# An equivalent mathematical program for games with random constraints

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**Abstract**

This paper shows that there exists a Nash equilibrium of an $n$-player chance-constrained game for elliptically symmetric distributions. For a certain class of payoff functions, we suitably construct an equivalent mathematical program whose global maximizer is a Nash equilibrium.

*Keywords:* Chance-constrained game, Elliptical distribution, Mathematical program, Nash equilibrium.

## 1. Introduction

The Nash equilibrium is one of the most desired solution concept which is used to study the competition among several selfish and rational players. It is a strategy profile of the players where there is no incentive for unilateral deviation by any player. The theory of games involving Nash equilibrium started with the paper by John Nash [17] where he showed that there exists a mixed strategy Nash equilibrium for finite strategic games. Later it has been shown that a Nash equilibrium of a general non-cooperative game exists under certain conditions on payoff functions and strategy sets of the players [8, 9]. The games considered in these papers are deterministic in nature, i.e., the players' strategy

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sets and payoff functions are defined using real valued functions. However, in practical situations the decision making process usually faces various types of uncertainties due to which payoff functions or strategy sets are modeled using random variables [7, 16, 18]. The expected value approach is used to model the uncertainties when the decision makers are risk neutral [19]. For risk averse players, the payoff criterion with the risk measure CVaR [12, 19] and the variance was considered in the literature [6]. Singh et al. [21, 22] introduced chance-constrained games by considering a risk averse payoff criterion based on the chance constraint programming for finite strategic games with random payoffs. For elliptically distributed random payoffs, the authors showed the existence of a Nash equilibrium for a chance-constrained game [21], and proposed an equivalent mathematical program to compute the Nash equilibria of the game [22]. There are several zero-sum chance-constrained games studied in past literature [1, 3, 4, 5].

The chance-constrained games in the above-mentioned papers model the payoffs' uncertainties using chance constraints. The literature on games with chance-constrained based strategy sets is rather limited [20, 23]. Singh and Lisser [23] considered a two player zero-sum matrix game where strategy set of each player is defined using individual chance constraints. They showed that the saddle point equilibria of the game can be computed by solving a primal-dual pair of second order cone programs when the row vectors defining constraints follow elliptical distributions. Peng et al. [20] considered an $n$-player game with joint chance constraints and showed that there exists a Nash equilibrium of the game if the row vectors are independent and follow multivariate normal distributions. In this paper, we consider an $n$-player game where the strategy sets are defined by individual chance constraints. Under standard quasi-concavity and continuity conditions on the payoff functions, we show that there exists a Nash equilibrium of the game if the row vectors associated with chance constraints follow elliptical distributions. Further, we consider a specific payoff function for each player which satisfy the standard conditions. The first term of a player's payoff function is multi-linear in all the players strategies and second term is a quadratic concave function of the player's strategies. Such types of payoff functions are often encountered in practical situations [18]. We propose an equivalent mathematical program for this class of games and show the one-to-one correspondence between a Nash equilibrium of the game and a global maximizer of the mathematical program.

The structure of rest of the paper is as follows. Section 2 contains the definition of a chance-constrained game. Section 3 presents the existence of a Nash equilibrium. The equivalent mathematical program is given in Section 4.

## 2. The model

We consider an $n$-player non-cooperative game defined by tuple $\big(I,(X^i)_{i\in I},(u_i)_{i\in I}\big)$, where $I=\{1,2,\ldots,n\}$ is the set of players, $X^i\subset\mathbb{R}^{m_i}$ is a strategy set of player $i$ and $u_i:\prod_{i\in I}X^i\to\mathbb{R}$ is a payoff function of player $i$. The strategy set $X_i$, $i\in I$, is a non-empty, convex and compact set. The product set $X=\prod_{i\in I}X^i$ is a set of all strategy profiles of the game. Let $X^{-i}=\prod_{j=1;j\neq i}^{n}X^j$ be the set of vectors of strategies of all the players but player $i$. The generic elements of $X^i$, $X^{-i}$, and $X$ are denoted by $x^i$, $x^{-i}$, and $x$ respectively. For $y^i\in X^i$, we define $(y^i,x^{-i})$ to be a strategy profile where player $i$ chooses a strategy $y^i$ and each player $j\in I$, $j\neq i$, chooses a strategy $x^j$. We consider the case where the strategies of player $i$ are further restricted by the following random linear constraints

$$\left(a_k^i\right)^T x^i\leq b_k^i,\quad k=1,2,\ldots,K_i, \tag{2.1}$$

where $a_k^i$ is an $m_i\times 1$ random vector defined on the probability space $(\Omega,\mathcal{F},\mathbb{P})$ and $b_k^i\in\mathbb{R}$; $T$ denotes the transposition. We consider the case where $k$th constraint of player $i$ given by (2.1) is satisfied with at least a given probability level $\alpha_k^i$. Let $\alpha^i=(\alpha_k^i)_{k=1}^{K_i}$ denotes the probability level vector. The chance constraints corresponding to random constraints (2.1) are given by

$$\mathbb{P}\Big\{\left(a_k^i\right)^T x^i\leq b_k^i\Big\}\geq\alpha_k^i,\quad k=1,2,\ldots,K_i. \tag{2.2}$$

Therefore, for a given probability level vector $\alpha^i$, the feasible strategy set of player $i$, $i \in I$, is defined by

$$S^i_{\alpha^i} = \Big\{ x^i \in X^i \mid \mathbb{P}\big\{ \big(a^i_k\big)^T x^i \leq b^i_k \big\} \geq \alpha^i_k, \quad k = 1, 2, \ldots, K_i \Big\}. \tag{2.3}$$

We call the game with payoff functions $(u_i(\cdot))_{i\in I}$ and strategy sets $(S^i_{\alpha_i})_{i\in I}$ a chance-constrained game. We assume that the set $S^i_{\alpha^i}$, $i \in I$, is non-empty, and the probability distribution of the random vector $(a^i_k)^{K_i}_{k=1}$, $i \in I$, and the probability level vector $(\alpha^i)_{i\in I}$ are known to all the players. Then, the above chance-constrained game is a non-cooperative game with complete information. For a given strategy profile $x^{-i}$ of other players, a set of best response strategies of player $i$ at probability level vector $\alpha^i$ is defined as

$$BR^i_{\alpha^i}(x^{-i}) = \big\{ \bar{x}^i \in S^i_{\alpha^i} \mid u_i(\bar{x}^i, x^{-i}) \geq u_i(x^i, x^{-i}), \ \forall\, x^i \in S^i_{\alpha^i} \big\}.$$

A strategy profile $x^*$ is said to be a Nash equilibrium of a chance-constrained game at probability level vector $(\alpha^i)_{i\in I}$ if and only if for each $i \in I$

$$u_i(x^{i*}, x^{-i*}) \geq u_i(x^i, x^{-i*}), \ \forall\, x^i \in S^i_{\alpha^i}.$$

It is clear that $x^*$ is a Nash equilibrium if and only if $x^{i*} \in BR^i_{\alpha^i}(x^{-i*})$ for all $i \in I$.

## 3. Existence of Nash equilibrium

We consider the case where the vector $(a^i_k)^{K_i}_{k=1}$, $i \in I$, follows a multivariate elliptically symmetric distribution. The class of multivariate elliptically symmetric distributions generalizes the multivariate normal distribution.

**Definition 3.1.** *A $d$-dimensional random vector $\xi$ follows an elliptically symmetric distribution $Ellip_d(\mu, \Gamma, \varphi)$ if its characteristic function is given by $\mathbb{E}e^{iz^T\xi} = e^{iz^T\mu}\varphi(z^T\Gamma z)$ where $\varphi$ is the characteristic generator function, $\mu$ is the location parameter, and $\Gamma$ is the scale matrix.*

Some famous distributions belonging to the family of elliptical distributions include normal distribution with $\varphi(t) = \exp\{-\frac{1}{2}t\}$, Student's $t$ distribution with $\varphi(t)$ varying with its degree of freedom [14], Cauchy distribution with $\varphi(t) = \exp\{-\sqrt{t}\}$, Laplace distribution with $\varphi(t) = (1 + \frac{1}{2}t)^{-1}$, and logistic distribution with $\varphi(t) = \frac{2\pi\sqrt{t}}{e^{\pi\sqrt{t}} - e^{-\pi\sqrt{t}}}$. It is well known that the family of elliptical distributions are closed under affine transformations. We summarize this result in Proposition 3.2.

**Proposition 3.2 (Fang et al. [10]).** *If a $d$-dimensional random vector $\xi$ follows an elliptical distribution $Ellip_d(\mu, \Gamma, \varphi)$, then for any $(N \times d)$-matrix $C$ and any $N \times 1$-vector $c$, $C\xi + c$ follows an $N$-dimensional elliptical distribution $Ellip_N(C\mu + c, C\Gamma C^T, \varphi)$.*

We assume that the random vector $a_k^i$ follows an elliptically symmetric distribution $Ellip_{m_i}(\mu_k^i, \Sigma_k^i, \varphi_k^i)$, where $\Sigma_k^i$ is a positive definite scale matrix; we denote it by $\Sigma_k^i \succ 0$. From Proposition 3.2, for a given $x^i$, $\left(a_k^i\right)^T x^i$ follows $Ellip\left((\mu_k^i)^T x^i, (x^i)^T \Sigma_k^i x^i, \varphi_k^i\right)$. We can write $\left\|(\Sigma_k^i)^{1/2} x^i\right\| = \sqrt{(x^i)^T \Sigma_k^i x^i}$, where $(\Sigma_k^i)^{1/2}$ is the unique positive definite square root of matrix $\Sigma_k^i$ and $\|\cdot\|$ is the Euclidean norm. Then, $\xi_k^i = \frac{\left(a_k^i\right)^T x^i - (\mu_k^i)^T x^i}{\left\|(\Sigma_k^i)^{1/2} x^i\right\|}$ follows a univariate standard elliptical distributions $Ellip(0, 1, \varphi_k^i)$.

A linear chance constraint is equivalent to a second-order cone constraint for the case of multivariate normal distribution [13, 15, 24], multivariate elliptically symmetric distribution [11], and radial distribution [2]. By using the second-order cone constraint reformulation from [11], the feasible strategy set $S_{\alpha^i}^i$ can be written as

$$S_{\alpha^i}^i = \left\{x^i \in X^i \mid (x^i)^T \mu_k^i + \Psi_{\xi_k^i}^{-1}(\alpha_k^i)||(\Sigma_k^i)^{\frac{1}{2}} x^i|| \leq b_k^i,\ \forall\, k = 1, 2, \ldots, K_i\right\},\ \ i \in I, \tag{3.1}$$

where $\Psi_{\xi_k^i}^{-1}(\cdot)$ is a quantile function of 1-dimensional distribution function induced by the characteristic function $\varphi_k^i(t^2)$. It is evident that the feasible strategy set $S_{\alpha^i}^i$ is a compact set. It follows from [11] that $S_{\alpha^i}^i$ is a convex set for all $\alpha^i \in (0.5, 1]^{K_i}$. If each random vector $a_k^i$, $k = 1, 2, \ldots, K_i$, has strictly positive density function, $S_{\alpha^i}^i$ is a convex set for all $\alpha^i \in [0.5, 1]^{K_i}$ [11].

**Theorem 3.3.** *Consider an $n$-player chance-constrained game defined in Section 2, where the payoff function of player $i$, $i \in I$, satisfies the following conditions*

*(i)* $u_i(x^i, x^{-i})$ *is a quasi-concave function of* $x^i$ *for all* $x^{-i} \in X^{-i}$.

*(ii)* $u_i(x)$ *is a continuous function of* $x$.

*Suppose the random vector* $a_k^i \sim Ellip_{m_i}(\mu_k^i, \Sigma_k^i, \varphi_k^i)$*, where* $\Sigma_k^i \succ 0$ *, for all* $k = 1, 2, \ldots, K_i$ *and* $i \in I$*. Then, there exists a Nash equilibrium for all* $\alpha \in (0.5, 1]^{K_1} \times (0.5, 1]^{K_2} \times \cdots \times (0.5, 1]^{K_n}$.

Proof. For elliptically symmetric distribution, the feasible strategy sets $S^i_{\alpha^i}$, $i \in I$, is a non-empty, convex and compact sets for all $\alpha^i \in (0.5, 1]^{K_i}$. Then, under the conditions given in Theorem 3.3 the existence of Nash equilibrium directly follows from [8, 9]. □

## 4. Mathematical programming formulation

We consider a class of $n$-player chance-constrained games which satisfy the conditions $(i)$ and $(ii)$ of Theorem 3.3. For each $i \in I$, let $J^i = \{1, 2, \cdots, m_i\}$. Define the product sets $J = \prod_{i \in I} J^i$ and $J^{-i} = \prod_{k \in I; k \neq i} J^k$. Consider a vector $(r^i(s))_{s \in J}$, where $s = (s_1, s_2, \cdots, s_n)$ with $s_i \in J^i$. The payoff function of player $i$ is given by

$$u_i(x) = \sum_{s \in J} \prod_{j=1}^{n} r^i(s) x^j_{s_j} - \frac{1}{2}(x^i)^T Q_i x^i, \tag{4.1}$$

where $Q_i$ is a positive definite matrix. The first term of the payoff function of player $i$ is linear in $i$th player's strategies for a fixed strategy profile of rest of the players and the second term only depends on the strategies of player $i$ and is quadratic in nature. Such a payoff function appears in various applications where second term can be considered as a quadratic cost incurred by player $i$, e.g., in electricity market it can be the production cost [18], and the first term represents the payoff received due to the interaction among all the players.

For every $s_i \in J^i$, define $r^i(s_i, x^{-i}) = \sum_{s_{-i} \in J^{-i}} \prod_{j \in I; j \neq i} r^i(s_i, s_{-i}) x^j_{s_j}$. Then, $u_i(x^i, x^{-i}) = \sum_{s_i \in J^i} r^i(s_i, x^{-i}) x^i_{s_i}$. The strategy set $X^i$, $i \in I$, is a bounded polyhedron and it is defined as

$$X^i = \{x^i \in \mathbb{R}^{m_i} \mid C^i x^i = d^i, x^i \geq \mathbf{0}\}, \tag{4.2}$$

where $C^i \in \mathbb{R}^{L_i \times m_i}$, $d^i \in \mathbb{R}^{L_i}$ and $\mathbf{0}$ is an $m^i \times 1$ zero vector. In this paper, we identify $\mathbf{0}$ as a zero vector of appropriate dimension.

**Assumption 4.1.** *For each $i \in I$, the set $S^i_{\alpha_i}$ is strictly feasible, i.e., there exists an $x^i \in \mathbb{R}^{m_i}$ such that $x^i$ is a feasible point of $S^i_{\alpha_i}$ and all its inequality constraints are strictly feasible.*

The condition given in Assumption 4.1 is a Slater condition which is sufficient for strong duality in convex optimization problem. We use these conditions in order to derive equivalent mathematical program for the chance-constrained game.

*4.1. Best response convex programs*

For a given $x^{-i} \in S^{-i}_{\alpha^{-i}}$, a best response strategy of player $i$ is obtained by solving the following convex optimization problem

$$\begin{aligned}
[\mathrm{P}_i] \qquad & \max_{x^i, (t^i_k)_{k=1}^{K_i}} \sum_{s_i \in J^i} r^i(s_i, x^{-i}) x^i_{s_i} - \frac{1}{2}(x^i)^T Q_i x^i \\
\text{s.t} \ \ & (i) \ C^i x^i = d^i \\
& (ii) \ (x^i)^T \mu^i_k + \Psi^{-1}_{\xi^i_k}(\alpha^i_k) ||t^i_k|| \leq b^i_k, \ \forall \ k = 1, 2, \ldots, K_i \\
& (iii) \ t^i_k = (\Sigma^i_k)^{\frac{1}{2}} x^i, \ \forall \ k = 1, 2, \ldots, K_i \\
& (iv) \ x^i \geq \mathbf{0}.
\end{aligned}$$

The Lagrangian dual of best response convex optimization problem $[\mathrm{P}_i]$ is given by

$$\min_{(\gamma_k^i)_{k=1}^{K_i},\delta^i,\lambda^i\geq\mathbf{0},\beta^i\geq\mathbf{0}} \max_{(t_k^i)_{k=1}^{K_i},x^i\in\mathbb{R}^{m_i}} \Bigg[\sum_{s_i\in J^i} r^i(s_i,x^{-i})x_{s_i}^i - \frac{1}{2}(x^i)^T Q_i x^i$$
$$+(\delta^i)^T\big(d^i - C^i x^i\big) + \sum_{k=1}^{K_i}(\gamma_k^i)^T\big(t_k^i - (\Sigma_k^i)^{\frac{1}{2}}x^i\big)$$
$$+\sum_{k=1}^{K_i}\lambda_k^i\Big(b_k^i - (x^i)^T\mu_k^i - \Psi_{\xi_k^i}^{-1}(\alpha_k^i)||t_k^i||\Big) + (\beta^i)^T x^i\Bigg],$$

where $\delta^i \in \mathbb{R}^{L_i}$ and $\lambda^i \in \mathbb{R}_+^{K_i}$, $\gamma_k^i \in \mathbb{R}^{m_i}$, $k = 1,2,\ldots,K_i$, $\beta^i \in \mathbb{R}_+^{m_i}$ are the vectors of Lagrange multipliers corresponding to constraints $(i)$, $(ii)$, $(iii)$ and $(iv)$, respectively. For a given $x^{-i}$, define a vector $R^i(x^{-i}) = (r^i(s_i,x^{-i}))_{s_i\in J^i}$. Then, for a fixed $(\gamma_k^i)_{k=1}^{K_i}$, $\delta^i$ and $\lambda^i \geq \mathbf{0}$, $\beta^i \geq \mathbf{0}$, we have

$$\max_{(t_k^i)_{k=1}^{K_i},x^i\in\mathbb{R}^{m_i}} \Bigg[\sum_{s_i\in J^i} r^i(s_i,x^{-i})x_{s_i}^i\big) - \frac{1}{2}(x^i)^T Q_i x^i + (\delta^i)^T\big(d^i - C^i x^i\big)$$
$$+\sum_{k=1}^{K_i}(\gamma_k^i)^T\big(t_k^i - (\Sigma_k^i)^{\frac{1}{2}}x^i\big) + \sum_{k=1}^{K_i}\lambda_k^i\Big(b_k^i - (x^i)^T\mu_k^i - \Psi_{\xi_k^i}^{-1}(\alpha_k^i)||t_k^i||\Big) + (\beta^i)^T x^i\Bigg]$$
$$= \max_{x^i\in\mathbb{R}^{m_i}}\Bigg[-\frac{1}{2}(x^i)^T Q_i x^i + (x^i)^T\Bigg(R^i(x^{-i}) - (C^i)^T\delta^i - \sum_{k=1}^{K_i}(\Sigma_k^i)^{\frac{1}{2}}\gamma_k^i$$
$$-\sum_{k=1}^{K_i}\lambda_k^i\mu_k^i + \beta^i\Bigg)\Bigg] + \max_{(t_k^i)_{k=1}^{K_i}}\Bigg[\sum_{k=1}^{K_i}\Big((\gamma_k^i)^T t_k^i - \Psi_{\xi_k^i}^{-1}(\alpha_k^i)\lambda_k^i||t_k^i||\Big)\Bigg] + (\delta^i)^T d^i + \sum_{k=1}^{K_i}\lambda_k^i b_k^i$$

The first max is given by

$$\frac{1}{2}(P_i)^T Q_i^{-1} P_i,$$

where $P_i = R^i(x^{-i}) - (C^i)^T\delta^i - \sum_{k=1}^{K_i}(\Sigma_k^i)^{\frac{1}{2}}\gamma_k^i - \sum_{k=1}^{K_i}\lambda_k^i\mu_k^i + \beta^i$, and the second max problem is unbounded unless

$$||\gamma_k^i|| \leq \Psi_{\xi_k^i}^{-1}(\alpha_k^i)\lambda_k^i, \ \forall\, k = 1,2,\cdots,K_i.$$

Therefore, the dual of [P$_i$] is given by

$$[\mathrm{D}_i] \quad \min_{(\gamma_k^i)_{k=1}^{K_i},\delta^i,\lambda^i,\beta^i} \ \frac{1}{2}(P_i)^T Q_i^{-1} P_i + (\delta^i)^T d^i + \sum_{k=1}^{K_i} \lambda_k^i b_k^i$$

s.t.

$(i)$ $||\gamma_k^i|| \leq \Psi_{\xi_k^i}^{-1}(\alpha_k^i)\lambda_k^i, \ \forall\ k = 1, 2, \cdots, K_i$

$(ii)$ $\lambda^i \geq \mathbf{0}, \ \beta^i \geq \mathbf{0}.$

*4.2. Mathematical Program*

We construct a mathematical program by combining $n$ primal-dual pair [P$_i$]-[D$_i$] of best response convex programs. Then, we characterize the Nash equilibria of the chance-constrained game using the global optimal points of the mathematical program. Let $\zeta = \Big(x^i, (t_k^i)_{k=1}^{K_i}, \big(\gamma_k^i\big)_{k=1}^{K_i}, \delta^i, \lambda^i, \beta^i\Big)_{i\in I}$, and $\phi(\zeta)$ denote the decision variables and objective function of the mathematical program. We have the following characterization.

**Theorem 4.2.** *Consider an $n$-player chance-constrained game defined in Section 2, where the payoff function $u_i(x)$ and the strategy set $X_i$ of player $i$, $i \in I$, is given by* (4.1) *and* (4.2)*, respectively. Suppose the random vectors $a_k^i \sim Ellip_{m_i}(\mu_k^i, \Sigma_k^i, \varphi_k^i)$, where $\Sigma_k^i \succ 0$ , for all $k = 1, 2, \ldots, K_i$ and $i \in I$. Let Assumption 4.1 holds. Then, for an $\alpha \in (0.5, 1]^{K_1} \times (0.5, 1]^{K_2} \times \cdots \times (0.5, 1]^{K_n}$*

1. *If $(x^{i*})_{i\in I}$ is a Nash equilibrium of the chance-constrained game, there exists a vector $\zeta^* = \Big(x^{i*}, (t_k^{i*})_{k=1}^{K_i}, \big(\gamma_k^{i*}\big)_{k=1}^{K_i}, \delta^{i*}, \lambda^{i*}, \beta^{i*}\Big)_{i\in I}$ such that it is a global maximizer of the following mathematical program [MP]*

$$[\text{MP}] \qquad \max_{\zeta} \sum_{i\in I}\left[\sum_{s_i\in J^i} r^i(s_i,x^{-i})x^i_{s_i} - \frac{1}{2}(x^i)^T Q_i x^i \right.$$
$$\left. - \frac{1}{2}(P_i)^T Q_i^{-1} P_i - (\delta^i)^T d^i - \sum_{k=1}^{K_i} \lambda^i_k b^i_k \right]$$

*s.t.*

$(i)\ ||\gamma^i_k|| \leq \Psi^{-1}_{\xi^i_k}(\alpha^i_k)\lambda^i_k,\ \forall\ k=1,2,\cdots,K_i,\ i\in I$

$(ii)\ (x^i)^T\mu^i_k + \Psi^{-1}_{\xi^i_k}(\alpha^i_k)||(\Sigma^i_k)^{\frac{1}{2}}x^i|| \leq b^i_k,\ \forall\ k=1,2,\ldots,K_i,\ i\in I$

$(iii)\ C^i x^i = d^i,\ i\in I$

$(iv)\ x^i \geq \boldsymbol{0},\ \lambda^i \geq \boldsymbol{0},\ \beta^i \geq \boldsymbol{0},\ i\in I.$

*with objective function value* $\phi(\zeta^*) = 0$.

2. *If* $\zeta^* = \left(x^{i*}, (t^{i*}_k)_{k=1}^{K_i}, \left(\gamma^{i*}_k\right)_{k=1}^{K_i}, \delta^{i*}, \lambda^{i*}, \beta^{i*}\right)_{i\in I}$ *is a global maximizer of the mathematical program [MP],* $(x^{i*})_{i\in I}$ *is a Nash equilibrium of the chance-constrained game.*

Proof. 1. Let $(x^{i*})_{i\in I}$ be a Nash equilibrium. Then, for each $i\in I$, $\left(x^{i*}, (t^{i*}_k)_{k=1}^{K_i}\right)$ is an optimal solution of [P$_i$] for the fixed $x^{-i*}$. Under Assumption 4.1 strong duality holds. Therefore, there exists $\left(\left(\gamma^{i*}_k\right)_{k=1}^{K_i}, \delta^{i*}, \lambda^{i*}, \beta^{i*}\right)$ which is an optimal solution of [D$_i$] such that for each $i\in I$

$$\sum_{s_i\in J^i} r^i(s_i,x^{-i*})x^{i*}_{s_i} - \frac{1}{2}(x^{i*})^T Q_i x^{i*} = \frac{1}{2}P^*_i Q_i^{-1} P^*_i + (\delta^{i*})^T d^i + \sum_{k=1}^{K_i} \lambda^{i*}_k b^i_k, \tag{4.3}$$

where $P^*_i = R^i(x^{-i*}) - (C^i)^T\delta^{i*} - \sum_{k=1}^{K_i}(\Sigma^i_k)^{\frac{1}{2}}\gamma^{i*}_k - \sum_{k=1}^{K_i}\lambda^{i*}_k\mu^i_k + \beta^{i*}$. Therefore, $\zeta^* = \left(x^{i*}, (t^{i*}_k)_{k=1}^{K_i}, \left(\gamma^{i*}_k\right)_{k=1}^{K_i}, \delta^{i*}, \lambda^{i*}\right)_{i\in I}$ is a feasible point of [MP] and $\phi(\zeta^*) = 0$. Let $\zeta$ be an arbitrary feasible point of [MP]. Then, for each $i\in I$, $(x^i, (t^i_k)_{k=1}^{K_i})$ and $((\gamma^i_k)_{k=1}^{K_i}, \delta^i, \lambda^i, \beta^i)$ will be feasible points of [P$_i$] and [D$_i$], respectively. Then, from weak duality theorem $\phi(\zeta) \leq 0$ for all feasible point $\zeta$. Therefore, $\zeta^*$ is a global maximizer of [MP].

2. Let $\zeta^*$ be a global maximizer of [MP]. From the proof of first part it follows that $\phi(\zeta^*) = 0$. The primal-dual pair [P$_i$]-[D$_i$], $i \in I$, of convex programs are feasible at $\zeta^*$. Therefore, it follows from weak-duality that each part of the objective function is non-negative at $\zeta^*$. Hence, (4.3) holds at $\zeta^*$. For every $x^i \in S^i_{\alpha^i}$, take $t^i_k = (\Sigma^i_k)^{\frac{1}{2}} x^i$, $k = 1, 2, \ldots, K_i$. Then, $(x^i, (t^i_k)_{k=1}^{K_i})$ is a feasible solution of [P$_i$]. Again, from weak duality we have

$$\sum_{s_i \in J^i} r^i(s_i, x^{-i*}) x^i_{s_i} - \frac{1}{2}(x^i)^T Q_i x^i \le \frac{1}{2}(P_i^*)^T Q_i^{-1} P_i^* + (\delta^{i*})^T d^i + \sum_{k=1}^{K_i} \lambda_k^{i*} b_k^i,$$

for all $x^i \in S^i_{\alpha^i}$, From (4.3), for each $i \in I$, we have

$$u_i(x^{i*}, x^{-i*}) \ge u_i(x^i, x^{-i*}), \ \forall \ x^i \in S^i_{\alpha^i}.$$

Hence, $(x^{i*})_{i \in I}$ is a Nash equilibrium of the chance-constrained game.

□

**Acknowledgement**

This research was supported by DST/CEFIPRA Project No. IFC/4117/DST-CNRS-5th call/2017-18/2 and CNRS Project No. AR/SB:2018-07-440.

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