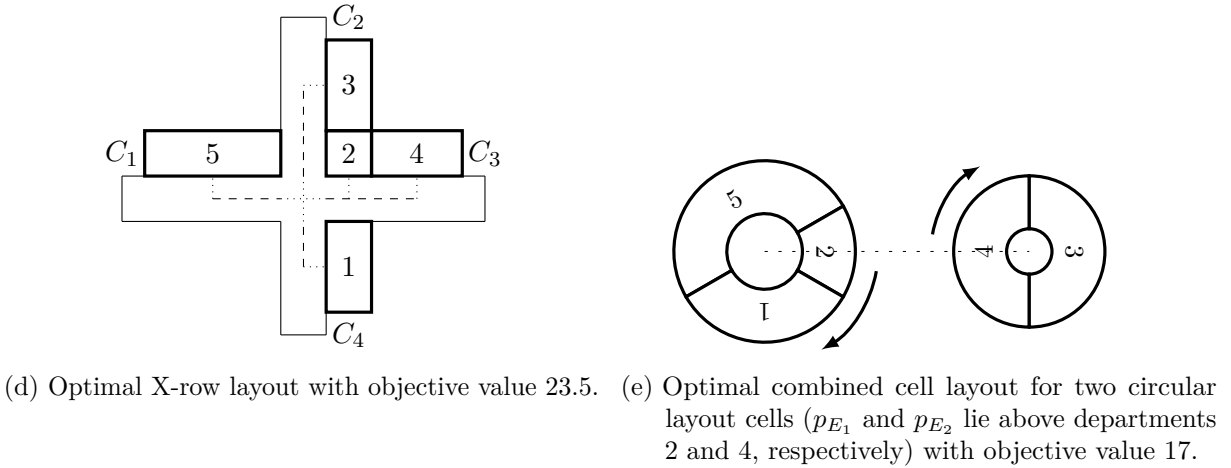
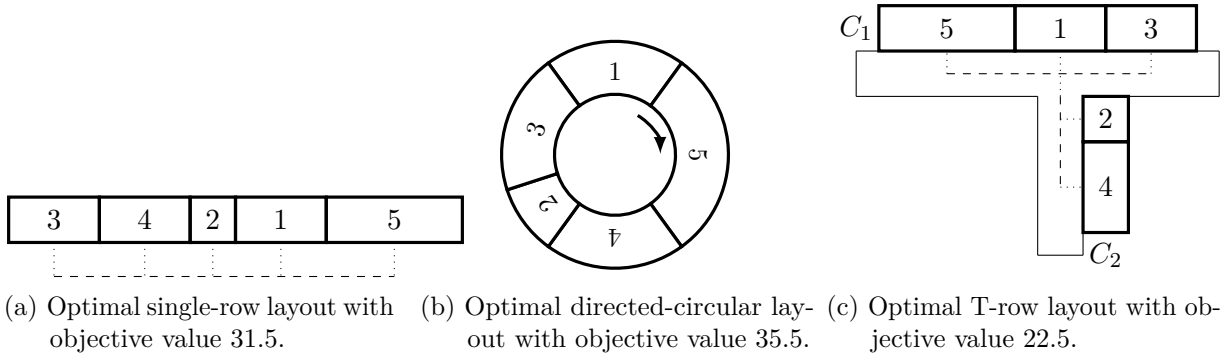
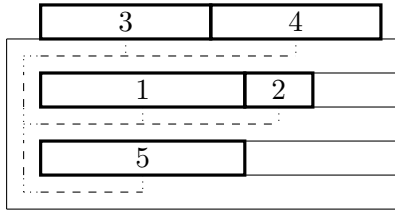


Figure 1: We are given an instance with $n = 5$ departments with lengths $\ell_2 = 1$, $\ell_1 = \ell_3 = \ell_4 = 2$, $\ell_5 = 3$ and non-zero weights $w_{15} = w_{24} = w_{31} = 1$, $w_{21} = w_{23} = w_{52} = 2$, $w_{42} = 3$. The inter-cell distances are set to zero. Illustration of optimal layouts and the associated distance calculations for the SRFLP, the DCFLP, the TRFLP, the XRFLP, the CCLP (0, 2) and the CCLP (1, 1). Detailed calculations of the objective values are given in Example 1. Dotted lines are neglected in the distance calculations.

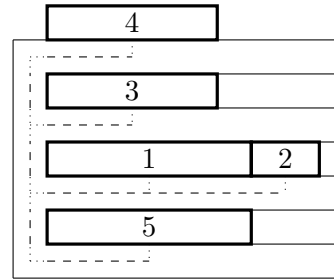
1.1 Literature review

There are several facility layout problems studied in the literature, see, e. g., [12, 30] for two recent surveys. In the following we give an overview of existing solution approaches for facility layout problems considered in this work as well as related ones:

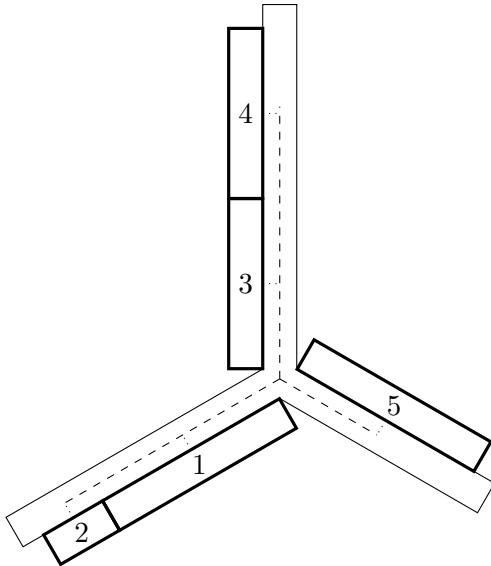
- The SRFLP is one of a few layout types for which strong lower and upper bounds for even large-sized instances exist which are based on Semidefinite Programming (SDP) and Integer Linear Programming (ILP) formulations. The strongest SDP approach [35, 36] is able to solve one instance with 42 departments to optimality while the current best ILP approach [5], based on betweenness variables, is able to solve instances with up to 35 departments. Several heuristic approaches were presented, see, e. g., [25, 43, 49], (see [24]



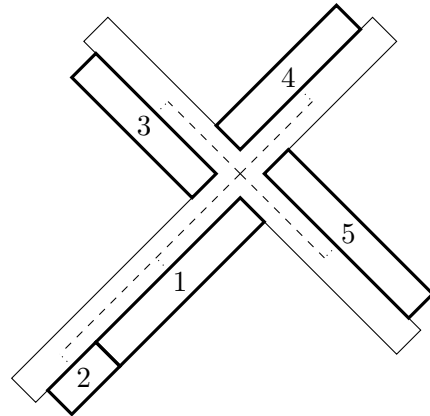
(a) Optimal 3-Bay layout with objective value 45.5.



(b) Optimal 4-Bay layout with objective value 42.5.



(c) Optimal 3-Pier-Type layout with objective value 44.5.



(d) Optimal 4-Pier-Type layout with objective value 40.5.

Figure 2: We are given an instance with $n = 5$ departments with lengths $l_1 = l_5 = 6$, $l_3 = l_4 = 5$, $l_2 = 2$ and non-zero weights $w_{12} = 2$, $w_{13} = w_{14} = w_{15} = w_{34} = w_{35} = 1$, $w_{path} = 1$. We illustrate optimal layouts and the associated distance calculations for the 3-BFLP, the 4-BFLP, the 3-PMFP and the 4-PMFP. Dotted lines are neglected in these calculations. For details on the distances we refer to Example 2.

for a correction of the proof for the main result the heuristic of [49] is based on). A recent survey is given in [40].

- The Checkpoint Ordering Problem (COP) asks for a space-free non-overlapping arrangement of the departments in one cell such that the sum of the weighted distances of the centers of the departments to a checkpoint whose position is given in advance is minimized. In [31] a dynamic programming algorithm and an ILP approach is suggested for solving the COP. Further, [34] proposed the Multiple Checkpoint Ordering Problem, which generalizes the COP to an arbitrary but fixed number of checkpoints.
- Let a set of nodes V with $|V| = n$ and weights w_{ij} and w_{ji} , $i, j \in V$, $i < j$, be given. The Linear Ordering Problem (LOP) looks for a bijective mapping $\sigma: [n] \rightarrow [n]$ such that $\sum_{\substack{i, j \in V \\ \sigma^{-1}(i) < \sigma^{-1}(j)}} w_{ij}$ is maximized, see, e. g., [27, 28]. The DCFLP can be modeled as an LOP, and hence the DCFLP can be solved faster in practice than the SRFLP, see [37, 38], which was so far considered as the simplest available layout type. An SDP and an ILP approach are given in [38] as well as heuristic approaches such that tight lower and upper bounds for instances with up to 100 departments are provided. We refer to [38] for an overview of further circular layout problems.
- For the FC-CCLP where additionally in each cell one department is fixed as the loading station, the ILP model of [15] outperformed the SDP approach of [32]. To the best of our knowledge, our paper is the first that considers the CCLP without fixing one department as loading station and without pre-assigning departments to given cells.
- There are several two-stage procedure heuristics for the MBFLP, see, e. g., [19, 20, 47], where at first, the departments are assigned to the cells and second, the order of the departments within each cell is determined. In [26] an ILP model for the MBFLP with fixed cell assignment was proposed and optimal solutions for instances with up to 25 departments and up to 5 rows are obtained within one second. Further, a Mixed-Integer Linear Programming (MILP) model for solving the PMFP can be found in [21]. The current best approach for the 3-BFLP as well as for the TRFLP is given in [23] where instances with up to 18 departments are solved to optimality and tight lower bounds for the 3-BFLP with up to 24 departments are provided.
- The Multi-Row Facility Layout Problem (MRFLP) consists of a set of m non-overlapping parallel cells of type SRFLP where free-space between neighboring departments in the same cell may arise and where the distance between the departments equals their horizontal distance. The special case with two cells is denoted by Double-Row Facility Layout Problem (DRFLP). There are several MILP approaches for the DRFLP, see, [6, 9, 22, 51, 55], and an SDP approach for the MRFLP [33]. The current best approach is an enumerative approach by [26], which is able to solve double-row instances with up to 16 departments and multi-row instances with up to 5 cells and 13 departments in reasonable time. Recently a two-stage approach for the MRFLP was presented in [11] which allows to derive good solutions quickly for $m \geq 3$.

1.2 Our contribution

The main contributions of this paper are the following:

- We present a new exact approach for the CCLP where we enumerate over all cell assignments of the departments and then solve several FC-CCLP. We show how to reduce the number of cell assignments that have to be considered significantly. Indeed, given a CCLP instance where all cells of type SRFLP have the same inter-cell distances and let an assignment of

the departments to the cells be given, then we can merge two cells of type SRFLP. If the number of cells of type SRFLP is even, this result allows us to halve the number of cells of type SRFLP.

- We extend the previous mentioned results to the 3-BFLP and to the 4-BFLP with positive inter-cell distances. We introduce the XRFLP, which is a realistic extension of the 4-BFLP with a more complex path structure, and we show that we can use our main result for this problem as well even with positive inter-cell distances.
- In [15, 32] a fixed department deals as (un-) loading station. We omit this assumption by adding a dummy department with appropriate length and weights to each cell which deals as (un-) loading station. Considering cells of type DCFLP, one department may overlap with the dummy department and for the arising optimization problem we present a new MILP model which outperforms an associated enumerative approach.
- At first, we present a theoretical study between the relationship of the optimal values of several facility layout problems, see Section 3. Then, in a computational study, see Section 6, we compare the optimal values of these layout problems on instances from the literature in order to support the decision maker to choose the layout of a factory which is built from the ground up. We also display the running time for solving these problems as this might influence the decision.
- Our approach outperforms the current best approach for the 3-BFLP as well as the 3-PMFP and 4-PMFP, and hence we partially answer a research question of [21] to derive a more efficient exact solution approach for the PMFP.

2 Summary of exact approaches for the SRFLP and the DCFLP

In this section we summarize MILP models for the SRFLP and the DCFLP. Let \mathcal{D} denote a set of departments. For single-row instances with up to 20 departments the MILP model of [5] based on betweenness variables allows to calculate optimal layouts faster than other approaches from the literature, see, e. g. [35, 36]. For solving the DCFLP, we choose the MILP formulation suggested in [15, 37, 38].

2.1 The Single-Row Facility Layout Problem

In the following we recall the respective MILP formulation of [5]. First, we make use of betweenness variables x_{ikj} , $i, j, k \in \mathcal{D}$, $|\{i, j, k\}| = 3$, $i < j$, where \mathcal{D} represents a set of departments (a subset of $[n]$ plus partially some added dummy departments in the following where we assume that w, ℓ are known), with the interpretation

$$x_{ikj} = \begin{cases} 1, & \text{if } k \text{ lies between } i \text{ and } j, \\ 0, & \text{otherwise.} \end{cases}$$

Then the MILP model reads as follows, where we neglect the constant weights

$$W^S = \sum_{\substack{i, j \in \mathcal{D} \\ i < j}} (w_{ij} + w_{ji}) \frac{\ell_i + \ell_j}{2}.$$

$$\min \sum_{\substack{i, j \in \mathcal{D} \\ i < j}} (w_{ij} + w_{ji}) \sum_{k \in \mathcal{D} \setminus \{i, j\}} \ell_k x_{ikj} \quad (1)$$

$$\text{s. t. } x_{ikj} + x_{jik} + x_{ijk} = 1, \quad i, j, k \in \mathcal{D}, i < j < k, \quad (2)$$

$$-x_{ihj} + x_{ihk} + x_{jhk} \geq 0, \quad i, j, k, h \in \mathcal{D}, i < j < k, |\{i, j, k, h\}| = 4, \quad (3)$$

$$x_{ihj} - x_{ihk} + x_{jhk} \geq 0, \quad i, j, k, h \in \mathcal{D}, i < j < k, |\{i, j, k, h\}| = 4, \quad (4)$$

$$x_{ihj} + x_{ihk} - x_{jhk} \geq 0, \quad i, j, k, h \in \mathcal{D}, i < j < k, |\{i, j, k, h\}| = 4, \quad (5)$$

$$x_{ihj} + x_{ihk} + x_{jhk} \leq 2, \quad i, j, k, h \in \mathcal{D}, i < j < k, |\{i, j, k, h\}| = 4, \quad (6)$$

$$x_{ikj} \in \{0, 1\}, \quad i, j, k \in \mathcal{D}, i < j, |\{i, j, k\}| = 3. \quad (7)$$

Note that for the SRFLP there always exists an optimal layout without free-spaces between neighboring departments, hence, neglecting W^S the distance between two departments equals the lengths of the departments between them, see (1). Equations (2) ensure that for three departments $i, j, k \in \mathcal{D}$, $i < j < k$, exactly one of the departments lies between the other two. In a feasible arrangement of the departments \mathcal{D} for each subset $\{i, j, k, h\} \subseteq \mathcal{D}$, $|\{i, j, k, h\}| = 4$, the following holds: if department h lies between departments i and j , then department h additionally lies between a) departments i and k or b) departments j and k , see Inequalities (3), but not both a) and b), see Inequalities (6). Hence, Inequalities (3)–(6) ensure transitivity and according to [5, 50] together with the binarity of the x -variables (7) we obtain a feasible ordering of the departments.

2.2 The Directed Circular Facility Layout Problem

The approach of [15, 37, 38] makes use of distance variables d_{ij} , $i, j \in \mathcal{D}$, $i \neq j$, measuring the center-to-center distances between i and j in clockwise direction. Given a directed-circular layout, we interpret this layout as a single-row layout with a different distance calculation by splitting the circle at one department and unwinding it. Indeed, in order to obtain an ordering of the departments with respect to the splitting department, we choose an arbitrary department $f \in \mathcal{D}$ to be the leftmost department and then we say that $i \in \mathcal{D} \setminus \{f\}$ is to the left of $j \in \mathcal{D} \setminus \{f\}$, $i < j$, if $d_{fi} < d_{fj}$. Therefore we use ordering variables

$$z_{ij} = \begin{cases} 1, & \text{if } i \text{ lies to the left of } j, \\ 0, & \text{otherwise,} \end{cases}$$

$i, j \in \mathcal{D} \setminus \{f\}$, $i < j$. Note that f is the leftmost department, and hence we neglect ordering variables containing f . Further, we can exclude some constants, let $L := \sum_{k \in \mathcal{D}} \ell_k$. If both w_{ij} and w_{ji} , $i, j \in \mathcal{D}$, $i < j$, are greater than zero, then we set $\tilde{w}_{ij} = w_{ij} - \min\{w_{ij}, w_{ji}\}$ and $\tilde{w}_{ji} = w_{ji} - \min\{w_{ij}, w_{ji}\}$ and add the constant $W := \sum_{\substack{i, j \in \mathcal{D} \\ i < j}} \min\{w_{ij}, w_{ji}\}L$ to the objective value, see [37]. In total, we neglect the constant weights $W^D = \sum_{\substack{i, j \in \mathcal{D} \\ i \neq j}} \tilde{w}_{ij} \frac{\ell_i + \ell_j}{2} + W$. Let D_{ij} , $i, j \in \mathcal{D} \setminus \{f\}$, $i < j$, denote the sum of the lengths of the departments $k \in \mathcal{D} \setminus \{f, i\}$ which are left of j minus the sum of the lengths of the departments $k \in \mathcal{D} \setminus \{f, j\}$ which are left of i . Note that D_{ij} , $i, j \in \mathcal{D}$, $i < j$, is negative if j is left to i . Then the MILP can be written as follows where we set $L_{ij} = L_{ji} = L - \ell_i - \ell_j$, $i, j \in \mathcal{D}$, $i < j$.

$$\min \sum_{\substack{i, j \in \mathcal{D} \\ i \neq j}} \tilde{w}_{ij} d_{ij}$$

$$\text{s. t. } 0 \leq z_{ij} + z_{jk} - z_{ik} \leq 1, \quad i, j, k \in \mathcal{D} \setminus \{f\}, i < j < k, \quad (8)$$

$$d_{fi} - \sum_{\substack{k \in \mathcal{D} \setminus \{f\} \\ k < i}} \ell_k z_{ki} + \sum_{\substack{k \in \mathcal{D} \setminus \{f\} \\ k > i}} \ell_k z_{ik} = \sum_{\substack{k \in \mathcal{D} \setminus \{f\} \\ k > i}} \ell_k, \quad i \in \mathcal{D} \setminus \{f\}, \quad (9)$$

$$d_{if} + d_{fi} = L_{fi}, \quad i \in \mathcal{D} \setminus \{f\}, \quad (10)$$

$$\begin{aligned} D_{ij} - \sum_{\substack{k \in \mathcal{D} \setminus \{f, i\} \\ k < j}} \ell_k z_{kj} + \sum_{\substack{k \in \mathcal{D} \setminus \{f, i\} \\ k > j}} \ell_k z_{jk} \\ + \sum_{\substack{k \in \mathcal{D} \setminus \{f, j\} \\ k < i}} \ell_k z_{ki} - \sum_{\substack{k \in \mathcal{D} \setminus \{f, j\} \\ k > i}} \ell_k z_{ik} = - \sum_{\substack{k \in \mathcal{D} \setminus \{f\} \\ i < k < j}} \ell_k, \quad i, j \in \mathcal{D} \setminus \{f\}, i < j, \end{aligned} \quad (11)$$

$$d_{ij} + L_{ij}z_{ij} - D_{ij} = L_{ij}, \quad i, j \in \mathcal{D} \setminus \{f\}, i < j, \quad (12)$$

$$d_{ji} - L_{ij}z_{ij} + D_{ij} = 0, \quad i, j \in \mathcal{D} \setminus \{f\}, i < j, \quad (13)$$

$$d_{ij} \geq 0, \quad i, j \in \mathcal{D}, i \neq j,$$

$$z_{ij} \in \{0, 1\}, \quad i, j \in \mathcal{D} \setminus \{f\}, i < j.$$

Inequalities (8) are the well-known three-cycle-inequalities, which ensure transitivity and suffice to induce a feasible ordering of the departments, see, e. g., [15, 37, 38]. Via Equations (9)–(10) we determine for each department $i \in \mathcal{D} \setminus \{f\}$ the sum of the lengths of the departments which lie between f and i and between i and f , respectively (calculated in clockwise direction). Equations (11) compute for each pair of departments i and j with $i, j \in \mathcal{D} \setminus \{f\}$, the sum of the lengths of the departments $k \in \mathcal{D} \setminus \{f, i\}$ which are left of j minus the sum of the lengths of the departments $k \in \mathcal{D} \setminus \{f, j\}$ which are left of i . By Equations (12) and (13) we obtain the correct distance between i and j with $i, j \in \mathcal{D} \setminus \{f\}$, $i < j$, i. e., if i lies left to j we get $d_{ij} = D_{ij}$ and $d_{ji} = L_{ij} - D_{ij}$ and otherwise $d_{ij} = L_{ij} + D_{ij}$ and $d_{ji} = -D_{ij}$ with negative values of D_{ij} in the latter case.

3 Relation between the optimal values of several facility layout problems

After repeating and introducing several facility layout problems we want to study in this section the relation of the optimal objective values of the considered layout problems. We start with a comparison of the SRFLP and the DCFLP. Let d_{ij}^{DCFLP} , $i, j \in [n]$, $i \neq j$, denote the center-to-center distances between i and j measured in clockwise direction in the DCFLP and let $d_{ij}^{\text{SRFLP}} = d_{ji}^{\text{SRFLP}}$, $i, j \in [n]$, $i < j$, denote the horizontal center-to-center distance between i and j in the SRFLP. It holds that $d_{ij}^{\text{SRFLP}} < \sum_{k \in [n]} \ell_k$, $i, j \in [n]$, $i < j$, in the SRFLP and $d_{ij}^{\text{DCFLP}} + d_{ji}^{\text{DCFLP}} = \sum_{k \in [n]} \ell_k$ in the DCFLP. Hence, for an instance with symmetric weights, i. e., if $w_{ij} = w_{ji}$ for all $i, j \in [n]$, $i < j$, the optimal value of the SRFLP is less than the optimal value of the DCFLP if $n \geq 3$ (if not all weights are equal to zero). In contrast to this, consider an instance with lengths $\ell_i = 1$, $i \in [n]$, and non-zero (asymmetric) weights $w_{i(i+1)} = 1$, $i \in [n-1]$, and $w_{n1} = 1$. Then, the optimal value of the DCFLP equals $C := \sum_{i, j \in [n], i \neq j} w_{ij} \frac{\ell_i + \ell_j}{2} = n$ and for the SRFLP one obtains $C + n - 2$. We illustrate two optimal single-row layouts for $n = 7$ in Figure 3. This shows that it is not possible to provide a general statement comparing the optimal values of the SRFLP and the DCFLP.

Next we extend our study to layout problems with more than one cell. In the following consideration let $w_{\text{path}} = w_{\text{path}}^1 = w_{\text{path}}^2$ and for the CCLP (2,0), the CCLP (1,1) and the CCLP (0,2) let $u_{12} = w_{\text{path}}$. The digraph $D = (O, A(O))$ illustrated in Figure 4 shows the relations of the optimal objective values of the considered layout problems. An arc $(i, j) \in A(O)$ from $i \in O := \{\text{SRFLP}, \text{DCFLP}, \text{CCLP (2,0)}, \text{CCLP (1,1)}, \text{CCLP (0,2)}, \text{TRFLP}, \text{3-BFLP}, \text{4-BFLP}, \text{3-PMFP}, \text{4-PMFP}, \text{XRFLP}\}$ to $j \in O$, $i \neq j$, is added if the optimal value of the layout problem i is at least as high as the optimal value of layout problem j . Note that transitive arcs are not illustrated. The displayed digraph is correct. This can be shown by considering the associated optimal layouts. Let $(i, j) \in A(O)$, then usually the optimal layout for problem $i \in O$ is a feasible layout for problem $j \in O$ and comparing bay and pier-type layouts just the distance calculation varies. It was mentioned in [23] that the optimal value of the TRFLP is not higher than the one for the 3-BFLP under the described conditions. One can construct instances such that each of the problems TRFLP, XRFLP and 4-PMFP has the smallest optimal value of these three layout problems. Additionally, one can construct instances such that the optimal value of the 4-BFLP is less than or equal to the optimal value of the XRFLP and vice versa. In Section 6 we will compare the objective values of the various layout problems for instances from the literature.

1	2	3	4	5	6	7
---	---	---	---	---	---	---

5	4	6	7	1	3	2
---	---	---	---	---	---	---

- (a) Optimal single-row layout with objective value 12 and distances, e. g., $d_{71} = 6$ and $d_{i(i+1)} = 1$, $i \in [6]$.
(b) Optimal single-row layout with objective value 12 and distances, e. g., $d_{12} = d_{56} = 2$, $d_{71} = 1$, $d_{34} = 4$, and $d_{i(i+1)} = 1$, $i \in \{2, 4, 6\}$.

Figure 3: We are given an SRFLP instance with $n = 7$, lengths $\ell_i = 1$, $i \in [7]$, and non-zero weights $w_{i(i+1)} = 1$, $i \in [6]$, $w_{71} = 1$. We illustrated two optimal single-row layouts.

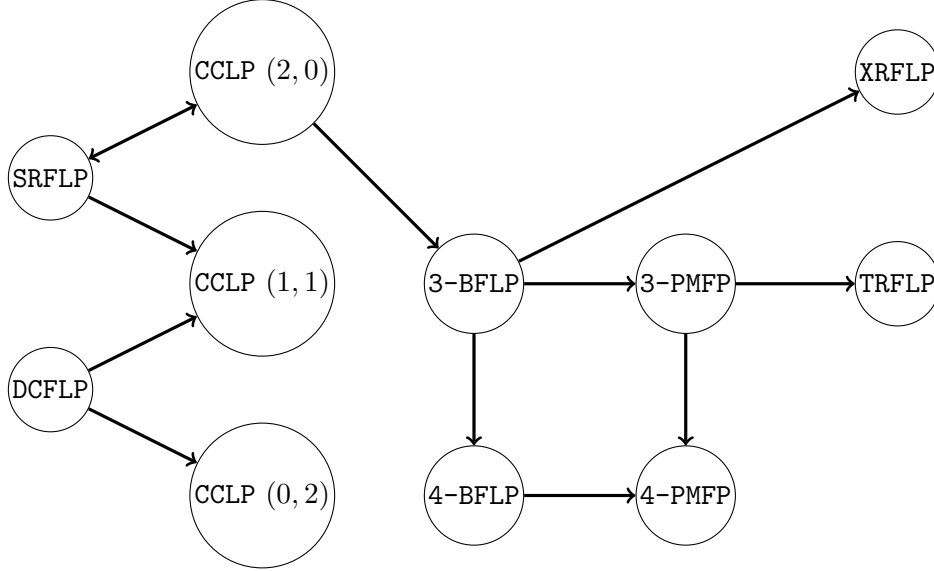


Figure 4: A comparison of the optimal values of several facility layout problems where an arc between $i \in O = \{\text{SRFLP}, \text{DCFLP}, \text{CCLP (0, 2)}, \text{CCLP (1, 1)}, \text{CCLP (2, 0)}, \text{TRFLP}, \text{3-BFLP}, \text{4-BFLP}, \text{3-PMFP}, \text{4-PMFP}, \text{XRFLP}\}$ and $j \in O$ indicates that the optimal value of the layout problem i is at least as high as the optimal value of the layout problem j . Transitive arcs are not illustrated.

4 The Combined Cell Layout Problem

Our goal is to solve the CCLP exactly by enumerating over all (distinguishable) cell assignments and then solving several FC-CCLP. In [47] it is stated that if the cell assignment is given, one can solve the cells in the MBFLP independently by adding appropriate dummy departments. We extend this result to the CCLP. In Section 4.1 we provide a significant reduction on the number of cell assignments of type SRFLP that have to be considered if the inter-cell distances u_{ko} are the same for all $k, o \in \mathcal{C}$, $k < o$, with $\{t(k), t(o)\} \cap \{\text{SRFLP}\} \neq \emptyset$, by merging two cells of type SRFLP, and hence solving larger single-row instances including an additional dummy department.

In Section 4.2 we show how to solve the problems associated with circular cells. Indeed, in the DCFLP one department may overlap with the dummy department and we prove that there always exists an optimal directed-circular layout where one department lies opposite the dummy department. Based on this result we present an MILP model for determining an optimal layout for the departments contained in a cell of type DCFLP including the additional dummy department. Alternatively one can use an enumerative approach instead of the MILP model. The running times of both variants are compared in Section 6.

4.1 Our Algorithm

Our goal is to solve the optimization problems in each cell of the FC-CCLP independently as done in [47]. Therefore we define the following problems where the weights of exactly one department

are adjusted appropriately.

Definition 3. Given an FC-CCLP instance and let $\mathcal{D} \subseteq [n+m']$, $m' \in \mathbb{N}_{\geq 0}$, be a set of departments where $n+1, n+2, \dots, n+m'$ are dummy departments with lengths $\ell_{n+1}, \ell_{n+2}, \dots, \ell_{n+m'} \in \mathbb{R}_{\geq 0}$ and weights $w_{iz} = w_{zi} = 0$, $i \in [n+m']$, $z \in \{n+1, \dots, n+m'\}$, $i \neq z$. For the department $s_M \in \mathcal{D}$ we set

$$w_{is_M} \leftarrow w_{is_M} + \sum_{j \in [n] \setminus \mathcal{D}} w_{ij}, \quad i \in \mathcal{D} \setminus \{s_M\}, \quad (14)$$

$$w_{s_M i} \leftarrow w_{s_M i} + \sum_{j \in [n] \setminus \mathcal{D}} w_{ji}, \quad i \in \mathcal{D} \setminus \{s_M\}, \quad (15)$$

all other weights remain the same. Then the aim of the optimization problem $W_{(s_M, p_z)}^A(\mathcal{D})$ for the updated weights is to find an optimal layout of the departments \mathcal{D} respective the structure of the cell $A \in \{\text{SRFLP}, \text{DCFLP}\}$. Additionally, p_z , $z \in \{a, b\}$, specifies the position of s_M , where p_a denotes that the position of s_M can be chosen arbitrarily and p_b expresses that s_M has to lie at the leftmost position of the layout. Let $A = \text{DCFLP}$ and $s_M \in \{n+1, \dots, n+m'\}$, then one department of the set $\mathcal{D} \setminus \{s_M\}$ may overlap with s_M .

Given an FC-CCLP instance, we add a dummy department to each cell with length zero and weights as described above, and hence we obtain the following result.

Lemma 4. Given a fixed-cell combined cell layout instance where C_k , $k \in \mathcal{C}$, denotes the set of departments assigned to cell k and let the dummy department $n+k$ be added to cell k with length $\ell_{n+k} = 0$ and adapted weights for dummy department $n+k$ as described in (14)–(15) for $\mathcal{D} = C_k \cup \{n+k\}$ and $s_M = n+k$. Then, the FC-CCLP is equivalent to solving the problems

$$\begin{aligned} W_{(n+k, p_b)}^S(C_k \cup \{n+k\}), & \quad k \in \mathcal{C} \text{ with } t(k) = \text{SRFLP}, \\ W_{(n+k, p_a)}^D(C_k \cup \{n+k\}), & \quad k \in \mathcal{C} \text{ with } t(k) = \text{DCFLP}, \end{aligned}$$

and the sum of the optimal values (plus constant inter-cell weights) is equal to the optimal value of the FC-CCLP.

The result of Lemma 4 is stated in [47] for the MBFLP without a proof and this result was implicitly used in [15, 32]. For the convenience of the reader we present a proof.

Proof. Remark that inter-cell distances lead to constant weights $C := \sum_{\substack{k, o \in \mathcal{C} \\ k < o}} \sum_{\substack{i \in C_k \\ j \in C_o}} u_{ko} (w_{ij} + w_{ji})$ in the FC-CCLP, and thus we may exclude them. As we interpret the dummy departments as the (un-) loading stations of the cells, we can express the inter-cell distances by summing up inner-cell distances, i. e., $d_{ij} = d_{i(n+k)} + d_{(n+o)j}$, $k, o \in \mathcal{C}$, $k \neq o$, $i \in C_k$, $j \in C_o$. This proves the desired result. \square

Hence, the FC-CCLP can be divided into m sub-problems. In this paper we concentrate on cells of type SRFLP and DCFLP, but note that the result of Lemma 4 is independent of the layout type of the cells, and thus our approach can be extended to cells of other types such as the DRFLP.

We provide the following result which breaks some symmetries and which allows us to reduce the number of cell assignments of type SRFLP that have to be considered significantly if $u_{ko} = c$ for some constant $c \in \mathbb{R}_{\geq 0}$ for all $k, o \in \mathcal{C}$, $k < o$, with $\{t(k), t(o)\} \cap \{\text{SRFLP}\} \neq \emptyset$.

Theorem 5. The CCLP (m_1, m_2) with $u_{ko} = c \in \mathbb{R}_{\geq 0}$ for all $k, o \in \mathcal{C}$, $k < o$, with $\{t(k), t(o)\} \cap \{\text{SRFLP}\} \neq \emptyset$ is equivalent to enumerate over $\frac{\binom{\lceil \frac{m_1}{2} \rceil + m_2}{\lfloor \frac{m_1}{2} \rfloor}^n}{\lfloor \frac{m_1}{2} \rfloor! m_2!}$ cell assignments and solve the following optimization problems for a fixed cell assignment exactly

$$\begin{aligned} W_{(n+k, p_a)}^S(C_k \cup C_{k+1} \cup \{n+k\}), & \quad k = 1, 3, \dots, h, \\ W_{(n+m_1, p_b)}^S(C_{m_1} \cup \{n+m_1\}), & \quad \text{if } m_1 \text{ is odd,} \\ W_{(n+k, p_a)}^D(C_k \cup \{n+k\}), & \quad k = m_1 + 1, \dots, m, \end{aligned}$$

with $h = m_1 - 1$ if m_1 is even and $h = m_1 - 2$ if m_1 is odd and the departments C_k are assigned to cell $k \in [m]$. Apart from this the dummy department $n + k$ is added to cell k for $k = 1, 3, \dots, h$, $k = m_1$ if m_1 odd and $k = m_1 + 1, \dots, m$. Additionally, we have to compute some constants such that inter-cell distances are calculated correctly.

Further, the SRFLP is equivalent to the CCLP (2,0).

In [23] it is shown that the TRFLP is a generalization of the 3-BFLP. Therefore, an additional dummy department $n + 1$ with lengths $\ell_{n+1} = 2w_{path}$ and weights $w_{i(n+1)} = 0$, $i \in [n]$, is added to the TRFLP and its center position is fixed on position p_{E_1} in row 1. In such an optimal T-row layout, the departments left (right) to $n + 1$ are assigned to cell 3 (cell 1) in the 3-BFLP and the departments in cell 2 are assigned to cell 2 in the 3-BFLP without changing the order of the departments in the same row. We use this idea in the following proof.

Proof. Let C_k denote the set of departments assigned to cell $k \in [m]$ in the FC-CCLP. By Lemma 4 the CCLP is equivalent to enumerate over $\frac{(m_1+m_2)^n}{m_1!}$ distinguishable cell assignments and solve the problems

$$\begin{aligned} W_{(n+k,p_b)}^S(C_k \cup \{n+k\}), & \quad k = 1, 2, \dots, m_1, \\ W_{(n+k,p_a)}^D(C_k \cup \{n+k\}), & \quad k = m_1 + 1, \dots, m, \end{aligned}$$

where the dummy department $n + k$ is assigned to cell k . Note that by our assumptions on u_{ko} with $k, o \in \mathcal{C}$, $k < o$, such that $\{t(k), t(o)\} \cap \{\text{SRFLP}\} \neq \emptyset$ it is sufficient to determine which departments lie in a common cell of type SRFLP. Hence, we can divide $(m_1 + m_2)^n$ by $m_1!$.

For the improved formula, let us first consider cells of type DCFLP. If we know which departments should be together in one cell of type DCFLP, it remains to determine the exact cell of each of the departments. For this note that the associated inter-cell weights do not have to be the same. So, given C_{m_1+1}, \dots, C_m , we calculate $m_2!$ constants and determine a best bijection $\pi: [m] \setminus [m_1] \rightarrow [m] \setminus [m_1]$ minimizing the associated inter-cell distances $\sum_{k,o \in [m] \setminus [m_1], k < o} u_{ko} \sum_{i \in C_{\pi(k)}, j \in C_{\pi(o)}} w_{ij}$. Thus, we can break the symmetries with respect to the cells of type DCFLP and divide the number of cell assignments that have to be checked by $m_2!$.

It remains to consider cells of type SRFLP. If $m_1 = 1$, we are done. So, let $m_1 \geq 2$. We exclude the constant $(w_{ij} + w_{ji}) \frac{c}{2}$ if $i \in [n]$ lies in a cell of type SRFLP and $j \in [n]$ lies in a cell of type DCFLP. Let $k, k + 1 \in [2 \lfloor \frac{m_1}{2} \rfloor]$ and we consider the departments $C_k \cup C_{k+1}$ without changing the cell assignments of the remaining departments. We merge the cells k and $k + 1$ and the resulting cell is called *merged* cell in the following. We add a dummy department $n + k$ to the merged cell with length $\ell_{n+k} = c$ and weights $w_{i(n+k)} = \sum_{j \in [n] \setminus (C_k \cup C_{k+1})} w_{ij}$, $w_{(n+k)i} = \sum_{j \in [n] \setminus (C_k \cup C_{k+1})} w_{ji}$ (see (14)–(15)). Then, enumerating over all possible cell assignments of the departments $C_k \cup C_{k+1}$ to two cells of type SRFLP is equivalent to solve the $W_{(n+k,p_a)}^S(C_k \cup C_{k+1} \cup \{n+k\})$ because in an optimal layout of the $W_{(n+k,p_a)}^S(C_k \cup C_{k+1} \cup \{n+k\})$ one can assign all departments left (right) to $n + k$ to cell k (cell $k + 1$) and vice versa. Therefore, we solve the $W_{(n+k,p_a)}^S(C_k \cup C_{k+1} \cup \{n+k\})$ and we obtain a new cell assignment \tilde{C}_k (\tilde{C}_{k+1}), which contains all departments to the left (right) of $n + k$. Note that the inter-cell distances are taken into account by the length of the dummy department $n + k$ as well as the excluded constant.

So, if m_1 is even, it is sufficient to consider $\frac{m_1}{2}$ cells of type SRFLP. If m_1 is odd, we have to take one additional cell into account and solve the $W_{(n+m_1,p_b)}^S(C_{m_1} \cup \{n+m_1\})$ with dummy department $(n + m_1)$ with $\ell_{n+m_1} = 0$ and weights $w_{i(n+m_1)} = \sum_{j \in [n] \setminus C_{m_1}} w_{ij}$, $w_{(n+m_1)i} = \sum_{j \in [n] \setminus C_{m_1}} w_{ji}$ (see (14)–(15)) and we additionally exclude the constant

$$\sum_{\substack{i \in C_{m_1} \\ j \in [n] \setminus C_{m_1}}} (w_{ij} + w_{ji}) \frac{c}{2}.$$

Breaking again the symmetries concerning the merged SRFLP cells we have to consider $\frac{(\lceil \frac{m_1}{2} \rceil + m_2)^n}{\lceil \frac{m_1}{2} \rceil! m_2!}$ cell assignments.

One immediate consequence of these considerations is that the SRFLP is equivalent to the CCLP (2,0) with $u_{12} = 0$. If $u_{12} > 0$, then there exists an optimal solution for the CCLP (2,0) where all departments are arranged in one cell, and hence the SRFLP is equivalent to the CCLP (2,0). \square

The last result leads to the following definition which allows us to specify the (generalized) cell assignments which have to be considered more precisely.

Definition 6. *Let a CCLP (m_1, m_2) instance with $u_{ko} = c \in \mathbb{R}_{\geq 0}$ for all $k, o \in \mathcal{C}$, $k < o$, with $\{t(k), t(o)\} \cap \{\text{SRFLP}\} \neq \emptyset$ be given. We denote by $\tilde{m} := \lceil \frac{m_1}{2} \rceil + m_2$ the number of generalized cells. Then, a generalized cell assignment $\tilde{c}: [n] \rightarrow [\tilde{m}]$ is called proper if the following conditions are satisfied where $\tilde{C}_k = \{i \in [n]: \tilde{c}(i) = k\}, k \in [\tilde{m}]$.*

$$\begin{aligned} \inf\{i \in \tilde{C}_k\} &\leq \inf\{i \in \tilde{C}_l\}, & k, l \in [\lceil \frac{m_1}{2} \rceil], k < l, \\ \inf\{i \in \tilde{C}_k\} &\leq \inf\{i \in \tilde{C}_l\}, & k, l \in [\tilde{m}] \setminus [\lceil \frac{m_1}{2} \rceil], k < l. \end{aligned}$$

Indeed, it suffices to consider proper generalized cell assignments where the concrete cell of departments in cells of type DCFLP might have to be determined in our algorithm.

Our approach for solving the CCLP is summarized in Algorithm 1. If the upper bound v^* is exceeded, we neglect the current generalized cell assignment and go to the next one. We solve cells of type DCFLP first since the DCFLP is in practice easier to solve than the SRFLP, see [37, 38], and the results of [15, 32] indicate that in general the optimal values of cells of type DCFLP are higher than the optimal values of cells of type SRFLP, and hence we hope to exceed the upper bound u earlier such that we can neglect the current cell assignment.

We describe in Section 5 how to include inter-cell distances for the 3-BFLP and the 4-BFLP. Considering cells of type SRFLP, it remains to present our approach for solving $W_{(n+\tilde{m}_1, p_b)}^S(\tilde{C}_{\tilde{m}_1} \cup \{n + \tilde{m}_1\})$ if m_1 is odd with $\tilde{m}_1 = \lceil \frac{m_1}{2} \rceil$. We can simply fix the dummy department $n + \tilde{m}_1$ at the border, i. e.,

$$x_{i(n+\tilde{m}_1)j} = 0, \quad i, j \in C_{\tilde{m}_1}, i < j,$$

and then solve a single-row instance with departments $C_{\tilde{m}_1} \cup \{n + \tilde{m}_1\}$ and these additional equations.

4.2 Circular Cells in the Fixed-Cell Combined Cell Layout Problem

In this section we focus on the subproblems of the FC-CCLP concerning circular cells. Let C_k denote the set of departments assigned to some cell k with $t(k) = \text{DCFLP}$. By the following proposition we can use techniques from the directed-circular literature, see Section 2.2, for solving the $W_{(n+k, p_a)}^D(C_k \cup \{n + k\})$ to optimality.

Proposition 7. *Let an FC-CCLP instance be given where $C_k, k \in \mathcal{C}$, denotes the set of departments assigned to the cell k with $t(k) = \text{DCFLP}$. Let dummy department $n + k$ be assigned to cell k with $\ell_{n+k} = 0$ and $w_{i(n+k)} = \sum_{j \in [n] \setminus C_k} w_{ij}$, $w_{(n+k)i} = \sum_{j \in [n] \setminus C_k} w_{ji}$ (see (14), (15)) and let $V_k := \{i \in C_k: w_{i(n+k)} + w_{(n+k)i} > 0\} \neq \emptyset$. Further, let v_{n+k}^* denote the optimal value of the $W_{(n+k, p_a)}^D(C_k \cup \{n + k\})$ and let $v_{s_M}^*$ denote the optimal value of the $W_{(s_M, p_a)}^D(C_k)$, $s_M \in V_k$, where in $W_{(s_M, p_a)}^D(C_k)$ the weights of s_M are adjusted according to (14), (15). Then*

$$v_{n+k}^* = \min\{v_{s_M}^* : s_M \in V_k\} \quad (16)$$

The proof is related to a proof in [24]. Given a DRFLP instance with objective function $\min \sum_{j \in S} w_{ij} d_{ij}$, $i \in [n]$, $S \subseteq [n] \setminus \{i\}$, there exists an optimal layout where some $j \in S$ lies directly opposite i .

Algorithm 1: Exact approach for the CCLP (m_1, m_2)

Input : instance of the CCLP (m_1, m_2) with departments $[n]$, weights w , lengths ℓ and inter-cell distances $u_{ko} = c \in \mathbb{R}_{\geq 0}$ for $k, o \in \mathcal{C}$, $k < o$, with $\{t(k), t(o)\} \cap \{\text{SRFLP}\} \neq \emptyset$ and arbitrary $u_{ko} \in \mathbb{R}_{\geq 0}$ otherwise.

Output : Optimal value v^* of the CCLP (m_1, m_2) .

$v^* \leftarrow \infty$ or v^* set to some known upper bound.

$z \leftarrow 0$, $\hat{C}^* \leftarrow \emptyset$.

$\tilde{m} \leftarrow \lceil \frac{m_1}{2} \rceil + m_2$, $\tilde{m}_1 \leftarrow \lceil \frac{m_1}{2} \rceil$.

1 for $\tilde{c} = (\tilde{c}_1, \dots, \tilde{c}_n) \in [\tilde{m}]^n$ **with** \tilde{c} **proper do**

$\tilde{C}_k \leftarrow \{i \in [n] : \tilde{c}_i = k\}$, $k \in [\tilde{m}]$.

Compute constant $C \leftarrow \sum_{i \in [\tilde{m}_1]} \sum_{j \in [\tilde{m}] \setminus [\tilde{m}_1]} \sum_{k \in \tilde{C}_i} \sum_{l \in \tilde{C}_j} (w_{kl} + w_{lk}) \frac{c}{2}$.

2 if m_1 **is odd then**

$C \leftarrow C + \sum_{i \in \tilde{C}_{\tilde{m}_1}} \sum_{j \in [n] \setminus \tilde{C}_{\tilde{m}_1}} (w_{ij} + w_{ji}) \frac{c}{2}$.

3 if $m_2 \geq 2$ **then**

Determine an assignment $\hat{c}^* : \bigcup_{i=\tilde{m}_1+1}^{\tilde{m}} \tilde{C}_i \rightarrow [m] \setminus [m_1]$ such that $\hat{c}^*(i) = \hat{c}^*(j)$ if and only if $i, j \in \tilde{C}_k$ for some $k \in [m] \setminus [\tilde{m}_1]$ and such that \hat{c}^* is a minimizer of

$$\min_{\hat{c}} \left\{ \sum_{\substack{k, o \in [m] \setminus [m_1] \\ k < o}} \sum_{i \in \tilde{C}_k} \sum_{j \in \tilde{C}_o} (w_{ij} + w_{ji}) u_{ko} \right\} \text{ where}$$

$\hat{C}_k = \{j \in \bigcup_{i=\tilde{m}_1+1}^{\tilde{m}} \tilde{C}_i : \hat{c}(j) = k\}$, $k \in [m] \setminus [m_1]$ with optimal value \hat{C}^* .

$u \leftarrow v^* - C - \hat{C}^*$, $v \leftarrow C + \hat{C}^*$.

for $k = \tilde{m}_1 + 1, \dots, \tilde{m}$ **do**

Compute optimal value z of the $W_{(n+k, p_a)}^D(\tilde{C}_k \cup \{n+k\})$ (dummy department $n+k$ with $\ell_{n+k} = 0$ and weights $w_{i(n+k)} = \sum_{j \in [n] \setminus \tilde{C}_k} w_{ij}$, $w_{(n+k)i} = \sum_{j \in [n] \setminus \tilde{C}_k} w_{ji}$) with the additional constraint that the optimal value is at most u (othw. $z \leftarrow \infty$).

$v \leftarrow v + z$, $u \leftarrow u - z$.

4 if m_1 **is odd then**

Compute optimal value z of the $W_{(n+\tilde{m}_1, p_b)}^S(\tilde{C}_{\tilde{m}_1} \cup \{n+\tilde{m}_1\})$ (dummy department $n+\tilde{m}_1$ with $\ell_{n+\tilde{m}_1} = 0$ and weights $w_{i(n+\tilde{m}_1)} = \sum_{j \in [n] \setminus \tilde{C}_{\tilde{m}_1}} w_{ij}$, $w_{(n+\tilde{m}_1)i} = \sum_{j \in [n] \setminus \tilde{C}_{\tilde{m}_1}} w_{ji}$) with the additional constraint that the optimal value is at most u (othw. $z \leftarrow \infty$).

$v \leftarrow v + z$, $u \leftarrow u - z$.

for $k = 1, \dots, \lfloor \frac{m_1}{2} \rfloor$ **do**

Compute optimal value z of the $W_{(n+k, p_a)}^S(\tilde{C}_k \cup \{n+k\})$ (dummy department $n+k$ with $\ell_{n+k} = c$ and weights $w_{i(n+k)} = \sum_{j \in [n] \setminus \tilde{C}_k} w_{ij}$, $w_{(n+k)i} = \sum_{j \in [n] \setminus \tilde{C}_k} w_{ji}$) with the additional constraint that the optimal value is at most u (othw. $z \leftarrow \infty$).

$v \leftarrow v + z$, $u \leftarrow u - z$.

5 if $v < v^*$ **then**

$v^* \leftarrow v$.

6 return v^* .

Proof. Let $V_k \neq \emptyset$, $k \in \mathcal{C}$, with $t(k) = \text{DCFLP}$, and assume that an optimal layout of the $W_{(n+k,p_a)}^D(C_k \cup \{n+k\})$ is given where no department of the set V_k lies directly opposite $n+k$. Then, shifting all departments to the, w.l.o.g., right by some small ε does not change the distances between departments $i, j \in C_k$, $i \neq j$, and hence influences the objective value by

$$\sum_{j \in V_k} \varepsilon (w_{(n+k)j} - w_{j(n+k)}). \quad (17)$$

By the optimality of the layout we do not change the objective value by shifting all departments, w.l.o.g., to the right until the first department contained in V_k lies directly opposite $n+k$. Thus, there always exists an optimal solution of $W_{(n+k,p_a)}^D(C_k \cup \{n+k\})$ where an $s_M \in V_k$ has the same position as $n+k$ and (16) follows for the described weight adjustment. \square

If $V_k = \emptyset$ for some $k \in \mathcal{C}$ with $t(k) = \text{DCFLP}$ (hence, the departments in this cell do not have relations to departments in other cells), we simply neglect the dummy department $n+k$ and then the $W_{(n+k,p_a)}^D(C_k \cup \{n+k\})$ is equivalent to the DCFLP with departments C_k . If $V_k \neq \emptyset$, one can fix one department $i \in V_k$ opposite $n+k$ and enumerate over each department fixed opposite $n+k$, see Proposition 7. In summary, we obtain an optimal layout of the $W_{(n+k,p_a)}^D(C_k \cup \{n+k\})$ by solving $\max\{1, |V_k|\}$ directed-circular instances.

However, our goal is to reduce the number of directed-circular instances that have to be solved, and therefore we set up an MILP model for the $W_{(n+k,p_a)}^D(C_k \cup \{n+k\})$, $C_k \subseteq [n]$, $k \in \mathcal{C}$, with dummy department $n+k$ with $\ell_{n+k} = 0$, $w_{i(n+k)} = \sum_{j \in [n] \setminus C_k} w_{ij}$, $w_{(n+k)i} = \sum_{j \in [n] \setminus C_k} w_{ji}$. An advantage of an MILP model is to obtain good lower bounds quickly, and hence to exclude unbalanced cell assignments earlier in Algorithm 1. Therefore, we use the following binary variables

$$y_i = \begin{cases} 1, & \text{if } i \text{ lies opposite } n+k, \\ 0, & \text{otherwise,} \end{cases}$$

for $i \in C_k$. Let $L = \sum_{i \in C_k} \ell_i$ and $\hat{w}_{ij} = w_{ij} - \min\{w_{ij}, w_{ji}\}$, $i, j \in C_k$, $i \neq j$, $\hat{w}_{i(n+k)} = w_{i(n+k)}$, $\hat{w}_{(n+k)i} = w_{(n+k)i}$, $i \in C_k$. We exclude the constant

$$W_k^D = \sum_{\substack{i, j \in C_k \cup \{n+k\} \\ i \neq j}} \hat{w}_{ij} \frac{\ell_i + \ell_j}{2} + \sum_{\substack{i, j \in C_k \\ i < j}} \min\{w_{ij}, w_{ji}\} L.$$

We define $L_i = \sum_{j \in C_k \setminus \{i\}} \ell_j$ and then our MILP with $V_k \neq \emptyset$ reads as follows

$$\min \sum_{\substack{i, j \in C_k \cup \{n+k\} \\ i \neq j}} \hat{w}_{ij} d_{ij} \quad (18)$$

s. t. (8), (11)–(13),

$$\mathcal{D} := C_k \cup \{n+k\}, \quad f := n+k,$$

$$\sum_{i \in C_k} y_i = 1, \quad (18)$$

$$z_{ij} - y_i \geq 0, \quad i, j \in C_k, \quad i < j, \quad (19)$$

$$z_{ij} + y_j \leq 1, \quad i, j \in C_k, \quad i < j, \quad (20)$$

$$y_i = 0, \quad i \in C_k \setminus V_k, \quad (21)$$

$$d_{(n+k)i} - \sum_{\substack{j \in C_k \\ j < i}} \ell_j z_{ji} + \sum_{\substack{j \in C_k \\ j > i}} \ell_j z_{ij} + \sum_{j \in C_k} \frac{\ell_j}{2} y_j = \sum_{\substack{j \in C_k \\ j > i}} \ell_j, \quad i \in C_k, \quad (22)$$

$$d_{i(n+k)} + d_{(n+k)i} + (L_i + \ell_i) y_i = L_i, \quad i \in C_k, \quad (23)$$

$$d_{ij} \geq 0, \quad i, j \in C_k, \quad i \neq j, \quad (24)$$

$$d_{ij} \geq -\frac{\ell_i}{2}, \quad i, j \in C_k \cup \{n+k\}, \quad i \neq j, \quad (25)$$

$$z_{ij} \in \{0, 1\}, \quad |\{i, j\} \cap \{n+k\}| = 1, \quad (26)$$

$$y_i \in \{0, 1\}, \quad i \in C_k, \quad i < j, \quad (27)$$

Let $k \in \mathcal{C}$. Equations (18) ensure that exactly one department lies opposite $n+k$, see Proposition 7. According to the z -variables we obtain an ordering of the departments, see Section 2.2, and we ensure via Inequalities (19)–(20) that the department $i' \in C_k$ with $y_{i'} = 1$ is the leftmost department. By Proposition 7 and our assumption $V_k \neq \emptyset$, we set y_i , $i \in C_k \setminus V_k$, equal to zero, see Equations (21). The distance calculation in Equations (22) is similar to the distance calculation in Equations (9) where we additionally subtract half the length of the department that is fixed opposite $n+k$ because we excluded the constant W_k^D . By Equations (23) and Inequalities (25) we obtain $d_{i(n+k)} = d_{(n+k)i} = -\frac{\ell_i}{2}$, $i \in C_k$, if i lies opposite $n+k$. In this case the distances $d_{i(n+k)}$ as well as $d_{(n+k)i}$, $i \in C_k$, are negative since we excluded the constant $\sum_{i \in C_k} (w_{i(n+k)} + w_{(n+k)i}) \frac{\ell_i}{2}$. If $i \in C_k$ does not lie opposite $n+k$, then we obtain $d_{i(n+k)} + d_{(n+k)i} = L_i$, see Equations (23).

Proposition 8. *Let C_k denote the set of departments assigned to cell $k \in \mathcal{C}$ with $t(k) = DCFLP$. Then*

$$\begin{aligned} \min \quad & \sum_{\substack{i, j \in C_k \cup \{n+k\} \\ i < j}} \hat{w}_{ij} d_{ij} \\ \text{s. t.} \quad & (8), (11)–(13), (18)–(27) \end{aligned}$$

is a MILP model for the $W_{(n+k, p_a)}^D(C_k \cup \{n+k\})$.

Proof. Let $i' \in C_k$ such that $y_{i'} = 1$, see Equation (18). By Inequalities (8) together with the binary constraints we obtain a feasible ordering of the departments and the distances d_{ij} , $i, j \in C_k \setminus \{i'\}$, are calculated correctly by Equations (11)–(13), see [15, 37, 38]. According to the z -variables we obtain an ordering of the departments and by Inequalities (19)–(20) i' is the leftmost department. We neglect the constant $\sum_{i \in C_k} (\hat{w}_{i(n+k)} + \hat{w}_{(n+k)i}) \frac{\ell_i}{2}$ and by Equations (22) and Inequalities (25) we obtain $d_{i'(n+k)} = d_{(n+k)i'} = -\frac{\ell_{i'}}{2}$ if $y_{i'} = 1$, $i' \in C_k$, and $d_{i(n+k)} + d_{(n+k)i} = L_i$ in the case $y_i = 0$, $i \in C_k$. Hence, the distance calculation between $i \in C_k$ and $n+k$ is correct as well, see Equations (22)–(23). \square

We conclude this section by pointing out how further realistic extensions can be included in our approach.

Remark 9. 1) *Our approach presented above can be combined with further aspects relevant in practice. Let the size of the cells be equal to $F \in \mathbb{R}_{\geq 0}$ (restriction on the sum of the lengths of all departments in one cell). We can neglect all cell assignments where the sum of the lengths of the departments in one cell exceeds F , see [17, 26]. However, a solution of the $W_{(n+k, p_a)}^D(C_k \cup \{n+k\})$ contains free-spaces if the sum of the lengths of the departments assigned to cell k is smaller than F . By the same shifting argument as used in the proof of Proposition 7 there exists an optimal layout where the free-space is interruption-free, i. e., it is sufficient to add one additional dummy department $n+k'$ with $\ell_{n+k'} = F - \sum_{i \in C_k} \ell_i$, $w_{i(n+k')} = w_{(n+k')i} = 0$, $i \in C_k \cup \{n+k\}$, as done in [14] for the MREFLP and solve the $W_{(n+k, p_a)}^D(C_k \cup \{n+k, n+k'\})$ without spaces. Thus, one can use the methods presented above. In cells of type SRFLP, there always exists an optimal layout where the possible free-space arises only at the borders of the layout. Hence, it suffices to restrict the horizontal center-to-center distances of all departments to the loading stations, i. e., $d_{i(n+k)} \leq F - \frac{\ell_i}{2}$, $i \in C_k$, $k \in [m]$, $t(k) = SRFLP$. Clearly, the length of the dummy department $n+k$ can be neglected.*

- 2) In the facility layout planning literature it is a standard assumption that the (un-) loading points of the departments are fixed at their centers, see, e. g., [5, 7, 12]. However, if the input and output positions are fixed on the left (or right) border of the departments, one can treat $n + k$ as an ordinary department, i. e., no department may overlap with $n + k$, and we just need to solve one directed-circular instance with departments $C_k \cup \{n + k\}$ in order to solve the $W_{(n+k, p_a)}^D(C_k \cup \{n + k\})$.

5 Extensions of the Multi-Bay Facility Layout Problem

In this section we describe the adaption of our approach presented in Section 4 to the 3-BFLP, the XRFLP and the 4-BFLP. Adding appropriate dummy departments we are able to use the main ideas of Algorithm 1. Even for the problems with four cells we have to consider at most 2^n distinguishable generalized cell assignments of the departments. Considering the 3-BFLP and the 4-BFLP, we are able to further reduce the number of cell assignments that have to be considered.

5.1 The 3-Bay Facility Layout Problem

In [23] it is shown that the 3-BFLP is equivalent to the TRFLP where a dummy department $n + 1$ is fixed on the (un-) loading station with length $\ell_{n+1} = 2w_{path}$ and weights $w_{i(n+1)} = w_{(n+1)i} = 0$, $i \in [n]$. In order to reduce the number of distinguishable cell assignments that have to be considered, we present a different approach. We merge cell 1 and cell 2 as a new cell and interpret cell 3 as new cell 2. Let $C_1, C_2 \subseteq [n]$, $C_1 \cap C_2 = \emptyset$, such that $C_1 \dot{\cup} C_2 = [n]$ and let C_1 (C_2) denote the set of departments assigned to the merged cell 1 (cell 2). The dummy department $n + 1$ ($n + 2$) is added to the merged cell 1 (cell 2) with $\ell_{n+1} = 0$ ($\ell_{n+2} = 0$) and weights $w_{i(n+1)} = \sum_{j \in [n] \setminus C_1} w_{ij}$, $w_{(n+1)i} = \sum_{j \in [n] \setminus C_1} w_{ji}$ ($w_{i(n+2)} = \sum_{j \in [n] \setminus C_2} w_{ij}$, $w_{(n+2)i} = \sum_{j \in [n] \setminus C_2} w_{ji}$) (see (14)–(15)). We exclude the constant $\sum_{i \in C_1, j \in C_2} (w_{ij} + w_{ji}) \cdot w_{path}$ and we fix an additional dummy department $n + 3$ to the merged cell 1 neighboring $n + 1$ with lengths $\ell_{n+3} = w_{path}$ and weights $w_{i(n+3)} = w_{(n+3)i} = 0$, $i \in C_1 \cup \{n + 1\}$. Then, we obtain an optimal solution for the 3-BFLP by solving the $W_{(n+1, p_a)}^S(C_1 \cup \{n + 1, n + 3\})$ and the $W_{(n+2, p_b)}^S(C_2 \cup \{n + 2\})$. Hence, we can solve the 3-BFLP similar to the approach summarized in Algorithm 1.

As done in [23] one can use symmetry breaking for the 3-BFLP. If $w_{path} = 0$, we can fix two departments to C_1 and one has to consider 2^{n-2} generalized cell assignments. If $w_{path} > 0$, one department can be fixed to C_1 , and hence we have to consider 2^{n-1} generalized cell assignments. In the 3-PMFP and the 4-PMFP one can fix two departments to the merged cell C_1 , and hence it is sufficient to consider 2^{n-2} generalized cell assignments for $w_{path} \geq 0$.

In contrast to the previous presented approaches, the computation time of our approach for the TRFLP is slightly higher than the approach of [23]. Thus, we summarize our approach for the TRFLP in the appendix.

5.2 The X-Row and the 4-Bay Facility Layout Problem

Next we consider layout problems with four original cells in more detail. Recall that we assume that the height of each department equals one. The following proposition is essential to solve the cells independently, and thus we derive a result similar to Theorem 5

Proposition 10. *Given an XRFLP instance. There always exists an optimal X-row layout where some $i \in [n]$ is contained in cell 3 and $d_{i p_{E_3}} = \frac{\ell_i}{2}$.*

Proof. Let an optimal X-row layout be given. We denote by C_a the set of departments assigned to cell $a \in [4]$ and we assume that $d_{i p_{E_3}} > \frac{\ell_i}{2}$ for all $i \in C_3$. If all departments are contained in $C_a \cup C_b$, $a, b \in [4]$, $a < b$, then we arrange all departments space-free in cell 3 respecting the

order of the departments in cell a and cell b such that the department in cell a which is closest to p_{E_a} is neighboring the department in cell b which is closest to p_{E_b} . Then we can shift the departments such that afterwards one department $i \in [n]$ satisfies $d_{ip_{E_3}} = \frac{\ell_i}{2}$ and clearly, we do not increase the objective value by this method.

Next, let $|C_a|, |C_b|, |C_c| \geq 1$, $a, b, c \in [4]$, $|\{a, b, c\}| = 3$. If there exists $i \in C_3$ with $d_{ip_{E_3}} < \frac{\ell_i}{2} + 1$, $i \in C_3$, then we shift all departments in cell 3 to the left until the left border of the leftmost department reaches the position p_{E_3} and the resulting layout is feasible since the previous layout was feasible. The objective value of the layout is not increased by this method. Further, we assume there exists $i \in C_2$ with $d_{ip_{E_2}} < \frac{\ell_i}{2} + 1$ or $d_{ip_{E_2}} \geq \frac{\ell_i}{2} + 1$ for all $i \in C_2$ and $C_3 = \emptyset$. Then, we shift all departments in cell 2 to the left in the direction to p_{E_2} until the leftmost department $j \in C_2$ satisfies $d_{jp_{E_2}} = \frac{\ell_j}{2}$. Then we arrange the departments in cell 3 (cell 2) to cell 2 (cell 3) space-free respecting the order of the departments. We do not increase the objective value by this method since $u_{12} = u_{13}$ and $u_{24} = u_{34}$. It remains to consider the case $d_{ip_{E_3}} \geq \frac{\ell_i}{2} + 1$, $i \in C_3 \neq \emptyset$, and $d_{ip_{E_2}} \geq \frac{\ell_i}{2} + 1$, $i \in C_2$. Then we shift all departments in cell 3 to the left until the first department has its left border on position p_{E_3} . Clearly, we do not increase the objective value by this method. \square

Given an X-row instance, we want to ensure that departments in cell 2 and cell 3 do not overlap. Therefore, we fix an additional dummy department $n + 3$ at the border of cell 2 with lengths $\ell_{n+3} = 1$ and weights $w_{i(n+3)} = w_{(n+3)i} = 0$, $i \in [n + 2]$, see Proposition 10. If $w_{path}^1 = w_{path}^2 = 0$, then one can solve the cells independently, see Lemma 4, and hence we can apply Theorem 5 with $u_{ko} = 0$, $k, o \in [4]$, $k < o$, and interpret cell 1 and cell 3 as well as cell 2 and cell 4 as a new (larger) cell. The dummy department $n + 1$ ($n + 2$) is added to the merged cell 1 (cell 2) with $\ell_{n+1} = 0$ ($\ell_{n+2} = 0$) and weights as described in (14)–(15) and the merged cell 2 additionally contains the dummy department $n + 3$ which is neighboring $n + 2$.

It remains to include inter-cell distances in our approach for the XRFLP. Let $C_1, C_2 \subseteq [n]$, $C_1 \cap C_2 = \emptyset$, such that $C_1 \cup C_2 = [n]$ and let C_1 (C_2) denote the set of departments assigned to the merged cell 1 (cell 2). We add the additional dummy department $n + 4$ to the merged cell 1 with lengths $\ell_{(n+4)} = w_{path}^1$ and weights $w_{i(n+4)} = w_{(n+4)i} = 0$, $i \in [n + 3]$, and then we solve the $W_{(n+1, p_a)}^S(C_1 \cup \{n + 1, n + 4\})$ where $n + 4$ is neighboring $n + 1$. In the merged cell 2 we avoid adding another dummy department by adapting the lengths of the dummy departments $n + 2$ and $n + 3$ such that $\ell_{n+2} = 2 \cdot \min\{1, w_{path}^2\}$ and $\ell_{n+3} = |w_{path}^2 - 1|$. In both cases, the distance of $n + 2$ and the departments to the, w. l. o. g., left of $n + 2$ is at least w_{path}^2 and to the departments to the right of $n + 2$ is at least 1, and thus the distances are calculated correctly. We refer to Figure 5 for an illustration. This leads to the following result.

Corollary 11. *Given an X-row instance with $w_{path}^1, w_{path}^2 \geq 1$, and we set $\ell_{n+1} = 0$, $\ell_{n+2} = 2 \cdot \min\{w_{path}^2, 1\}$, $\ell_{n+3} = |w_{path}^2 - 1|$, $\ell_{n+4} = w_{path}^1$, and $n + 1$ ($n + 2$) is neighboring $n + 4$ ($n + 3$). We obtain an optimal X-row layout by enumerating over all assignments of the departments to the cells $C = \{1, 2\}$ and solving the $W_{(n+1, p_a)}^S(C_1 \cup \{n + 1, n + 4\})$ and the $W_{(n+2, p_a)}^S(C_2 \cup \{n + 2, n + 3\})$ where C_k , $k \in [2]$, denotes the set of departments assigned to (merged) cell k and the weights of the additional dummy departments are set to $w_{i(n+3)} = w_{(n+3)i} = w_{i(n+4)} = w_{(n+4)i} = 0$, $i \in [n + 2]$.*

Proof. By adding dummy departments $n + 3$ and $n + 4$ with length $\ell_{n+4} = w_{path}^1$, $\ell_{n+3} = |w_{path}^2 - 1|$ as well as $\ell_{n+2} = 2 \cdot \min\{w_{path}^2, 1\}$, we obtain the correct distance calculation. Thus, similar to Theorem 5 it is sufficient to consider two cells. \square

In Algorithm 1 one has to consider 2^n cell assignments to obtain an optimal X-row layout since we use symmetry breaking in the proof of Proposition 10. In our algorithm we solve the $W_{(n+2, p_a)}^S(C_2 \cup \{n + 2, n + 3\})$ first if $|C_2| \leq |C_1| + 1$ and otherwise we solve the $W_{(n+1, p_a)}^S(C_1 \cup \{n + 1, n + 4\})$ first with the idea to exclude unbalanced cell assignments earlier. For the whole

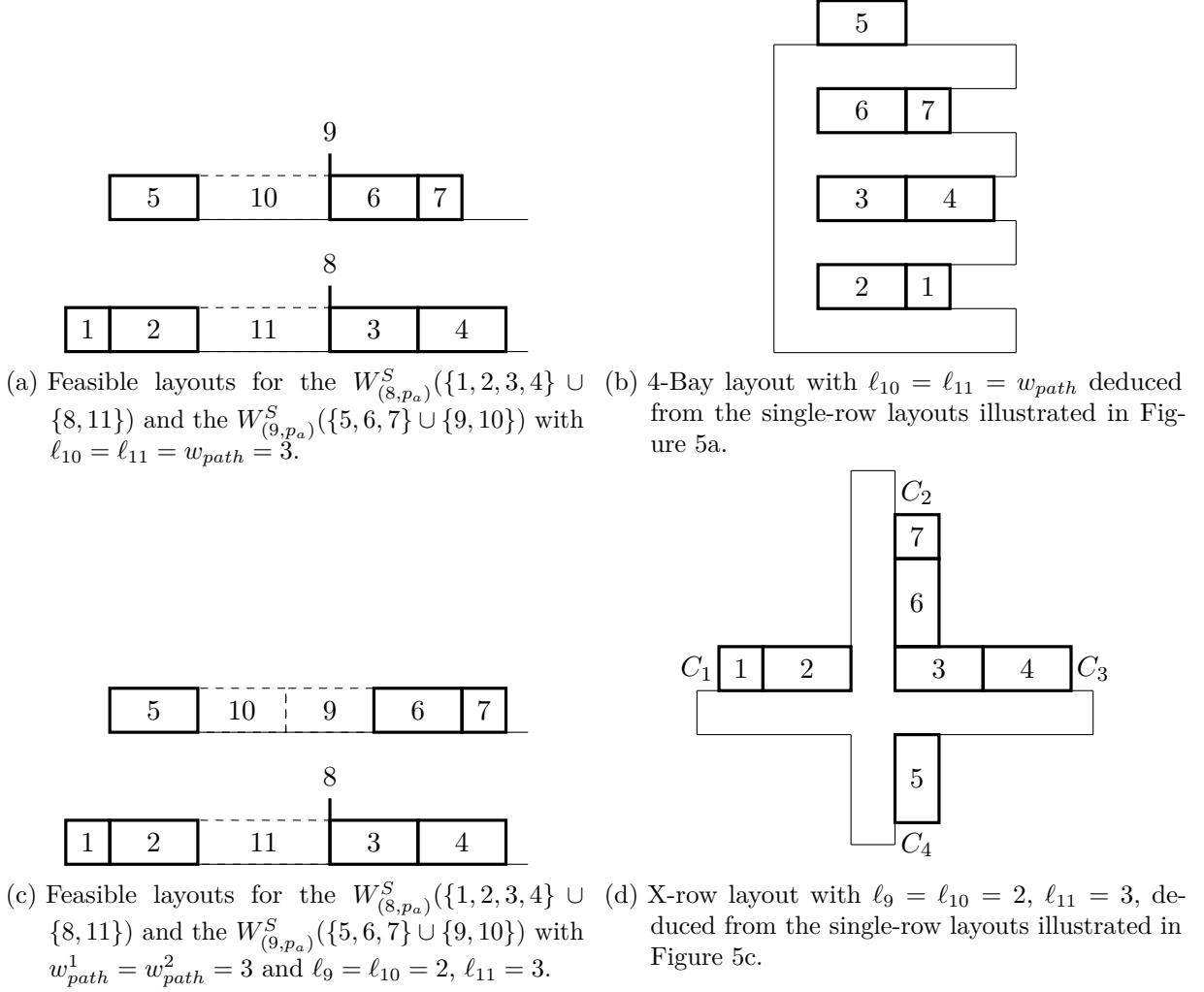


Figure 5: Visualization of obtained 4-Bay and X-row layouts with positive inter-cell distances by solving appropriate single-row instances. In order to construct, e. g., a 4-Bay layout, with $n = 7$ departments, the departments left (right) to the dummy department 8 are assigned to cell 1 (cell 2) in reversed (the same) order and the departments left (right) to the dummy department 9 are assigned to cell 4 (cell 3) in reversed (the same) order. The distances between each pair of departments in the layouts illustrated in Figure 5a and in Figure 5b as well as in Figure 5c and in Figure 5d are the same.

problem we exclude the constant

$$W := \sum_{\substack{i,j \in [n] \\ i < j}} (w_{ij} + w_{ji}) \frac{\ell_i + \ell_j}{2} + \sum_{i \in C_2} (w_{i(n+2)} + w_{(n+2)i}) \frac{\ell_{n+2}}{2}.$$

We neglect the dummy department $n+4$ if $w_{path}^1 = 0$. However, we additionally add the following constraints

$$\begin{aligned} x_{ji(n+2)} - x_{ji(n+3)} &= 0, & i, j \in [n], i \neq j, \\ x_{ji(n+1)} - x_{ji(n+4)} &= 0, & i, j \in [n], i \neq j, \end{aligned} \quad (28)$$

where inequalities (28) are only added if $w_{path}^1, w_{path}^2 > 0$.

Next we describe how to include inter-cell distances to the 4-BFLP in Algorithm 1. At first, we merge cell 1 and cell 2 as well as cell 3 and cell 4. Hence, let C_1 (C_2) denote the set of departments

assigned to the merged cell 1 (cell 2). We exclude the constant $\sum_{i \in C_1, j \in C_2} (w_{ij} + w_{ji}) \cdot w_{path}$ and solve the $W_{(n+k, p_a)}^S(C_k \cup \{n+k, n+k+2\})$, $k \in \{1, 2\}$, with an additional dummy department $n+k+2$ with length $\ell_{n+k+2} = w_{path}$ and weights $w_{i(n+k+2)} = w_{(n+k+2)i} = 0$, $i \in C_k \cup \{n+k\}$, where $n+k+2$ is a neighboring department of $n+k$. By Corollary 11 we obtain an optimal solution for the 4-BFLP by this method. For an illustration we refer to Figure 5. In our algorithm we solve the $W_{(n+2, p_a)}^S(C_2 \cup \{n+2, n+3\})$ first if $|C_2| \leq |C_1|$ and otherwise we solve the $W_{(n+1, p_a)}^S(C_1 \cup \{n+1, n+4\})$ first. Considering the 4-BFLP with $w_{path} = 0$, we can fix two departments (randomly chosen) to cell 1, and hence it is sufficient to consider 2^{n-2} cell assignments. If $w_{path} > 0$, one can fix one department to cell 1, and hence it is sufficient to consider 2^{n-1} cell assignments.

6 Computational results

In this section we present our computational results. The computational experiments are implemented in C++ and we use Cplex 12.10 as an MILP Solver [39]. All results were conducted on a 2.10GHz quad-core using Virtual Box 6 and running on Debian GNU/Linux 8 in single processor mode.

6.1 Computational experiments

In our computational results we focus on optimization problems which consist of at most 2 merged cells in view of our results derived in Theorem 5. The instances from the literature are symmetric, see e. g., [10, 13]. Hence, let w_{ij} , $i, j \in [n]$, $i < j$, be given, we choose uniformly at random $\tilde{w}_{ij} \in \{0, 1, \dots, w_{ij}\}$, $i, j \in [n]$, $i < j$, and set $\tilde{w}_{ji} = w_{ij} - \tilde{w}_{ij}$, $i, j \in [n]$, $j < i$, to obtain asymmetric instances. All instances can be downloaded from <https://tinyurl.com/instances-DaFiHuMa20>. At first, we present results for the FC-CCLP, see Table 1. We assign the departments $1, \dots, \lfloor \frac{n}{2} \rfloor$ to the first cell and the remaining departments to the second cell. For the FC-CCLP (1,1) we assume that the cell of type SRFLP is the first cell. The inter-cell distance is constant, and hence set to zero. In the first column in Table 1 the instance names are given, and the first number in the names indicates the number of the departments. Instances with second number equal to 1 are equidistant instances, and hence we omit them here. In the next three columns we display the optimal values and in the last five columns the corresponding running times are given where “enu” describes the enumerative approach and “MILP” describes our MILP model for cells of type DCFLP. We write < 1 if the running time is less than one second. For all instances tested here the optimal value of the FC-CCLP (2,0) is smaller than the optimal value of the FC-CCLP (1,1) which is smaller than the optimal value of the FC-CCLP (0,2). Hence, it follows that in our tests a cell of type SRFLP has a smaller optimal value than a cell of type DCFLP with the same departments. One can observe that even large-sized instances were solved quickly and it seems that a cell of type SRFLP is easier to solve than a single-row instance with the same number of departments. The reason is the additionally dummy department which has usually high weights to all departments. However, cells of type DCFLP could be solved to optimality in a few seconds while cells of type SRFLP partially need a few minutes.

In order to compare our approach for the TRFLP, the 3-BFLP and the 3-PMFP with the approach of [23], we make use of the heuristically determined upper bounds in [23]. In Table 2 we compare the optimal values of several facility layout problems where the inter-cell distances are set to zero and in Table 3 the inter-cell distances are set to three. In the column “Source” we display the source of the symmetric instances. We observe that the optimal value of the TRFLP is smaller than the optimal value of the 3-BFLP and the 3-PMFP. The optimal value of the 4-BFLP is slightly smaller than the optimal value of the XRFLP if the inter-cell distances are set to zero, and if $w_{path} = w_{path}^1 = w_{path}^2 = 3$, then the optimal value of XRFLP is smaller than the optimal value of the 4-BFLP and greater than the optimal value of the 4-PMFP, see Table 3. For all instances the

Instance	Source	FC-CCLP (2,0)	FC-CCLP (1,1)	FC-CCLP (0,2)	FC-CCLP (2,0)	FC-CCLP (1,1)		FC-CCLP (0,2)	
		opt value	opt value	opt value	time	enu	MILP	enu	MILP
AV25-2	[10]	42745.5	47933.5	51169.5	<1	<1	<1	<1	<1
AV25-3	[10]	27602.0	29890.0	31001.0	<1	<1	<1	<1	<1
AV25-4	[10]	55549.5	62776.5	67306.5	<1	<1	<1	<1	<1
AV25-5	[10]	18044.0	19692.0	20937.0	<1	<1	<1	<1	<1
AV30-2	[10]	24397.5	25505.5	27447.5	<1	<1	<1	<1	<1
AV30-3	[10]	52018.0	56298.0	61122.0	1	<1	<1	<1	<1
AV30-4	[10]	65516.5	71443.5	78201.5	1	<1	<1	<1	<1
AV30-5	[10]	134026.0	141672.0	162364.0	1	1	<1	<1	<1
ste36-2	[13]	181508.0	223338.0	307066.0	5	4	4	1	<1
ste36-3	[13]	101644.0	128985.0	206897.0	8	7	7	1	<1
ste36-4	[13]	95805.5	111559.0	186153.0	7	6	5	1	<1
ste36-5	[13]	91651.5	114715.0	175358.0	4	4	3	1	<1
sko42-2	[13]	249986.0	278212.0	308533.0	43	32	23	4	<1
sko42-3	[13]	198270.0	220596.0	242858.0	5	4	2	3	<1
sko42-4	[13]	154057.0	172482.0	190150.0	22	5	4	2	<1
sko42-5	[13]	287194.0	318344.0	351982.0	20	12	10	3	<1
sko49-2	[13]	459140.0	519684.0	558166.0	3:32	2:26	2:51	7	<1
sko49-3	[13]	357705.0	410297.0	441037.0	2:25	1:29	1:54	13	<1
sko49-4	[13]	262364.0	292652.0	315012.0	2:51	45	52	9	<1
sko49-5	[13]	737087.0	785207.0	860911.0	15:00	59	56	14	4

Table 1: Optimal values (“opt value”) and running times (in sec or min:sec) for different variants of the FC-CCLP, where $\lfloor \frac{n}{2} \rfloor$ departments are assigned to the first cell and the remaining departments to the second cell.

optimal value of the XRFLP is smaller than the optimal value of the TRFLP and for all instances the 4-PMFP (the 4-BFLP if $w_{path} = 0$) has the smallest optimal value. Recall that the SRFLP is equivalent to the CCLP (2,0), see Theorem 5. The optimal value of the SRFLP is smaller than the optimal value of the CCLP (0,2) here but the CCLP (1,1) has the smallest optimal value of these three problems in our tests. The optimal values of the CCLP are greater than the ones of the 3-BFLP. The CCLP (0,2) with up to 18 departments could be solved in less than 30 minutes, see Table 4, by using our MILP approach. Our MILP approach clearly outperforms the enumerative approach on the instances considered. Therefore, our approach outperforms the approach of [15, 32] as well because their models do not contain dummy departments, and hence one would have to use the enumerative approach.

The approach of [23] can easily be extended to the 3-PMFP. The optimal values derived by the approach of [23] are optimal neglecting computational accuracy, we refer to [23] for details. Considering the 3-BFLP with positive inter-cell distance as well as with inter-cell distance of zero we outperform the approach of [23], see Table 4 and Table 5. With our approach the TRFLP and the 4-BFLP could be solved with up to 17 departments within a time limit of 8 hours and the XRFLP with up to 16 departments. Our MILP model for the optimization problem in cell 1 in the TRFLP leads to a smaller running time than the corresponding enumerative approach, see Table 4 and Table 5. However, the approach of [23] can even solve one instance with 18 departments and is for most instances faster than our approach. The 3-PMFP (4-PMFP) is solved faster than the 3-BFLP and the TRFLP (the 4-BFLP and the XRFLP) since the number of distinguishable cell assignments is significantly smaller, see Table 5.

7 Conclusion and Future work

In this paper we presented a new exact approach for the Combined Cell Layout Problem (CCLP) and we focus on the special cases of the CCLP, i.e., the Multi-Bay Facility Layout Problem (MBFLP) and the Pier-Type Material Flow Pattern (PMFP) with $m = 3$ and $m = 4$ denoted by (3-BFLP), (3-PMFP), (4-BFLP), (4-PMFP). Further, we considered the T-Row Facility Layout Problem (TRFLP) and we introduced a new layout problem, the so called X-Row Facility Layout Problem (XRFLP), which is a generalization of the 4-BFLP with a more complex path structure.

Instances	Source	SRFLP	CCLP (1,1)	CCLP (0,2)	3-BFLP	TRFLP	4-BFLP	XRFLP
<i>Am11a</i>	[9]	10630.5	9840.0	11178.5	8466.5	8407.0	6899.5	7038.5
<i>Am11b</i>	[9]	7375.5	6802.5	7262.0	5694.5	5665.0	4864.5	4990.5
<i>Am12a</i>	[6, 7]	2901.0	2702.5	3266.5	2382.0	2354.5	1994.0	2047.0
<i>Am12b</i>	[6, 7]	3280.5	3042.5	3389.5	2557.5	2539.5	2172.5	2234.5
<i>Am13a</i>	[6]	4902.5	4404.5	5283.5	3863.5	3836.0	3258.5	3327.5
<i>Am13b</i>	[6]	5698.0	5046.0	6029.0	4376.0	4362.5	3642.0	3702.0
<i>Am14_1</i>	[26]	5481.5	5132.0	5766.5	4370.5	4350.5	3557.5	3634.5
<i>Am14a</i>	[51]	5673.0	5263.0	6619.0	4475.0	4446.5	3773.0	3872.0
<i>Am14b</i>	[51]	5595.0	5166.0	5725.0	4451.0	4430.5	3749.0	3838.0
<i>Am15</i>	[3]	6305.0	5961.5	6899.0	5093.0	5071.0	4237.0	4319.0
<i>HK15</i>	[29]	33220.0	30880.0	37440.0	26290.0	26125.0	21810.0	21891.0
<i>P16a</i>	[8]	14829.0	14087.0	15125.0	11999.0	11943.0	10076.0	10194.0
<i>P16b</i>	[8]	11878.5	11360.0	12768.5	9499.5	9469.5	7805.5	7921.5
<i>P17a</i>	[23]	14436.5	14066.0	15930.0	11551.5	11524.5	9574.5	-
<i>P17b</i>	[23]	15682.0	14870.5	16034.0	12389.0	12317.0	10174.0	-
<i>Am17</i>	[4]	9254.0	8604.0	10896.0	7345.0	7315.0	6044.0	-
<i>P18a</i>	[23]	16118.5	15043.5	17904.0	12528.5	-	10266.5	-
<i>P18b</i>	[23]	17716.5	16733.0	18022.5	14138.5	-	-	-
<i>Am18</i>	[23]	10650.5	10050.5	12274.5	8446.5	8413.5	6914.5	-

Table 2: We illustrated the optimal values of several facility layout problems for instances from the literature where the inter-cell distances are set to zero. Instances marked with “-” could not be solved to optimality within the time limit of 8 hours.

Instances	SRFLP	CCLP (1,1)	CCLP (0,2)	3-BFLP	3-PMFP	TRFLP	4-BFLP	4-PMFP	XRFLP
<i>Am11a</i>	10630.5	10075.5	11727.5	9453.5	9237.5	8902.0	8261.5	7763.5	7882.5
<i>Am11b</i>	7375.5	7240.5	7739.0	6619.5	6444.5	6118.5	6152.5	5723.5	5793.5
<i>Am12a</i>	2901.0	2782.0	3468.0	2748.0	2672.0	2552.0	2519.0	2333.0	2374.0
<i>Am12b</i>	3280.5	3213.0	3584.5	2959.5	2872.5	2740.5	2733.5	2565.5	2593.5
<i>Am13a</i>	4902.5	4573.5	5532.5	4337.5	4232.5	4077.0	3926.5	3706.5	3741.5
<i>Am13b</i>	5698.0	5262.0	6260.0	4820.0	4727.0	4581.5	4293.0	4065.0	4102.0
<i>Am14_1</i>	5481.5	5278.5	6117.5	4940.5	4802.5	4642.0	4313.5	4049.5	4106.5
<i>Am14a</i>	5673.0	5386.0	6964.0	5093.0	4967.0	4751.0	4641.0	4365.0	4408.0
<i>Am14b</i>	5595.0	5400.0	6094.0	5059.0	4939.0	4739.5	4623.0	4325.0	4379.0
<i>Am15</i>	6305.0	6102.0	7298.0	5687.0	5546.0	5378.0	5047.0	4768.0	4822.0
<i>HK15</i>	33220.0	31174.0	37773.0	26905.0	26755.0	26446.0	22680.0	22368.0	22429.0
<i>P16a</i>	14829.0	14447.0	15551.0	12866.0	12668.0	12381.0	11276.0	10844.0	10940.0
<i>P16b</i>	11878.5	11638.5	13194.5	10324.5	10147.5	9882.5	8974.5	8561.5	8645.5
<i>P17a</i>	14436.5	14237.0	16443.0	12429.5	12232.5	11956.5	10811.5	10361.5	-
<i>P17b</i>	15682.0	15341.5	16526.0	13298.0	13094.0	12779.0	11446.0	10969.0	-
<i>Am17</i>	9254.0	8814.0	11328.0	8245.0	8046.0	7767.5	7292.0	6836.0	-
<i>P18a</i>	16118.5	15529.5	18480.0	13506.5	13272.5	-	-	11155.5	-
<i>P18b</i>	17716.5	17300.0	18544.5	15110.5	14885.5	-	-	12531.5	-
<i>Am18</i>	10650.5	10272.5	12823.5	9430.5	9202.5	8911.5	-	7791.5	-

Table 3: Optimal values of facility layout problems for instances from the literature with inter-cell distances $w_{path} = w_{path}^1 = w_{path}^2 = 3$ and for the CCLP (1,1) and CCLP (0,2) we set $u_{12} = 3$. Instances marked with “-” could not be solved to optimality within the time limit of 8 hours.

Instances	CCLP (1,1)		CCLP (0,2)		3-BFLP		TRFLP		4-BFLP	XRFLP	
	enu	MILP	enu	MILP	our	[23]	our	MILP			[23]
<i>Am11a</i>	21	11	14	5	5	14	28	29	33	5	30
<i>Am11b</i>	20	13	14	4	4	19	28	23	20	5	30
<i>Am12a</i>	45	25	32	9	12	29	1:35	1:03	51	14	1:25
<i>Am12b</i>	41	27	29	12	13	30	1:36	1:00	42	15	1:24
<i>Am13a</i>	1:46	1:05	1:06	24	34	1:13	4:38	3:16	2:17	45	6:12
<i>Am13b</i>	1:46	1:07	1:04	22	31	59	4:26	3:07	2:17	40	4:17
<i>Am14_1</i>	4:44	2:57	2:47	1:08	2:00	3:11	15:16	11:09	9:11	2:47	17:24
<i>Am14a</i>	4:27	2:39	3:17	52	1:37	2:53	13:11	8:23	6:00	2:41	16:39
<i>Am14b</i>	4:58	3:04	3:09	1:03	1:49	7:06	13:46	9:07	6:51	3:08	17:49
<i>Am15</i>	12:07	7:15	7:05	2:16	4:43	7:40	41:22	25:46	15:31	9:55	1:18:18
<i>HK15</i>	12:01	7:12	7:52	1:56	4:08	11:32	39:03	23:10	13:48	7:15	1:00:47
<i>P16a</i>	33:09	19:51	18:42	4:53	17:24	25:14	2:20:09	1:36:50	12:31	45:09	4:12:20
<i>P16b</i>	33:19	19:56	19:33	5:01	15:48	19:36	2:10:25	1:27:46	1:09:58	43:12	3:51:49
<i>P17a</i>	1:25:18	51:20	48:48	12:22	44:37	56:59	6:02:53	3:57:38	3:32:27	2:06:24	-
<i>P17b</i>	1:27:18	52:56	47:25	12:07	57:11	51:41	6:14:47	4:02:21	3:37:59	2:17:39	-
<i>Am17</i>	1:15:16	42:34	46:13	11:00	44:06	30:33	5:46:36	3:52:34	2:19:20	1:47:45	-
<i>P18a</i>	3:36:22	2:07:54	1:58:28	28:04	2:11:53	<i>MS</i>	-	-	-	7:22:52	-
<i>P18b</i>	3:48:57	2:13:47	1:58:29	26:45	3:08:51	3:48:07	-	-	-	-	-
<i>Am18</i>	3:12:23	1:46:18	1:57:58	26:24	2:06:25	6:55:33	-	-	6:01:58	5:58:33	-

Table 4: Computational results for various layout types. The running times are given in sec, min:sec or h:min:sec for instances from the literature. We write “_” if the time limit of 8 hours is exceeded and “*MS*” if we run out of memory storage. We set the inter-cell distance $w_{path} = 0$ and for the CCLP (1,1) and CCLP (0,2) we set $u_{12} = 0$.

Instances	CCLP (1, 1)			CCLP (0, 2)			3-BFLP			3-PMFP			TRFLP			4-BFLP	4-PMFP	XRFLP
	enu	MILP	[23]	enu	MILP	[23]	our	[23]	our	[23]	enu	MILP	[23]	4-BFLP	4-PMFP			
<i>Am11a</i>	17	10	15	5	14	20	4	9	35	28	28	21	5	49				
<i>Am11b</i>	18	13	14	7	10	25	3	8	33	24	16	23	4	42				
<i>Am12a</i>	40	31	31	13	32	1:17	13	40	1:30	57	51	56	14	1:45				
<i>Am12b</i>	40	55	22	12	28	1:40	11	19	1:30	1:07	40	56	13	1:42				
<i>Am13a</i>	1:53	1:02	1:00	29	1:32	2:41	29	1:35	4:34	2:58	2:01	2:59	37	6:32				
<i>Am13b</i>	1:43	1:02	1:14	26	1:18	3:22	27	38	4:26	2:51	1:54	2:27	35	5:08				
<i>Am14_1</i>	4:40	2:42	2:35	58	4:25	32:21	1:34	3:26	14:57	9:57	8:53	10:38	2:07	20:02				
<i>Am14a</i>	4:21	2:27	2:47	1:03	3:50	4:16	1:21	3:55	13:03	7:40	5:35	9:56	1:54	18:41				
<i>Am14b</i>	4:45	2:49	2:48	1:07	3:57	15:50	1:22	2:09	13:32	8:07	6:47	10:14	2:04	19:16				
<i>Am15</i>	11:42	6:35	6:54	2:26	10:10	36:55	3:44	8:13	40:07	22:49	15:53	32:29	6:10	1:16:00				
<i>HK15</i>	11:46	6:48	7:06	2:26	9:18	36:45	3:57	3:51	38:52	22:09	13:04	26:16	6:57	1:12:02				
<i>P16a</i>	32:42	18:59	18:34	6:11	39:43	1:12:27	14:47	19:37	2:17:57	1:30:34	1:18:48	2:41:17	34:25	5:15:04				
<i>P16b</i>	32:33	18:38	17:56	6:15	33:07	1:31:40	12:53	19:55	2:07:20	1:19:18	1:12:04	2:26:58	32:58	4:18:53				
<i>P17a</i>	1:22:53	47:32	47:13	15:52	1:31:28	5:52:05	35:33	1:20:49	5:53:04	3:33:50	3:57:30	7:06:58	1:27:54	-				
<i>P17b</i>	1:26:35	50:15	46:27	14:30	1:59:14	<i>MS</i>	46:37	57:34	6:06:13	3:57:31	3:06:40	7:38:45	1:40:30	-				
<i>Am17</i>	1:13:34	39:08	45:33	13:23	1:30:51	1:45:44	32:04	1:38:18	5:37:08	3:17:52	2:20:52	6:29:01	1:03:49	-				
<i>P18a</i>	3:28:28	1:59:51	1:56:37	33:59	4:34:46	-	1:42:05	2:45:40	-	-	-	-	5:44:03	-				
<i>P18b</i>	3:58:24	2:06:57	1:56:57	33:27	5:36:26	<i>MS</i>	2:25:24	4:06:51	-	-	-	-	7:53:52	-				
<i>Am18</i>	3:06:06	1:37:50	1:55:47	31:06	4:07:26	6:06:49	1:30:36	2:11:02	-	-	5:41:57	-	4:04:54	-				

Table 5: Computational results for various layout types. Running times are given in sec, min:sec or h:min:sec for instances from the literature with the inter-cell distance $w_{path} = w_{path}^1 = w_{path}^2 = 3$ and for the CCLP (1,1) and CCLP (0,2) with $u_{12} = 3$. We write “-” if the time limit of 8 hours is exceeded and “*MS*” if we run out of memory storage.

The CCLP, the TRFLP and the XRFLP have several applications such as in heavy manufacturing and semiconductor fabrication.

We extend the approaches known from the literature in various ways. Given a CCLP instance where all cells of type SRFLP have the same inter-cell distances, we proved that by enumerating over all cell assignments and solving the CCLP with fixed-cell assignment, one can merge two cells of type SRFLP and therefore reduce the number of cell assignments that have to be considered significantly. Further, we omit fixing one department on the loading station by adding a dummy department with appropriate lengths and weights to each cell dealing as loading and unloading station. Considering cells of type DCFLP, one department overlaps with the dummy department and for the arising optimization problem a Mixed-Integer Linear Programming (MILP) model is presented. The main result is adapted to the 3-BFLP, 4-BFLP, TRFLP and the XRFLP with positive inter-cell distances which are leveled by dummy departments of appropriate lengths. The optimal values of the optimization problems are studied from a theoretical point of view and compared in our computational study to support the decision maker to choose the layout of a factory which is built from the ground up. The CCLP consisting of two cells can be solved fast and our MILP model for cells of type DCFLP works well and clearly outperforms the enumerative approach. Hence, we can solve instances with up to 18 departments in at most 30 minutes if both cells are of type DCFLP. For the 3-BFLP as well as the 3-PMFP we outperform the current best approach from the literature.

It remains for future work to apply our approach on facility layout problems consisting of a higher amount of cells as well as to consider cells of other types, e. g., cells of type Double-Row Facility Layout Problem. A further realistic extension is to consider two-dimensional departments with varying widths. Then it might be harder to ensure in the XRFLP that departments in cell 2 and cell 3 do not overlap.

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Appendix

In the appendix we summarize our approach for the TRFLP. We denote by C_1 (C_2) the set of departments assigned to cell 1 (cell 2) and if the cell assignment of the departments is known, we denote the resulting problem as Fixed-Row T-Row Facility Layout Problem (FR-TRFLP). Given an FR-TRFLP instance, by Lemma 4 we obtain the problems $W_{(n+2,p_b)}^S(C_2 \cup \{n+2\})$ and the $W_{(n+1,p_a)}^S(C_1 \cup \{n+1\})$ with the dummy department $n+2$ or $n+1$, respectively, where additionally one department in cell 1 may overlap with the dummy department $n+1$. We denote the obtained problem by $W_{(n+1,p_a,o)}^S(C_1 \cup \{n+1\})$. For solving the $W_{(n+2,p_b)}^S(C_2 \cup \{n+2\})$ we refer to Section 4.2. Thus, it remains to study the $W_{(n+1,p_a,o)}^S(C_1 \cup \{n+1\})$.

We study new exact methods for solving the $W_{(n+1,p_a,o)}^S(C_1 \cup \{n+1\})$. In [23] the following result is proven for the TRFLP and this result is valid for the FR-TRFLP as well.

Proposition 12. *Given an FR-TRFLP instance where $C_1 \neq \emptyset$ (C_2) denotes the set of departments assigned to cell 1 (cell 2) and let $V_1 := \{i \in C_1 : \sum_{j \in C_2} (w_{ij} + w_{ji}) > 0\} \neq \emptyset$. Let the dummy department $n+1$ be assigned to cell 1 with $\ell_{n+1} = 0$ and $w_{i(n+1)} = \sum_{j \in [n] \setminus C_1} w_{ij}$, $w_{(n+1)i} = \sum_{j \in [n] \setminus C_1} w_{ji}$ (see (14), (15)) and let the optimal value of the $W_{(n+1,p_a,o)}^S(C_1 \cup \{n+1\})$*

be denoted by v_{n+1}^* . Further, we denote by $v_{s_M}^*$ the optimal value of the $W_{(s_M, p_a)}^S(C_1)$ where the weights of $s_M \in C_1$ are adjusted according to (14), (15). Then,

$$v_{n+1}^* = \min\{v_{s_M}^* : s_M \in V_1\}.$$

Further, there always exists an optimal T-row layout with $C_1 \neq \emptyset$. If $V_1 = \emptyset$, we neglect the dummy department $n+1$ and then the $W_{(n+1, p_a, o)}^S(C_1 \cup \{n+1\})$ is equivalent to the SRFLP with the set of departments C_1 . Given an FR-TRFLP instance, we fix $s_M \in C_1$ directly opposite the (un-) loading station, and hence we have to solve the $W_{(s_M, p_a)}^S(C_1 \cup \{n+1\})$. By this method one has to consider $2^n - 1$ cell assignments for the TRFLP and solve the $W_{(n+2, p_b)}^S(C_2 \cup \{n+2\})$ once and the $W_{(s_M, p_a)}^S(C_1 \cup \{n+1\})$, $s_M \in C_1$, has to be solved $\max\{1, |V_1|\}$ times for every cell assignment.

Hence, we present an MILP model for solving the $W_{(n+1, p_a, o)}^S(C_1)$ with $V_1 \neq \emptyset$ in order to solve this problem faster as well as to obtain good lower bounds and neglect unbalanced cell assignments early in our algorithm. The idea is to split the dummy department $n+1$ into two dummy departments $n+3$ and $n+4$ with lengths $\ell_{n+3} = \ell_{n+4} = 0$ and weights $w_{i(n+3)} = w_{i(n+4)} = \frac{w_{i(n+1)}}{2}$, $w_{(n+3)i} = w_{(n+4)i} = \frac{w_{(n+1)i}}{2}$, $i \in C_1$. We ensure that exactly one department lies between $n+3$ and $n+4$ and this department lies on position p_{E_1} . Neglecting the constant weights $W = \sum_{\substack{i, j \in [n] \\ i < j}} (w_{ij} + w_{ji}) \frac{\ell_i + \ell_j}{2}$ for the whole problem our MILP model for the subproblem in cell 1 reads as follows

$$\min \sum_{\substack{i, j \in C_1 \\ i < j}} (w_{ij} + w_{ji}) \sum_{k \in C_1 \setminus \{i, j\}} \ell_k x_{ikj} \quad (29)$$

$$+ \sum_{\substack{i \in C_1 \\ j \in \{n+3, n+4\}}} (w_{ij} + w_{ji}) \left(\sum_{k \in C_1 \setminus \{i\}} \ell_k x_{ikj} - \frac{\ell_i}{2} x_{(n+3)i(n+4)} \right)$$

$$(2)-(6),$$

$$\mathcal{D} = C_1 \cup \{n+3, n+4\},$$

$$\sum_{i \in C_1} x_{(n+3)i(n+4)} = 1, \quad (30)$$

$$x_{(n+3)i(n+4)} = 0, \quad i \notin V_1, \quad (31)$$

$$x_{ikj} \in \{0, 1\}, \quad i, j, k \in C_1 \cup \{n+3, n+4\},$$

$$|\{i, j, k\}| = 3, \quad i < j.$$

By Equation (30) exactly one department is arranged between the dummy departments $n+3$ and $n+4$ and if $i \in C_1 \setminus V_1$, then i does not lie between the dummy departments, see Equations (31). In the objective function (29) the distances between $i \in C_1$ and $j \in C_1 \cup \{n+3, n+4\}$, $i < j$, are measured similar as in the SRFLP if $x_{(n+3)i(n+4)} = 0$. Otherwise, if $x_{(n+3)i(n+4)} = 1$, then the distance between i and $j \in \{n+3, n+4\}$ equals $-\frac{\ell_i}{2}$ because we excluded the value $(w_{ij} + w_{ji}) \frac{\ell_i}{2}$ in the calculation of the constant W . Measuring the distance between $j \in C_1 \setminus \{i\}$ and the dummy departments we subtract $\frac{\ell_i}{2}$, and thus we do not take the length ℓ_i into account, as desired by the calculation of the constant W . Hence, our distance calculation and the MILP model are correct. In our algorithm we solve the $W_{(n+2, p_b)}^S(C_2 \cup \{n+2\})$ first and then we solve the $W_{(n+1, p_a)}^S(C_1 \cup \{n+1\})$ because the $W_{(n+2, p_b)}^S(C_2 \cup \{n+2\})$ can often be solved faster, see Table 1 and the corresponding conclusions. Note that in the FR-TRFLP the inter-cell distance u_{12} leads only to constant weights, and thus we can exclude them.