A study of the relation between the single-row and the double-row facility layout problem

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The NP-hard Multi-Row Facility Layout Problem (MRFLP) consists of a set of one-dimensional departments and pairwise transport weights between them. It asks for a non-overlapping arrangement of the departments along a given number of rows such that the weighted sum of the horizontal center-to-center distances between the departments is minimized. We mainly focus on the MRFLP with exactly two rows, the so called Double-Row Facility Layout Problem (DRFLP), and on the case with exactly one row, the so called Single-Row Facility Layout Problem (SRFLP). Although the MRFLP has wide applications in factory planning, only small instances can be solved to optimality in reasonable time for the MRFLP with at least two rows while provably good or optimal solutions for the SRFLP can be derived very fast. In the equidistant case, where all departments have the same size, we prove that the optimal value of the MRFLP is less than or equal to the optimal value of the SRFLP divided by the number of rows of the MRFLP. We derive equidistant double-row layouts satisfying this property in a very short time and we improve some of the best known upper bounds for the equidistant DRFLP. Given a double-row instance with arbitrary department lengths we provide a formula for the relation of the optimal value of the DRFLP and the SRFLP and provide an example which shows that this bound is tight. In addition, we present heuristic approaches for the DRFLP based on good or optimal single-row layouts. For instances with up to 40 departments we obtain small gaps to the best known upper bounds and for even larger instances we improve the best known upper bounds. Our approaches are significantly faster than the ones in the literature.

Key words. Facilities planning and design; Row Layout Problem; Heuristic

1 Introduction

Given a set of departments $\{1, \dots, n\} =: [n], n \in \mathbb{N}$, with length $\ell_i, i \in [n]$, and pairwise non-negative weights $w_{ij} = w_{ji}, i, j \in [n], i < j$, the Multi-Row Facility Layout Problem (`MRFLP`) asks for an assignment of the departments to the rows $\mathcal{R} := [m] \in \mathbb{N}, m \geq 1$, such that departments in the same row do not overlap and such that the weighted sum of the horizontal center-to-center distances between the departments is minimized. So we look for an assignment $r\colon [n] \to \mathcal{R}$ of the

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departments to the m rows and for a vector $p \in \mathbb{R}^n$ of the center positions of the departments such that

$$\begin{aligned} \min_{r \in \mathcal{R}^n,\, p \in \mathbb{R}^n} \; & \sum_{\substack{i,j \in [n] \\ i<j}} w_{ij} |p_i - p_j| \\ \text{s. t. } & |p_i - p_j| \geq \frac{\ell_i + \ell_j}{2}, \qquad i,j \in [n],\; i<j, \text{ if } r_i = r_j. \end{aligned}$$

The special case of the MRFLP with $m = 1$ is called Single-Row Facility Layout Problem (SRFLP) and is well-known to be $\mathcal{NP}$-hard [3, 29, 47]. The SRFLP arises in factory planning, in the arrangement of departments in office buildings, hospitals or supermarkets [50] as well as of books on a shelf [4]. Further applications are the assignment of files to disk cylinders in computer storage and the design of warehouse layouts [38, 47]. There always exists an optimal single-row layout without spaces between neighboring departments, so the SRFLP is equivalent to finding a permutation of the departments that minimizes the weighted sum of the horizontal center-to-center distances. Most often exact approaches for the SRFLP are based on integer linear programming (ILP) models, see, e. g., [1, 2, 3, 7], or semidefinite optimization (SDP) formulations, see, e. g., [10, 11, 14, 34, 35]. The current fastest approach of [34, 35] is able to solve one instance with 42 departments to optimality in less than 2 hours and obtains gaps of less than 2 % for instances with up to 110 departments in 400 hours. Additionally, several heuristic algorithms have been suggested that are able to obtain high quality solutions [20, 24, 40, 41, 44, 46]. The heuristic in [24] is applied on instances from the literature with 60 to 80 departments. Almost half of their solutions were improved by [40, 41] and their heuristic is applied on instances with up to 110 departments. Afterwards, [44] presented a heuristic which derived layouts of the same quality but the running time was reduced and hence instances with up to 300 departments were considered. One of the leading heuristics is given in [46], where a multi-start simulated annealing heuristic obtains the best known solutions or small gaps for instances from the literature with 60 to 80 departments. This heuristic is applied on instances with up to 1000 departments. A recent survey on the SRFLP is given in [37].

The SRFLP with departments of equal length is denoted by Single-Row Equidistant Facility Layout Problem (SREFLP) and is a special case of the Quadratic Assignment Problem (QAP). In [32] it is shown that the best method for the SRFLP is better than methods especially designed for the SREFLP, see, e. g., [43], and methods for the QAP. For a heuristic approach we refer to [45].

For $m = 2$ the MRFLP is called Double-Row Facility Layout Problem (DRFLP). Problems in factory layout planning can often be decomposed, see [23], and hence most often real factory layout problems reduce to a combination of single-row and double-row layouts. Determining good solutions is important since the costs of the production are highly influenced by the layout of the departments, see, e. g., [16, 31, 51]. In contrast to the SRFLP, the DRFLP is very challenging even for instances with a small number of departments. Several mixed-integer linear programming (MILP) models have been developed for the DRFLP and the MRFLP, see, e. g. [4, 5, 13, 18, 19, 49] (see [53] for a correction of [19]). However, the current fastest exact approach for the DRFLP and the MRFLP in [27, 28] is able to solve double-row instances with up to 16 departments in less than 12 hours and multi-row instances with up to 13 departments and 5 rows in at most 7 hours. Heuristics for the MRFLP and the DRFLP, partially handling some extended versions, are given in [6, 13, 15, 19, 30, 42, 48, 52, 54, 55]. A large disadvantage of the heuristic presented in [6], which performs well with respect to the obtained solution values, is its large computational effort and so the long running times. The question arises whether one can get good solutions quickly. The quality of heuristically determined solution can be judged to some extend using a method for calculating lower bounds on the optimal value of the DRFLP [22].

The special case of the MRFLP where all departments have the same length is called Multi-Row Equidistant Facility Layout Problem (MREFLP) and Double-Row Equidistant Facility Layout Problem (DREFLP) if $m = 2$, considered in [8, 9, 33]. In the current fastest approach [9] instances with up to 25 departments and up to 5 rows are solved to optimality within a time limit of 3 hours. For further papers on facility layout planning we refer to the recent surveys [12, 25].

In this paper we mainly focus on the SRFLP and the DRFLP, so we present an example in order to compare the distance calculation. Note that in an optimal double-row layout one might obtain free-spaces between neighboring departments.

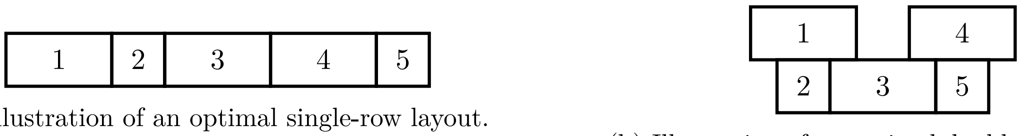
Example 1. *Given an instance with 5 departments and lengths* $\ell_1 = \ell_3 = \ell_4 = 2$, $\ell_2 = \ell_5 = 1$ *and non-zero weights* $w_{12} = w_{45} = 3, w_{23} = w_{34} = 1$.

- *An optimal single-row layout is depicted in Figure 1a with objective value*

$$3 \cdot 1.5 + 1 \cdot 1.5 + 1 \cdot 2 + 3 \cdot 1.5 = 12.5.$$

- *An optimal double-row layout is illustrated in Figure 1b with objective value*

$$3 \cdot 0 + 1 \cdot 1.5 + 1 \cdot 1.5 + 3 \cdot 0 = 3.$$



(a) Illustration of an optimal single-row layout.

(b) Illustration of an optimal double-row layout.

Figure 1: We consider an instance with 5 departments, lengths $\ell_1 = \ell_3 = \ell_4 = 2$, $\ell_2 = \ell_5 = 1$ and non-zero weights $w_{12} = w_{45} = 3, w_{23} = w_{34} = 1$.

1.1 Main contribution

Our main contribution in this paper is the following. We provide a theoretical study of the relationship between optimal values of the MREFLP and the SREFLP as well as between the optimal values of the DRFLP and the SRFLP. Therefore, we present the following results.

- Given an equidistant instance, we prove that $v_m^* \leq \frac{v_1^*}{m}$ where v_m^* denotes the optimal value of the MREFLP with m rows, see Section 2.1.
- Given a (general) double-row instance with n departments, we prove that $v_2^* \leq \frac{n-2}{n-1} v_1^*$ where v_1^* (v_2^*) denotes the optimal value of the SRFLP (DRFLP). Further we present an example which shows that this bound is tight, see Section 3.1.

In addition, we present new heuristic approaches for the MREFLP and the DRFLP based on a good or optimal single-row layout π with objective value v_1, see Sections 2.2 and 3.2.

- Given an equidistant instance, we derive a multi-row layout with objective value less than or equal to $\frac{v_1}{m}$. The constructed layout contains free-spaces only at the left or right border of the layout. We set up an MILP model which simplifies to a linear programming (LP) model to include free-spaces, and thus to further improve the quality of the determined layouts. We improve some of the best known upper bounds, in particular, for large-sized instances. Additionally, given a good or optimal single-row layout and combining our approach with some improvement heuristics, good double-row layouts can be derived in a few minutes while the SDP lower bounding approach, which includes a construction heuristic, in [9] had a time limit of three hours.
- We present two heuristics for the DRFLP which are extensions of the heuristic in the equidistant case. For instances with more than 40 departments we outperform the approach of [19] in combination with a shorter running time. Considering instances with 30 or 40 departments we obtain tight gaps, i. e., less than 1 %, to the upper bounds derived in [6] while our approaches are significantly faster.

2 The multi-row equidistant facility layout problem

In Section 2.1 we study the relation of the optimal solution values of the `MREFLP` and the `SREFLP`. Therefore, let v_m^* denote the optimal value of the `MREFLP` with m rows. One of our main results in this paper is that $v_m^* \leq \frac{v_1^*}{m}$. Further, we show that this bound is tight. Several heuristics are able to compute good or optimal single-row layouts for large-sized instances, see, e. g., [20, 34, 35, 46]. We show how to construct (at least) one equidistant multi-row layout with objective value v_m based on a single-row layout with objective value v_1 such that $v_m \leq \frac{v_1}{m}$. All constructed layouts contain free space only at the border of the layout. We further can improve the layouts by allowing free spaces. In Section 2.2 we present an `ILP`, which simplifies to some `LP`, to determine the exact positions of the departments and include possible free space. Further improvement heuristics from the literature can be applied.

2.1 Relation between equidistant single- and multi-row layouts

In this paper we study the relation between the optimal solution values of single-row and double-row layouts, but in the equidistant case all result can be extended to the `MREFLP`, so we present our results for the `MREFLP` in this part. First, we repeat a combinatorial property of the `MREFLP` given in [33].

Theorem 2. *Given an `MREFLP` instance, there always exists an optimal multi-row layout where the departments are arranged on the grid.*

We say that $i \in [n]$ lies in column $j \in [n]$ if the center of i is located at the j^{th} grid point. In order to study the relation of the equidistant single-row and double-row layouts, we consider unweighted distances first. Let $d_{ij} = d_{ji}, i, j \in [n], i < j$, denote the horizontal center-to-center distance between i and j.

Proposition 3. *The following properties hold independent of the order of the departments:*

1. *The sum of the distances between all pairs of departments in (space-free) equidistant single-row layouts satisfies* $\sum_{\substack{i,j\in[n]\\ i<j}} d_{ij} = \frac{(n+1)n(n-1)}{6} =: f(n)$.
2. *The sum of the distances between all pairs of departments in a space-free equidistant double-row layouts satisfies*

$$\sum_{\substack{i,j\in[n]\\ i<j}} d_{ij} = \begin{cases} \frac{(n+1)n(n-1)}{12}, & \text{for } n \text{ odd,} \\ \frac{(n+2)n(n-2)}{12}, & \text{for } n \text{ even.} \end{cases}$$

Proof. 1. For equidistant single-row layouts the result follows directly from the clique equations in [7] using the fact that all department lengths are equal to one, i. e., $\frac{1}{6}\left((\sum_{i=1}^n 1)^3 - \sum_{i=1}^n 1^3\right) = \frac{1}{6}(n^3 - n) = \frac{(n+1)n(n-1)}{6} = f(n)$.

2. For space-free equidistant double-row layouts with n even we have

$$\sum_{\substack{i,j\in[n]\\ i<j}} d_{ij} = 4f(\tfrac{n}{2}) = \frac{(n+2)n(n-2)}{12}$$

because each row contains $\frac{n}{2}$ departments and we count the inner- as well as the inter-row horizontal distances. For space-free equidistant double-row layouts with n odd we have

$$\sum_{\substack{i,j\in[n]\\ i<j}} d_{ij} = 4f(\tfrac{n-1}{2}) + 2 \cdot \frac{\left(\frac{n-1}{2}\right)\left(\frac{n-1}{2}+1\right)}{2} = \frac{(n+1)n(n-1)}{12}.$$

In comparison to the even case we first arrange $n-1$ departments without spaces in two rows and then add the distance of the remaining department to all others.

□

Note that in the equidistant case we cannot hope to reduce the sum of the (unweighted) distances by going from single- to double-row layouts by more than one half because for n odd the sum of the pairwise distances is reduced exactly by one half.

Definition 4. *We are given an equidistant multi-row instance and an equidistant single-row layout π. Then, the equidistant multi-row layout $L_k(\pi), k \in [m]$, is constructed by assigning the first k departments in the order of π to the first column and totally filling up all other columns with the remaining departments in the order of π. The objective value of layout $L_k(\pi), k \in [m]$, is denoted by $v_{L_k(\pi)}$.*

Let π be an equidistant single-row layout. Note that the layouts $L_k(\pi), k \in [m]$, contain possible spaces only in the first and last column. We refer to Figure 2 for an illustration of the special case $m = 2$. In the following theorem we provide a relation between the objective value of an equidistant single-row layout π and the objective value of the associated multi-row layouts $L_1(\pi), \ldots, L_m(\pi)$.

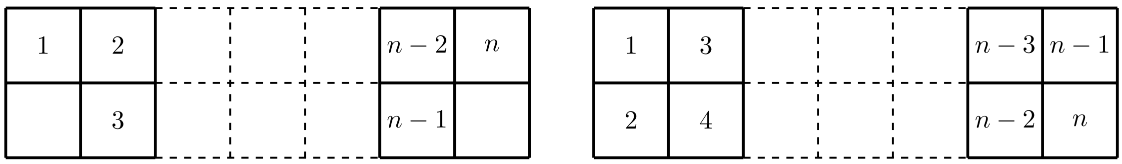


Figure 2: Illustration of the equidistant double-row layouts $L_1(\pi)$ on the left-hand side and $L_2(\pi)$ on the right-hand side deduced from the single-row layout $\pi = (1, \ldots, n)$. Note that in this drawing we assume, w. l. o. g., that n is even.

Theorem 5. *Given an equidistant multi-row instance and an equidistant single-row layout π with objective value v_1 and let $v_{L_k(\pi)}$ denote the objective value of layout $L_k(\pi), k \in [m]$. Then*

$$\sum_{k \in [m]} v_{L_k(\pi)} = v_1.$$

Proof. Given an equidistant single-row layout π with, w. l. o. g., $\pi = (1, \ldots, n)$, and the layouts $L_1(\pi), \ldots, L_m(\pi)$. Let $i, j \in [n], i < j$, and let d_{ij}^k and d_{ij}^π denote the horizontal center-to-center distance between i and j in layout $L_k(\pi), k \in [m]$, and layout π, respectively. We get $d_{ij}^\pi = j - i$ and our aim is to show that

$$\sum_{k \in [m]} d_{ij}^k = d_{ij}^\pi, \qquad i, j \in [n], i < j. \tag{1}$$

So let $i, j \in [n], i < j$. For calculating the distances $d_{ij}^k, k \in [m]$, we start with the special case $j - i = 1$ $(= 2)$. Then, in $m-1$ $(m-2)$ of the layouts $L_1(\pi), \ldots, L_m(\pi)$ the departments i and j lie in the same column and in one (two) layout (layouts) they lie in neighboring columns. So we obtain:

a) Let $j - i \leq m$. Then, i and j are in $m - j + i$ of the layouts $L_1(\pi), \ldots, L_m(\pi)$ in the same column, and hence their distance is zero. In the remaining $j - i$ layouts, i and j are in neighboring columns and hence we obtain $\sum_{k \in [m]} d_{ij}^k = j - i$.

b) Let $j - i > m$. Then, we choose $j' = j - zm, z \in \mathbb{N}_{\geq 1}$, such that $j' > i$ and $j' - i \leq m$. By the result in a) it follows that

$$\sum_{k \in [m]} d_{ij}^k = \sum_{k \in [m]} \left(d_{ij'}^k + z \right) = zm + j' - i = j - i.$$

This proves equations (1). The desired result follows immediately: Let $v_{L_k(\pi)}, k \in [m]$, and v_1, denote the objective value of layout $L_k(\pi)$ and layout π, respectively. Then

$$\sum_{k\in[m]} v_{L_k(\pi)} = \sum_{\substack{i,j\in[n]\\ i<j}} w_{ij} \sum_{k\in[m]} d_{ij}^k = \sum_{\substack{i,j\in[n]\\ i<j}} w_{ij} d_{ij}^{\pi} = v_1.$$

□

Note that the result derived by equations (1) is even stronger than the result stated in Theorem 5. In view of Proposition 3 and Theorem 5 we present the relation between v_1^* and v_m^* in the equidistant case.

Corollary 6. *Given an equidistant multi-row instance and let v_m^* (v_1^*) denote the optimal value of the* `MREFLP` *(*`SREFLP`*). Then*

$$v_m^* \leq \frac{v_1^*}{m}. \tag{2}$$

Given an equidistant multi-row instance and a single-row layout π with objective value v_1, then Corollary 6 and Theorem 5 provide an easy way to construct an equidistant multi-row layout with objective value v_m based on π that satisfies $v_m \leq \frac{v_1}{m}$ by computing the layouts $L_1(\pi), \ldots, L_m(\pi)$ and choosing one layout with minimal objective value.

2.2 Heuristics for the `MREFLP` building on combinatorial properties

Let an equidistant single-row layout π be given with objective value v_1 and we assume, w. l. o. g., $\pi = (1, \ldots, n)$. One can determine the layouts $L_i(\pi), i \in [m]$, easily, see Corollary 6, but the question arises whether one can improve the layouts $L_i(\pi)$ by including free-space not only in the first and last column. For determining such a layout, we use an `ILP` approach based on the following variables

$$x_i = \begin{cases} 1, & \text{the } i\text{-th department in the single-row layout } \pi \text{ opens a new column,} \\ 0, & \text{otherwise,} \end{cases}$$

$i \in [n]$. Then, the `ILP` approach reads as follows

$$\min \sum_{\substack{i,j\in[n]\\ i<j}} w_{ij} \sum_{k=i+1}^{j} x_k \tag{3}$$

$$\sum_{j=i}^{i+m-1} x_j \geq 1, \qquad i \in [n-m+1], \tag{4}$$

$$x_1 = 1, \tag{5}$$

$$x_i \in \{0,1\}, \qquad i \in [n]. \tag{6}$$

The distance of i and j, $i, j \in [n], i < j$, equals the number of columns between i and j plus one if i and j lie in distinct columns (3). Inequalities (4) ensure that at most m departments are assigned to each column. Of course, the first department opens a new column, see (5). The matrix corresponding to inequalities (4) satisfies the consecutive ones property, i. e., the ones in each column appear consecutively, and so the integrality conditions (6) can be replaced by $x_i \in [0,1], i \in [n]$, see, e. g., [17], and our `ILP` simplifies to some `LP`. We illustrate the usage of this `LP` by the following example.

Example 7. *Given an instance with $\ell_i = 1$, $i \in [8]$, with non-zero weights $w_{12} = w_{34} = w_{56} = w_{67} = 1$ and $w_{23} = w_{45} = w_{78} = 5$. Then, an optimal single-row layout is $\pi^* = (1, 2, 3, 4, 5, 6, 7, 8)$ with $v_1^* = 19$. The layouts $L_1(\pi^*)$ and $L_2(\pi^*)$ have objective value 8 and 11, respectively. Applying the* LP (3)–(5) *with $x_i \in [0, 1], i \in [n]$, with $m = 2$, we obtain the double-row layout illustrated in Figure 3 with objective value* 4 *and this layout has free-space in columns 1 and 4.*

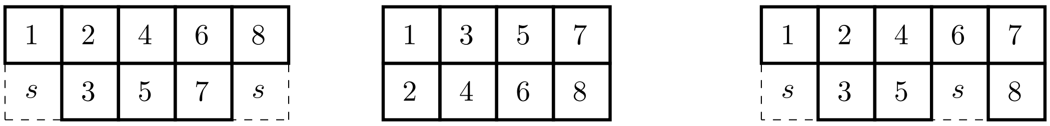


Figure 3: We consider an instance with $\ell_i = 1$, $i \in [8]$, non-zero weights $w_{12} = w_{34} = w_{56} = w_{67} = 1$ and $w_{23} = w_{45} = w_{78} = 5$ and an optimal single-row layout $\pi^* = (1, 2, 3, 4, 5, 6, 7, 8)$. Then, layout $L_1(\pi^*)$ illustrated on the left-hand side has objective value 8, layout $L_2(\pi^*)$ illustrated in the middle has objective value 11 and a double-row layout obtained by solving the LP (3)–(5) with $x_i \in [0, 1], i \in [n]$, has objective value 4, where free space between two neighboring departments in the same row is illustrated by some dashed rectangle denoted by s.

A layout obtained by solving the LP (3)–(5) with $x_i \in [0, 1], i \in [n]$, is our initial start layout for further heuristic approaches. At first, we insert $n - 2$ dummy departments with length $\ell_k = 1, k = n + 1, \ldots, 2n - 2$, and weights $w_{ik} = 0$, $i \in [2n - 2], k = n + 1, \ldots, 2n - 2, i \neq k$, and we fix them on the free spaces in the initial layout. Remaining dummy departments, if they exist at all, are assigned uniformly to the left and to the right border of the layout. It is sufficient to add $n - 2$ dummy departments such that the resulting equidistant double-row layout is space-free.

Then, we try to improve the layout by using exchange heuristics. The exchange heuristics 2-opt, 3-opt, 1-column-opt, 2-column-opt and 3-column-opt were used in a related version in [9]. Given a double-row layout with $n - 1$ columns, we say that $i \in [2n - 2]$ which lies in column $k \in [n-1]$ is on position $q = k$ if i is in row 1 and on position $q = k+n-1$ if i is in row 2. At first, we apply a 1-opt heuristic, where we place $i \in [n]$ on every possible position $q = 1, \ldots, 2n - 2$ on the grid. If there is a dummy department on position q, we place the dummy department on the previous position of i. Otherwise, let $j \in [n]$ be on position q. We shift j to the right or to the left, depending on the case whether i was to the right of to the left of position q before. We assume, w. l. o. g., we shift j to the right. If there is a dummy department on position $q + 1$, we are done. Otherwise we shift the department which was previously on position $q + 1$ to the right. We continue in this manner until we reach a dummy department or until every department to the right of j (which is in the same row as j) was shifted.

We say that column $i \in [n - 1]$ is on column position $q' = i$. Similar to the 1-opt, we use a 1-column-opt, where we arrange the departments in column $i, i = 1, \ldots, n - 1$, on every possible position $q', q' = 1, \ldots, n - 1$, and shift the departments in the column on position q' to the right or left, depending on the previous position of column i. Furthermore, we apply a 2-opt and a 3-opt algorithm, where we simply change the position of 2 or 3 departments respectively, and a 2-column-opt as well as a 3-column-opt, where we swap the position of 2 or 3 columns. In the 3-opt algorithm at most one of the considered three departments may be a dummy department while we neglect dummy departments in the 2-opt algorithm. During our improvement algorithm, we compute the objective value of the space-free layouts, since we added dummy departments. Whenever we obtain a better solution, we swap the departments and we apply each opt-algorithm until the solution is k-optimal for $k = 1, 2, 3$. The opt-algorithms are applied in increasing order of k, $k = 1, 2, 3$, i. e., in the order 1-column-opt, 1-opt, 2-column-opt, 2-opt, 3-column-opt, 3-opt.

3 The double-row facility layout problem

In Section 3.1 we study the relation between the optimal values of the SRFLP and the DRFLP. So let v_1^* (v_2^*) denote the optimal value of the SRFLP (DRFLP). At first, we prove that $\frac{v_2^*}{v_1^*} \leq \frac{n-2}{n-1}$. Let $n \geq 3$. One of our main results in this paper is that there exists a constant $C > 0$ such that for every $0 < \varepsilon \leq \frac{1}{10}$ there exists a δ with $0 < \delta \leq C\varepsilon$ such that there exists an instance with $\frac{v_2^*}{v_1^*} > \frac{n-2}{n-1} - \delta$. This shows that our bound is tight.

However, there are three reasons for constructing double-row layouts based on single-row layouts. First, good or optimal single-row layouts can be obtained very fast, see, e. g., [20, 34, 44, 46]. Second, our computational results, see Section 4, indicate that by going from single-row layouts to double-row layouts the objective value is approximately halved. A third reason is that these double-row layouts can be calculated very fast, in particular, for heuristically determined single-row layouts. So in Section 3.2.1 we present a generalization of the equidistant layouts $L_1(\pi)$ and $L_2(\pi)$ with the aim to construct a balanced double-row layout, i. e., a layout where the sum of the lengths of the departments in row 1 is almost equal to the sum of the lengths of the departments in row 2. In Section 3.2.2 we present a new heuristic based on the idea that the ordering of departments in the same row is given by the single-row layout and it remains to determine the row assignment of the departments as well as their exact positions.

3.1 Relation between optimal single-row and double-row layouts

We construct double-row layouts based on given single-row layouts in the following way.

Definition 8. *Let a single-row layout π be given, then we construct layouts $\tilde{L}_i(\pi), i \in [n-1]$, in the following way: We assign i to row 1 if $\ell_i > \ell_{i+1}$ and otherwise we assign i to row 2. Then, we fix $i+1$ directly opposite i. The remaining departments are assigned space-free to row 2 respecting the order of π.*

We refer to Figure 4 for an illustration. The optimal value of the DRFLP is less than or equal to the optimal value of the SRFLP since every single-row layout is also a valid double-row layout. Using the layouts $\tilde{L}_i, i \in [n-1]$, we provide the following result:

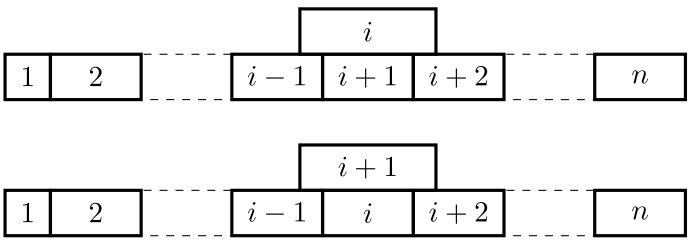


Figure 4: Let a single-row layout $\pi = (1, \ldots, n)$ be given. The layout $\tilde{L}_i(\pi), i \in [n-1]$, is illustrated in the two cases $\ell_i > \ell_{i+1}$ and $\ell_i \leq \ell_{i+1}$.

Proposition 9. *Let v_1^* (v_2^*) denote the optimal value of the SRFLP (DRFLP). Then we get*

$$(n-1)v_2^* \leq (n-2)v_1^*. \tag{7}$$

Proof. This result is clear for $n = 1$, so let $n \geq 2$. Let, w. l. o. g., $\pi^* = (1, \ldots, n)$ be an optimal single-row layout, and hence the distance between $i \in [n]$ and $j \in [n], i < j$, with respect to π^* simplifies to

$$d_{ij}^{\pi^*} := \frac{\ell_i + \ell_j}{2} + \sum_{\substack{h \in [n] \\ i < h < j}} \ell_h. \tag{8}$$

Further, we define $d_{ij}^k, i,j \in [n], i < j, k \in [n-1]$, as the distance between i and j in layout $\tilde{L}_k(\pi^*)$. At first, we will show that

$$\sum_{k\in[n-1]} d_{ij}^k \leq (n-2) d_{ij}^{\pi^*}, \qquad\qquad i,j \in [n], i < j. \tag{9}$$

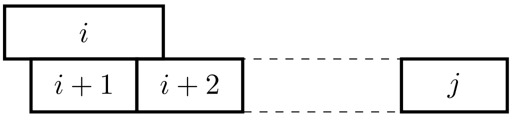
Let two departments $i,j \in [n], i < j$, be fixed. If $j = i+1$, then we obtain $d_{i(i+1)}^i = 0$ as well as $d_{i(i+1)}^k \leq d_{i(i+1)}^{\pi^*}, k \in [n-1], k \neq i$, and hence inequalities (9) are satisfied in this case. So let $j > i+1$. We distinguish between the following four cases to calculate an upper bound for $d_{ij}^k, k \in [n-1]$.

a) $k < i$ or $k \geq j$: k or $k+1$ is assigned to row 1. For $k < i-1$ and $k > j$ we get $d_{ij}^k = d_{ij}^{\pi^*}$ and if $k = i-1$ or $k = j$, we obtain $d_{ij}^k \leq d_{ij}^{\pi^*}$.

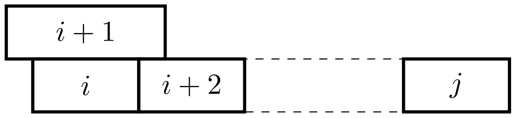
b) $k = i$: i or $i+1$ is assigned to row 1 in layout $\tilde{L}_i(\pi^*)$. Then we get $d_{ij}^i = d_{ij}^{\pi^*} - \ell_{i+1} - \frac{\ell_i}{2} + \frac{\min\{\ell_i,\ell_{i+1}\}}{2}$. We refer to Figure 5a and Figure 5b for an illustration.

c) $k = j-1$: Similar to b) we get $d_{ij}^{j-1} = d_{ij}^{\pi^*} - \frac{\ell_j}{2} - \ell_{j-1} + \frac{\min\{\ell_{j-1},\ell_j\}}{2}$.

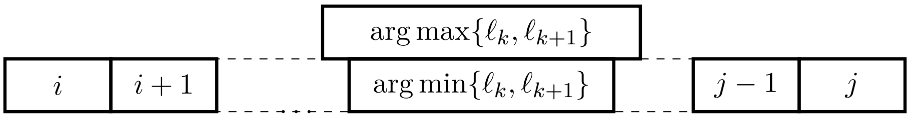
d) $i < k < j-1$: k or $k+1$ is assigned to row 1. Then it follows $d_{ij}^k = d_{ij}^{\pi^*} - \max\{\ell_k, \ell_{k+1}\}$. An illustration is given in Figure 5c.



(a) Illustrated layout $\tilde{L}_i(\pi^*)$ with $\ell_i > \ell_{i+1}$ and $d_{ij}^i = d_{ij}^{\pi^*} - \frac{\ell_i+\ell_{i+1}}{2}$



(b) Illustrated layout $\tilde{L}_i(\pi^*)$ with $\ell_{i+1} \geq \ell_i$ and $d_{ij}^i = d_{ij}^{\pi^*} - \ell_{i+1}$



(c) Layout $\tilde{L}_k(\pi^*)$ is illustrated and if $i < k < j-1$, we obtain $d_{ij}^k = d_{ij}^{\pi^*} - \max\{\ell_k, \ell_{k+1}\}$

Figure 5: Illustration of layouts $\tilde{L}_k(\pi^*), k \in [n-1]$, with $\pi^* = (1,\dots,n)$ where we only illustrate departments $i,\dots,j,\ i,j \in [n], i+1 < j$.

Since $\frac{\min\{\ell_i,\ell_{i+1}\}+\min\{\ell_{j-1},\ell_j\}}{2} \leq \max\{\ell_{i+1},\ell_{j-1}\}$ we obtain the desired inequalities (9) by summing up the distances in all layouts $\tilde{L}_1(\pi^*),\dots,\tilde{L}_{n-1}(\pi^*)$ and using (8)

$$\begin{aligned}
\sum_{k\in[n-1]} d_{ij}^k \leq &\sum_{k\in[n-1]} d_{ij}^{\pi^*} - \ell_{i+1} - \ell_{j-1} - \frac{\ell_i+\ell_j}{2} + \frac{\min\{\ell_i,\ell_{i+1}\}+\min\{\ell_{j-1},\ell_j\}}{2} \\
&- \sum_{k=i+1}^{j-2} \max\{\ell_k,\ell_{k+1}\} \\
\leq &\sum_{k\in[n-1]} d_{ij}^{\pi^*} - \left(\sum_{k=i+1}^{j-2} \max\{\ell_k,\ell_{k+1}\}\right) - \min\{\ell_{i+1},\ell_{j-1}\} - \frac{\ell_i+\ell_j}{2} \\
\leq &(n-1)d_{ij}^{\pi^*} - d_{ij}^{\pi^*} = (n-2)d_{ij}^{\pi^*}.
\end{aligned}$$

Let $v_{\tilde{L}_k(\pi^*)}$, $k \in [n-1]$, denote the objective value of layout $\tilde{L}_k(\pi^*)$, and let v_1^* (v_2^*) denote the optimal value of the SRFLP (DRFLP). By inequalities (9) we obtain

$$(n-1)v_2^* \leq \sum_{k\in[n-1]} v_{\tilde{L}_k(\pi^*)} = \sum_{\substack{i,j\in[n]\\ i<j}} w_{ij} \sum_{k\in[n-1]} d_{ij}^k \leq \sum_{\substack{i,j\in[n]\\ i<j}} w_{ij}(n-2)d_{ij}^{\pi^*} = (n-2)v_1^*.$$

□

The optimality of the single-row layout π^* is not used in the proof of Proposition 9. Therefore, the result can be extended such that $v_2 \leq \frac{n-2}{n-1} v_1$ where v_1 denotes the objective value of a given single-row layout π and v_2 denotes the minimum value of the objective values of the layouts $\tilde{L}_1(\pi), \ldots, \tilde{L}_{n-1}(\pi)$. This result can be used to derive a double-row layout with a slightly smaller objective value than the corresponding single-row layout π.

However, we show that inequality (7) is tight. We start with instances consisting of $n = 3, 4, 5$ departments and afterwards we enlarge the instances recursively.

Example 10. *a) We are given 3 departments with length $\ell_1 = \ell_2 = \ell_3 = \varepsilon > 0$ and non-zero weights $w_{12} = w_{23} = 1$. Then, we get $v_1^* = 2\varepsilon$ and $v_2^* = \varepsilon$, and hence inequality $\frac{v_2^*}{v_1^*} > \frac{1}{2} - \delta$ is satisfied for $\delta > 0$.*

b) Now we add a fourth department with length $\ell_4 = 2 - \varepsilon$ and non-zero weight $w_{34} = \varepsilon$. Clearly, $v_1^* = 3\varepsilon$ and an optimal double-row layout for $\varepsilon \leq \frac{1}{10}$ is depicted in Figure 6 and has objective value $v_2^* = \varepsilon + (1 - \varepsilon)\,\varepsilon$. Hence we have $\frac{v_2^*}{v_1^*} = \frac{\varepsilon(2-\varepsilon)}{3\varepsilon} = \frac{2-\varepsilon}{3} > \frac{2}{3} - \delta$ if $\varepsilon < 3\delta$.

c) Next we add a fifth department and we choose $0 < \varepsilon \leq \frac{1}{10}$ such that $\ell_5 = \frac{2}{\varepsilon} - \ell_4 = \frac{2}{\varepsilon} - 2 + \varepsilon$ and non-zero weight $w_{45} = \varepsilon^2$. Then $v_1^* = 4\varepsilon$. An optimal double-row layout can be obtained by arranging the first 4 departments as good as possible as illustrated in Figure 6 and then additionally arrange 5 such that the centers of departments 4 and 5 are as close as possible, see Figure 6. Then $d_{45}w_{45} = (p_5 - p_4)w_{45} = (\frac{\ell_5}{2} + \ell_3 - \frac{\ell_4}{2})\varepsilon^2 = \varepsilon - 2\varepsilon^2 + 2\varepsilon^3$ and thus $v_2^* = \varepsilon + (1 - \varepsilon)\,\varepsilon + \varepsilon\,(1 - 2\varepsilon + 2\varepsilon^2)$ and the desired inequality $\frac{v_2^*}{v_1^*} > \frac{3}{4} - \delta$ is satisfied for $\delta > \frac{3\varepsilon - 2\varepsilon^2}{4}$.

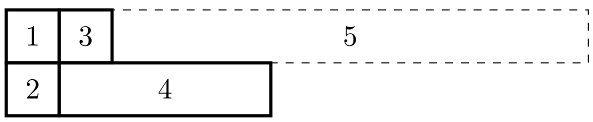


Figure 6: We are given an instance with $n = 4$ departments with lengths $\ell_1 = \ell_2 = \ell_3 = \varepsilon > 0$, $\ell_4 = 2 - \varepsilon$, and non-zero weights $w_{12} = w_{23} = 1$, $w_{34} = \varepsilon$. An optimal single-row layout is $\pi^* = (1, 2, 3, 4)$ and has objective value $v_1^* = 3\varepsilon$. The structure of an optimal double-row layout for $\varepsilon \leq \frac{1}{10}$ is depicted above and has objective value $v_2^* = \varepsilon + (1 - \varepsilon)\,\varepsilon$. One can add a fifth department with $\ell_5 = \frac{2}{\varepsilon} - \ell_4$ and the non-zero weight $w_{45} = \varepsilon^2$. Then, we obtain $v_1^* = 4\varepsilon$ and $v_2^* = \varepsilon + (1 - \varepsilon)\,\varepsilon + \varepsilon\,(1 - 2\varepsilon + 2\varepsilon^2)$ and thus $\frac{v_2^*}{v_1^*} > \frac{3}{4} - \delta$ for $\delta > \frac{3\varepsilon - 2\varepsilon^2}{4}$.

In the following we prove that the double-row layouts constructed in Example 10 are optimal and we show how recursively enlarged double-row instances can be solved to optimality.

Lemma 11. *Let an instance with n departments with lengths $\ell_1 = \ell_2 = \ell_3 = \varepsilon$ and $\ell_k = \frac{2}{\varepsilon^{k-4}} - \ell_{k-1}, k \in [n], k \geq 4$, and non-zero weights $w_{12} = 1$ and $w_{(k-1)k} = \varepsilon^{k-3}, k \in [n], k \geq 3$ be given with $0 < \varepsilon \leq \frac{1}{10}$. Then an optimal double-row layout is obtained by arranging the departments in the order $1, \ldots, n$ in an alternating manner to the rows. In each step, one department is assigned at the rightmost possible position such that the layout is space-free and such that 1 lies directly opposite 2.*

Proof. Let d_{ij} (d'_{ij}, d''_{ij} and $\tilde{d}_{ij}$), $i, j \in [n], i < j$, denote the horizontal center-to-center distance between i and j in layout L which is constructed as described in the statement of Lemma 11 (in layout L', layout L'' and layout $\tilde{L}$ with its construction described below) and let $p_i, i \in [n]$, (p'_i, p''_i and $\tilde{p}_i$) denote the center position of department i in layout L (layout L', layout L'' and layout $\tilde{L}$). Further, let $r_i \in \{1,2\}$ ($r'_i, r''_i, \tilde{r}_i \in \{1,2\}$) denote the row assignment of $i \in [n]$ in layout L (L', L'' and $\tilde{L}$). We divide the proof into three parts. In the first two parts we assume that 1 lies opposite 2 with $p_1 = p_2 = \frac{1}{2}$ and then we show in part 1) that the departments to the, w. l. o. g., right of 1 or 2 are assigned in an alternating manner with respect to the rows, at the rightmost possible position such that the resulting layout is space-free in an increasing order. In this step there might be departments right and left to 1 and 2. In part 2) we show that in an optimal layout all the departments $3, \ldots, n$ are to the, w. l. o. g., right of 1 or 2 and we complete the proof by showing in part 3) that in an optimal layout 1 lies opposite 2.

1) Let a double-row layout be given where 1 lies directly opposite 2 and let $S \subseteq \{3, \ldots, n\}$ denote the set of departments to the right of 1 or 2. We show that in an optimal layout the departments in S are arranged in an alternating manner to the rows in increasing order. In each step, one department is assigned at the rightmost possible position such that the layout is space-free. The layout is denoted by $\tilde{L}$ and we assume, w. l. o. g., $\tilde{p}_1 = \tilde{p}_2 = \frac{1}{2}$. An illustration is given in Figure 6 for $n = 5$ and $S = \{3, 4, 5\}$. Assume, on the contrary, there exists an optimal layout L' where 1 lies directly opposite 2 (with $p'_1 = p'_2 = \frac{1}{2}$) where exactly the departments in S lie right of 1 or 2 and the others lie left of 1 or 2 and at least one department in S is not arranged as in $\tilde{L}$. Let $k \in S$ be the department with its left border closest to 1 in layout L' with $\tilde{d}_{1k} \neq d'_{1k}$ (if two departments satisfy this property, we choose the department with the smaller index). Let k be to the right of $h \in S \cup \{1,2\}$ in the same row with possible free-space between h and k such that no department lies between h and k.

   At first, we show that in layout L' the departments $\{z \in S \colon z \leq h\}$ have the same position as in layout $\tilde{L}$. This in particular shows that in an optimal layout L' it holds that $h < k$. Let $i := \min\{z \in S \colon p'_z \neq \tilde{p}_z\}$ and we assume, on the contrary, $i \leq h$. Then, by the definition of k, h, i there is free-space around $\tilde{p}_i$ in L' such that one can arrange i on position $\tilde{p}_i$ in layout L' without overlapping other departments (the positions of all other departments remain the same). We denote the resulting layout by L''. Note that either $i - 1 \in S \cup \{1,2\}$ and $p'_{i-1} = \tilde{p}_{i-1}$ or $i - 1$ is to the left of 1 or 2 in layouts L' and L''. So it follows that $p''_i > p''_{i-1}$. Let $\delta = p'_i - p''_i > 0$. Then, we obtain $d''_{(i-1)i} + \delta = d'_{(i-1)i}$ and $d''_{i(i+1)} - \delta \leq d'_{i(i+1)}$. The remaining weighted distances in layouts L' and L'' are equal. Since $w_{(i-1)i} > w_{i(i+1)}$, the objective value of layout L'' is smaller than the objective value of layout L', a contradiction. Thus, we have $p'_z = \tilde{p}_z, z \in S, z \leq h$.

   We distinguish now between the following two cases where the first case has two subcases in the description below depending on h. For the first case, let $k = \min\{z \in S \colon z \geq h + 1\}$ if $h \geq 2$, and $k = \min\{z \in S \colon z \geq 3\}$ if $h = 1$. If $h \geq 2$ ($h = 1$), we obtain either $k - 1 = h$ ($k = 3$) or $k - 1 > h$ ($k > 3$) and $k - 1$ is to the left of 1 or 2. By the construction of the layout and the definition of k, at most one department $j \in S \cup \{1,2\}$ overlaps with h. We obtain $j < h$ if $h \neq 1$ and $j = 2$ if $h = 1$. Therefore, department j satisfies $\tilde{p}_j = p'_j$ and $p'_j + \frac{\ell_j}{2} < p'_k - \frac{\ell_k}{2}$. Note that if $j \in \{1,2\}$, then there is free-space between h and k. Therefore, we obtain a feasible layout by changing the row assignment of the departments $z \in S$ with $p'_z > p'_h$ without changing their positions. Then, we shift k without spaces to the right of j, if j exists, and to the right of 1 or 2 otherwise. We denote the resulting layout by L'' and we obtain $p''_h < p''_k$ because we have $\tilde{p}_z = p''_z, z \in S, z \leq h$ and $\ell_h < \ell_k$ or $k = 3$. Let $\delta := p'_k - p''_k > 0$. Therefore, we obtain $d''_{(k-1)k} + \delta = d'_{(k-1)k}$ and, if $k + 1 \in [n]$, $d''_{k(k+1)} - \delta \leq d'_{k(k+1)}$. The remaining weighted distances in layout L'' are the same as in layout L'. Since $w_{(k-1)k} > w_{k(k+1)}$ if $k + 1 \in [n]$, the objective value of layout L'' is smaller than the objective value of layout L', a contradiction.

Now we consider the case of $k > \min\{z \in S \colon z \geq h+1\}$ and we set

$$j := \begin{cases} 2, & h = 1, \\ 1, & h = 2, \\ \min\{z \in S \colon z \geq h+1\}, & h \geq 3. \end{cases}$$

We aim to show that $p'_j = \tilde{p}_j$. This result is clear if $j \in \{1, 2\}$, so let $h \geq 3$. Then, either $j - 1 = h$ or $j - 1$ is left to 1 or 2. By the definition of k we can arrange j on position $\tilde{p}_j$ either in row 1 or row 2 such that the resulting layout L'' is feasible. Let $\delta := |p'_j - \tilde{p}_j| > 0$. If $\delta \leq \ell_j$, then j is in layout L' and L'' in the same row (since $\ell_k > \ell_j$) and layout L'' is feasible since $p''_z = \tilde{p}_z$ for all $z \in S, z \leq h$. We obtain $d''_{(j-1)j} + \delta = d'_{(j-1)j}$ and $d''_{j(j+1)} - \delta \leq d'_{(j-1)j}$. By $w_{(j-1)j} > w_{j(j+1)}$ it follows that the objective value of layout L'' is smaller than the objective value of layout L', a contradiction. Now, let $\delta > \ell_j$. Then, to avoid the overlapping of departments, we shift all departments which are in layout L'' to the right of j in the same row to the right by the smallest possible value such that a feasible layout is obtained. The departments are shifted at most by the value ℓ_j, so we obtain $d''_{(j-1)j} + \delta = d'_{(j-1)j}$ and $d''_{z(z+1)} \leq d'_{z(z+1)} + \delta$, $z \in [n-1], z \geq j$. Note that for $0 < \varepsilon \leq \frac{1}{10}$ and $o \in [n-1], o \geq 3$, we obtain

$$w_{(o-1)o} > 2w_{o(o+1)} + \sum_{z=o+1}^{n-1} w_{z(z+1)}. \tag{10}$$

Hence, the objective value of layout L'' is smaller than the objective value of layout L', a contradiction. Therefore, we have $p'_j = \tilde{p}_j$.

Now we consider $o := \min\{q \in S \colon q \geq \max\{h, j\} + 1\}$. We obtain either $\max\{h, j\} = o - 1$ or $\max\{h, j\} < o - 1$ and $o - 1$ is to the left of 1 or 2. Let $k = o$, then there is free-space between h and k because of the choice of k. We simply shift k to the left such that k is to the right of h without free-space and the resulting layout is denoted by L''. By construction of the layout we obtain $p''_k > p''_{k-1}$ and we set $\delta = p'_k - p''_k > 0$. We obtain $d''_{(k-1)k} + \delta = d'_{(k-1)k}$ and, if $k + 1 \in [n]$, $d''_{k(k+1)} - \delta \leq d'_{k(k+1)}$. The remaining weighted distances are equal in layout L' and layout L''. Since $w_{(k-1)k} > w_{k(k+1)}$ if $k + 1 \in [n]$, the objective value of layout L'' is smaller than the objective value of layout L', a contradiction. So let $k > o$ and let $\delta > 0$ denote the horizontal distance between the right border of h and the left border of o. We distinguish between the following two cases.

a) Let $\delta \geq \ell_o$, then we arrange o space-free to the right of h and, if necessary, we shift the departments which are to the right of h or j to the right by the smallest possible value such that o and k do not overlap. Note that these departments are shifted at most by ℓ_o, we refer to Figure 7a and Figure 7b for an illustration. The resulting layout is denoted by L'' and we obtain $p''_o > p''_{o-1}$. Comparing the distances in layout L' and L'', we obtain $d''_{(o-1)o} + \delta = d'_{(o-1)o}$ and $d''_{o(o+1)} \leq d'_{o(o+1)} + \delta + \ell_o$. Additionally, we obtain $d''_{z(z+1)} \leq d'_{z(z+1)} + \ell_o$, $z \in [n-1], z \geq o+1$. The remaining weighted distances in layout L'' and L' are equal. By (10) the following inequality is satisfied

$$\delta w_{(o-1)o} > (\delta + \ell_o)\, w_{o(o+1)} + \ell_o \sum_{z=o+1}^{n-1} w_{z(z+1)}.$$

So the objective value of layout L'' is smaller than the objective value of layout L', a contradiction.

b) It remains to consider the case $\delta < \ell_o$. Since $k > o$ and $\ell_k > \ell_o$ it follows that o and j are neighboring (with possible free-space), see Figure 7c. Let δ' denote the horizontal distance of the left border of k and the right border of j if k and j overlap and otherwise we set $\delta' = 0$. Then we change the row assignment of all departments to the right of h or j (without changing the order of these departments), and we shift these departments (without o) by $\delta' > 0$ to the right such that j and k do not overlap. Then we shift o to the left by the value δ such that there is no free-space between h and o, see Figure 7d. By construction of the layout, we obtain $\delta' \leq \delta$. The resulting layout is denoted by L''. We compare the weighted distances in layout L'' and L', similar as done above, and it turns out that the objective value of layout L'' is smaller than the objective value of layout L' if

$$\delta w_{(o-1)o} > (\delta + \delta') w_{o(o+1)} + \delta' \sum_{z=o+1}^{n-1} w_{z(z+1)}.$$

This inequality is satisfied for $0 < \varepsilon \leq \frac{1}{10}$, see inequalities (10), a contradiction.

2) Let 1 lie directly opposite 2. We prove that the departments $3, \ldots, n$ are to the, w. l. o. g., right of 1 or 2. Assume, on the contrary, there exists an optimal double-row layout L' where 1 lies directly opposite 2 with $p'_1 = p'_2 = \frac{1}{2}$ and the departments $S \subset [n], S \neq \emptyset$, are to the right of 1 or 2 and the departments $T \subset [n], T \neq \emptyset$, are to the left of 1 or 2 such that $S \,\dot\cup\, T = \{3, \ldots, n\}, S \cap T = \emptyset$. Since L' is an optimal double-row layout, the departments in S and T are arranged as described in 1). Let $i \in S$, $i+1, \ldots, j \in T$ and $j+1 \notin T$. Note that, if $i \in T$ and $i+1, \ldots, j \in S, j+1 \notin S$, we simply arrange the departments left (right) to 1 or 2 to the right (left) of 1 or 2 without changing the row assignment and the order of the departments in S (T), and hence we obtain $i \in S$ and $i+1, \ldots, j \in T, j+1 \notin T$. Our goal is to show that $\sum_{z=i}^{j-1} w_{z(z+1)} d_{z(z+1)} < \sum_{z=i}^{j-1} w_{z(z+1)} d'_{z(z+1)}$. At first, we obtain

$$d'_{i(i+1)} > \frac{\ell_i + \ell_{i+1}}{2} > d_{i(i+1)}.$$

If $j = i+1$, the desired inequality is satisfied. Otherwise, we obtain

$$d'_{i(i+1)} + d'_{(i+1)(i+2)} > d_{i(i+1)} + d_{(i+1)(i+2)} = \frac{\ell_i + \ell_{i+2}}{2} \tag{11}$$

where the last equation follows from $p_i < p_{i+1} < p_{i+2}$. We refer to Figure 8 for an illustration. We continue in this manner and we obtain

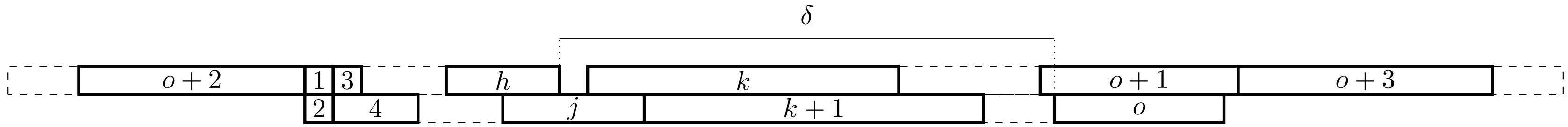
$$d'_{z(z+1)} + d'_{(z+1)(z+2)} = d_{z(z+1)} + d_{(z+1)(z+2)}, \tag{12}$$

$z, z+1, z+2 \in T, z \geq i+1$. For i even we obtain, $d_{(i+1)(i+2)} = \sum_{\substack{z \geq 4\\ z \text{ even}\\ z \leq i}} \ell_z - \sum_{\substack{z \geq 3\\ z \text{ odd}\\ z \leq i}} \ell_z + \frac{\ell_{i+2} - \ell_{i+1}}{2}$. Note that, if $i+1, i+2 \in T$, then $i+1$ and $i+2$ lie in distinct rows. We obtain

$$d'_{(i+1)(i+2)} - d_{(i+1)(i+2)} = \sum_{\substack{z \in T\\ z \leq i-1\\ r'_z = r'_{i+2}}} \ell_z - \sum_{\substack{z \in T\\ z \leq i-1\\ r'_z = r'_{i+1}}} \ell_z + \sum_{\substack{z \geq 3\\ z \text{ odd}\\ z \leq i}} \ell_z - \sum_{\substack{z \geq 4\\ z \text{ even}\\ z \leq i}} \ell_z < 0 \tag{13}$$

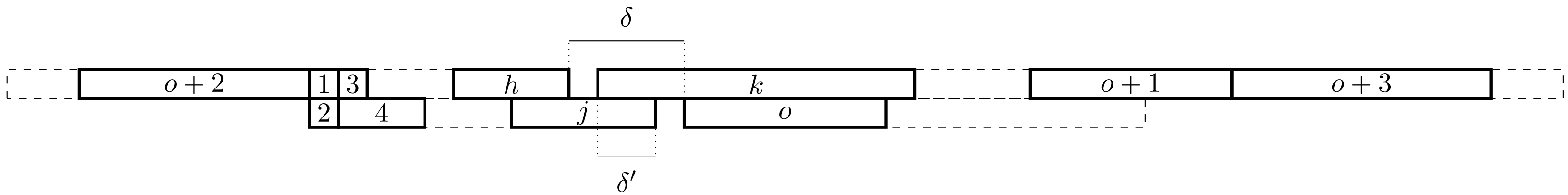
since for $0 < \varepsilon \leq \frac{1}{10}$ we have $\ell_i > 2 \sum_{z \in [i-1]} \ell_z$. Now, let i be odd. We get $d_{(i+1)(i+2)} = \sum_{\substack{z \geq 3\\ z \text{ odd}\\ z \leq i}} \ell_z - \sum_{\substack{z \geq 4\\ z \text{ even}\\ z \leq i}} \ell_z + \frac{\ell_{i+2} - \ell_{i+1}}{2}$, and thus we get

$$d'_{(i+1)(i+2)} - d_{(i+1)(i+2)} = \sum_{\substack{z \in T\\ z \leq i-1\\ r'_z = r'_{i+2}}} \ell_z - \sum_{\substack{z \in T\\ z \leq i-1\\ r'_z = r'_{i+1}}} \ell_z - \sum_{\substack{z \geq 3\\ z \text{ odd}\\ z \leq i}} \ell_z + \sum_{\substack{z \geq 4\\ z \text{ even}\\ z \leq i}} \ell_z < 0 \tag{14}$$

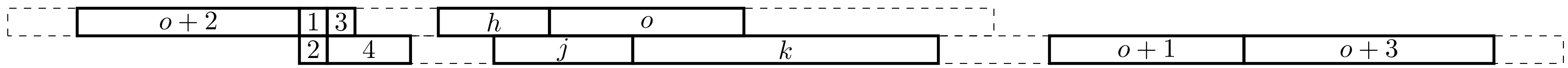


(a) Let a double-row layout L' be given where $k \in S$ is assigned differently than described in Lemma 11 with $\delta \geq \ell_o$.

(b) Given a layout with $\delta \geq \ell_o$ as illustrated in Figure 7a, we shift o to the right of h without free-space and we shift the departments which are to the right of h or j (without o) to the right by the smallest possible value (which is less than or equal to ℓ_o) such that o and k do not overlap. The objective value of the layout is reduced by this method.

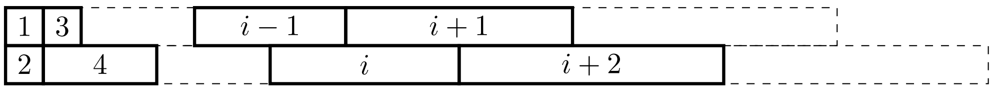


(c) Let a double-row layout be given where $k \in S$ is assigned differently than described in Lemma 11 with $\delta < \ell_o$.

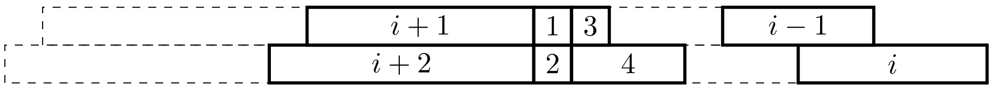


(d) We are given the layout illustrated in Figure 7c. We change the row assignment of all departments to the right of h or j and we shift these departments (without o) to the right by the value δ' such that j and k do not overlap, o is arranged without free-space to the right of h. The objective value of the layout is reduced by this method.

Figure 7: We are given the instance described in Lemma 11 and we illustrate layouts considered in part 1) of the proof of Lemma 11. Let $h, j, k, k+1, o, o+1, o+3 \in S \subseteq [n], h < j < o < k, |\{k, k+1, o+1, o+3\}| = 4$, with $o := \min\{z \in S \colon z \geq \max\{h, j\}+1\}$ and $o+2 \in [n] \setminus S$. Further, let δ denote the horizontal distance between the right border of h and the left border o and δ' the horizontal distance of the left border of k and the right border of j if k and j overlap and otherwise we set $\delta' = 0$. The objective values of the layouts illustrated in Figure 7a and Figure 7c which are not constructed as described in part 1) of the proof of Lemma 11 can be reduced, see Figure 7b and Figure 7d, respectively.



(a) Given a double-row layout L constructed as described in Lemma 11 we obtain $d_{i(i+1)} + d_{(i+1)(i+2)} = \frac{\ell_i + \ell_{i+2}}{2}$.



(b) Given a double-row layout L' where 1 lies opposite 2 and the departments $1, \dots, i, i \in [n], i \geq 3$, are to the right of 1 or 2 and the departments $i+1, \dots, j \in [n]$, are to the left of 1 or 2 (with $i + 1 \neq j$ here) we obtain $d'_{i(i+1)} + d'_{(i+1)(i+2)} > d_{i(i+1)} + d_{(i+1)(i+2)}$.

Figure 8: We are given the instance described in Lemma 11 and we compare the distances in layout L and L'. We obtain $d_{i(i+1)} \leq d'_{i(i+1)} - \delta$, and $d_{(i+1)(i+2)} = d'_{(i+1)(i+2)} + \delta$, for some $\delta > 0$. We continue in this manner and since $w_{z(z+1)} > w_{(z+1)(z+2)}, z \in [n-2], z \geq 2$, the objective value of layout L is smaller than the objective value of layout L'.

and for $0 < \varepsilon \leq \frac{1}{10}$ this inequality is satisfied since $\ell_i > 2 \sum_{z \in [i-1]} \ell_z$. So we obtain $d'_{(i+1)(i+2)} = d_{(i+1)(i+2)} - \delta$ for some $\delta > 0$ and by inequalities (11) we obtain $d'_{i(i+1)} > d_{i(i+1)} + \delta$. Further, by equations (12) $d_{(i+z)(i+z+1)} = d'_{(i+z)(i+z+1)} - \delta, \frac{z}{2} \in \mathbb{Z}, z \geq 2, i + z + 1 \leq j$, and $d_{(i+z+1)(i+z+2)} = d'_{(i+z+1)(i+z+2)} + \delta, \frac{z}{2} \in \mathbb{Z}, z \geq 2, i + z + 2 \leq j$. Recall that $w_{z(z+1)} > w_{(z+1)(z+2)}$, $z + 2 \in [n], z \geq 2$, so the desired inequality is satisfied. One can continue in this manner if further departments k and $k+1$ are in distinct sets S and $T, k \in [n-1], k \geq j$. So the objective value of layout L is smaller than the objective value of layout L', a contradiction.

3) It remains to show that 1 lies directly opposite 2 in an optimal double-row layout. Assume, on the contrary, there exists an optimal double-row layout L' where 1 does not lie directly opposite 2. We divide this proof into three cases.

   a) At first, let $d'_{23} < \varepsilon$, and we assume, w. l. o. g., $p'_2 \leq p'_3$. We calculate a lower bound for such a layout by allowing the departments $4, \dots, n$ to overlap with 1 in the same row (correct sublayout with respect to 1, 2, 3). Then, the departments $4, \dots, n$ are arranged in an alternating manner to the rows such that 4 is arranged in the same row as 2 and space-free at the rightmost position (since $p'_2 \leq p'_3$) in increasing order, the proof of this result is similar to the proof of 1) and 2). We denote the resulting (in general not feasible) layout with departments $1, \dots, n$ by L'' and clearly, the objective value of layout L'' is less than or equal to the objective value of layout L'. We obtain $d_{12} + d_{23} \leq d''_{12} + d''_{23}$. Further, let $\delta := \varepsilon - d''_{23} > 0$ be the length of the line segment at which 2 and 3 overlap, then we get $d_{(2+i)(2+i+1)} = d''_{(2+i)(2+i+1)} - \delta, i \in [n-3], i$ odd, and $d_{(2+i+1)(2+i+2)} = d''_{(2+i+1)(2+i+2)} + \delta$, $i \in [n-4], i$ odd. Similar to 2) it follows that the objective value of layout L'' is greater than the objective value of layout L, a contradiction.

   b) Let $d'_{13} < \varepsilon$ and $d'_{23} \geq \varepsilon$, then one can swap the positions of 1 and 2 and the objective value of the layout is reduced, a contradiction.

   c) It remains to consider an optimal double-row layout L' with $d'_{13} \geq \varepsilon$ and $d'_{23} \geq \varepsilon$. At first, let $0 < d'_{12} < \varepsilon$ and we assume, w. l. o. g., $p'_1 < p'_2$. If $p'_3 < p'_1$, we shift all departments in the same row as 1 to the right until 1 and 2 lie directly opposite. The resulting layout is denoted by L'' and we set $d'_{12} := \delta > 0$. Then, we obtain $d''_{12} = 0$, $d''_{23} \leq d'_{23}$ and $d''_{z(z+1)} \leq d'_{z(z+1)} + \delta, z \in [n-1], z \geq 3$. Since $w_{(z-1)z} > w_{z(z+1)}, z \in [n-1], z \geq 3, w_{12} = w_{23}$, the objective value of layout L'' is smaller than the objective value of layout L', a contradiction. Now, let $p'_3 > p'_2$. Then we shift 3 to the right of 1 without free-space and, if necessary, we shift all departments now lying to the right of 3 or 2 in the same row to the right to avoid overlapping. The resulting layout has a smaller objective value than layout L', the proof is similar to 1a) and 1b) with $o = 3$, a contradiction.

So, let $d'_{12} \geq \varepsilon$. We calculate a lower bound on a layout containing the departments $3, \ldots, n$ and then we add $w_{12}d'_{12} + w_{23}d'_{23} \geq 2\varepsilon$ to this lower bound. To calculate a lower bound for the layout of the departments $3, \ldots, n$, we distinguish between two cases. If 3 and 4 overlap in layout L', then 3 and 4 lie directly opposite as described above for 1 and 2. So let 3 and 4 lie directly opposite. Then the departments $5, \ldots, n$ are arranged in an alternating manner space-free at the rightmost position in increasing order to the rows (this result is similar to the proof of 1) and 2)). The resulting layout is denoted by L'' and the objective value of layout L'' is less than or equal to the objective value of layout L'. We obtain $w_{12}d_{12} + w_{23}d_{23} + w_{34}d_{34} < 2\varepsilon$, see Example 10. Furthermore, we obtain $d''_{z(z+1)} = d_{z(z+1)} + \delta, z \in [n-1], z \geq 4, \frac{z}{2} \in \mathbb{Z}$, and $d''_{z(z+1)} = d_{z(z+1)} - \delta, z \in [n-1], z \geq 5, \frac{z+1}{2} \in \mathbb{Z}$, for some $\delta > 0$. So the objective value of layout L is smaller than the objective value of layout L''.

Now we assume that 3 and 4 do not overlap in layout L'. Then, we calculate a lower bound for the double-row instance consisting of the departments $4, \ldots, n$ and add $\sum_{i\in[3]} w_{i(i+1)}d'_{i(i+1)} \geq 3\varepsilon$ to this lower bound. We distinguish between the two cases whether 4 and 5 overlap or not. We continue as described above and this lower bound exceeds the objective value of layout L.

□

So for these instances we obtain an optimal double-row layout as described above. Therefore, we obtain our desired result.

Theorem 12. *Let $n \geq 3$. Then, there exists a constant $C > 0$ such that for every $0 < \varepsilon \leq \frac{1}{10}$ there exists a δ with $0 < \delta \leq C\varepsilon$ such that*

$$\frac{v_2^*}{v_1^*} > \frac{n-2}{n-1} - \delta, \tag{15}$$

where v_1^* (v_2^*) denotes the optimal value of the `SRFLP` (`DRFLP`) with n departments and lengths $\ell_1 = \ell_2 = \ell_3 = \varepsilon$ and $\ell_k = \frac{2}{\varepsilon^{k-4}} - \ell_{k-1}, k \in [n], k \geq 4$, and non-zero weights $w_{12} = w_{23} = 1$ and $w_{(k-1)k} = \varepsilon^{k-3}, k \in [n], k \geq 3$.

Proof. We prove this result by induction. For $n = 3, 4$ and $n = 5$ we refer to Example 10. So let $n \in \mathbb{N}, n \geq 6$, and we consider instances as described in the statement of the theorem and considered in Lemma 11. We obtain $\frac{\ell_{k-1}+\ell_k}{2} w_{(k-1)k} = \varepsilon, k \in [n], k \geq 2$, and thus the single-row layout $\pi^* = (1, \ldots, n)$ has objective value $v^*_{1,n} := (n-1)\varepsilon$ which proves that π^* is optimal because its objective value equals the constant $\tilde{C} := \sum_{\substack{i,j\in[n]\\ i<j}} w_{ij} \frac{\ell_i+\ell_j}{2}$ which is a lower bound on the optimal value of the `SRFLP`.
An optimal double-row layout can be obtained as described in Lemma 11 and it remains to compare the optimal value of the double-row instance with $v^*_{1,n}$. Therefore, let p_n and p_{n-1} denote the center position of n and $n-1$ measured from the left border of the double-row layout. We assume, w. l. o. g., that n is odd and hence n is in the same row as 1.

$$\begin{aligned}(p_n - p_{n-1})w_{(n-1)n} &= \left(\left(\frac{\ell_n}{2} + \ell_{n-2} + \ldots + \ell_3 + \ell_1\right) - \left(\frac{\ell_{n-1}}{2} + \ell_{n-3} + \ldots + \ell_4 + \ell_2\right)\right)\varepsilon^{n-3} \\ &= \left(\frac{1}{\varepsilon^{n-4}} - \ell_{n-1} + \ell_{n-2} - \ell_{n-3} + \ell_{n-4} - \ldots - \ell_4 + \ell_3\right)\varepsilon^{n-3}\end{aligned}$$

$$= \left(\frac{1}{\varepsilon^{n-4}} - \frac{2}{\varepsilon^{n-5}} + 2\ell_{n-2} - \ell_{n-3} + \ell_{n-4} - \ldots - \ell_4 + \ell_3 \right) \varepsilon^{n-3}$$
$$= \left(\frac{1}{\varepsilon^{n-4}} + \sum_{z=1}^{n-4} \left((-1)^z \frac{2z}{\varepsilon^{n-4-z}} \right) + (n-3)\ell_3 \right) \cdot \varepsilon^{n-3}$$
$$= \varepsilon + \sum_{z=1}^{n-4} \left((-1)^z 2z\varepsilon^{1+z} \right) + (n-3)\varepsilon^{n-2}.$$

We set $\delta'' = (-1)\frac{\sum_{z=1}^{n-4}\left((-1)^z 2z\varepsilon^{1+z}\right)+(n-3)\varepsilon^{n-2}}{(n-1)\varepsilon}$ and for small $\varepsilon > 0$ we obtain $0 < \delta'' \leq \varepsilon$. Let $v^*_{2,n}$ and $v^*_{2,n-1}$ denote the optimal value of an optimal double-row layout with n and $n-1$ departments, respectively. By assumption there exists a constant $C' > 0$ such that there exists a δ' with $0 < \delta^{'} \leq C'\varepsilon$, such that $\frac{v^*_{2,n-1}}{(n-2)\varepsilon} > \frac{n-3}{n-2} - \delta^{'}$. Then, we obtain

$$\frac{v^*_{2,n}}{v^*_{1,n}} = \frac{v^*_{2,(n-1)} + (p_n - p_{n-1})w_{n(n-1)}}{(n-1)\varepsilon}$$
$$> \frac{n-3}{n-1} - \delta^{'}\frac{n-2}{n-1} + \frac{1}{n-1} - \delta''$$
$$= \frac{n-2}{n-1} - \delta^{'}\frac{n-2}{n-1} - \delta''.$$

We set $C = C'\frac{n-2}{n-1} + 1, \delta = \delta^{'}\frac{n-2}{n-1} + \delta''$ and we obtain $\delta \leq C\varepsilon$. □

3.2 Heuristic approaches for the DRFLP

In Sections 3.2.1 and 3.2.2 we describe two heuristics for the DRFLP based on good or optimal single-row layouts. Considering the instances described in Lemma 11, both heuristics determine an optimal double-row layout given the optimal single-row layout $\pi = (1, \ldots, n)$. Further, both heuristics are extensions of the heuristics presented in Section 2.2. Similar as before both variants use a given single-row layout, especially the order of the departments, to construct a double-row layout. Because of the arbitrary department lengths in the DRFLP adaptations are needed. In Section 3.2.3 we describe exchange heuristics to further improve given double-row layouts. In particular, we set up an MILP model for deriving a 1-optimal solution.

3.2.1 Balanced rows

Given a single-row layout $\pi = (\pi_1, \ldots, \pi_n)$, we first determine the row assignment of the departments. In the order of π we assign the current department to a row where the sum of the lengths of the departments already assigned to that row is minimal. The order in π is then used as the order of the departments in each of the two rows. Given the assignments of the departments to the rows and the order of the departments in each row, we determine the exact positions of the departments by solving an LP, see, e. g., [4]. This approach is motivated by the construction of $L_1(\pi), L_2(\pi)$ in the equidistant case.

3.2.2 Mincut heuristic

Let a single-row layout π be given and we assume, w. l. o. g., $\pi = (1, \ldots, n)$. We present a new heuristic based on the idea that the sorting of the departments in the same row is given via π, so it remains to determine the row assignment of the departments as well as their exact positions. We are given $n_{min}, n_{max} \in \mathbb{N}, n_{min} \leq n_{max}$, and in each step we add a set S of departments which contains, if possible, at least n_{min} departments and at most n_{max} departments.

Let the set of departments $[h], 0 \leq h \leq n, h \in \mathbb{N}_0$, be already added to the double-row layout (we start with $h = 0$ and we stop if $h = n$). If $h + n_{max} \geq n$, all remaining departments are added and we set $S = \{h+1, \dots, n\}$. Otherwise, we interpret the departments $H = \{h+1, \dots, n\}$ as nodes in a complete graph with weights $w_{ij}, i, j \in H, i < j$. Our goal is to determine some k' and an associated set $S := \{h+1, h+2, \dots, k'\}$ such that the sum of the total transport weights between S and $[n] \setminus [k']$ is small. So we detect which departments should be considered together in the next step. We set

$$k' := \begin{cases} \arg\min_{\substack{h+n_{min} \le k \le h+n_{max} \\ k \in \mathbb{N}}} \sum_{\substack{i=h+1,\dots,k \\ j=k+1,\dots,n}} w_{ij}, & h+n_{max} < n, \\ n, & h+n_{max} \ge n. \end{cases}$$

So $|S| \le n_{max}$, and, if $h + n_{min} \le n$, then $|S| \ge n_{min}$. The calculation of k' is related to the calculation of a constrained minimum cut in the graph described above.

Then, we add the dummy department $n+1$ ($n+2$) to row 1 (row 2) with length $\ell_{n+1} = \ell_{n+2} = 0$ and weights $w_{i(n+1)} = w_{(n+1)i} = w_{i(n+2)} = w_{(n+2)i} = \frac{1}{2} \sum_{\substack{i=h+1,\dots,k' \\ j=k'+1,\dots,n}} w_{ij}$ such that $n+1$ ($n+2$) is the rightmost department in row 1 (row 2). Knowing k' and so S, our goal is to determine a row assignment of the departments S such that departments in the same row are sorted according to π and such that $\sum_{\substack{i,j \in [k'] \cup \{n+1,n+2\} \\ i<j}} w_{ij} d_{ij}$ is minimized, i. e., we have to solve a (small) double-row instance where the order of the departments in the same row is known. For solving this problem we apply the approach of [28] and enumerate over all distinguishable assignments of the departments S to the rows. Knowing the order of the departments in the rows, each subproblem reduces to some `LP` with $k'+2$ departments. We choose one of the row assignments for S where the layout has minimal objective value. In the last step, when $n \in S$, the solution of the `LP` corresponds to a double-row layout including possible free-spaces. The algorithm stops after returning this layout. We denote this heuristic by $\texttt{mc}(n_{min}, n_{max})$.

Considering the `DREFLP` and choosing $n_{min} = n_{max} = n$ in the approach here the resulting objective value is less than or equal to the objective value obtained by solving the `LP` (3)–(5) with $x_i \in [0,1], i \in [n]$. However, a huge number of row assignments would have to be checked.

3.2.3 Exchange algorithms

The layouts derived in Section 3.2.1 and Section 3.2.2 are our initial start solutions. We denote a given double-row layout by $\sigma = (\sigma_1, \dots, \sigma_k, f, \sigma_{k+1}, \dots, \sigma_n)$, where f indicates that the departments which arise in the order of σ before f are assigned to row 1 and the remaining departments are assigned to row 2 (in the order of σ). In [22] a 1-opt algorithm is used where in each step one department is arranged at every possible position in σ, and then an `LP` is solved to determine the exact position of the departments. The department is arranged on a position which leads to a minimal objective value.

Instead of using this enumerative approach, we set up an `MILP` model for the 1-opt algorithm. Recall that $d_{ij} = d_{ji}, i, j \in [n], i < j$, denotes the horizontal center-to-center distance between i and j and $p_i, i \in [n]$, denotes the position of the center of i. Given a double-row layout, we remove $t \in [n]$, so let, w. l. o. g., the departments $1, \dots, k, k \in [n-1]$, be in row 1 and $k+1, \dots, n-1$ be in row 2 in this sorting and $t = n$. Then, we use the following variables

$$z_i = \begin{cases} 1, & t \text{ is arranged on position } i \text{ in } \sigma, \\ 0, & \text{otherwise,} \end{cases}$$

$i = 1, \dots, n+1$, where $z_1 = 1$ corresponds to t left to 1, $z_{k+1} = 1$ corresponds to t right to k, $z_{k+2} = 1$ corresponds to t left to $k+1$. Then our `MILP` model reads as follows with $M = \sum_{i \in [n]} \ell_i$

$$\min \sum_{\substack{i,j \in [n] \\ i<j}} w_{ij} d_{ij}$$

$$p_{i+1} - p_i \geq \frac{\ell_{i+1} + \ell_i}{2}, \qquad i \in [n-2] \setminus \{k\}, \tag{16}$$

$$\sum_{i \in [n+1]} z_i = 1, \tag{17}$$

$$p_i - p_t - M z_i \geq \frac{\ell_i + \ell_t}{2} - M, \qquad i \in [k], \tag{18}$$

$$p_t - p_i - M z_{i+1} \geq \frac{\ell_i + \ell_t}{2} - M, \qquad i \in [k], \tag{19}$$

$$p_i - p_t - M z_{i+1} \geq \frac{\ell_i + \ell_t}{2} - M, \qquad i \in [n-1] \setminus [k], \tag{20}$$

$$p_t - p_i - M z_{i+2} \geq \frac{\ell_i + \ell_t}{2} - M, \qquad i \in [n-1] \setminus [k], \tag{21}$$

$$d_{ij} - p_j + p_i \geq 0, \qquad i, j \in [k], i < j, \tag{22}$$

$$d_{ij} - p_j + p_i \geq 0, \qquad i, j \in [n-1] \setminus [k], i < j, \tag{23}$$

$$d_{ij} - p_j + p_i \geq 0, \qquad i \in [k] \cup \{t\}, j \in ([n-1] \setminus [k]) \cup \{t\}, i \neq j, \tag{24}$$

$$d_{ij} - p_i + p_j \geq 0, \qquad i \in [k] \cup \{t\}, j \in ([n-1] \setminus [k]) \cup \{t\}, i \neq j, \tag{25}$$

$$p_i \geq \frac{\ell_i}{2}, \qquad i \in [n], \tag{26}$$

$$d_{ij} = d_{ji} \geq 0, \qquad i, j \in [n], i < j, \tag{27}$$

$$z_i \in \{0, 1\}, \qquad i \in [n+1]. \tag{28}$$

By inequalities (16) we ensure that departments in the same row do not overlap, recall that the sorting of the departments is known. By equation (17) we ensure that t is assigned to exactly one position in σ. If t is assigned to row 1, then we ensure by inequalities (18) that t and the department to the right of t do not overlap and by inequalities (19) we ensure that t does not overlap with the department to the left of t (if t is assigned to row 2, see inequalities (20)–(21)). The distance between departments $i, j \in [n-1], i < j$, in the same row is calculated by inequalities (22)–(23) and distances between departments in different rows as well as distances between $i \in [n-1]$ and t are calculated by inequalities (24)–(25).

Afterwards we use a 2-opt algorithm where we swap the position of two departments in σ. We only accept changes if the objective value is reduced and the 1-opt (2-opt) algorithm is applied until the double-row layout cannot be improved by a 1-opt (2-opt) step.

4 Computational results

In this section we present our computational results. The computational experiments are based on a C++ implementation which uses Cplex 12.10 as an `MILP` solver [36]. All results were conducted on a 2.10GHz quad-core using Virtual Box 6 and running on Debian GNU/Linux 8 in single processor mode. As often done in the literature, in our computational experiments we focus on the `DREFLP` and the `DRFLP`. We apply our heuristics based on optimal or best known single-row layouts, see, e. g., [39, 40, 41], and we use a heuristic for the `SRFLP` with a short running time and which is easy to implement, i. e., we start with a random single-row layout and apply a 1-opt algorithm and a 2-opt algorithm until the single-row layout cannot be improved by a 1-opt or 2-opt step, respectively. The single-row layouts and all considered instances are available from the authors.

4.1 Results for the `DREFLP`

We start our computational study with the equidistant case. In Table 1 we display in column two (column three) the objective value of an optimal or best known (heuristically determined) single-row layout denoted by “Best known” (“Heuristic”). The objective value is marked with a

	SREFLP		Start layout		Exchange			Time	
Instance	Best known	Heuristic	H_{Best}	H_{Heur}	H_{Best}	H_{Heur}	[9]	H_{Best}	H_{Heur}
Y20	12185*	12185*	6047	6047	6046*	6046*	6046*	<1	<1
Y25	20357*	20434	10170	10206	10170	10206	10170	<1	<1
Y30	27673*	27704	13801	13801	13800	13801	13790	<1	<1
Y35	38194*	38290	19093	19141	19087	19141	19087	1	1
Y40	47561*	47604	23737	23762	23732	23759	23739	3	2
Y45	62904	63357	31447	31671	31442	31671	31442	4	4
Y50	83127	83179	41523	41538	41523	41538	41517	7	7
Y60	112126	112735	56017	56330	56010	56328	55986	32	17
sko42-1	25525	25525	12749	12749	12743	12743	12731	2	2
sko49-1	40967	42469	20477	21226	20470	21224	20512	6	6
sko56-1	64024	66083	31975	33011	31972	32932	31988	11	24
sko64-1	96883	98122	48418	49052	48409	49004	48574	23	45
sko72-1	139150	143317	69535	71607	69531	71603	69621	41	42
sko81-1	205106	208554	102549	104263	102549	104067	102793	1:10	4:36
sko100-1	378234	384049	189062	191982	189056	191964	-	3:31	3:56

Table 1: Heuristically determined upper bounds for equidistant double-row instances from the literature [9]. Our heuristics are based on optimal or best known single-row layouts as well as heuristically determined single-row layouts where known optimal solution values are marked with a "*". The running times of our heuristics are given in sec or min:sec.

"*" if the associated single-row layout is known to be an optimal layout. The objective value of the start layout and of the final layout after applying our exchange algorithm is denoted by H_{Best} (H_{Heur}) which is based on the best known single-row layout (heuristically determined single-row layout). The current best upper bounds for these instances are derived by the semidefinite optimization approach of [9] and are given in column eight. This approach mainly focuses on determining strong lower bounds and the time limit is set to 3 hours. In the last two columns we summarize the running times of our heuristics, see Section 2.2, given in sec or min:sec. The best known or optimal single-row layouts are available at `https://www.philipphungerlaender.com/benchmark-libraries/layout-lib/row-layout-instances/`.

We observe that for all instances in Table 1 our heuristic based on optimal or best known single-row layouts is better than the one based on heuristically determined single-row layouts. Note that only for the instance Y20 our single-row heuristic derived an optimal solution, but the obtained gaps for the remaining instances are rather small and the running time is at most one minute, even for $n = 100$. For the instance Y20 our heuristic derives an optimal double-row layout based on an optimal (heuristically determined) single-row layout. For larger instances the optimal solutions are not known. Considering the Y-instances with $25 \leq n \leq 60$ and given a best known single-row layout, we obtain three times the same objective value as the approach of [9], once we can even improve the best known upper bound and three times we obtain small gaps to the best known upper bounds. For all large sko-instances with $n \geq 49$ and given some best known single-row layout, we improve the previously known best upper bounds in [9] with a significantly smaller running time. Using our approach based on heuristically determined single-row layouts, we obtain small gaps to the approach of [9], however, these layouts can be calculated in a few minutes, including the calculation of the corresponding single-row layout. Comparing the best solution values of the `SREFLP` and the `DREFLP` one can see that the value of the `DREFLP` is strictly less than halve the value of the `SREFLP`, but rather close to this value in our tests.

4.2 Results for the `DRFLP`

In the following, we denote the balanced row heuristic by `br`. Given a single-row layout $\pi = (\pi_1, \dots, \pi_n)$, we calculate a start layout via our heuristics `br` and `mc` and afterwards we calculate again a start layout based on the inverted order $(\pi_n, \dots, \pi_1)$. In Tables 2–5 we display the minimum value of these two layouts and we apply our improvement algorithm only on one double-row layout with minimal objective value. In our test we used $n_{min} = 8$ and $n_{max} = 14$ in

Instance	SRFLP	Start layout br	Start layout mc(8, 14)	Exchange br	Exchange mc(8, 14)	DRFLP
*Am*14_1	5481.5*	2873.5	2743.5	2738.5	2738.5	2738.5*
*Am*14*a*	5673.0*	2981.0	2907.0	2907.0	2907.0	2904.0*
*Am*14*b*	5595.0*	2773.0	2736.0	2736.0	2736.0	2736.0*
*Am*15_1	6305.0*	3236.0	3211.0	3223.0	3211.0	3195.0*
*HK*15	33220.0*	17040.0	16895.0	16740.0	16600.0	16570.0*
*P*16*a*	14829.0*	7427.0	7365.5	7416.0	7365.5	7365.5*
*P*16*b*	11878.5*	5944.5	5870.5	5928.5	5870.5	5870.5*

Table 2: Results for instances from the literature where optimal DRFLP solution values are known [28]. Given an optimal single-row layout, the mc heuristic derived four times an optimal double-row layout and the br heuristic two times. The start layouts were only slightly improved by the exchange algorithm.

the mc heuristic.

In Table 2 we consider double-row instances with known optimal solution values. Optimal single-row layouts can be derived very fast for these small instances via, e. g., the approach of [3], so we apply our heuristics only on optimal single-row layouts in Table 2. The mc heuristic determined four optimal double-row layouts and the br heuristic calculated two optimal double-row layouts. If an optimal single-row layout was given, the running time of the br heuristic was less than one second and the mc heuristic needed less than one minute (including the improvement algorithm). Considering the values of the start layouts derived with the mc and the br heuristics, the mc heuristic was always better. The start layouts were only slightly improved by the exchange algorithm.

In Table 3–5 we display in column two (three) the objective value of an optimal or best known (heuristically determined) single-row layout. The objective value is marked with a “*” if the single-row layout is optimal. The objective value of the start layout and the final layout is denoted by $\texttt{br}_{\text{opt}}$ and $\texttt{mc}(8,14)_{\text{opt}}$ or $\texttt{br}_{\text{Best}}$ and $\texttt{mc}(8,14)_{\text{Best}}$ ($\texttt{br}_{\text{Heur}}$ and $\texttt{mc}(8,14)_{\text{Heur}}$) if the heuristics are based on an optimal or best known (heuristically determined) single-row layout. The optimal single-row layouts in Table 3–4 are obtained by the approach of [26] and are available from the authors.

In [6] four heuristics are presented and each of them is applied ten times. The minimum value is displayed in Table 3 in the twelfth column. Each single run for $n = 40$ needs about an hour, which leads to a high total running time of this approach. The equidistant instance N40-1 is neglected in Table 3. Let a single-row layout be given, then our heuristics run in a few minutes. For all instances in Table 3, the mc heuristic based on an optimal single-row layout obtains gaps of less than 0.5 % to the results of [6] and for one instance our heuristic obtains a layout with the same objective value. This shows that our approach is able to derive high-quality double-row layouts even for larger instances and our approach is much faster than the methods in the literature. For these instances the running time of the br and the mc heuristic hardly depends of the start layout, so we only displayed the running times based on optimal single-row layouts. Note that the start layouts are only slightly improved by the exchange algorithm and considering the mc heuristic, the final layout often equals the start layout. The quality of our start layouts depends on the quality of the single-row layouts. Given an optimal (heuristically determined) single-row layout, then the mc heuristic is better than the br heuristic. However, the br heuristic based on optimal single-row layouts is often better than the mc heuristic based on heuristically determined single-row layouts.

In Table 4-5 we compare our approach with the heuristic presented in [19] where five start layouts are computed and the minimal objective value is displayed. Then, our exchange algorithm is applied on one layout with minimal objective value. We consider randomly generated instances from the literature [22] with $n \in \{20, 30, 40, 50\}$. The transport density is set to 10 %, 50 % and 100 % and integer transport weights are chosen randomly between 1 and 10. The integral lengths

	SRFLP		Start layout				Exchange					Time	
Instance	optimal	Heuristic	$\texttt{br}_{\text{opt}}$	$\texttt{br}_{\text{Heur}}$	$\texttt{mc}(8,14)_{\text{opt}}$	$\texttt{mc}(8,14)_{\text{Heur}}$	$\texttt{br}_{\text{opt}}$	$\texttt{br}_{\text{Heur}}$	$\texttt{mc}(8,14)_{\text{opt}}$	$\texttt{mc}(8,14)_{\text{Heur}}$	[6]	$\texttt{br}_{\text{opt}}$	$\texttt{mc}(8,14)_{\text{opt}}$
N30-2	21582.5*	22308.5	10824.5	11156.5	10784.0	11122.0	10781.5	11136.5	10774.5	11122.0	10771.0	6	1:23
N30-3	45449.0*	45473.0	22736.0	22757.0	22702.5	22721.5	22733.0	22711.5	22702.5	22710.0	22692.0	6	2:03
N30-4	56873.5*	59155.5	28475.5	29614.5	28451.5	29593.5	28445.5	29610.5	28451.0	29593.5	28390.0	6	3:33
N30-5	115268.0*	115916.0	57521.0	58107.0	57399.5	57733.5	57521.0	57811.5	57393.5	57733.5	57393.5	3	1:48
A40-1	199534.0*	204108.0	99828.0	101785.0	99631.0	101617.0	99828.0	101624.0	99545.0	101617.0	99525.5	11	6:29
A40-2	602650.0*	617210.0	301232.0	309056.0	301084.0	308376.0	301126.0	308424.0	301084.0	308374.0	300973.5	31	4:56
A40-3	833326.0*	840843.0	416575.0	420022.0	416325.0	419946.0	416372.0	419972.0	416308.0	419886.0	416257.0	22	7:22
A40-4	415226.0*	422797.0	207846.0	211310.0	207694.0	211265.0	207774.0	211144.0	207694.0	211265.0	207510.0	24	6:48
A40-5	387819.0*	390057.0	193993.0	195225.0	193945.0	195052.0	193842.0	195034.0	193816.0	195050.0	193748.0	21	7:04
A40-6	3764907.5*	3775995.5	1882761.5	1887577.0	1881382.5	1886636.5	1881637.5	1885342.5	1881382.5	1886070.0	1881277.0	30	7:45
A40-7	1090764.0*	1107962.0	548760.0	555732.0	546898.0	554223.0	548364.0	554388.0	546898.0	554202.0	545239.0	14	3:58

Table 3: Results of the br and mc heuristic for double-row instances from the literature [6] based on optimal and heuristically determined single-row layouts. The quality of the start layouts depends on the quality of the single-row layouts. Given an optimal single-row layout, the mc heuristic derives small gaps to the approach of [6] and for one instance the same objective value is obtained. The running time of our heuristics are given in sec and min:sec. Even for instances with $n = 40$, the br heuristic runs in less than one minute and the mc heuristic needs a few minutes, including the exchange algorithm.

of the departments are chosen randomly between 1 and 15 as well as between 5 and 10. For each type ten instances are tested and the average values are displayed in Table 4. We denote the instances by $n_{k,\ell_{min}-\ell_{max}}$ where n is the number of departments, k is the transport density and ℓ_{min} and ℓ_{max} describe the upper and the lower bound of the integral department lengths.

Considering sparse instances in Table 4, the heuristically determined layouts have large gaps to optimal single-row layouts and the corresponding double-row layouts of the `br` and `mc` heuristic have large gaps to the layouts based on optimal single-row layouts. So the quality of the start layout and of the final layout of the `br` and `mc` heuristic depend highly on the quality of the single-row layout. Note that it would be possible to improve the `SRFLP` solutions by using the fact that the sparse instances with $n = 20$ and $n = 30$ could often be divided into two or more independent smaller instances. Given an optimal single-row layout, then the objective values of the start layout and of the final layout of the `br` and `mc` heuristic are smaller than the corresponding objective values of the approach of [19]. The exchange algorithms only slightly improve the start layouts of the `br` and `mc` heuristic, however, the start layout of [19] is improved significantly by the exchange algorithms. Note again that the `mc` heuristic is better than the `br` heuristic if both heuristics are based on optimal or heuristically determined single-row layouts.

In Table 5 we only consider sko-instances where good heuristically determined single-row layouts are available at `https://www.philipphungerlaender.com/benchmark-libraries/layout-lib/row-layout-instances/`. The instance sko56-5 marked with a "°" is not the best known single-row layout but the best known single-row layout with objective value 592294.5 is not available online, so we decided to choose this single-row layout. Looking at the results for the sko-instances in Table 5 all solutions derived using the `br` and the `mc` heuristic based on best known single-row layouts are better than the results of [19]. If we use the two heuristics in combination with our simple single-row heuristic, we could improve 5 out of 9 upper bounds in comparison to the approach in [19]. As seen in Tables 3-4, for most instances the `mc` heuristic is slightly better than the `br` heuristic based on best known (heuristically determined) single-row layouts. For the `mc` heuristic based on best known single-row layouts, the exchange algorithm only slightly improves the start layout.

In Table 6 we compare the running times of the heuristics applied on the sko-instances. The `br` heuristic is the fastest heuristic for best known and heuristically determined single-row layouts. In column three we display the running time of the `br` heuristic based on a best known single-row layout, where the 1-opt algorithm is calculated by an enumerative approach instead of using our `MILP` approach. For 8 out of 9 instances the running time is at least halved by using our `MILP` approach. Altogether, the running times of the `br` and the `mc` heuristics are not highly influenced by choosing a best known or heuristically determined single-row layout. The `mc` heuristic is for both, best known and heuristically determined single-row layouts, a bit slower than the heuristic approach of [19].

5 Conclusion and future work

In this paper we studied the relationship of the `MREFLP` and the `SREFLP` and we proved that $v_m^* \leq \frac{v_1^*}{m}$ where v_m^* denotes the optimal value of the `MREFLP` with $m \in \mathbb{N}$ rows. Given an equidistant single-row layout with value v_1, we presented an easy way to derive an equidistant multi-row layout whose value v_m is at most $\frac{v_1}{m}$. We can further improve such a layout by the inclusion of free spaces via a new `ILP` model which simplifies to some `LP` model. For the `DREFLP` we improved some of the best known upper bounds and we significantly reduced the running time for calculating these layouts.

We proved that the optimal solution value of the `SRFLP` and the `DRFLP` for the same instance might be close. In particular, we showed that the following inequality holds $(n-1)v_2^* \leq (n-2)v_1^*$, where v_1^*, v_2^* denote the optimal solution values of the two problems. Additionally, we presented an example where this bound is tight. Nonetheless, good or optimal single-row layouts can be a

	SRFLP		Start layout					Exchange				
Instance	optimal	Heuristic	$\texttt{br}_{\text{opt}}$	$\texttt{br}_{\text{Heur}}$	$\texttt{mc}(8,14)_{\text{opt}}$	$\texttt{mc}(8,14)_{\text{Heur}}$	[19]	$\texttt{br}_{\text{opt}}$	$\texttt{br}_{\text{Heur}}$	$\texttt{mc}(8,14)_{\text{opt}}$	$\texttt{mc}(8,14)_{\text{Heur}}$	[19]
$20_{10,1-15}$	1282.5	1496.3	634.8	746.9	588.0	699.0	676.3	585.1	708.9	581.1	694.6	632.6
$20_{50,1-15}$	16801.7	17069.6	8426.2	8567.1	8360.8	8520.3	8918.4	8385.4	8543.1	8360.8	8472.4	8516.4
$20_{100,1-15}$	41686.4	42007.1	20866.1	21028.8	20822.2	20979.0	21350.6	20831.3	20959.4	20819.6	20978.8	20932.9
$20_{10,5-10}$	1260.2	1516.6	592.0	738.0	580.7	710.5	675.1	577.1	702.0	575.6	690.3	636.1
$20_{50,5-10}$	17558.5	18184.6	8754.5	9067.6	8722.8	9044.1	9388.3	8739.5	9048.9	8719.5	9038.7	8867.1
$20_{100,5-10}$	45208.8	45456.1	22520.6	22650.2	22490.8	22619.3	23010.7	22499.4	22634.2	22488.0	22618.9	22610.7
$30_{10,1-15}$	4768.4	5256.6	2435.3	2647.1	2317.5	2547.4	2801.6	2336.7	2566.4	2311.2	2543.5	2540.3
$30_{50,1-15}$	60662.1	62126.1	30417.8	31140.8	30324.8	31042.6	32511.4	30357.8	31076.4	30319.2	31037.0	30864.2
$30_{100,1-15}$	138962.0	139331.0	69490.1	69670.0	69410.0	69633.3	71086.1	69436.2	69628.7	69407.2	69632.7	70037.8
$30_{10,5-10}$	5228.7	6261.2	2588.4	3113.0	2536.4	3058.1	3234.7	2550.0	3059.8	2532.1	3016.8	2928.1
$30_{50,5-10}$	63616.1	65644.3	31755.9	32788.7	31722.6	32757.2	34295.5	31732.8	32778.1	31718.6	32754.7	32347.8
$30_{100,5-10}$	155948.0	157316.0	77843.3	78530.7	77808.7	78493.6	79509.2	77815.1	78471.9	77808.7	78488.1	78394.7
$40_{10,1-15}$	14376.2	16442.6	7261.0	8294.9	7115.0	8143.0	8972.5	7162.3	8113.8	7106.1	8090.0	8212.3
$40_{50,1-15}$	150331.0	153046.0	75316.9	76579.4	75193.2	76516.5	79604.1	75240.1	76490.6	75170.1	76489.6	76166.7
$40_{100,1-15}$	334031.0	337538.0	167059.0	168926.0	167008.0	168801.0	170402.0	167007.0	168624.0	166986.0	168668.0	167899
$40_{10,5-10}$	14597.2	17073.3	7259.4	8520.0	7217.6	8446.8	9059.0	7229.1	8461.6	7214.7	8406.3	8239.2
$40_{50,5-10}$	156889.0	160155.0	78419.4	80067.8	78363.7	80012.1	83163.5	78386.8	79569.2	78347.4	79822.2	79727.3
$40_{100,5-10}$	372806.0	375713.0	186279.0	187718.0	186196.0	187664.0	190317.0	186215.0	187667.0	186187.0	187642.0	187741.0
$50_{10,1-15}$	-	36562.5	-	18369.2	-	18201.5	20085.3	-	18022.8	-	18047.2	18148.4
$50_{50,1-15}$	-	300004.0	-	150123.0	-	150036.0	155466.0	-	150026.0	-	149965.0	149589.0
$50_{100,1-15}$	-	654842.0	-	327558.0	-	327388.0	332631.0	-	327387.0	-	327365.0	327817.0
$50_{10,5-10}$	-	37209.3	-	18592.3	-	18519.5	19563.3	-	18527.7	-	18505.1	17507.7
$50_{50,5-10}$	-	319605.0	-	159762.0	-	159714.0	166613.0	-	159706.0	-	159607.0	159668.0
$50_{100,5-10}$	-	739078.0	-	369379.0	-	369283.0	373090.0	-	369293.0	-	369261.0	367815.0

Table 4: Results of the br and mc heuristic for the randomly generated instances of [22] with integral department lengths between 1 and 15 as well as 5 and 10 where we show the average over 10 instances each. The transport density is set to 10 %, 50 % and 100 %. Our heuristics are based on optimal and heuristically determined single-row layouts and we compare our results with the heuristic approach of [19]. Given an optimal single-row layout, then for all instances the objective value of the start layout (final layout) of the br as well as the mc heuristic is smaller than the objective value of the start layout (final layout) of the approach of [19].

	SRFLP		Start layout					Exchange				
Instance	Best known	Heuristic	br_{Best}	br_{Heur}	$mc(8,14)_{Best}$	$mc(8,14)_{Heur}$	[19]	br_{Best}	br_{Heur}	$mc(8,14)_{Best}$	$mc(8,14)_{Heur}$	[19]
sko56-3	170449.0	171884.0	85252.5	85919.0	85267.0	85927.0	88326.5	85216.0	85904.0	85185.0	85905.5	86960.5
sko56-4	313388.0	316222.0	156780.0	158165.0	156772.0	158140.0	160847.0	156721.0	158115.0	156690.0	158118.0	156946.0
sko56-5	592299.5°	600080.0	296348.0	300052.0	296336.0	300138.0	307706.0	296272.0	300042.0	296306.0	300046.0	304137.0
sko64-3	414323.5	420810.0	207310.0	210460.0	207278.0	210392.0	214524.0	207223.0	210420.0	207160.0	210372.0	210634.0
sko64-4	297129.0	299193.0	148543.0	149629.0	148546.0	149559.0	154923.0	148541.0	149598.0	148508.0	149500.0	149844.0
sko72-2	711998.0	719188.0	355986.0	359531.0	355990.0	359486.0	373665.0	355980.0	359504.0	355975.0	359482.0	364886.0
sko72-3	1054110.5	1067699.5	527156.0	534050.0	527030.0	533922.0	539676.0	527127.0	533856.0	526956.0	533748.0	530166.0
sko72-4	919586.0	931922.0	460040.0	466278.0	459892.0	466200.0	476218.0	459900.0	466158.0	459824.0	466144.0	463502.0
sko72-5	428226.5	433700.0	214112.0	216806.0	214087.0	216798.0	219243.0	214094.0	216796.0	214080.0	216782.0	215986.0

Table 5: Results of the `br` and `mc` heuristics for double-row instances with up to 72 departments based on best known and heuristically determined single-row layouts. All solutions derived by using the `br` and the `mc` heuristic based on best known single-row layouts are better than the results of [19]. For most instances, the `mc` heuristic is better than the `br` heuristic based on best known and heuristically determined single-row layouts.

Instance	$\mathrm{br}_{\mathrm{Best}}$	$\mathtt{br}^{enu}_{\mathrm{Best}}$	$\mathtt{br}_{\mathrm{Heur}}$	$\mathtt{mc}(8,14)_{\mathrm{Best}}$	$\mathtt{mc}(8,14)_{\mathrm{Heur}}$	[19]
sko 56-3	2:13	7:49	2:20	19:02	26:38	9:44
sko 56-4	3:10	6:37	4:50	20:36	25:10	9:50
sko 56-5	3:16	7:07	1:33	21:02	23:52	9:02
sko 64-3	6:59	18:15	2:43	39:36	39:29	22:50
sko 64-4	2:35	9:51	4:35	39:28	36:17	24:24
sko 72-2	8:46	23:20	4:54	1:03:33	54:57	39:32
sko 72-3	10:36	20:13	11:38	1:06:41	1:22:41	53:26
sko 72-4	18:54	33:34	11:20	1:12:32	1:17:09	1:03:50
sko 72-5	8:44	23:17	9:03	56:28	1:02:30	40:46

Table 6: In this table we compare the running times of the heuristic approaches considered in Table 5. The running times are given in sec, min:sec and h:min:sec, respectively. In the third column the running time of the `br` heuristic is displayed with an enumerative 1-opt algorithm as used in [22]. One can see that the running time is significantly reduced by using our `MILP` approach for the 1-opt algorithm, displayed in the second column.

good starting point for deriving good double-row layouts. Indeed, we presented two heuristics for the `DRFLP`, which rely on the ideas used for the `MREFLP` and which can be calculated very fast. We obtained very small gaps to the best known upper bounds for instances with 30 and 40 departments, but derive these solutions much faster and for instances with more departments we outperform the heuristic of [19].

It remains for future work to set up heuristics for layout problems with more complicated path structures like the T-Row Facility Layout Problem [21]. Further it is interesting from a practical point of view to extend the current exact models and heuristics for the Single-Row and Double-Row Facility Layout Problem such that more aspects like individual input and output positions of the departments or certain clearance conditions between the departments can be taken into account.

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