

TIME-DOMAIN DECOMPOSITION FOR OPTIMAL CONTROL PROBLEMS GOVERNED BY SEMILINEAR HYPERBOLIC SYSTEMS

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ABSTRACT. In this article, we extend the time-domain decomposition method described by Lagnese and Leugering [15] to semilinear optimal control problems for hyperbolic balance laws with spatio-temporal varying coefficients. We provide the design of the iterative method applied to the global first-order optimality system, prove its convergence, and derive an a posteriori error estimate. The analysis is done entirely on the continuous level. A distinguishing feature of the method is that the decomposed optimality system can be interpreted as an optimality system of a local “virtual” optimal control problem. Thus, the iterative time-domain decomposition of the optimality system can be interpreted as an iterative parallel scheme for virtual optimal control problems on the subintervals. A typical example and further comments are given to show the range of potential applications. Moreover, we provide some numerical experiments to give a first interpretation of the role of the parameters involved in the iterative process.

1. INTRODUCTION

Time-domain decomposition for partial differential equations (PDEs) has been a subject of intense research in the past. Given a PDE with time domain $[0, T]$, the idea is to introduce a coarse time discretization of $[0, T]$ into a disjoint union of subintervals $I_k := [T_k, T_{k+1}]$ with $[0, T] = \text{cl}(\cup_{k=1}^K (T_k, T_{k+1}))$ and then to iteratively decouple the PDE such that on each subinterval I_k , the same PDE is solved together with conditions at the breakpoints T_k that couple the states at the current iteration $n + 1$ with those at iteration n . We may trace back the contributions to the seminal paper [21] by J. L. Lions et al., in which the so-called “parareal”-scheme has been introduced. In [24, 25], the authors further developed the scheme and applied it to quantum control problems. This scheme has then later been identified as a variant of the common multiple-shooting method; see, e.g., [8]. These methods, which consist of a coupling of coarse grain discrete-in-time solutions at the break points with a parallel computation of full (respectively, small grain) solutions on the subintervals, were first developed for the mere simulation of nonlinear PDEs. In the article [14], the authors, for the first time, considered the time-domain decomposition of optimal control problems for the time-dependent Maxwell system. Later, in [13], a broad number of such problems—even combined with a spatial domain decomposition for PDEs on networked domains—have been investigated. We also refer to [11] and [28], where methods related to multiple-shooting have been provided along with applications for the heat equation. During the last decade, there has been an increasing interest in applying time- and space-domain decomposition techniques to optimal control problems; see, e.g., [1, 6, 7, 23, 29, 30].

A distinguishing feature of the method in [13–15] is the fact that the iterative time-domain decomposition is applied to the optimality system of the original optimal control problem on the time domain $[0, T]$ in such a way that the decomposed problems are by themselves optimality systems corresponding to so-called virtual control problems on the subintervals I_k . Thus, the fully parallel iteration can be seen as one of optimal control problems on the subintervals. This can be utilized in applications, where solvers for the virtual control problems, possibly on small time intervals, are available. The analysis is provided at the continuous level, as an example of the first-optimize-then-discretize approach, leading inherently to mesh independence when it comes to numerical realizations. The method uses the fact that the state variables evolve forwardly in time, whereas the adjoint variable progresses backwardly. Therefore, on the subinterval level, one can design two-point boundary value problems w.r.t. the time variable.

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To the best of our knowledge, the contributions in the literature do not include time-varying systems so far. Moreover, time-domain decomposition for semilinear optimal control problems that possess this distinguishing feature do not seem to be tackled as well. The goal of this article is to extend the time-domain decomposition method of [13, 15] to optimal control problems involving semilinear hyperbolic systems of conservation laws where the coefficients may depend on time and space. We are particularly interested in applications that focus on processes on metric graphs or networks. As examples, we will focus on networks of semilinear strings or rods. Systems that are related to gas flow in pipe networks are easily seen to fit into the framework of this article—for the model; see, e.g., [10, 16]. Such problems on metric graphs, where the edges, which are representative of the spatial domains of the corresponding PDEs, are coupled at the vertices of the graph, can be transcribed into two-point initial boundary value problems with a possibly large number of state variables; see Example 2 for further explanation.

The issue of existence of solutions to optimal control problems, in particular in the context of possibly nonsmooth, say $L^2(0, T)$, boundary controls is not settled for the type of systems considered. For a class of distributed controls, where the spatial part is fixed, we refer to [27]. Using smooth controls, one can use the classic shifting of boundary data to the state equation to use the results of [27]. We may already add here, for the sake of clarity, that the problem is not with the well-posedness of the PDE systems per se, but rather with the regularity of controls interlinked with a sufficient sequential-lower-semicontinuity property of the cost function. As for the well-posedness of the optimality system, we refer to the work [4] of Brokate. The analysis provided in this article is based on the continuous PDE-level and hence follows the “first-optimize-then-discretize” paradigm. Thus, we obtain mesh-independent convergence results so that we do not aim for an efficient numerical implementation. For numerical evidence, we provide some simple examples to give a first idea on the behavior of the different parameters involved. For a full treatment, we have to refer to a further publication.

We also remark that the iterative time-domain decomposition considered here can be interpreted in a way in the context of the classic Robin-Robin-type Schwarz-method introduced by P. L. Lions [22], which has been interpreted by Glowinski and Le Tallec in [9] as a variant of their Uzawa-type saddle-point algorithm. This algorithm, in turn, results from an augmented Lagrangian formulation of the interface problem. Indeed, for the time-domain decomposition, say with two time intervals, one may regard the optimality system as a second-order boundary value problem in space and time. The analogue of the P. L. Lions algorithm with the addition of a damped-Richardson relaxation, see [3], is then related to our method. This will be made more explicit in due course.

The article is organized as follows. We begin with the problem statement in Section 2, where we include a detailed discussion of an example of a network of controlled strings or rods, in fact a star graph, where a possibly local nonlinear damping term is present along some or all strings involved together with nonlinear boundary conditions. In Section 3, we introduce the time-domain decomposition method for the overall optimality system into systems on the subintervals and show in which way these decomposed systems are themselves optimality systems for “virtual” optimal control problems on the subintervals. In Section 4, we discuss the convergence of the iteration for unconstrained controls, while the constrained case is tackled in Section 5—however, for the linear case only. As the nonlinearities are not assumed to be explicitly given by specific functions and the corresponding Nemytskij operators, we will rely on bounds, regularity, and smallness assumptions and use the control structure to compensate for nonlinear effects. Section 6 contains the development of an a posteriori error estimate for our iteration. In Section 7, we provide some numerical examples, which demonstrate first evidence for the behavior of the iterative time-domain decomposition approach. Finally, the paper closes with some concluding remarks in Section 8.

2. PROBLEM STATEMENT

We consider two-point boundary value problems for systems of hyperbolic semilinear equations of the form

$$\partial_t y + A(t, x) \partial_x y = f(t, x, u, y), \quad (t, x) \in (0, T) \times (0, 1), \quad (1a)$$

$$y_i(t, 0) = \sum_{j=1}^m g_{ij}^0(t, y_j(t, 0)), \quad i = m+1, \dots, d, \quad t \in (0, T), \quad (1b)$$

$$y_i(t, 1) = \sum_{j=m+1}^d g_{ij}^1(t, v_j(t), y_j(t, 1)), \quad i = 1, \dots, m, \quad t \in (0, T), \quad (1c)$$

$$y(0, x) = y_0(x), \quad x \in (0, 1), \quad (1d)$$

$$u(t) \in U_{\text{ad}}^d \quad \text{a. e. in } (0, T), \quad (1e)$$

$$v(t) \in U_{\text{ad}}^b \quad \text{a. e. in } (0, T), \quad (1f)$$

where $y(t, x) \in \mathbb{R}^d$, $t \in [0, T]$, $x \in [0, 1]$, denotes the state,

$$A(t, x) = \text{diag} (\lambda_1(t, x), \dots, \lambda_m(t, x), \lambda_{m+1}(t, x), \dots, \lambda_d(t, x)) \in \mathbb{R}^{d \times d}$$

with

$$\lambda_1(t, x) \leq \lambda_2(t, x) \leq \dots \leq \lambda_m(t, x) < 0 < \lambda_{m+1}(t, x) \leq \dots \leq \lambda_d(t, x)$$

for all $(t, x) \in [0, T] \times [0, 1]$ represents the physics of the system, taken in characteristic coordinates to make the mathematical description simpler. Moreover, f_j , $j = 1, \dots, d$, g_{ij}^0 , $i = m+1, \dots, d$, $j = 1, \dots, m$, and g_{ij}^1 , $i = 1, \dots, m$, $j = m+1, \dots, d$, are differentiable functions to be specified below, u and v are taken to represent distributed and boundary controls, respectively, where $u(t, x) \in \mathbb{R}^d$, $v(t) \in \mathbb{R}^m$, are constrained a. e. by closed and convex sets U_{ad}^d and U_{ad}^b . If $U_{\text{ad}}^d = \mathbb{R}^d$, $U_{\text{ad}}^b = \mathbb{R}^m$, then controls are unrestricted. Finally, $y_0(x) \in \mathbb{R}^d$ for $x \in [0, 1]$ denotes the initial data. We should note that the boundary conditions in (1) are in accordance with the standard formulation as in Chapter 6 of [2]. Under these conventions, System (1) is a controlled hyperbolic and semilinear system. In addition to (1), we consider the cost function

$$J(w, y) := \int_0^T \int_0^1 L(t, x, u, y) \, dx \, dt + \int_0^1 L_T(x, y(T, x)) \, dx + \int_0^T L_b(t, v(t), y(t, 1)) \, dt, \quad (2)$$

with $w = (u, v)$ and functions L , L_T , L_b that are measurable w.r.t. (t, x) , differentiable w.r.t. (u, y) , (v, y) , respectively, with bounded derivatives on bounded sets, and uniformly continuous w.r.t. (t, x) . The considered control problem is thus given by

$$\min_{w, y} J(w, y) \quad \text{s.t.} \quad (w, y) \text{ satisfies (1)}. \quad (3)$$

This problem, even on non-cylindrical domains, has already been considered by Brokate in [4].

In order to describe the corresponding first-order optimality system for (3), we use \mathcal{D}_{y_i} , \mathcal{D}_{u_i} , and \mathcal{D}_{v_i} to describe the corresponding derivatives of y_i , u , and v . Moreover, we use the Lagrangian

$$\mathcal{L}(w, y, p) = J(w, y) + \sum_{i=1}^d \int_0^T \int_0^1 (\partial_t y + A(t, x) \partial_x y - f(t, x, u, y))_i p_i \, dx \, dt. \quad (4)$$

Using integration by parts, the Lagrangian (4) reads as follows

$$\begin{aligned} \mathcal{L}(w, y, p) &= J(w, y) + \sum_{i=1}^d \int_0^1 y_i(T, x) p_i(T, x) dx - \sum_{i=1}^d \int_0^1 y_{i0}(x) p_i(0, x) dx \\ &\quad + \sum_{i=1}^m \int_0^T \left(|\lambda_i(t, 0)| y_i(t, 0) p_i(t, 0) - \sum_{j=m+1}^d |\lambda_j(t, 0)| g_{ji}^0(t, y_i(t, 0)) p_j(t, 0) \right) dt \\ &\quad + \sum_{i=m+1}^d \int_0^T \left(|\lambda_i(t, 1)| y_i(t, 1) p_i(t, 1) - \sum_{j=1}^m |\lambda_j(t, 1)| g_{ji}^1(t, v_i(t), y_i(t, 1)) p_j(t, 1) \right) dt \\ &\quad - \sum_{i=1}^d \int_0^T \int_0^1 (y_i (\partial_t p_i + \partial_x (\lambda_i(t, x) p_i)) + f_i(t, x, u, y) p_i) dx dt. \end{aligned}$$

We obtain the directional derivative of $\mathcal{L}(w, y, p)$ w.r.t. y in the direction of \hat{y} :

$$\begin{aligned} &\partial_y \mathcal{L}(w, y, p)(\hat{y}) \\ &= \partial_y J(w, y)(\hat{y}) + \sum_{i=1}^d \int_0^1 \hat{y}_i(T, x) p_i(T, x) dx \\ &\quad + \sum_{i=1}^m \int_0^T \left(|\lambda_i(t, 0)| \hat{y}_i(t, 0) p_i(t, 0) - \sum_{j=m+1}^d |\lambda_j(t, 0)| \partial_y g_{ji}^0(t, y_i(t, 0)) \hat{y}_i(t, 0) p_j(t, 0) \right) dt \\ &\quad + \sum_{i=m+1}^d \int_0^T \left(|\lambda_i(t, 1)| \hat{y}_i(t, 1) p_i(t, 1) - \sum_{j=1}^m |\lambda_j(t, 1)| \partial_y g_{ji}^1(t, v_i(t), y_i(t, 1)) \hat{y}_i(t, 1) p_j(t, 1) \right) dt \\ &\quad - \sum_{i=1}^d \int_0^T \int_0^1 \left(\hat{y}_i (\partial_t p_i + \partial_x (\lambda_i(t, x) p_i)) + \sum_{j=1}^d \partial_{y_i} f_j(t, x, u, y) \hat{y}_i p_j \right) dx dt. \end{aligned}$$

Going further, we now take variations and derive the following optimality conditions governing the adjoint variable p :

$$\begin{aligned} \partial_t p + A(t, x) \partial_x p &= \mathcal{D}_y L(t, x, u, y) - (\mathcal{D}_y f^\top(t, x, u, y) + \partial_x A(t, x)) p, \quad (t, x) \in (0, T) \times (0, 1), \\ |\lambda_i(t, 0)| p_i(t, 0) &= \sum_{j=m+1}^d |\lambda_j(t, 0)| \partial_y g_{ji}^0(t, y_i(t, 0)) p_j(t, 0), \quad i = 1, \dots, m, \quad t \in (0, T), \\ |\lambda_i(t, 1)| p_i(t, 1) &= \sum_{j=1}^m |\lambda_j(t, 1)| \partial_y g_{ji}^1(t, v_i(t), y_i(t, 1)) p_j(t, 1) \\ &\quad - \mathcal{D}_{y_i} L_b(t, v(t), y(t, 1)), \quad i = m+1, \dots, d, \quad t \in (0, T), \\ p(T, x) &= -\mathcal{D}_y L_T(x, y(T, x)), \quad x \in (0, 1). \end{aligned} \tag{5}$$

By taking the directional derivative of $\mathcal{L}(w, y, p)$ w.r.t. u in the direction \hat{u} we obtain

$$\partial_u \mathcal{L}(w, y, p)(\hat{u}) = \partial_u J(w, y)(\hat{u}) - \sum_{i=1}^d \int_0^T \int_0^1 \sum_{j=1}^d \partial_{u_i} f_j(t, x, u, y) \hat{u}_i p_j dx dt$$

and, thus, taking variations leads to

$$\sum_{i=1}^d \left(\partial_{u_i} L(t, x, u, y) - \sum_{j=1}^d \partial_{u_i} f_j(t, x, u, y) p_j \right) (u_i - \hat{u}_i) \geq 0 \tag{6}$$

a. e. in $(0, T) \times (0, 1)$. Similarly, for v we get

$$\sum_{i=m+1}^d \left(\partial_{v_i} L_b(t, v, y(t, 1)) - \sum_{j=1}^m |\lambda_j(t, 1)| \partial_v g_{ji}^1(t, v_i, y_i(t, 1)) p_j(t, 1) \right) (v_i - \hat{v}_i) \geq 0 \tag{7}$$

a. e. in $(0, T)$. We are now in the position to restate Theorem 3.4 of [4].

Theorem 1 (Brokate, 1985). *Under the assumptions above, the system consisting of (1) and (5)–(7) admits a unique solution (w, y, p) , where $p \in L^\infty((0, T) \times (0, 1))^d$ and (w, y) solves the optimal control problem (3).*

Example 1 (Tracking-type cost functional). *We are going to focus on semilinear problems as in (1) with quadratic cost functions*

$$\begin{aligned} J(w, y) := & \frac{\kappa}{2} \int_0^T \int_0^1 \|y - y_d\|_{\mathbb{R}^d}^2 dx dt + \frac{\rho}{2} \int_0^T \|y(t, 1) - y_1^d(t)\|_{\mathbb{R}^d}^2 dt \\ & + \frac{\nu}{2} \int_0^T \int_0^1 \|u\|_{\mathbb{R}^d}^2 dx dt + \frac{\mu}{2} \int_0^T \|v(t)\|_{\mathbb{R}^m}^2 dt. \end{aligned} \quad (8)$$

Hence,

$$\mathcal{D}_y L(t, x, u, y) = \kappa(y - y_d), \quad (9a)$$

$$\mathcal{D}_{y_i} L_b(t, v(t), y(t, 1)) = \rho(y_i(t, 1) - y_{i1}^d(t)), \quad (9b)$$

$$\mathcal{D}_y L_T(x, y(T, x)) = 0, \quad (9c)$$

$$\mathcal{D}_u L(t, x, u, y) = \nu u, \quad (9d)$$

$$\mathcal{D}_v L_b(t, v(t), y(t, 1)) = \mu v. \quad (9e)$$

Example 2 (A network of strings). *We consider a star-graph consisting of m strings or rods connected at a multiple node located at $x = 0$. The individual strings are stretched along an interval $[0, L]$. Each string is represented by a displacement $w_i(t, x)$ for $x \in [0, L]$ and $t \in [0, \infty)$. Indeed, we assume that there is a spatio-temporal axial loading $c_i(t, x)$. These strings or rods form a network located in the plane and $w_i(t, x)$ is either the out-of-the-plane displacement of the i th string or the longitudinal displacement of the i th rod. We assume that the strings (or rods) satisfy a semilinear damped wave equation such that at $x = 0$, the displacements are equal for all times and the sum of forces is 0. At the simple nodes, i.e., at $x = L$, the strings $i = 2, \dots, m$ are subject to dissipative controlled boundary conditions, while string $i = 1$ is clamped. The corresponding system can be written as*

$$\partial_{tt} w_i - \partial_x(c_i \partial_x w_i) + b_i(\partial_t w_i) = B_d u_i^d \quad \text{in } (0, T) \times (0, L), \quad i = 1, \dots, m, \quad (10a)$$

$$w_i(t, 0) = w_j(t, 0) \quad \text{in } (0, T), \quad i, j = 1, \dots, m, \quad (10b)$$

$$\sum_{i=1}^m c_i(t, 0) \partial_x w_i(t, 0) = 0 \quad \text{in } (0, T), \quad (10c)$$

$$w_1(t, L) = 0, \quad \text{in } (0, T), \quad (10d)$$

$$\partial_x w_i(t, L) + \beta_i(\partial_t w_i(t, L)) = v_i(t) \quad \text{in } (0, T), \quad i = 2, \dots, m, \quad (10e)$$

$$w_i(0, x) = w_{i0}(x) \quad \text{in } (0, L), \quad i = 1, \dots, m, \quad (10f)$$

$$\partial_t w_i(0, x) = w_{i1}(x) \quad \text{in } (0, L), \quad i = 1, \dots, m. \quad (10g)$$

We now transform (10) into the format (1). In a first step, we transform (10a) into a 2×2 -system. To this end, we set

$$z_{i1} := \frac{1}{2} (\partial_t w_i + \sqrt{c_i} \partial_x w_i), \quad z_{i2} := \frac{1}{2} (\partial_t w_i - \sqrt{c_i} \partial_x w_i).$$

Hence

$$\partial_t w_i = (z_{i1} + z_{i2}), \quad \partial_x w_i = \frac{1}{\sqrt{c_i}} (z_{i1} - z_{i2})$$

and, therefore

$$\partial_t \begin{pmatrix} z_{i1} \\ z_{i2} \end{pmatrix} + \begin{bmatrix} -\sqrt{c} & 0 \\ 0 & \sqrt{c} \end{bmatrix} \partial_x \begin{pmatrix} z_{i1} \\ z_{i2} \end{pmatrix} = \frac{1}{4c} \begin{bmatrix} \partial_t c + \sqrt{c} \partial_x c & -(\partial_t c + \sqrt{c} \partial_x c) \\ -(\partial_t c - \sqrt{c} \partial_x c) & \partial_t c - \sqrt{c} \partial_x c \end{bmatrix} \begin{pmatrix} z_{i1} \\ z_{i2} \end{pmatrix} - b_i \left(\frac{1}{2} (z_{i1} + z_{i2}) \right) \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

We define

$$y_i = z_{i1} \text{ for } i = 1, \dots, m, \quad y_i = z_{(i-m)2} \text{ for } i = m+1, \dots, d.$$

For the sake of simplicity, we assume that the tensions are equal at $x = 0$ for all times; i.e., $c_i(t, 0) = c_j(t, 0)$ holds for all $t \in [0, T]$. Then, the transmission conditions (10b) and (10c) can be equivalently written as

$$\begin{pmatrix} y_{m+1} \\ \vdots \\ y_d \end{pmatrix}(t, 0) = \frac{1}{m} \begin{bmatrix} m-2 & -2 & \cdots & -2 \\ -2 & m-2 & \cdots & -2 \\ \vdots & \vdots & \ddots & \vdots \\ -2 & -2 & \cdots & m-2 \end{bmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}(t, 0). \quad (11)$$

Notice that without the assumption on the axial loads at $x = 0$, the matrix on the right-hand side becomes non-symmetric which, in turn, is not a problem in principle. We introduce the matrix S such that

$$(S\varphi)_i := \left(\frac{2}{m} \sum_{j=1}^m \varphi_j - \varphi_i \right).$$

Thus, (11) reads as $Y^+(0) = SY^-(0)$. The matrix S has nice properties. It can be interpreted as a scattering matrix. In particular,

$$\sum_{i=1}^m (S\varphi)_i = \sum_{i=1}^m \varphi_i, \quad SS\varphi = \varphi$$

holds. At $x = L$, we have, at least formally for sufficiently regular states,

$$\partial_t w_1(t, L) = 0 \implies z_{11}(t, L) + z_{12}(t, L) = 0 \implies y_1(t, L) = -y_{m+1}(t, L)$$

for the clamped string and

$$\begin{aligned} \partial_x w_i(t, L) + \beta_i(\partial_t w_i(t, L)) &= v_i(t) \\ \implies z_{i1}(t, L) - z_{i2}(t, L) + \sqrt{c_i(t, L)} \beta_i(z_{i1}(t, L) + z_{i2}(t, L)) &= \sqrt{c_i(t, L)} v_i(t) \\ \implies (Id + \sqrt{c_i(t, L)} \beta_i(\cdot))(z_{i1}(t, L) + z_{i2}(t, L)) &= \sqrt{c_i(t, L)} v_i(t) + 2z_{i2}(t, L) \end{aligned}$$

for the other strings. Therefore, if we now assume that the functions β_i are monotone, then on the left-hand side of (10e), $Id + \sqrt{c_i(t, L)} \beta_i(\cdot)$ is invertible for each t with inverse $(Id + \sqrt{c_i(t, L)} \beta_i(\cdot))^{-1} =: h(t, \cdot)$ and the boundary condition (10e) can be rewritten as

$$\begin{aligned} z_{i1}(t, L) &= -z_{i2}(t, L) + h_i(t, 2z_{i2}(t, L) + v_i(t)), & i = 2, \dots, m, \\ y_i(t, L) &= -y_{m+i}(t, L) + h_i(t, 2y_{m+i}(t, L) + v_i(t)), & i = 2, \dots, m. \end{aligned}$$

This provides the boundary conditions at the end $x = L$:

$$\begin{aligned} y_1(t, L) &= -y_{m+1}(t, L), \\ y_i(t, L) &= -y_{m+i}(t, L) + h_i(t, 2y_{m+i}(t, L) + v_i(t)), & i = 2, \dots, m. \end{aligned}$$

Thus, our example (after normalization of the length) is of the format (1).

Remark 1. We remark that there are many more examples—in particular for systems of semilinear hyperbolic balance laws on metric graphs—that exactly fit into this framework. These are, e.g., networks of open channels with the dynamics governed by the shallow water equations with wall friction; see, e.g., [18], or networks of gas pipelines [16]. Moreover, networks of semilinear Timoshenko beams [17] can be written in the framework of (1) as well.

3. TIME-DOMAIN DECOMPOSITION

We now embark on time-domain decomposition of the optimal control problem by decomposing the optimality system (1), (5)–(7). The procedure pursued in this article is very much inspired by the work in [15] for the wave equation and the Maxwell equations in case of linear dynamics. The novelty of the current article lies in the fact that we deal with first-order semilinear equations of hyperbolic type with coefficients varying in space and time. The emphasis on the non-overlapping paradigm in time-domain decomposition and the particular focus on the method described in [15] is due to the fact that after the decomposition of the optimality system into subsystems on $[T_k, T_{k+1}] \times (0, 1)$ with

$$0 = T_0 < T_1 < \cdots < T_k < T_{k+1} < \cdots < T_K < T_{K+1} = T,$$

the subsystems themselves are optimality systems for suitably chosen optimal control problems on the subdomains. By this procedure, the original optimal control problem is iteratively decomposed into

optimal control problems on $[T_k, T_{k+1}] \times [0, 1]$, which, in turn, can be solved completely in parallel. Another distinguishing feature is the fact that the approach is formulated completely in the continuous space-time setting in which also a proof of convergence will be given. To the best of our knowledge, there is no other paper in the literature meeting these requirements.

We now formulate the time-domain decomposition procedure for the general optimality system (1), (5)–(7) and then focus on the case of unconstrained controls u, v and quadratic costs as in (8) for a proof of convergence in the next section. To this end, we denote the restrictions of y_k, p_k, u_k, v_k to $Q_k := I_k \times (0, 1)$ with $I_k := (T_k, T_{k+1})$ by

$$y_k := y|_{Q_k}, \quad p_k := p|_{Q_k}, \quad u_k := u|_{Q_k}, \quad v_k := v|_{I_k}.$$

The idea is to satisfy the continuity conditions

$$\begin{aligned} y_k(T_{k+1}) &= y_{k+1}(T_{k+1}), \quad k = 0, \dots, K-1, \\ p_k(T_{k+1}) &= p_{k+1}(T_{k+1}), \quad k = 0, \dots, K-1, \end{aligned} \quad (12)$$

in the limit of an iterative procedure. We therefore use the decoupling

$$\begin{aligned} y_k^{n+1}(T_{k+1}) + \beta p_k^{n+1}(T_{k+1}) &= \phi_{k,k+1}^n, \quad k = 0, \dots, K-1, \\ y_k^{n+1}(T_k) - \beta p_k^{n+1}(T_k) &= \phi_{k,k-1}^n, \quad k = 1, \dots, K, \end{aligned} \quad (13)$$

together with the update rule

$$\begin{aligned} \phi_{k,k+1}^n &= (1 - \varepsilon) (y_{k+1}^n(T_{k+1}) + \beta p_{k+1}^n(T_{k+1})) + \varepsilon (y_k^n(T_{k+1}) + \beta p_k^n(T_{k+1})), \quad k = 0, \dots, K-1, \\ \phi_{k,k-1}^n &= (1 - \varepsilon) (y_{k-1}^n(T_k) - \beta p_{k-1}^n(T_k)) + \varepsilon (y_k^n(T_k) - \beta p_k^n(T_k)), \quad k = 1, \dots, K, \end{aligned} \quad (14)$$

where $n = 0, 1, 2, \dots, \beta > 0, 0 \leq \varepsilon < 1$. As mentioned in the introduction, β is related to the corresponding parameter in the algorithm of P. L. Lions [22], while ε can be interpreted as a classic relaxation parameter; see [3].

Remark 2. *Suppose that the iteration (13), (14) converges for $n \rightarrow \infty$, where y_k, p_k, u_k, v_k solve (1), (5)–(7) on Q_k . Then, (13) holds without iteration indices n and $n+1$. As a result, the iteration updates (14) and the decoupling (13) reduce to*

$$\begin{aligned} (1 - \varepsilon) (y_k(T_{k+1}) + \beta p_k(T_{k+1})) &= (1 - \varepsilon) (y_{k+1}(T_{k+1}) + \beta p_{k+1}(T_{k+1})), \\ (1 - \varepsilon) (y_k(T_k) - \beta p_k(T_k)) &= (1 - \varepsilon) (y_{k-1}(T_k) - \beta p_{k-1}(T_k)), \end{aligned}$$

where we may divide by $(1 - \varepsilon)$ and shift the second equation by $k \rightarrow k+1$ to obtain

$$\begin{aligned} y_k(T_{k+1}) + \beta p_k(T_{k+1}) &= y_{k+1}(T_{k+1}) + \beta p_{k+1}(T_{k+1}), \\ y_{k+1}(T_{k+1}) - \beta p_{k+1}(T_{k+1}) &= y_k(T_{k+1}) - \beta p_k(T_{k+1}). \end{aligned}$$

Adding the last two equations leads to

$$y_k(T_{k+1}) = y_{k+1}(T_{k+1}), \quad p_k(T_{k+1}) = p_{k+1}(T_{k+1}).$$

Thus, (12) is satisfied and, in the limit, the continuity conditions hold. Therefore, the non-overlapping domain decomposition (13), (14) appears reasonable.

In view of this remark, we propose the time-domain decomposition

$$\begin{aligned}
\partial_t y_k^n + A(t, x) \partial_x y_k^n &= f_k(t, x, u_k^n, y_k^n), \quad (t, x) \in Q_k, \\
\partial_t p_k^n + A(t, x) \partial_x p_k^n &= \mathcal{D}_y L(t, x, u_k^n, y_k^n) - (\mathcal{D}_y f^\top(t, x, u_k^n, y_k^n) + \partial_x A(t, x)) p_k^n, \quad (t, x) \in Q_k, \\
y_{ki}^n(t, 0) &= \sum_{j=1}^m g_{ij}^0(t, y_{kj}^n(t, 0)), \quad i = m+1, \dots, d, \quad t \in I_k, \\
|\lambda_i(t, 0)| p_{ki}^n(t, 0) &= \sum_{j=m+1}^d |\lambda_j(t, 0)| \partial_y g_{ji}^0(t, y_{ki}^n(t, 0)) p_{kj}^n(t, 0), \quad i = 1, \dots, m, \quad t \in I_k, \\
y_{ki}^n(t, 1) &= \sum_{j=m+1}^d g_{ij}^1(t, v_{ki}^n(t), y_{kj}^n(t, 1)), \quad i = 1, \dots, m, \quad t \in I_k, \\
|\lambda_i(t, 1)| p_{ki}^n(t, 1) &= \sum_{j=1}^m |\lambda_j(t, 1)| \partial_y g_{ji}^1(t, v_{ki}^n(t), y_{ki}^n(t, 1)) p_{kj}^n(t, 1) \\
&\quad - \mathcal{D}_{y_{ki}^n} L_b(t, v_{ki}^n(t), y_{ki}^n(t, 1)), \quad i = m+1, \dots, d, \quad t \in I_k.
\end{aligned} \tag{15}$$

of System (5)–(7) together with (6), (7) and (13), (14), which have to be extended by

$$y_0^n(0, \cdot) = y_0, \quad p_K^n(T_{K+1}, \cdot) = 0.$$

Here, we use the sub-index notation y_{ki}^n to denote the i th element of y_k^n and denote the derivatives w.r.t. y on the distributed level by \mathcal{D}_y , while for the boundary terms, we use ∂_y .

For $k = 1, \dots, K-1$, we now introduce so-called virtual controls $g_{k,k-1}(x)$ for $x \in (0, 1)$ in the sense of [13, 15] and consider the following virtual control problem on Q_k :

$$\begin{aligned}
\min_{g_{k,k-1}, w_k, y_k} \quad & J_k(w_k, y_k) + \frac{1}{2\beta} \left(\|y_k(T_{k+1}) - \phi_{k,k+1}\|_{L^2(0,1)^d}^2 + \|g_{k,k-1}\|_{L^2(0,1)^d}^2 \right) \\
\text{s.t.} \quad & \partial_t y_k + A(t, x) \partial_x y_k = f_k(t, x, u_k, y_k), \quad (t, x) \in Q_k, \\
& y_{ki}(t, 0) = \sum_{j=1}^m g_{ij}^0(t, y_{kj}(t, 0)), \quad i = m+1, \dots, d, \quad t \in I_k, \\
& y_{ki}(t, 1) = \sum_{j=m+1}^d g_{ij}^1(t, v_{ki}(t), y_{kj}(t, 1)), \quad i = 1, \dots, m, \quad t \in I_k, \\
& y(T_k, x) = \phi_{k,k-1} + g_{k,k-1}, \quad x \in (0, 1), \\
& u(t) \in U_{\text{ad}}^d \quad \text{a. e. in } I_k, \\
& v(t) \in U_{\text{ad}}^b \quad \text{a. e. in } I_k,
\end{aligned} \tag{16}$$

where $J_k(w_k, y_k)$ is given by (2) restricted to Q_k , i.e.,

$$J_k(w_k, y_k) = \int_{T_k}^{T_{k+1}} \int_0^1 L(t, x, u_k, y_k) dx dt + \int_{T_k}^{T_{k+1}} L_b(t, v_k(t), y_k(t, 1)) dt. \tag{17}$$

Theorem 2. *Suppose that the controls $g_{k,k-1}$, u_k , v_k , and the state y_k are optimal for the virtual control problem (16). Then, there exists a unique adjoint state $p_k \in L^\infty(Q_k)^d$ such that y_k, p_k satisfy (6), (7) and (13), (14), (15). In particular, the optimal virtual control $g_{k,k-1}$ is given by*

$$g_{k,k-1} = \beta p_k(T_k).$$

Proof. The proof is essentially an adaption of the proof of Theorem 1 (cf. Theorem 3.4 in [4]), where the additional virtual control $g_{k,k-1}$ now appears in the initial data, while the extra term

$$\frac{1}{2\beta} \|y_k(T_{k+1}) - \phi_{k,k+1}\|_{L^2(0,1)^d}^2$$

is an adaption of L_T in the original cost function (2). The Lagrangian of Problem (16) is thus given by

$$\begin{aligned}
& \mathcal{L}_k(w_k, y_k, p_k) \\
&= J_k(w_k, y_k) + \frac{1}{2\beta} \left(\|y_k(T_{k+1}) - \phi_{k,k+1}\|_{L^2(0,1)^d}^2 + \|g_{k,k-1}\|_{L^2(0,1)^d}^2 \right) \\
&+ \sum_{i=1}^d \left(\int_0^1 y_{ki}(T_{k+1}) p_{ki}(t_{k+1}) \, dx - \int_0^1 (\phi_{k,k-1} + g_{k,k-1})_i p_{ki}(T_k) \, dx \right) \\
&+ \sum_{i=1}^m \int_{T_k}^{T_{k+1}} \left(|\lambda_i(t, 0)| y_{ki}(t, 0) p_{ki}(t, 0) - \sum_{j=m+1}^d |\lambda_j(t, 0)| g_{ji}^1(t, v_{ki}(t), y_{ki}(t, 0)) p_{kj}(t, 0) \right) dt \\
&+ \sum_{i=m+1}^d \int_{T_k}^{T_{k+1}} \left(|\lambda_i(t, 1)| y_{ki}(t, 1) p_{ki}(t, 1) - \sum_{j=1}^m |\lambda_j(t, 1)| g_{ji}^0(t, y_{ki}(t, 1)) p_{kj}(t, 1) \right) dt \\
&- \sum_{i=1}^d \int_{T_k}^{T_{k+1}} \int_0^1 (y_{ki} (\partial_t p_{ki} + \partial_x (\lambda_i(t, x) p_{ki})) + f_{ki}(t, x, u_k, y_k) p_{ki}) \, dx \, dt.
\end{aligned}$$

Taking the corresponding variations of \mathcal{L}_k w.r.t. y_k , u_k , v_k , and $g_{k,k-1}$, we arrive at the conclusion. \square

Remark 3. *The virtual control problems for $k = 1, \dots, K-1$ have to be complemented by a corresponding problem for $k = 0$ and $k = K$, respectively. Clearly, for $k = 0$, no additional virtual control is needed as*

$$y_0(T_0) = y_0(0) = y(0) = y_0$$

is given data, while at $k = K$,

$$p_K(T_{K+1}) = p_K(T) = p(T)$$

is prescribed and, therefore, no penalty term for the upper transmission condition is needed. We remark that the fact that the iteration procedure can be reformulated as an iteration involving parallel virtual optimal control problems on the sub-intervals is important for applications, where one wants to resort to solvers for this kind of standard tracking-type optimal control problems. We will dwell on this potential in a forthcoming publication in which the numerical realization is at the focus.

Remark 4. *As for the existence of optimal controls for problem (3), we reiterate the remark from the introduction that, in general, no such result seems to be published yet. However, for distributed controls where the spatial part is kept fixed, a rather general existence result is given in [27], where the method of characteristics is used. In case of time-independent coefficients, one can use semi-group theory to achieve well-posedness in a Sobolev-space setting for the semilinear problem such that the cost function is sequentially weakly lower semi-continuous w.r.t. sequences of pairs (u^m, y^m) . In the given case of time-varying coefficients, one may use the Kato framework of evolution operators; see, e.g., [12]. For the case of quasilinear equations, which we do not study here, we refer to [2]. We also note that the theory of semi-global classical solutions by Tatsien Li, in particular its extension to BV-solutions, might be used; see, e.g., [19, 20]. In this article, we do not embark on this issue, and refer instead to an upcoming publication.*

4. CONVERGENCE FOR THE CASE OF UNCONSTRAINED CONTROLS

In order to approach a convergence proof for the iterative time-domain decomposition (13)–(15), we assume, for the sake of simplicity, that the controls are unrestricted and appear linearly in (1). In the general nonlinear case, as in [4], one has to formulate appropriate conditions on the nonlinearities that guarantee coercivity w.r.t. controls. We also refrain from taking into account control-input operators that map the corresponding control spaces into the state space, but rather accept that where controls are present in the state equation or at the boundary, they have full access to state variables. Again, the general case can be handled as well but results in tedious details that are not of interest in the first place. The same remark applies to the terms in the cost function. We thus stick to the tracking-type case as in

Example 1. This means we consider

$$\partial_t y_i + \lambda_i(t, x) \partial_x y_i = f_i(t, x, y) + u_i, \quad i = 1, \dots, d, (t, x) \in Q, \quad (18a)$$

$$y_i(t, 0) = \sum_{j=1}^m g_{ij}^0(t) y_j(t, 0), \quad i = m+1, \dots, d, t \in (0, T), \quad (18b)$$

$$y_i(t, 1) = v_i(t) + \sum_{j=m+1}^d g_{ij}^1(t, y_j(t, 1)), \quad i = 1, \dots, m, t \in (0, T), \quad (18c)$$

$$y_i(0, x) = y_{0i}(x), \quad i = 1, \dots, d, x \in (0, 1), \quad (18d)$$

together with the cost function

$$\begin{aligned} J(w, y) := & \frac{\kappa}{2} \int_0^T \int_0^1 \|y - y_d\|_{\mathbb{R}^d}^2 dx dt + \frac{\rho}{2} \int_0^T \|y(t, 1) - y_d(t)\|_{\mathbb{R}^d}^2 dt \\ & + \frac{\nu}{2} \int_0^T \int_0^1 \|u\|_{\mathbb{R}^d}^2 dx dt + \frac{\mu}{2} \int_0^T \|v\|_{\mathbb{R}^m}^2 dt. \end{aligned} \quad (19)$$

The control problem considered from now on thus reads

$$\min_{w, y} J(w, y) \quad \text{s.t.} \quad (w, y) \text{ satisfies (18)}. \quad (20)$$

We go back to the decomposed optimality system (15), which reads

$$\partial_t y_k^n + A \partial_x y_k^n = f_k(t, y_k) + u_k^n, \quad (t, x) \in Q_k, \quad (21a)$$

$$\partial_t p_k^n + A \partial_x p_k^n = \kappa(y_k^n - y_d) - (\mathcal{D}_y f_k^\top + \partial_x A) p_k^n, \quad (t, x) \in Q_k, \quad (21b)$$

$$y_{ki}^n(t, 0) = \sum_{j=1}^d g_{ij}^0(t) y_{kj}^n(t, 0), \quad i = m+1, \dots, t \in T_k, \quad (21c)$$

$$y_{ki}^n(t, 1) = \sum_{j=m+1}^d g_{ij}^1(t, y_{kj}^n(t, 1)) + v_{ki}^n(t), \quad i = 1, \dots, m, t \in I_k, \quad (21d)$$

$$|\lambda_i(t, 0)| p_{ki}^n(t, 0) = \sum_{j=m+1}^d g_{ji}^0(t) |\lambda_j(t, 0)| p_{kj}^n(t, 0), \quad i = 1, \dots, m, t \in I_k, \quad (21e)$$

$$|\lambda_i(t, 1)| p_{ki}^n(t, 1) = \sum_{j=1}^m \partial_y (g_{ji}^1(t, y_{kj}^n(t, 1))) |\lambda_j(t, 1)| p_{kj}^n(t, 1) \quad (21f)$$

$$- \rho (y_{ki}^n(t, 1) - y_{ki}^d(t)), \quad i = m+1, \dots, t \in T_k, \quad (21g)$$

and

$$\begin{aligned} y_k^n(T_{k+1}) + \beta p_k^n(T_{k+1}) &= \phi_{k, k+1}^{n-1}, \quad k = 0, \dots, K-1, \\ y_k^n(T_k) - \beta p_k^n(T_k) &= \phi_{k, k-1}^{n-1}, \quad k = 1, \dots, K, \end{aligned} \quad (22)$$

together with (14) for $n-1$ as well as

$$\nu u_k^n = p_k^n, \quad \mu v_{ki}^n = |\lambda_i(t, 1)| p_{ki}^n(t, 1). \quad (23)$$

We introduce the errors

$$\tilde{y}_k^n := y_k^n - y_k, \quad \tilde{p}_k^n := p_k^n - p_k, \quad \tilde{u}_k^n := u_k^n - u_k, \quad \tilde{v}_k^n := v_k^n - v_k.$$

These errors solve the semilinear problems for each $k = 1, \dots, K - 1$, i.e.,

$$\partial_t \tilde{y}_k^n + A \partial_x \tilde{y}_k^n = f_k(y_k^n) - f_k(y_k) + \tilde{u}_k^n, \quad (t, x) \in Q_k, \quad (24a)$$

$$\partial_t \tilde{p}_k^n + \partial_x (A \tilde{p}_k^n) = \kappa \tilde{y}_k^n - (\mathcal{D}_y f(y_k^n))^\top \tilde{p}_k^n - (\mathcal{D}_y f_k(y_k^n))^\top - \mathcal{D}_y f_k(y_k)^\top p_k, \quad (t, x) \in Q_k, \quad (24b)$$

$$\tilde{y}_{ki}^n(t, 0) = \sum_{j=1}^m g_{ij}^0(t) \tilde{y}_{kj}^n(t, 0), \quad i = m+1, \dots, d, \quad t \in I_k, \quad (24c)$$

$$\tilde{y}_{ki}^n(t, 1) = \sum_{j=m+1}^d (g_{ij}^1(t, y_{kj}^n(t, 1)) - g_{ij}^1(t, y_{kj}(t, 1))) + \tilde{v}_{ki}^n(t), \quad i = 1, \dots, m, \quad t \in I_k, \quad (24d)$$

$$|\lambda_i(t, 0)| \tilde{p}_{ki}^n(t, 0) = \sum_{j=m+1}^d g_{ji}^0(t) |\lambda_j(t, 0)| \tilde{p}_{kj}^n(t, 0), \quad i = 1, \dots, m, \quad t \in I_k, \quad (24e)$$

$$|\lambda_i(t, 1)| \tilde{p}_{ki}^n(t, 1) = \sum_{j=1}^m |\lambda_j(t, 1)| \left(\partial_y g_{ji}^1(t, y_{ki}^n(t, 1)) p_{kj}^n(t, 1) - \partial_y g_{ji}^1(t, y_{ki}(t, 1)) p_{kj}(t, 1) \right) \quad (24f)$$

$$- \rho \tilde{y}_{ki}^n(t, 1), \quad i = m+1, \dots, d, \quad t \in I_k, \quad (24g)$$

and

$$\begin{aligned} \tilde{y}_k^n(T_{k+1}) + \beta \tilde{p}_k^n(T_{k+1}) &= \tilde{\phi}_{k,k+1}^{n-1}, \quad k = 0, \dots, K-1, \\ \tilde{y}_k^n(T_k) - \beta \tilde{p}_k^n(T_k) &= \tilde{\phi}_{k,k-1}^{n-1}, \quad k = 1, \dots, K, \end{aligned} \quad (25)$$

together with (14) for $n-1$ as well as

$$v \tilde{u}_k^n = \tilde{p}_k^n, \quad \mu \tilde{v}_{ki}^n = |\lambda_i(t, 1)| \tilde{p}_{ki}^n(t, 1). \quad (26)$$

In order to turn (24) into a semilinear problem just in $\tilde{y}_k^n, \tilde{p}_k^n$, we may rewrite

$$f_k(t, y_k^n) = f_k(t, \tilde{y}_k^n + y_k), \quad \mathcal{D}_y f_k(y_k^n) = \mathcal{D}_y f_k(\tilde{y}_k^n + y_k).$$

Notice that the decomposed optimality system (24), (25) for the errors is again the optimality system w.r.t. a corresponding virtual control problem in analogy to (16). This implies that the iteration is well-defined, according to Theorem 1 and 2. The following arguments follow the spirit of [15]. We introduce

$$\begin{aligned} X &:= \left((\phi_{k,k+1})_{k=0}^{K-1}, (\phi_{k,k-1})_{k=1}^K \right) \in \mathcal{X} = L^2(0, 1)^{2Kd}, \\ \|X\|_{\mathcal{X}}^2 &= \sum_{k=0}^{K-1} \|\phi_{k,k+1}\|_{L^2(0,1)^d}^2 + \sum_{k=1}^K \|\phi_{k,k-1}\|_{L^2(0,1)^d}^2. \end{aligned} \quad (27)$$

For given iteration histories $\phi_{k,k+1}$ and $\phi_{k,k-1}$, i.e., for given X , we consider the unique solution y_k^n, p_k^n of (24), (25) and then define $T : \mathcal{X} \rightarrow \mathcal{X}$ by

$$TX := \left((\tilde{y}_{k+1}^n(T_{k+1}) + \beta \tilde{p}_{k+1}^n(T_{k+1}))_{k=0}^{K-1}, (\tilde{y}_{k-1}^n(T_k) - \beta \tilde{p}_{k-1}^n(T_k))_{k=1}^K \right). \quad (28)$$

As the evaluation of the states and adjoint states involves a nonlinear process, the mapping T is nonlinear, even though the transmission conditions are linear. Thus, formally, in order to estimate the errors, we need to write $T(X^n) - T(X)$, but when we understand that in the arguments below we always consider the solutions of the semilinear problem, no ambiguities appear if we write $T(X^n - X)$ instead. Keeping this in mind, we replace in the sequel $X^n - X$ by \tilde{X}^n and in fact just by X^n . The key for our convergence proof is the fixed-point iteration

$$X^{n+1} = (1 - \varepsilon)TX^n + \varepsilon X^n, \quad \varepsilon \in [0, 1). \quad (29)$$

Thus, we have

$$\begin{aligned} X^n &= \left((\tilde{y}_k^n(T_{k+1}) + \beta \tilde{p}_k^n(T_{k+1}))_{k=0}^{K-1}, (\tilde{y}_k^n(T_k) - \beta \tilde{p}_k^n(T_k))_{k=1}^K \right), \\ TX^n &= \left((\tilde{y}_{k+1}^n(T_{k+1}) + \beta \tilde{p}_{k+1}^n(T_{k+1}))_{k=0}^{K-1}, (\tilde{y}_{k-1}^n(T_k) - \beta \tilde{p}_{k-1}^n(T_k))_{k=1}^K \right) \end{aligned} \quad (30)$$

at iteration n . Moreover, we define the energies

$$\mathcal{E}_k^n(t) := \|\tilde{y}_k^n(t)\|_{L^2(0,1)^d}^2 + \beta^2 \|\tilde{p}_k^n\|_{L^2(0,1)^d}^2, \quad \mathcal{E}^n := \sum_{k=0}^{K-1} \mathcal{E}_k^n(T_{k+1}) + \mathcal{E}_{k+1}^n(T_{k+1}). \quad (31)$$

Now, abbreviating the norm indices by L^2 in place of $L^2(0,1)^d$, we have that

$$\begin{aligned} \|X^n\|_{\mathcal{X}}^2 &= \sum_{k=0}^{K-1} \|\tilde{y}_k^n(T_{k+1}) + \beta \tilde{p}_k^n(T_{k+1})\|_{L^2}^2 + \sum_{k=1}^K \|\tilde{y}_k^n(T_k) - \beta \tilde{p}_k^n(T_k)\|_{L^2}^2 \\ &= \sum_{k=0}^{K-1} \left(\|\tilde{y}_k^n(T_{k+1})\|_{L^2}^2 + 2\beta \tilde{y}_k^n(T_{k+1}) \tilde{p}_k^n(T_{k+1}) + \beta^2 \|\tilde{p}_k^n(T_{k+1})\|_{L^2}^2 \right) \\ &\quad + \sum_{k=1}^K \left(\|\tilde{y}_k^n(T_k)\|_{L^2}^2 - 2\beta \tilde{y}_k^n(T_k) \tilde{p}_k^n(T_k) + \beta^2 \|\tilde{p}_k^n(T_k)\|_{L^2}^2 \right) \\ &= \mathcal{E}^n + 2\beta \left(\sum_{k=0}^{K-1} \tilde{y}_k^n(T_{k+1}) \tilde{p}_k^n(T_{k+1}) - \sum_{k=1}^K \tilde{y}_k^n(T_k) \tilde{p}_k^n(T_k) \right) \end{aligned} \quad (32)$$

holds. We now multiply (24a) by \tilde{p}_k^n and integration by parts then leads to

$$\begin{aligned} 0 &= \int_{T_k}^{T_{k+1}} \int_0^1 (\partial_t \tilde{y}_k^n + A \partial_x \tilde{y}_k^n - (f_k(t, y_k^n) - f_k(t, y_k)) - \tilde{u}_k^n) \tilde{p}_k^n \, dx \, dt \\ &= \int_0^1 (\tilde{y}_k^n(T_{k+1}) \tilde{p}_k^n(T_{k+1}) - \tilde{y}_k^n(T_k) \tilde{p}_k^n(T_k)) \, dx \\ &\quad + \int_{T_k}^{T_{k+1}} [A \tilde{y}_k^n \tilde{p}_k^n]_{x=0}^1 \, dt - \int_{T_k}^{T_{k+1}} \int_0^1 \tilde{u}_k^n \tilde{p}_k^n \, dx \, dt \\ &\quad - \int_{T_k}^{T_{k+1}} \int_0^1 \tilde{y}_k^n (\partial_t \tilde{p}_k^n + \partial_x A \tilde{p}_k^n) \, dx \, dt - \int_{T_k}^{T_{k+1}} \int_0^1 (f_k(t, y_k^n) - f_k(t, y_k)) \tilde{p}_k^n \, dx \, dt \\ &= \int_0^1 (\tilde{y}_k^n(T_{k+1}) \tilde{p}_k^n(T_{k+1}) - \tilde{y}_k^n(T_k) \tilde{p}_k^n(T_k)) \, dx \\ &\quad - \sum_{i=1}^m \int_{T_k}^{T_{k+1}} |\lambda_i(t, 1)|^2 \frac{1}{\mu} \tilde{p}_{ki}^n(t, 1)^2 \, dt - \sum_{i=m+1}^d \int_{T_k}^{T_{k+1}} \rho \tilde{y}_{ki}^n(t, 1)^2 \, dt \\ &\quad - \frac{1}{v} \int_{T_k}^{T_{k+1}} \int_0^1 \tilde{p}_k^n \tilde{p}_k^n \, dx \, dt - \kappa \int_{T_k}^{T_{k+1}} \int_0^1 \tilde{y}_k^n \tilde{y}_k^n \, dx \, dt \\ &\quad + \int_{T_k}^{T_{k+1}} \int_0^1 \tilde{y}_k^n ((\mathcal{D}_y f_k(t, y_k^n)^\top + \partial_x A) \tilde{p}_k^n + (\mathcal{D}_y f_k(t, y_k^n)^\top - \mathcal{D}_y f_k(t, y_k)^\top) p_k \\ &\quad - (f_k(t, y_k^n) - f_k(t, y_k)) \tilde{p}_k^n) \, dx \, dt \\ &\quad + \sum_{i=1}^m \sum_{j=m+1}^d \int_{T_k}^{T_{k+1}} |\lambda_i(t, 1)| \left(\tilde{y}_{kj}^n(t, 1) (\partial_y g_{ij}^1(y_{kj}^n(t, 1)) p_{ki}^n(t, 1) - \partial_y g_{ij}^1(y_{kj}(t, 1)) p_{ki}(t, 1)) \right. \\ &\quad \left. - \tilde{p}_{ki}^n(t, 1) (g_{ij}^1(y_{kj}^n(t, 1)) - g_{ij}^1(y_{kj}(t, 1))) \right) \, dt. \end{aligned} \quad (33)$$

Next, we use (33), $\tilde{y}_0^n(T_0) = 0$, and $p_K^n(T_{K+1}) = \tilde{p}_K^n(T) = 0$ to obtain

$$\begin{aligned}
\|X^n\|^2 &= \mathcal{E}^n + 2\beta \sum_{k=0}^K \left(\int_{T_k}^{T_{k+1}} \int_0^1 \kappa \|\tilde{y}_k^n\|_{\mathbb{R}^d}^2 + \frac{1}{\nu} \|\tilde{p}_k^n\|_{\mathbb{R}^d}^2 \, dx \, dt \right. \\
&\quad + \frac{1}{\mu} \int_{T_k}^{T_{k+1}} \sum_{i=1}^m |\lambda_i(t, 1)|^2 \tilde{p}_{ki}^n(t, 1)^2 \, dt + \rho \sum_{i=m+1}^d \int_{T_k}^{T_{k+1}} |\tilde{y}_{ki}^n(t, 1)| \, dt \\
&\quad + \int_{T_k}^{T_{k+1}} \int_0^1 (f_k(y_k^n) - f_k(y_k) - \mathcal{D}_y f(y_k^n) \tilde{y}_k^n) \tilde{p}_k^n \\
&\quad \quad - (\mathcal{D}_y f_k(y_k^n) - \mathcal{D}_y f_k(y_k)) \tilde{y}_k^n p_k \, dx \, dt - \int_{T_k}^{T_{k+1}} \int_0^1 \tilde{y}_k^n \partial_x A \, dx \tilde{p}_k^n \, dt \\
&\quad + \sum_{i=1}^m \sum_{j=m+1}^d \int_{T_k}^{T_{k+1}} |\lambda_i(t, 1)| \left(\tilde{y}_{kj}^n(t, 1) (\partial_y g_{ij}^1(y_{kj}^n(t, 1)) - \partial_y g_{ij}^1(y_{kj}(t, 1))) p_{ki}^n(t, 1) \right) \, dt \\
&\quad + \sum_{i=1}^m \sum_{j=m+1}^d \int_{T_k}^{T_{k+1}} |\lambda_i(t, 1)| \tilde{y}_{kj}^n(t, 1) (\partial_y g_{ij}^1(y_{kj}(t, 1)) \tilde{p}_{ki}^n(t, 1) \, dt \\
&\quad - \sum_{i=1}^m \sum_{j=m+1}^d \int_{T_k}^{T_{k+1}} |\lambda_i(t, 1)| \tilde{p}_{ki}^n(t, 1) (g_{ij}^1(y_{kj}^n(t, 1)) - g_{ij}^1(y_{kj}(t, 1))) \, dt \Big) \\
&=: \mathcal{E}^n + \mathcal{F}^n.
\end{aligned} \tag{34}$$

Similarly, we have

$$\|TX^n\|_X^2 = \mathcal{E}^n - \mathcal{F}^n. \tag{35}$$

Proposition 1 (Lemma 2 in [15]). *For any $\varepsilon \in [0, 1)$ and $n \in \mathbb{N}$ we have*

- (i) $\mathcal{E}^{n+1} + \mathcal{F}^{n+1} \leq \mathcal{E}^n - (1 - 2\varepsilon)\mathcal{F}^n$,
- (ii) $\mathcal{E}^{n+1} + \sum_{l=1}^{n+1} c_l(\varepsilon)\mathcal{F}^l \leq \mathcal{E}^1$ with $c_1(\varepsilon) = 1 - 2\varepsilon$, $c_{n+1}(\varepsilon) = 1$, and $c_l(\varepsilon) = 2(1 - \varepsilon)$ for $l = 2, \dots, n$.

Proof. As the proof is on the level of relations between \mathcal{E}^{n+1} , \mathcal{F}^{n+1} , \mathcal{E}^n , \mathcal{F}^n only, we refer to [15]. \square

In order to utilize the crucial property (ii) of Proposition 1, we need to establish a lower bound on \mathcal{F}^n , where \mathcal{F}^n is defined in (34). To this end, we need to compensate the nonlinear terms. In order to simplify the presentation below, we simply assume uniform norm bounds on the distributed and the boundary nonlinearity, respectively.

Assumption 1. *There exist bounds $L_f, L_g \geq 0$ such that for $t \in [T_k, T_{k+1}]$, $k = 0, \dots, K - 1$, we have*

- (i) $\|f_k(t, \cdot)\|_{L^2}^d, \|\mathcal{D}_y f_k(t, \cdot)\|_{L^2}^d \leq L_f, \|g^1(t, \cdot)\|, \|\partial_y g^1(t, \cdot)\| \leq L_g$,
- (ii) *and it holds*

$$\begin{aligned}
\|(\mathcal{D}_y f_k(t, y_k^n) - \mathcal{D}_y f_k(t, y_k)) \tilde{y}_k^n\| &\leq L_f \|\tilde{y}_k^n(t)\|_{L^2(0,1)^d}, \\
\|f_k(t, y_k^n) - f_k(t, y_k) - \mathcal{D}_y f_k(t, y_k^n) \tilde{y}_k^n\| &\leq L_f \|\tilde{y}_k^n(t)\|_{L^2(0,1)^d}, \\
|(\partial_y g_{ij}^1(y_{kj}^n(t, 1)) - \partial_y g_{ij}^1(y_{kj}(t, 1)))| &\leq L_g |\tilde{y}_{kj}^n(t, 1)|, \\
|(g_{ij}^1(y_{kj}^n(t, 1)) - g_{ij}^1(y_{kj}(t, 1)))| &\leq L_g |\tilde{y}_{kj}^n(t, 1)|.
\end{aligned}$$

Remark 5. *If we do not assume the boundedness of the derivatives of the nonlinear terms, we may use the ‘‘Stampacchia-trick’’ in the sense that we first extend the corresponding derivatives outside a given ball by constants and then show that for small enough data the solutions stay small and, hence, the extensions are not active. This procedure, however, would substantially extend the arguments and the length of this article as error estimates for the state and its traces would be in order. For the general concept however, see [5]. As this is very closely related to the issue of existence of optimal controls, we defer the analysis to a forthcoming publication. As of now, we therefore leave it to the reader to assess the validity of the remark.*

We rewrite the the last integrals and use Assumption 1 to obtain

$$\begin{aligned}
& \sum_{i=1}^m \sum_{j=m+1}^d \int_{T_k}^{T_{k+1}} |\lambda_i(t, 1)| \tilde{y}_{kj}^n(t, 1) \left(\partial_y g_{ij}^1(y_{kj}^n(t, 1)) \tilde{p}_{ki}^n - \partial_y g_{ij}^1(y_{kj}(t, 1)) p_{ki}(t, 1) \right) dt \\
& \quad - \sum_{i=1}^m \sum_{j=m+1}^d \int_{T_k}^{T_{k+1}} |\lambda_i(t, 1)| \tilde{p}_{ki}^n(t, 1) \left(g_{ij}^1(y_{kj}^n(t, 1)) - g_{ij}^1(y_{kj}(t, 1)) \right) dt \\
& = \sum_{i=1}^m \sum_{j=m+1}^d \int_{T_k}^{T_{k+1}} |\lambda_i(t, 1)| \left(\tilde{y}_{kj}^n(t, 1) \left(\partial_y g_{ij}^1(y_{kj}^n(t, 1)) - \partial_y g_{ij}^1(y_{kj}(t, 1)) \right) p_{ki}(t, 1) \right) dt \\
& \quad + \sum_{i=1}^m \sum_{j=m+1}^d \int_{T_k}^{T_{k+1}} |\lambda_i(t, 1)| \tilde{y}_{kj}^n(t, 1) (\partial_y g_{ij}^1(y_{kj}(t, 1)) \tilde{p}_{ki}^n(t, 1)) dt \\
& \quad - \sum_{i=1}^m \sum_{j=m+1}^d \int_{T_k}^{T_{k+1}} |\lambda_i(t, 1)| \tilde{p}_{ki}^n(t, 1) \left(g_{ij}^1(y_{kj}^n(t, 1)) - g_{ij}^1(y_{kj}(t, 1)) \right) dt \\
& \leq L_g(d-m) \int_{T_k}^{T_{k+1}} \sum_{i=1}^m |\lambda_i(t, 1)| |\tilde{p}_{ki}^n(t, 1)|^2 dt + L_g(\mu M + m \|\lambda\|_{L^\infty(0, T)^d}) \int_{T_k}^{T_{k+1}} \sum_{j=m+1}^d |\tilde{y}_{kj}^n(t, 1)|^2 dt \\
& \leq L_g \gamma_g \int_{T_k}^{T_{k+1}} \sum_{i=1}^m |\lambda_i(t, 1)| |\tilde{p}_{ki}^n(t, 1)|^2 dt + L_g \alpha_g \int_{T_k}^{T_{k+1}} \sum_{j=m+1}^d |\tilde{y}_{kj}^n(t, 1)|^2 dt,
\end{aligned} \tag{36}$$

where, $\gamma_g = d - m$, $M > 0$, is such that the original boundary controls $v_i(t)$ with $\mu v_i(t) = |\lambda_i(t, 1)| p_i(t, 1)$ satisfy

$$\sum_{i=1}^m |v_i(t)| \leq M \quad \text{and} \quad \alpha_g = \mu M + \|\lambda\|_{L^\infty(0, T)^d}.$$

Notice that due to the L^∞ -existence result in [4] w.r.t. time and space, such a number exists. We also need to estimate the contribution of the distributed nonlinearities (34), which corresponds to the Lipschitz constants L_f . We have

$$\begin{aligned}
& \sum_{k=0}^K \int_{T_k}^{T_{k+1}} \int_0^1 \left((f_k(t, y_k^n) - f_k(t, y_k) - \mathcal{D}_y f_k(t, y_k^n) \tilde{y}_k^n) \tilde{p}_k^n - (\mathcal{D}_y f_k(t, y_k^n) - \mathcal{D}_y f_k(t, y_k)) \tilde{y}_k^n p_k \right) dx dt \\
& \leq \sum_{k=0}^K \int_{T_k}^{T_{k+1}} \int_0^1 \left(L_f \alpha_f \|\tilde{y}_k^n\|_{\mathbb{R}^d}^2 + L_f \gamma_f \|\tilde{p}_k^n\|_{\mathbb{R}^d}^2 \right) dx dt,
\end{aligned} \tag{37}$$

with $\alpha_f = \frac{1}{2} + \|p\|_\infty$ and $\gamma_f = \frac{3}{2}$. As in the case of nonlinear boundary conditions, with $vu = p$, and the L^∞ -existence result of [4], $\|p\|_\infty \leq M$ holds. We also recall the estimate (36) for the boundary contributions, which used together with (37) in (34) leads to

$$\begin{aligned}
\mathcal{F}^l & \geq 2\beta \sum_{k=0}^K \left(\int_{T_k}^{T_{k+1}} \int_0^1 \left((\kappa - L_f \alpha_f) \|\tilde{y}_k^l\|^2 + \left(\frac{1}{\nu} - L_f \gamma_f \right) \|\tilde{p}_k^l\|^2 \right) dx dt \right. \\
& \quad \left. + \left(\frac{1}{\mu} - L_g \gamma_g \right) \int_{T_k}^{T_{k+1}} \sum_{i=1}^m |\lambda_i(t, 1)| |\tilde{p}_{ki}^l(t, 1)|^2 dt + (\rho - L_g \alpha_g) \sum_{i=m+1}^d \int_{T_k}^{T_{k+1}} |\tilde{y}_{ki}^2(t, 1)| dt \right).
\end{aligned}$$

Assume now that the parameters κ are chosen sufficiently large and μ, ν sufficiently small such that $\delta_1 := \kappa - L_f \alpha_f > 0$, $\delta_2 := \frac{1}{\nu} - L_f \gamma_f > 0$, $\delta_3 := \frac{1}{\mu} - L_g \gamma_g > 0$, and $\delta_4 := \rho - L_g \alpha_g > 0$ holds, and/or L_f, L_g are sufficiently small. Then, Proposition 1 (ii) yields

$$\begin{aligned}
\mathcal{E}^{n+1} & + \sum_{l=1}^{n+1} c_l(\varepsilon) \sum_{k=0}^K \left(\int_{T_k}^{T_{k+1}} \int_0^1 \delta_1 \|\tilde{y}_k^l\|_{\mathbb{R}^d}^2 + \delta_2 \|\tilde{p}_k^l\|_{\mathbb{R}^d}^2 dx dt \right. \\
& \quad \left. + \delta_3 \int_{T_k}^{T_{k+1}} \sum_{i=1}^m |\lambda_i(t, 1)| |\tilde{p}_{ki}^l(t, 1)|^2 dt + \int_{T_k}^{T_{k+1}} \sum_{i=m+1}^d \delta_4 |\tilde{y}_{ki}^l(t, 1)|^2 dt \right) \leq \mathcal{E}^1, \quad n = 1, 2, \dots,
\end{aligned} \tag{38}$$

and, in turn, (38) provides

$$\begin{aligned} \mathcal{E}^n \text{ is bounded,} \\ \tilde{y}_k^l \rightarrow 0, \tilde{p}_k^l \rightarrow 0 \text{ in } L^2(I_k; L^2(0, 1)), \\ \tilde{p}_{ki}^l(t, 1) \rightarrow 0, i = 1, \dots, m, \tilde{y}_{ki}^l(t, 1) \rightarrow 0, i = m + 1, \dots, d \text{ in } L^2(I_k), \end{aligned} \quad (39)$$

as $l \rightarrow \infty$ and for $\varepsilon \in [0, 1)$. We see from (39) that $\tilde{u}_k^n \rightarrow 0$ and $\tilde{v}_k^n \rightarrow 0$ in $L^2(0, T; L^2(0, 1))^d$ or $L^2(0, T)^m$, respectively. Due to the continuity of the nonlinear functions f_k and g^1 , we obtain vanishing right-hand sides in the state and adjoint equations and homogeneous boundary conditions in the limit on the entire sequence. Due to the uniqueness of the solution of the optimality system, the initial and final data $\tilde{y}_k^n(T_k)$, $\tilde{y}_k^n(T_{k+1})$, $\tilde{p}_k^n(T_k)$, and $\tilde{p}_k^n(T_{k+1})$ converge to zero in $L^2(0, 1)^d$. As the functions $y_k(\cdot)$ and $p_k(\cdot)$ satisfy Conditions (12), in the limit, the transmission conditions hold. This is true even for $\varepsilon = 0$.

Theorem 3. *The iteration (21)–(23) with $\varepsilon \in [0, 1)$ converges to (1), (5), (26) in the sense that the solutions (y_k^n, p_k^n) of (21)–(23) strongly converge in $L^2(0, T; L^2(0, 1))$ to (y_k, p_k) , which is the solution of (16), (17) for $k = 0, \dots, K$.*

Remark 6. *For $\varepsilon \in (0, 1)$, we can derive the convergence of the initial and final data for y_k^n, p_k^n at T_k, T_{k+1} directly. To this end, we consider the identity*

$$\begin{aligned} TX^n X^n &= \sum_{k=0}^{K-1} (\tilde{y}_k^n(T_{k+1}) + \beta \tilde{p}_k^n(T_{k+1})) (\tilde{y}_{k+1}^n(T_{k+1}) + \beta \tilde{p}_{k+1}^n(T_{k+1})) \\ &\quad + \sum_{k=1}^K (\tilde{y}_k^n(T_k) - \beta \tilde{p}_k^n(T_k)) (\tilde{y}_{k-1}^n(T_k) - \beta \tilde{p}_{k-1}^n(T_k)) \\ &= 2 \sum_{k=0}^{K-1} (\tilde{y}_k^n(T_{k+1}) \tilde{y}_{k+1}^n(T_{k+1}) + \beta^2 \tilde{p}_k^n(T_{k+1}) \tilde{p}_{k+1}^n(T_{k+1})). \end{aligned}$$

Thus,

$$\begin{aligned} \frac{1}{2} \|TX^n - X^n\|_X^2 &= \mathcal{E}^n - TX^n X^n \\ &= \sum_{k=0}^{K-1} \left(\|\tilde{y}_k^n(T_{k+1}) - \tilde{y}_{k+1}^n(T_{k+1})\|_{L^2}^2 + \beta^2 \|\tilde{p}_k^n(T_{k+1}) - \tilde{p}_{k+1}^n(T_{k+1})\|_{L^2}^2 \right). \end{aligned} \quad (40)$$

On the other hand, with $T_\varepsilon := \varepsilon I + (1 - \varepsilon)T$, we obtain that

$$\|T_\varepsilon^n X^1 - T_\varepsilon^{n-1} X^1\|_X = (1 - \varepsilon) \|TX^n - X^n\|_X \rightarrow 0$$

holds according to Schaefer's fixed-point theorem [26] for $\varepsilon \in (0, 1)$. This directly shows the desired convergence.

Remark 7. *It is apparent from the proof of Theorem 3 that the convergence of the iterative process (i.e., the fixed-point iteration) and its rate depend on a proper choice of the parameters κ, μ, ν, ρ determining the cost function and β, ε which are defined in the iteration procedure, as well as on the bounds in Assumption 1. In order to get a general idea of the role and the choice of β and the relaxation parameter ε , we refer to the analysis in [9] for β and to, e.g., [3] for ε . As can be seen immediately from the proof of the theorem, $\beta > 0$ is essential, as it is the factor of the dissipative part \mathcal{F} . Nevertheless, for $\beta = 0$, the iteration is on the state, only, and is similar to a non-overlapping Schwarz algorithm for the state variables, while the adjoint variables are not affected and, therefore, one cannot expect convergence at the interface. The question of how to choose the set of parameters optimally is open on the general level and has to be answered via specific numerical examples. A rigorous treatment is, therefore, not possible in the current article. However, an analysis as in [6], which was conducted for a semi-discretization of a parabolic problem, may be applied for a very particular scenario, namely, for a simple optimal control problem for a linear harmonic oscillator. Again, the analysis is beyond the scope of this article. In Section 7, we provide some insights into the choice of the parameters, where we focus on the wave equation as in our Example 2 (for a single link, i.e., $d = 2$) with costs as in Example 1.*

Remark 8. We see from the proof of Theorem 3 that with the given nonlinearity f_k , we need the distributed control u_k to compensate the appearance of the nonlinearity in the estimates. We also need the tracking term with $\kappa > 0$ being distributed over space and time. On the other hand, in this setting, we obtain stronger convergence results than in [13, 15].

Remark 9. Distributed control with full access to the state are typically hard to implement in practice. For boundary controls, on the other hand, full access is a not critical issue. As our convergence proofs reveals, full access to the state is however essential to compensate the distributed nonlinear term. In this respect, we add that one may replace the distributed control by yet another virtual control, however, at the expense of introducing an approximation to the adjoint variable appearing in the optimality system on the decomposed level. We do not have the space to elaborate on that variant here in detail but refer to a further publication.

5. CONVERGENCE IN THE PRESENCE OF CONTROL CONSTRAINTS AND LINEAR DYNAMICS

In this section, we consider pointwise constraints on u and v , i.e.,

$$u(t) \in U_{\text{ad}}^{\text{d}}, \quad v(t) \in U_{\text{ad}}^{\text{b}} \quad \text{a. e. in } (0, T). \quad (41)$$

However, we do not take into account nonlinearities. It turns out that the interaction of the control bounds with the bounds on the nonlinearities is rather complicated and not fully explored up to now. Nevertheless, we provide the convergence proof as also this extension is new in the context of optimal control for linear hyperbolic systems. We, thus, consider the tracking-type optimal control problem (20) together with the constraints (41). We notice that this has not been considered even in the context of [13, 15] and, thus, extends the literature in this direction. With the cost function given by (19), the optimality conditions (6), (7) are given by

$$\sum_{i=1}^d (vu_i - p_i(t, x))(\hat{u}_i - u_i(t, x)) \geq 0 \quad \text{a. e. in } (0, T) \times (0, 1), \quad (42a)$$

$$\sum_{i=1}^m (\mu v_i - |\lambda_i| p_i(t, 1))(\hat{v}_i - v_i(t)) \geq 0 \quad \text{a. e. in } (0, T) \quad (42b)$$

for all $\hat{u}_i \in U_{\text{ad}}^{\text{d}}, \hat{v}_i \in U_{\text{ad}}^{\text{b}}$, or

$$(vu - p)(t, x)(\hat{u} - u)(t, x) \geq 0,$$

where the product is understood as the scalar product. We also have

$$(vu_k - p_k)(t, x)(\hat{u}_k - u_k)(t, x) \geq 0$$

for the solutions on $I_k \times (0, 1)$ and, by similar arguments, for u_k^n, p_k^n . Moreover, we also have

$$\int_{T_k}^{T_{k+1}} \int_0^1 (vu_k^n - p_k^n)(\hat{u}_k - u_k^n) dx dt \geq 0 \quad \text{for all } \hat{u}_k \in U_{\text{ad}}^{\text{d}}. \quad (43)$$

For the corresponding errors we obtain (with $u_k \in U_{\text{ad}}^{\text{d}}$)

$$\begin{aligned} 0 &\leq \int_{T_k}^{T_{k+1}} \int_0^1 (v\tilde{u}_k^n - \tilde{p}_k^n + vu_k - p_k)(u_k - u_k^n) dx dt \\ &= \int_{T_k}^{T_{k+1}} \int_0^1 (v\tilde{u}_k^n - \tilde{p}_k^n)(-\tilde{u}_k^n) dx dt - \int_{T_k}^{T_{k+1}} \int_0^1 (vu_k - p_k)(u_k^n - u_k) dx dt \end{aligned}$$

and, therefore,

$$\int_{T_k}^{T_{k+1}} \int_0^1 (v\tilde{u}_k^n - \tilde{p}_k^n)(-\tilde{u}_k^n) dx dt \geq \int_{T_k}^{T_{k+1}} \int_0^1 (vu_k - p_k)(u_k^n - u_k) dx dt \geq 0.$$

Hence,

$$\int_{T_k}^{T_{k+1}} \int_0^1 \tilde{u}_k^n \tilde{p}_k^n dx dt \geq v \int_{T_k}^{T_{k+1}} \int_0^1 \tilde{u}_k^n \tilde{u}_k^n dx dt \quad (44)$$

holds. By the same argument, we obtain

$$\int_{T_k}^{T_{k+1}} \sum_{i=1}^m |\lambda_i(t, 1)| \tilde{p}_{ki}^n(t, 1) \tilde{v}_{ki}^n(t) dt \geq \mu \int_{T_k}^{T_{k+1}} \sum_{i=1}^m \tilde{v}_{ki}^n(t)^2 dt, \quad (45)$$

$$\int_{T_k}^{T_{k+1}} |\lambda(t, 1)| \tilde{p}_k^n(t, 1) \tilde{v}_k^n(t) dt \geq \mu \int_{T_k}^{T_{k+1}} \tilde{v}_k^n(t) \tilde{v}_k^n(t) dt, \quad (46)$$

where (u_k^n, v_k^n, p_k^n) and $(\tilde{u}_k^n, \tilde{v}_k^n, \tilde{p}_k^n)$ solve (21) and (24), respectively. We recall (33) and rewrite it as

$$\begin{aligned} & \int_0^1 \tilde{y}_k^n(T_{k+1}) \tilde{p}_k^n(T_{k+1}) - \tilde{y}_k^n(T_k) \tilde{p}_k^n(T_k) dx \\ &= \int_{T_k}^{T_{k+1}} \sum_{i=1}^m |\lambda_i(t, 1)| \tilde{v}_{ki}^n(t) \tilde{p}_{ki}^n(t, 1) dt + \rho \int_{T_k}^{T_{k+1}} \sum_{i=m+1}^d |\tilde{y}_{ki}^n|^2 dt \\ & \quad + \int_{T_k}^{T_{k+1}} \int_0^1 \tilde{u}_k^n \tilde{p}_k^n dx dt + \kappa \int_{T_k}^{T_{k+1}} \int_0^1 \|\tilde{y}_k^n\|_{\mathbb{R}^d}^2 dx dt. \end{aligned}$$

Then, according to (32) and (34), we obtain

$$\begin{aligned} \mathcal{F}^n &= 2\beta \sum_{k=0}^K \left(\int_{T_k}^{T_{k+1}} \sum_{i=1}^m |\lambda_i(t, 1)| \tilde{v}_{ki}^n(t) \tilde{p}_{ki}^n(t, 1) dt + \rho \int_{T_k}^{T_{k+1}} \sum_{i=m+1}^d |\tilde{y}_{ki}^n|^2 dt \right. \\ & \quad \left. + \int_{T_k}^{T_{k+1}} \int_0^1 \tilde{u}_k^n \tilde{p}_k^n dx dt + \kappa \int_{T_k}^{T_{k+1}} \int_0^1 \|\tilde{y}_k^n\|_{\mathbb{R}^d}^2 dx dt \right). \end{aligned} \quad (47)$$

Next, using (44) and (46), we can estimate \mathcal{F} from below by

$$\begin{aligned} \mathcal{F}^n &\geq 2\beta \sum_{k=0}^K \left(\int_{T_k}^{T_{k+1}} \int_0^1 \kappa \|\tilde{y}_k^n\|_{\mathbb{R}^d}^2 dx dt + \int_{T_k}^{T_{k+1}} \int_0^1 v \|\tilde{u}_k^n\|_{\mathbb{R}^d}^2 dx dt \right. \\ & \quad \left. + \int_{T_k}^{T_{k+1}} \mu \sum_{i=1}^m |\tilde{v}_{ki}^n|^2 dt + \rho \int_{T_k}^{T_{k+1}} \sum_{i=m+1}^d |\tilde{y}_{ki}^n|^2 dt \right). \end{aligned}$$

The definitions in (27)–(31) stay unchanged also in the case under consideration, whereas (34) now reads $\|X^n\|^2 = \mathcal{E}^n + \mathcal{F}^n$, where now \mathcal{F}^n is given by (47). By the same argument, we have $\|TX^n\|_X = \mathcal{E}^n - \mathcal{F}^n$ and Proposition 1 as well as (37) hold true as well. Thus, we arrive at the conclusion that

\mathcal{E}^n is bounded,

$$\begin{aligned} \tilde{y}_k^n &\rightarrow 0 \text{ in } L^2(I_k, L^2(0, 1)), \\ \tilde{p}_k^n &\rightarrow 0 \text{ in } L^2(I_k, L^2(0, 1)), \\ \tilde{v}_{ki}^n &\rightarrow 0 \text{ in } L(I_k), \end{aligned} \quad (48)$$

as $n \rightarrow \infty$.

Theorem 4. *Suppose that the controls u and v satisfy the pointwise constraints $u(t) \in U_{\text{ad}}^d$, $v(t) \in U_{\text{ad}}^b$, where $U_{\text{ad}}^d \subset L^2(0, 1)^d$ and $U_{\text{ad}}^b \subset \mathbb{R}^m$ are convex and closed. Further, let the iterates be defined as solutions (y_k^n, p_k^n) of (21), (22) with (43), (45). Then, these iterates converge in the sense of (48) to the corresponding solutions of (1), (5)–(7) together with (9). In fact, we have (42).*

6. A POSTERIORI ERROR ESTIMATES

We now embark on a posteriori error estimates for the iterates discussed in Section 4. Such estimates are important for the decisions on the choice of the breaks points T_k . Moreover, following the intention of this article, these estimates are developed on the continuous level. In the interest of space, we develop the estimates only for the case of unconstrained controls and nonlinearities in the state equation. Boundary nonlinearities can be handled similarly. In order to provide the corresponding information,

we introduce measures for global and local errors, pointwise in time, as well as measures of mismatch at the break points. In particular, we introduce the accumulated global error

$$\begin{aligned}
e^n := & \max_{0 \leq k \leq K} \int_0^1 \|\tilde{y}_k^n\|_{L^\infty(I_k)}^2 dx + \max_{0 \leq k \leq K} \int_0^1 \|\tilde{p}_k^n\|_{L^\infty(I_k)}^2 dx \\
& + \sum_{l=0}^K \int_{T_l}^{T_{l+1}} \sum_{i=1}^m |\lambda_i(s, 0)| \tilde{y}_{li}^n(s, 0)^2 + |\lambda_i(s, 1)| \tilde{p}_{li}^n(s, 1)^2 \\
& + \sum_{i=m+1}^d |\lambda_i(s, 1)| \tilde{y}_{ii}^n(s, 1)^2 + |\lambda_i(s, 0)| \tilde{p}_{ii}^n(s, 0)^2 ds
\end{aligned} \tag{49}$$

and the pointwise local error on I_k

$$\begin{aligned}
e_k^n(t) := & \int_0^1 \|\tilde{y}_k^n(t)\|_{\mathbb{R}^d}^2 + \|\tilde{p}_k^n(t)\|_{\mathbb{R}^d}^2 dx \\
& + \int_{T_k}^t \sum_{i=1}^m |\lambda_i(s, 0)| \tilde{y}_{ki}^n(s, 0)^2 + \sum_{i=m+1}^d |\lambda_i(s, 1)| \tilde{y}_{ki}^n(s, 1)^2 ds \\
& + \int_t^{T_{k+1}} \sum_{i=1}^m |\lambda_i(s, 1)| \tilde{p}_{ki}^n(s, 1)^2 + \sum_{i=m+1}^d |\lambda_i(s, 0)| \tilde{p}_{ki}^n(s, 0)^2 ds.
\end{aligned} \tag{50}$$

We further introduce the mismatch at the break point T_{k+1} at iteration n via

$$\mathcal{E}_{k,k+1}^n(T_{k+1}) := \|y_k^n(T_{k+1}) - y_{k+1}^n(T_{k+1})\|_{L^2}^2 + \|p_k^n(T_{k+1}) - p_{k+1}^n(T_{k+1})\|_{L^2}^2 \tag{51}$$

and between two consecutive iterates n and $n+1$ as

$$\begin{aligned}
\mathcal{E}_{k,k+1}^{n,n+1}(T_{k+1}) & := \|y_k^n(T_{k+1}) - y_{k+1}^{n+1}(T_{k+1})\|_{\mathbb{R}^d}^2 + \|p_k^n(T_{k+1}) - p_{k+1}^{n+1}(T_{k+1})\|_{\mathbb{R}^d}^2, \\
e^{n,n+1} & := \max_{0 \leq k \leq K} \|e_k^n + e_k^{n+1}\|_{L^\infty(I_k)}.
\end{aligned}$$

We further need energy estimates for y , \tilde{y}_k^n , p , and \tilde{p}_k^n . To this end, we multiply the state equation (18a) by y and integrate to obtain

$$\begin{aligned}
0 & = \int_0^t \int_0^1 (\partial_t y + A \partial_x y - f(t, y) - u) y dx ds \\
& = \frac{1}{2} \int_0^1 \|y\|_{\mathbb{R}^d}^2 \Big|_0^t dx + \int_0^t \frac{1}{2} \sum_{i=1}^d \lambda_i y_i^2 ds - \frac{1}{2} \int_0^t \int_0^1 \sum_{i=1}^d \partial_x \lambda_i \|y_i\|_{\mathbb{R}^d}^2 dx ds \\
& \quad - \int_0^t \int_0^1 f(s, y) y dx ds - \int_0^t \int_0^1 u y dx ds.
\end{aligned} \tag{52}$$

Moreover, we have

$$\begin{aligned}
& \frac{1}{2} \sum_{i=1}^d \lambda_i(t, 1) y_i(t, 1)^2 - \lambda_i(t, 0) y_i(t, 0)^2 \\
& = -\frac{1}{2} \sum_{i=1}^m |\lambda_i(t, 1)| \left(\sum_{j=m+1}^d g_{ij}^1(t) y_j(t, 1) + v_i(t) \right)^2 + \frac{1}{2} \sum_{i=m+1}^d |\lambda_i(t, 1)| y_i(t, 1)^2 \\
& \quad + \frac{1}{2} \sum_{i=1}^m |\lambda_i(t, 0)| y_i(t, 0)^2 - \frac{1}{2} \sum_{i=m+1}^d |\lambda_i(t, 0)| \left(\sum_{j=1}^m g_{ij}^0(t) y_j(t, 0) \right)^2
\end{aligned} \tag{53}$$

and (53) turns (52) into

$$\begin{aligned}
& \frac{1}{2} \int_0^1 \|y\|_{\mathbb{R}^d}^2(t) \, dx - \int_0^t \int_0^1 f(x, y)y \, dx \, ds + \frac{1}{2} \sum_{i=1}^m \int_0^t |\lambda_i(s, 0)| y_i(s, 0)^2 \, ds \\
& \quad + \frac{1}{2} \int_0^t \sum_{i=m+1}^d |\lambda_i(s, 1)| y_i(s, 1)^2 \, ds \\
& = \frac{1}{2} \int_0^1 \|y\|_{\mathbb{R}^d}^2(0) \, dx + \int_0^t \int_0^1 u y \, dx \, ds + \frac{1}{2} \int_0^t \int_0^1 \sum_{i=1}^d \partial_x \lambda_i \|y_i\|_{\mathbb{R}^d}^2 \, dx \, ds \\
& \quad + \frac{1}{2} \int_0^t \sum_{i=1}^m |\lambda_i(s, 1)| \left(\sum_{j=m+1}^d g_{ij}^1(s) y_j(s, 1) + v_i(s) \right)^2 \, ds \\
& \quad + \frac{1}{2} \int_0^t \sum_{i=m+1}^d |\lambda_i(s, 0)| \left(\sum_{j=1}^m g_{ij}^0(s) y_j(s, 0) \right)^2 \, ds.
\end{aligned} \tag{54}$$

Furthermore, we have the estimates

$$\begin{aligned}
& \frac{1}{2} \int_0^t \sum_{i=1}^m |\lambda_i(s, 1)| \left(\sum_{j=m+1}^d g_{ij}^1(s) y_j(s, 1) + v_i(s) \right)^2 \, ds \\
& \leq \frac{1}{2} \sum_{i=1}^m \int_0^t |\lambda_i(s, 1)| \left(2 \left(\sum_{j=m+1}^d g_{ij}^1(s) y_j(s, 1) \right)^2 + v_i(s)^2 \right) \, ds \\
& \leq \sum_{i=1}^m \int_0^t |\lambda_i(s, 1)| \sum_{j=m+1}^d g_{ij}^1(s)^2 \sum_{j=m+1}^d y_j(s, 1)^2 + v_i(s)^2 \, ds \\
& = \int_0^t \sum_{i=1}^m \left(|\lambda_i(s, 1)| \sum_{j=m+1}^d g_{ij}^1(s)^2 \right) \sum_{j=m+1}^d y_j(s, 1)^2 + \sum_{i=1}^m |\lambda_i(s, 1)| v_i(s)^2 \, ds.
\end{aligned} \tag{55}$$

Similarly,

$$\frac{1}{2} \int_0^t \sum_{i=m+1}^d |\lambda_i(s, 0)| \left(\sum_{j=1}^m g_{ij}^0(s) y_j(s, 0) \right)^2 \, ds \leq \frac{1}{2} \int_0^t \left(\sum_{i=m+1}^d |\lambda_i(s, 0)| \sum_{j=1}^m g_{ij}^0(s)^2 \right) \sum_{j=1}^m y_j(s, 0)^2 \, ds \tag{56}$$

holds. In order to absorb the terms (55), (56) into (54), we assume

$$\sum_{i=m+1}^d \sum_{j=1}^m g_{ji}^1(s)^2 |\lambda_j(s, 1)| \leq \frac{\delta}{2}, \quad \sum_{i=1}^m \sum_{j=m+1}^d g_{ji}^0(s)^2 |\lambda_j(s, 0)| \leq \frac{\delta}{2}, \quad 1 > \delta > 0. \tag{57}$$

Then, (54) becomes

$$\begin{aligned}
& \frac{1}{2} \int_0^1 \|y(t)\|_{\mathbb{R}^d}^2 \, dx - \int_0^t \int_0^1 f(s, y)y \, dx \, ds \\
& \quad + \frac{1-\delta}{2} \int_0^t \sum_{i=1}^m |\lambda_i(s, 0)| y_i(s, 0)^2 \, ds + \frac{1-\delta}{2} \int_0^t \sum_{i=m+1}^d |\lambda_i(s, 1)| y_i(s, 1)^2 \, ds \\
& \leq \frac{1}{2} \int_0^1 \|y(0)\|_{\mathbb{R}^d}^2 \, dx + \int_0^t \int_0^1 u y \, dx \, ds + \sum_{i=1}^m \int_0^t |\lambda_i(s, 1)| v_i(s)^2 \, ds \\
& \quad + \frac{1}{2} \int_0^t \int_0^1 \sum_{i=1}^d \partial_x \lambda_i \|y_i\|_{\mathbb{R}^d}^2 \, dx \, ds.
\end{aligned}$$

We now extend the estimate to the errors \tilde{y}_k . To this end, we use Assumption 1 and obtain

$$\begin{aligned}
& \int_0^1 \|\tilde{y}_k^n\|_{\mathbb{R}^d}^2 dx + (1-\delta) \int_{T_k}^t \sum_{i=1}^m |\lambda_i(s,0)| \tilde{y}_{ki}^n(s,0)^2 ds + (1-\delta) \int_{T_k}^t \sum_{i=m+1}^d |\lambda_i(s,1)| \tilde{y}_{ki}^n(s,1)^2 ds \\
& \leq \int_0^1 \|\tilde{y}_k^n(T_k)\|_{\mathbb{R}^d}^2 dx + \int_{T_k}^t \int_0^1 (\tilde{u}_k^n)^2 dx ds + 2 \sum_{i=1}^m \int_{T_k}^t |\lambda_i(s,t)| v_i(s)^2 ds \\
& \quad + \int_{T_k}^t \int_0^1 \left(L + 1 + \max_{s,x} |\partial_x \lambda_i| \right) \|\tilde{y}_{ki}^n\|_{\mathbb{R}^d}^2 dx ds.
\end{aligned} \tag{58}$$

Multiplying the adjoint equation (21b) by \tilde{p}_k^n , which satisfy

$$\tilde{p}_k^n(T_{k+1}) = \frac{1}{\beta} \left(\tilde{\phi}_{k,k+1}^n - \tilde{y}_k^n(T_{k+1}) \right),$$

and integrating from t to T_{k+1} , we obtain

$$\begin{aligned}
0 &= \int_t^{T_{k+1}} \int_0^1 \left(\partial_t \tilde{p}_k^n \tilde{p}_k^n + A \partial_x \tilde{p}_k^n \tilde{p}_k^n + (\mathcal{D}_y f_k(y_k^m)^\top + \partial_x A) \tilde{p}_k^n \tilde{p}_k^n \right. \\
& \quad \left. + (\mathcal{D}_y f_k(y_k^m)^\top - \mathcal{D}_y f_k(y_k)^\top) p_k \tilde{p}_k^n + \kappa \tilde{y}_k^n \tilde{p}_k^n \right) dx dt \\
&= \frac{1}{2} \int_0^1 \|\tilde{p}_k^n(s)\|_{\mathbb{R}^d}^2 \Big|_t^{T_{k+1}} dx + \frac{1}{2} \int_t^{T_{k+1}} \sum_{i=1}^d \lambda_i(s,x) \tilde{p}_{ki}^n \tilde{p}_{ki}^n \Big|_0^1 ds \\
& \quad + \int_t^{T_{k+1}} \int_0^1 \frac{1}{2} \partial_x A \tilde{p}_k^n \tilde{p}_k^n dx + \kappa \tilde{y}_k^n \tilde{p}_k^n dx ds \\
& \quad + \int_t^{T_{k+1}} \int_0^1 \mathcal{D}_y f_k(y_k^n)^\top \tilde{p}_k^n \tilde{p}_k^n + (\mathcal{D}_y f_k(y_k^m)^\top - \mathcal{D}_y f_k(y_k)^\top) p_k \tilde{p}_k^n dx ds
\end{aligned} \tag{59}$$

Similar to (55)–(57), we need boundary estimates

$$\begin{aligned}
& \frac{1}{2} \sum_{i=1}^d \int_t^{T_{k+1}} \lambda_i(s,1) \tilde{p}_{ki}^n(s,1)^2 - \lambda_i(s,0) \tilde{p}_{ki}^n(s,0)^2 ds \\
&= \frac{1}{2} \int_t^{T_{k+1}} - \sum_{i=1}^m |\lambda_i(s,1)| \tilde{p}_{ki}^n(s,1)^2 + \sum_{i=m+1}^d \left(\sum_{j=1}^m |\lambda_j(s,1)| g_{ji}^1(s) \tilde{p}_{kj}^n(s,1) \right)^2 \\
& \quad + \sum_{i=1}^m \sum_{j=m+1}^d |\lambda_j(s,0)| g_{ji}^0(s) \tilde{p}_{kj}^n(s,0)^2 - \sum_{i=m+1}^d |\lambda_i(s,0)| \tilde{p}_{ki}^n(s,0)^2 ds.
\end{aligned} \tag{60}$$

We have

$$\sum_{i=m+1}^d \left(\sum_{j=1}^m |\lambda_j(s,1)| g_{ji}^1(s) \tilde{p}_{kj}^n(s,1) \right)^2 \leq \sum_{i=m+1}^d \sum_{j=1}^m \left(\lambda_j(s,1) g_{ji}^1(s) \right)^2 \sum_{j=1}^m \tilde{p}_{kj}^n(s,1)^2 \tag{61}$$

and

$$\sum_{i=1}^m \left(\sum_{j=m+1}^d \lambda_j(s,0) g_{ji}^0(s) \tilde{p}_{kj}^n(s,0) \right)^2 \leq \sum_{i=1}^m \sum_{j=m+1}^d \left(\lambda_j(s,0) g_{ji}^0(s) \right)^2 \sum_{j=m+1}^d \tilde{p}_{kj}^n(s,0)^2. \tag{62}$$

Recalling (57), we obtain from (59) together with (60)–(62) that

$$\begin{aligned}
& \int_0^1 \|\tilde{p}_k^n\|_{\mathbb{R}^d}^2 dx + (1-\delta) \int_t^{T_{k+1}} \sum_{i=1}^m |\lambda_i(s,1)| \tilde{p}_{ki}^n(s,1)^2 ds + (1-\delta) \int_t^{T_{k+1}} \sum_{i=m+1}^d |\lambda_i(s,0)| \tilde{p}_{ki}^n(s,0)^2 ds \\
& \leq \int_t^{T_{k+1}} \int_0^1 \frac{1}{2} \sum_{i=1}^d \partial_x \lambda_i(s,x) (\tilde{p}_{ki}^n)^2 dx ds + \int_0^1 \|\tilde{p}_k^n(T_{k+1})\|_{\mathbb{R}^d}^2 dx \\
& \quad + \int_t^{T_{k+1}} \int_0^1 \mathcal{D}_y f_k(y_k^n)^\top \tilde{p}_k^n \tilde{p}_k^n + (\mathcal{D}_y f_k(y_k^n)^\top - \mathcal{D}_y f_k(y_k)^\top) p_k \tilde{p}_k^n + \kappa \tilde{y}_k^n \tilde{p}_k^n dx ds \\
& \leq \int_0^1 \|\tilde{p}_k^n(T_{k+1})\|_{\mathbb{R}^d}^2 dx + C \int_t^{T_{k+1}} \int_0^1 \|\tilde{p}_k^n\|_{\mathbb{R}^d}^2 + \|\tilde{y}_k^n\|_{\mathbb{R}^d}^2 dx ds
\end{aligned} \tag{63}$$

holds for some $C > 0$ that depends on the Lipschitz constant L in Assumption 1. Since in this section, as it was the case in Section 4 as well, we deal with unconstrained controls. Hence, (26) holds. Therefore, (58) reads

$$\begin{aligned}
\int_0^1 \|\tilde{y}_k^n(t)\|_{\mathbb{R}^d}^2 dx & \leq \int_0^1 \|\tilde{y}_k^n(T_k)\|_{\mathbb{R}^d}^2 dx + \frac{1}{v^2} \int_{T_k}^t \int_0^1 \|\tilde{p}_k^n\|_{\mathbb{R}^d}^2 ds dx + \frac{2}{\mu^2} \int_{T_k}^t \sum_{i=1}^m |\lambda_i(s,1)|^3 \tilde{p}_{ki}^n(s,1)^2 ds \\
& \quad + \int_{T_k}^t \int_0^1 \sum_{i=1}^d (L+1 + \max|\partial_x \lambda_i|) \|\tilde{y}_{ki}^n\|_{\mathbb{R}^d}^2 dx ds \\
& \quad - (1-\delta) \int_{T_k}^t \sum_{i=1}^m |\lambda_i(s,0)| \tilde{y}_{ki}^n(s,0)^2 + \sum_{i=m+1}^d |\lambda_i(s,1)| \tilde{y}_{ki}^n(s,1)^2 ds.
\end{aligned} \tag{64}$$

Clearly,

$$\int_0^1 \|\tilde{y}_k^n(T_k)\|_{\mathbb{R}^d}^2 dx = \int_0^1 \|\tilde{y}_k^n(T_k)\|_{\mathbb{R}^d}^2 dx - \int_0^1 \|\tilde{y}_{k-1}^n(T_k)\|_{\mathbb{R}^d}^2 dx + \int_0^1 \|\tilde{y}_{k-1}^n(T_k)\|_{\mathbb{R}^d}^2 dx. \tag{65}$$

We now apply (64) to the index $k-1$ and evaluate at T_k , which gives

$$\begin{aligned}
\int_0^1 \|\tilde{y}_{k-1}^n(T_k)\|_{\mathbb{R}^d}^2 dx & \leq \int_0^1 \|\tilde{y}_{k-1}^n(T_{k-1})\|_{\mathbb{R}^d}^2 dx \\
& \quad - (1-\delta) \int_{T_{k-1}}^{T_k} \sum_{i=1}^m |\lambda_i(s,0)| \tilde{y}_{k-1,i}^n(s,0)^2 + \sum_{i=m+1}^d |\lambda_i(s,1)| \tilde{y}_{k-1,i}^n(s,1)^2 ds \\
& \quad + \frac{1}{v^2} \int_{T_{k-1}}^{T_k} \int_0^1 \|\tilde{p}_{k-1}^n\|_{\mathbb{R}^d}^2 dx ds + \frac{2}{\mu^2} \int_{T_{k-1}}^{T_k} \sum_{i=1}^m |\lambda_i(s,1)|^3 \tilde{p}_{k-1,i}^n(s,1)^2 ds \\
& \quad + \int_{T_{k-1}}^{T_k} \int_0^1 \sum_{i=1}^d (L+1 + \max|\partial_x \lambda_i|) \|\tilde{y}_{k-1,i}^n\|_{\mathbb{R}^d}^2 dx ds.
\end{aligned} \tag{66}$$

With $\omega := L+1 + \max|\partial_x \lambda_i|$, (64) and (65) imply

$$\begin{aligned}
& \int_0^1 \|\tilde{y}_k^n(t)\|_{\mathbb{R}^d}^2 dx + (1-\delta) \left(\int_{T_{k-1}}^{T_k} \sum_{i=1}^m |\lambda_i(s,0)| \tilde{y}_{k-1,i}^n(s,0)^2 + \sum_{i=m+1}^d |\lambda_i(s,1)| \tilde{y}_{k-1,i}^n(s,1)^2 ds \right. \\
& \quad \left. + \int_{T_k}^t \sum_{i=1}^m |\lambda_i(s,0)| \tilde{y}_{ki}^n(s,0)^2 + \sum_{i=m+1}^d |\lambda_i(s,1)| \tilde{y}_{ki}^n(s,1)^2 ds \right) \\
& \leq \int_0^1 \|\tilde{y}_k^n(T_k)\|_{\mathbb{R}^d}^2 dx - \|\tilde{y}_{k-1}^n(T_k)\|_{\mathbb{R}^d}^2 dx + \|\tilde{y}_{k-1}^n(T_{k-1})\|_{\mathbb{R}^d}^2 dx \\
& \quad + \int_{T_{k-1}}^{T_k} \int_0^1 \frac{1}{v^2} \|\tilde{p}_{k-1}^n\|_{\mathbb{R}^d}^2 + \omega \|\tilde{y}_{k-1}^n\|_{\mathbb{R}^d}^2 dx ds + \int_{T_k}^t \int_0^1 \frac{1}{v} \|\tilde{p}_k^n\|_{\mathbb{R}^d}^2 + \omega \|\tilde{y}_k^n\|_{\mathbb{R}^d}^2 dx ds \\
& \quad + \int_{T_{k-1}}^{T_k} \frac{2}{\mu^2} \sum_{i=1}^m |\lambda_i(s,1)|^3 \tilde{p}_{k-1,i}^n(s,1)^2 ds + \int_{T_k}^t \frac{2}{\mu^2} \sum_{i=1}^m |\lambda_i(s,1)|^3 \tilde{p}_{ki}^n(s,1)^2 ds.
\end{aligned} \tag{67}$$

We can now iterate (67) by replacing the term $\|\tilde{y}_{k-1}^n(T_{k-1})\|_{\mathbb{R}^d}^2$ using (65) for $k-1$ and then (66). This procedure leads to

$$\begin{aligned}
& \int_0^1 \|\tilde{y}_k^n(t)\|_{\mathbb{R}^d}^2 dx + (1-\delta) \left(\sum_{l=0}^{k-1} \int_{T_l}^{T_{l+1}} \sum_{i=1}^m |\lambda_i(s,0)| \tilde{y}_{li}^n(s,0)^2 + \sum_{i=m+1}^d |\lambda_i(s,1)| \tilde{y}_{li}^n(s,1)^2 ds \right. \\
& \quad \left. + \int_{T_k}^t \sum_{i=1}^m |\lambda_i(s,0)| \tilde{y}_{ki}^n(s,0)^2 + \sum_{i=m+1}^d |\lambda_i(s,1)| \tilde{y}_{ki}^n(s,1)^2 ds \right) \\
& \leq \sum_{l=0}^{k-1} \int_0^1 \|\tilde{y}_{l+1}^n(T_{l+1})\|_{\mathbb{R}^d}^2 - \|\tilde{y}_l^n(T_{l+1})\|_{\mathbb{R}^d}^2 dx \\
& \quad + \sum_{l=0}^{k-1} \int_{T_l}^{T_{l+1}} \int_0^1 \frac{1}{v^2} \|\tilde{p}_l^n\|_{\mathbb{R}^d}^2 + \omega \|\tilde{y}_l^n\|_{\mathbb{R}^d}^2 dx ds + \int_{T_k}^t \int_0^1 \frac{1}{v^2} \|\tilde{p}_k^n\|_{\mathbb{R}^d}^2 + \omega \|\tilde{y}_k^n\|_{\mathbb{R}^d}^2 dx ds \\
& \quad + \sum_{l=0}^{k-1} \int_{T_l}^{T_{l+1}} \frac{2}{\mu^2} \sum_{i=1}^m |\lambda_i(s,1)|^3 \tilde{p}_{li}^n(s,1)^2 ds + \int_{T_k}^t \frac{2}{\mu^2} \sum_{i=1}^m |\lambda_i(s,1)| \tilde{p}_{ki}^n(s,1)^2 ds.
\end{aligned} \tag{68}$$

In order to arrive at an estimate for $e_k^n(t)$ in (50), we take the maximum in (68) w.r.t. k :

$$\begin{aligned}
& \max_{0 \leq k \leq K} \int_0^1 \|\tilde{y}_k^n(t)\|_{\mathbb{R}^d}^2 dx \\
& \quad + (1-\delta) \sum_{l=0}^K \int_{T_l}^{T_{l+1}} \sum_{i=1}^m |\lambda_i(s,0)| \tilde{y}_{li}^n(s,0)^2 + \sum_{i=m+1}^d |\lambda_i(s,1)| \tilde{y}_{li}^n(s,1)^2 ds \\
& \leq \sum_{k=0}^K \int_{T_k}^{T_{k+1}} \int_0^1 \frac{1}{v^2} \|\tilde{p}_k^n\|_{\mathbb{R}^d}^2 + \omega \|\tilde{y}_k^n\|_{\mathbb{R}^d}^2 dx ds + \sum_{k=0}^K \int_{T_k}^{T_{k+1}} \frac{2}{\mu^2} \sum_{i=1}^m |\lambda_i(s,1)|^3 \tilde{p}_{ki}^n(s,1)^2 ds \\
& \quad + \sum_{k=0}^{K-1} \int_0^1 \|\tilde{y}_k^n(T_{k+1}) - \tilde{y}_{k+1}^n(T_{k+1})\|_{\mathbb{R}^d} (\|\tilde{y}_k^n(T_{k+1})\|_{\mathbb{R}^d} + \|\tilde{y}_{k+1}^n(T_{k+1})\|_{\mathbb{R}^d}) dx_{\mathbb{R}^d} \\
& \leq \sqrt{2} \left(\sum_{k=0}^{K-1} \int_0^1 \|\tilde{y}_k^n(T_{k+1}) - \tilde{y}_{k+1}^n(T_{k+1})\|_{\mathbb{R}^d} dx \right)^{\frac{1}{2}} \left(\sum_{k=0}^{K-1} \int_0^1 \|\tilde{y}_k^n(T_{k+1})\|_{\mathbb{R}^d}^2 + \|\tilde{y}_{k+1}^n(T_{k+1})\|_{\mathbb{R}^d}^2 dx \right)^{\frac{1}{2}} \\
& \quad + \sum_{k=0}^K \int_{T_k}^{T_{k+1}} \int_0^1 \frac{1}{v^2} \|\tilde{p}_k^n\|_{\mathbb{R}^d}^2 + \omega \|\tilde{y}_k^n\|_{\mathbb{R}^d}^2 dx ds + \sum_{k=0}^K \int_{T_k}^{T_{k+1}} \frac{2}{\mu^2} \sum_{i=1}^m |\lambda_i(s,1)| \tilde{p}_{ki}^n(s,1)^2 ds.
\end{aligned} \tag{69}$$

We now go back to (63) and focus on the adjoint variable. We proceed in the same way and obtain

$$\begin{aligned}
& \int_0^1 \|\tilde{p}_k^n(T_{k+1})\|_{\mathbb{R}^d}^2 dx \int_0^1 \|\tilde{p}_k^n(T_{k+1})\|_{\mathbb{R}^d}^2 - \|\tilde{p}_{k+1}^n(T_{k+1})\|_{\mathbb{R}^d}^2 + \|\tilde{p}_{k+1}^n(T_{k+1})\|_{\mathbb{R}^d}^2 dx \\
& \leq \int_0^1 \|\tilde{p}_k^n(T_{k+1})\|_{\mathbb{R}^d}^2 - \|\tilde{p}_{k+1}^n(T_{k+1})\|_{\mathbb{R}^d}^2 dx + \int_0^1 \|\tilde{p}_{k+1}^n(T_{k+2})\|_{\mathbb{R}^d}^2 dx \\
& \quad + C \int_{T_{k+1}}^{T_{k+2}} \int_0^1 \|\tilde{p}_{k+1}^n\|_{\mathbb{R}^d}^2 + \|\tilde{y}_{k+1}^n\|_{\mathbb{R}^d}^2 dx ds \\
& \quad - (1-\delta) \int_{T_{k+1}}^{T_{k+2}} \sum_{i=1}^m |\lambda_i(s,1)| \tilde{p}_{k+1}^n(s,1)^2 + \sum_{i=m+1}^d |\lambda_i(s,0)| \tilde{p}_{k+1,i}^n ds.
\end{aligned} \tag{70}$$

Putting the last expression on the other side and iterating (70), we arrive at

$$\begin{aligned} & \int_0^1 \|\tilde{p}_k^n(t)\|_{\mathbb{R}^d}^2 dx + \int_t^{T_{k+1}} \sum_{i=1}^m |\lambda_i(s, 1)| \tilde{p}_{ki}^n(s, 1)^2 + \sum_{i=m+1}^d |\lambda_i(s, 0)| \tilde{p}_{ki}^n(s, 0)^2 ds \\ & + \sum_{l=k+1}^K \int_{T_l}^{T_{l+1}} \sum_{i=1}^m |\lambda_i(s, 1)| \tilde{p}_{li}^n(s, 1)^2 + \sum_{i=m+1}^d |\lambda_i(s, 0)| \tilde{p}_{li}^n(s, 0)^2 ds \\ & \leq C \sum_{l=k}^K \int_{T_l}^{T_{l+1}} \int_0^1 \|\tilde{p}_l^n\|_{\mathbb{R}^d}^2 + \|\tilde{y}_l^n\|_{\mathbb{R}^d}^2 dx ds + \sum_{l=k}^K \int_0^1 \|\tilde{p}_l^n(T_{l+1})\|_{\mathbb{R}^d}^2 - \|\tilde{p}_{l+1}^n(T_{l+1})\|_{\mathbb{R}^d}^2 dx \end{aligned}$$

for $k = 0, \dots, K$. Again, we take the maximum w.r.t. k and use the Cauchy–Schwartz inequality as before and get

$$\begin{aligned} & \max_{0 \leq k \leq K} \int_0^1 \tilde{p}_k^n(t) dx + \sum_{k=0}^K \int_{T_k}^{T_{k+1}} \sum_{i=1}^m |\lambda_i(s, 1)| \tilde{p}_{ki}^n(s, 1)^2 + \sum_{i=m+1}^d |\lambda_i(s, 0)| \tilde{p}_{ki}^n(s, 0)^2 ds \\ & \leq \sqrt{2} \left(\sum_{k=0}^{K-1} \int_0^1 \|\tilde{p}_k^n(T_{k+1}) - \tilde{p}_{k+1}^n(T_{k+1})\|_{\mathbb{R}^d}^2 dx \right)^{\frac{1}{2}} \left(\sum_{k=0}^{K-1} \int_0^1 \|\tilde{p}_k^n(T_{k+1})\|_{\mathbb{R}^d}^2 + \|\tilde{p}_{k+1}^n(T_{k+1})\|_{\mathbb{R}^d}^2 dx \right)^{\frac{1}{2}} \\ & + C \sum_{k=0}^K \int_{T_k}^{T_{k+1}} \int_0^1 \|\tilde{p}_k^n\|_{\mathbb{R}^d}^2 + \|\tilde{y}_k^n\|_{\mathbb{R}^d}^2 dx ds \end{aligned} \quad (71)$$

We add (69) and (71) to achieve, according to (50) and (49),

$$\begin{aligned} e^n & \leq \sqrt{2} \left(\sum_{k=0}^{K-1} \int_0^1 \|\tilde{y}_k^n(T_{k+1}) - \tilde{y}_{k+1}^n(T_{k+1})\|_{\mathbb{R}^d}^2 dx \right)^{\frac{1}{2}} \cdot \left(\sum_{k=0}^{K-1} \int_0^1 \|\tilde{y}_k^n(T_{k+1})\|_{\mathbb{R}^d}^2 + \|\tilde{y}_{k+1}^n(T_{k+1})\|_{\mathbb{R}^d}^2 dx \right)^{\frac{1}{2}} \\ & + \sqrt{2} \left(\sum_{k=0}^{K-1} \int_0^1 \|\tilde{p}_k^n(T_{k+1}) - \tilde{p}_{k+1}^n(T_{k+1})\|_{\mathbb{R}^d}^2 dx \right)^{\frac{1}{2}} \left(\sum_{k=0}^{K-1} \int_0^1 \|\tilde{p}_k^n(T_{k+1})\|_{\mathbb{R}^d}^2 + \|\tilde{p}_{k+1}^n(T_{k+1})\|_{\mathbb{R}^d}^2 dx \right)^{\frac{1}{2}} \\ & + C \sum_{k=0}^K \int_{T_k}^{T_{k+1}} \int_0^1 \|\tilde{p}_k^n\|_{\mathbb{R}^d}^2 + \|\tilde{y}_k^n\|_{\mathbb{R}^d}^2 dx ds + \sum_{k=0}^K \int_{T_k}^{T_{k+1}} \int_0^1 \frac{1}{v^2} \|\tilde{p}_k^n\|_{\mathbb{R}^d}^2 + \omega \|\tilde{y}_k^n\|_{\mathbb{R}^d}^2 dx ds \\ & + \sum_{k=0}^K \int_{T_k}^{T_{k+1}} \frac{2}{\mu^2} \sum_{i=1}^m |\lambda_i(s, 1)| \tilde{p}_{ki}^n(s, 1)^2 ds \\ & \leq 2\sqrt{2} \max\left(1, \frac{1}{\beta}\right) \sqrt{\mathcal{E}^n} \left(\sum_{k=0}^{K-1} \mathcal{E}_{k,k+1}^n(T_{k+1}) \right)^{\frac{1}{2}} + \sum_{k=0}^K \int_{T_k}^{T_{k+1}} \frac{2}{\mu^2} \sum_{i=1}^m |\lambda_i(s, 1)|^3 \tilde{p}_{ki}^n(s, 1)^2 ds \\ & + \sum_{k=0}^K \int_{T_k}^{T_{k+1}} \int_0^1 \left(C + \frac{1}{v^2} \right) \|\tilde{p}_k^n\|_{\mathbb{R}^d}^2 + (C + \omega) \|\tilde{y}_k^n\|_{\mathbb{R}^d}^2 dx ds. \end{aligned} \quad (72)$$

There is a computable constant $\tilde{C} > 0$ such that (72) yields

$$e^n \leq 2\sqrt{2} \max\left(1, \frac{1}{\beta}\right) \sqrt{\mathcal{E}^n} \left(\sum_{k=0}^{K-1} \mathcal{E}_{k,k+1}^n(T_{k+1}) \right)^{\frac{1}{2}} + \tilde{C} \mathcal{F}^n.$$

We now need an estimate of \mathcal{F}^n w.r.t. \mathcal{E}^n and $\mathcal{E}_{k,k+1}^n(T_{k+1})$. To this end, recall (40), i.e.,

$$\|TX^n - X^n\|_{\mathcal{X}}^2 = 2(\mathcal{E}^n - TX^n X^n) \leq 2 \max(1, \beta^2) \sum_{k=0}^{K-1} \mathcal{E}_{k,k+1}^n(T_{k+1}).$$

On the other hand, according to Proposition 1 (i), (50), and (35), we have that

$$2(1 - \varepsilon) \mathcal{F}^n \leq \mathcal{E}^n + \mathcal{F}^n - (\mathcal{E}^{n+1} + \mathcal{F}^{n+1}) \leq \|X^n - \mathcal{F}X^n\|_{\mathcal{X}} 2(1 - \varepsilon) \|X^n\|_{\mathcal{X}} \quad (73)$$

holds. Here, we used the definition of X^{n+1} in (29) and the fact that $\|X^n\|_{\mathcal{X}}$ is non-increasing. Now, (73) shows

$$\mathcal{F}^n \leq \sqrt{2} \max(1, \beta) (\mathcal{E}^n + \mathcal{F}^n)^{\frac{1}{2}} \left(\sum_{k=0}^{K-1} \mathcal{E}_{k,k+1}^n(T_{k+1}) \right)^{\frac{1}{2}}.$$

Theorem 5. *In addition to Assumption 1, we assume (57) to hold for the system data with $\varepsilon \in [0, 1)$. Then, the iterates defined in Section 3 satisfy the a posteriori estimate*

$$e^n \leq C \sqrt{\mathcal{E}^n + \mathcal{F}^n} \left(\sum_{k=0}^{K-1} \mathcal{E}_{k,k+1}^n(T_{k+1}) \right)^{\frac{1}{2}}, \quad C = \sqrt{2} \left(2 \max\left(1, \frac{1}{\beta}\right) + \tilde{C} \max(1, \beta) \right),$$

where $\mathcal{E}_{k,k+1}(T_{k+1})$ and e^n are given by (51) and (49), respectively. Moreover, \tilde{C} is explicitly computable in terms of the Lipschitz constant L in Assumption 1.

Corollary 1. *Under the conditions in Theorem 5, $\mathcal{E}^n + \mathcal{F}^n$ is bounded and, hence, there exists a constant $C > 0$ depending in a computable way on the initial and tracking data, as well as on the parameters β , K , and the initial values of the transmission data μ, η such that*

$$e^n \leq C \left(\sum_{k=0}^{K-1} \mathcal{E}_{k,k+1}^n(T_{k+1}) \right)^{\frac{1}{2}}$$

holds for $\varepsilon \in [0, 1)$ and $n = 1, 2, \dots$

Proof. For the sake of brevity, we refer the reader to the proof of Corollary 6.3.3.1 in [13], where the dependence of C on the data is made explicit in a special case. \square

Using similar arguments as in the proof of Theorem 6.3.3.2 in [13], we can prove estimates w.r.t. the measures $\mathcal{E}_{k,k+1}^{n,n+1}(T_{k+1})$ and $e^{n,n+1}$ between two iterates.

Theorem 6. *Under the conditions in Theorem 5, for $\varepsilon \in [0, \frac{1}{2})$, we have*

$$e^{n,n+1} \leq C_\varepsilon \sqrt{\mathcal{E}^{n+1} + \mathcal{E}^n} \left(\mathcal{E}_{k,k+1}^{n,n+1}(T_{k+1}) + \mathcal{E}_{k+1,k}^{n,n+1}(T_{k+1}) \right)^{\frac{1}{2}}$$

with a computable bound C_ε .

Corollary 2. *Under the conditions in Theorem 5 and $\varepsilon \in [0, \frac{1}{2})$, there exists a computable constant C_ε such that*

$$e^{n,n+1} \leq C_\varepsilon \sum_{k=0}^{K-1} \left(\mathcal{E}_{k,k+1}^{n,n+1}(T_{k+1}) + \mathcal{E}_{k+1,k}^{n,n+1}(T_{k+1}) \right)$$

holds.

For the application of the a posteriori error estimates w.r.t. the choice of the sub-intervals I_k and the actual numerical realization we have to refer to a forthcoming publication.

7. NUMERICAL EXPERIMENTS

We close the mathematical analysis in this article with some numerical examples, which should provide some first evidence for the behavior of the iterative time-domain decomposition w.r.t. the choice of the parameters and the presence of nonlinear terms. It is clear that this setup is only suitable to provide first evidence. A full numerical treatment will be the subject of a forthcoming publication.

We consider the wave equation as in Example 2 for a single link, i.e., $d = 2$. Moreover, we take $f(s) = \alpha|s|s$, $\alpha \geq 0$, as the nonlinearity in the damping. We do not consider boundary controls but assume, in fact, homogeneous Dirichlet boundary conditions. We apply a distributed control without constraints and take two sub-intervals $I_1 = [0, 1)$ and $I_2 = (1, 2]$. For the space discretization, we use the standard approximations corresponding to the standard discrete Dirichlet-Laplacian A_h on the second-order level in space. The corresponding global optimality system on the entire interval $[0, 2]$ and the local optimality systems on each time interval I_1 and I_2 are treated as a boundary value problems w.r.t. the time variable. Then, we solve these problems using the MATLAB solver `bvp4c` with tolerance 10^{-8} . For any initial data and tracking term, the system governing the errors \tilde{y}_k and

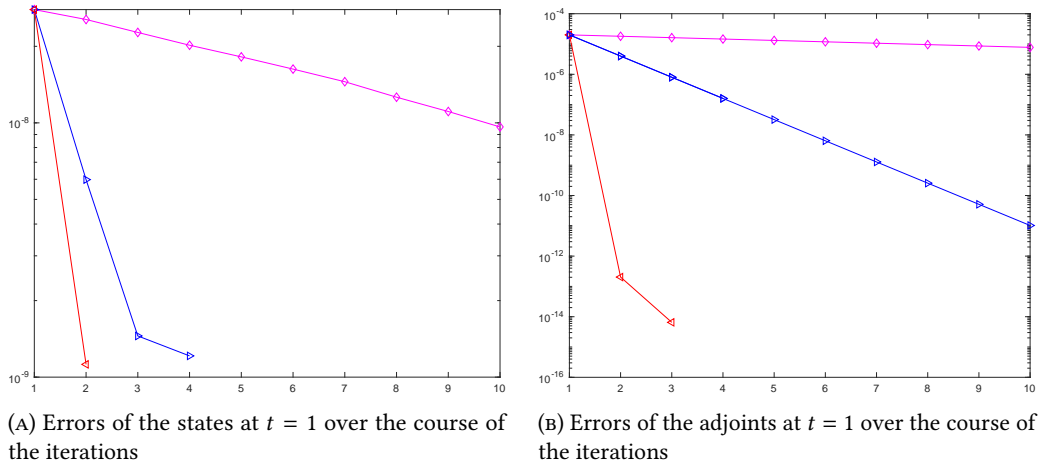


FIGURE 1. Left-most curve: $\varepsilon = 0.5$; middle curve: $\varepsilon = 0.4$ or 0.6 ; right-most curve $\varepsilon = 0.95$ or 0.05

\tilde{p}_k is homogeneous and, hence, the local optimality systems are homogeneous up to the errors at the transmission boundary $T = 1$. In particular, for vanishing initial data and target, the global optimality system has zero as the unique solution. We take $n = 10$ discretization points w.r.t. the space variable and choose $\beta = 10^5$, $\kappa = 10^3$, $\nu = 10^3$, and $\varepsilon = 0.5$. We observe very fast convergence for that particular choice of ε (almost two-step-convergence).

In order to get an idea about the role of the under-relaxation parameter ε , we repeat the calculations with all parameters fixed, but with $\varepsilon \in \{0.95, 0.6, 0.4, 0.05\}$. It turned out that the symmetric choices ($\varepsilon = 0.95$ or 0.05 and $\varepsilon = 0.4$ or 0.6) produced numerically identical results. In the plots of Figures 1, we show the errors of the state and the adjoint for the different choices of ε . The results show that for a given set of parameters κ , ν , β , there is an “optimal” under-relaxation parameter ε , in fact, the value $\varepsilon = 0.5$. We notice that the size of β balances between the errors of the state and those of the adjoint variables. The large value of β chosen here effects the stronger decay of the adjoint errors at the interface. We also included a nonlinear damping as in the Example 2, which did not show an adverse effect. In fact, it has dissipative nature, which may even enhance the convergence properties as pointed out in [13]. As noted above, a detailed treatment on the numerical realization, including a convergence analysis on the semi-discrete level, a full treatment of nonlinearities, more break-points T_k , constrained controls, more relevant examples on the level of networks as in Example 2, the use of the a posteriori error estimates, and, finally, the treatment of the virtual control paradigm is far beyond this article and has to be presented in a forthcoming publication. Fast convergence cannot be expected, however, in general as the iteration is related to an Uzawa-type saddle-point method [9].

8. CONCLUSION

In conclusion, we designed a time-domain decomposition for time-varying systems of semilinear hyperbolic equations. We proved convergence of the iterates and provided an a posteriori error estimate. The method extends the one originally given in [14, 15] for linear time-invariant elliptic and hyperbolic equations. Even though the main ideas are similar, the proofs are substantially different. A complete numerical realization of the iterative time-domain decomposition method is beyond the scope of this article and, instead, we refer to a forthcoming publication. However, to give some first insights into the interplay of the various parameters involved in the iteration process, we already provided some first numerical experiments in this paper.

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REFERENCES

- [1] A. T. Barker and M. Stoll. “Domain decomposition in time for PDE-constrained optimization.” In: *Computer Physics Communications* 197 (2015), pp. 136–143. DOI: [10.1016/j.cpc.2015.08.025](https://doi.org/10.1016/j.cpc.2015.08.025).
- [2] G. Bastin and J.-M. Coron. *Stability and boundary stabilization of 1-D hyperbolic systems*. Vol. 88. Progress in Nonlinear Differential Equations and their Applications. Subseries in Control. Birkhäuser/Springer, 2016, pp. xiv+307. DOI: [10.1007/978-3-319-32062-5](https://doi.org/10.1007/978-3-319-32062-5).
- [3] H. Bialy. “Iterative Behandlung linearer Funktionalgleichungen.” In: *Archive for Rational Mechanics and Analysis* 4 (1959), pp. 166–176. DOI: [10.1007/BF00281385](https://doi.org/10.1007/BF00281385).
- [4] M. Brokate. “Necessary optimality conditions for the control of semilinear hyperbolic boundary value problems.” In: *SIAM Journal on Control and Optimization* 25.5 (1985), pp. 1353–1369. DOI: [10.1137/0325074](https://doi.org/10.1137/0325074).
- [5] E. Casas and L. A. Fernández. “A Green’s formula for quasilinear elliptic operators.” In: *Journal of Mathematical Analysis and Applications* 142.1 (1989), pp. 62–73. DOI: [10.1016/0022-247X\(89\)90164-9](https://doi.org/10.1016/0022-247X(89)90164-9).
- [6] M. J. Gander and F. Kwok. “Schwarz methods for the time-parallel solution of parabolic control problems.” In: *Domain decomposition methods in science and engineering XXII*. Vol. 104. Lecture Notes in Computational Science and Engineering. Springer, Cham, 2016, pp. 207–216. DOI: [10.1007/978-3-319-18827-0_19](https://doi.org/10.1007/978-3-319-18827-0_19).
- [7] M. J. Gander, F. Kwok, and J. Salomon. “ParaOpt: a parareal algorithm for optimality systems.” In: *SIAM Journal on Scientific Computing* 42.5 (2020), A2773–A2802. DOI: [10.1137/19M1292291](https://doi.org/10.1137/19M1292291).
- [8] M. J. Gander and S. Vandewalle. “Analysis of the parareal time-parallel time-integration method.” In: *SIAM Journal on Scientific Computing* 29.2 (2007), pp. 556–578. DOI: [10.1137/05064607X](https://doi.org/10.1137/05064607X).
- [9] R. Glowinski and P. Le Tallec. “Augmented Lagrangian interpretation of the nonoverlapping Schwarz alternating method.” In: *Third International Symposium on Domain Decomposition Methods for Partial Differential Equations (Houston, TX, 1989)*. SIAM, Philadelphia, PA, 1990, pp. 224–231.
- [10] F. M. Hante, G. Leugering, A. Martin, L. Schewe, and M. Schmidt. “Challenges in Optimal Control Problems for Gas and Fluid Flow in Networks of Pipes and Canals: From Modeling to Industrial Applications.” In: *Industrial Mathematics and Complex Systems: Emerging Mathematical Models, Methods and Algorithms*. Ed. by P. Manchanda, R. Lozi, and A. H. Siddiqi. Industrial and Applied Mathematics. 2017, pp. 77–122. DOI: [10.1007/978-981-10-3758-0_5](https://doi.org/10.1007/978-981-10-3758-0_5).
- [11] M. Heinkenschloss. “A time-domain decomposition iterative method for the solution of distributed linear quadratic optimal control problems.” In: *Journal of Computational and Applied Mathematics* 173.1 (2005), pp. 169–198. DOI: [10.1016/j.cam.2004.03.005](https://doi.org/10.1016/j.cam.2004.03.005).
- [12] T. Kato. “Linear and quasi-linear equations of evolution of hyperbolic type.” In: *Hyperbolicity*. Vol. 72. C.I.M.E. Summer Schools. Springer, Heidelberg, 2011, pp. 125–191. DOI: [10.1007/978-3-642-11105-1_4](https://doi.org/10.1007/978-3-642-11105-1_4).
- [13] J. E. Lagnese and G. Leugering. *Domain decomposition methods in optimal control of partial differential equations*. Vol. 148. International Series of Numerical Mathematics. Birkhäuser Verlag, Basel, 2004, pp. xiv+443. DOI: [10.1007/978-3-0348-7885-2](https://doi.org/10.1007/978-3-0348-7885-2).
- [14] J. E. Lagnese and G. Leugering. “Time domain decomposition in final value optimal control of the Maxwell system.” In: *ESAIM: Control, Optimisation and Calculus of Variations* 8 (2002), pp. 775–799. DOI: [10.1051/cocv:2002042](https://doi.org/10.1051/cocv:2002042).
- [15] J. E. Lagnese and G. Leugering. “Time-domain decomposition of optimal control problems for the wave equation.” In: *Systems & Control Letters* 48.3-4 (2003), pp. 229–242. DOI: [10.1016/S0167-6911\(02\)00268-2](https://doi.org/10.1016/S0167-6911(02)00268-2).
- [16] G. Leugering, A. Martin, M. Schmidt, and M. Sirvent. “Nonoverlapping domain decomposition for optimal control problems governed by semilinear models for gas flow in networks.” In: *Control and Cybernetics* 46.3 (2017), pp. 191–225.
- [17] G. Leugering and C. Rodriguez. *Boundary feedback stabilization for the intrinsic geometrically exact beam model*. Tech. rep. 2020. URL: <https://arxiv.org/abs/1912.02543>.
- [18] G. Leugering and J. G. Schmidt. “On the modelling and stabilization of flows in networks of open canals.” In: *SIAM Journal on Control and Optimization* 41.1 (2002), pp. 164–180. DOI: [10.1137/S0363012900375664](https://doi.org/10.1137/S0363012900375664).

- [19] T. Li and L. Yu. “Local exact one-sided boundary null controllability of entropy solutions to a class of hyperbolic systems of balance laws.” In: *SIAM Journal on Control and Optimization* 57.1 (2019), pp. 610–632. doi: [10.1137/18M1187052](https://doi.org/10.1137/18M1187052).
- [20] T. Li and L. Yu. “One-sided exact boundary null controllability of entropy solutions to a class of hyperbolic systems of conservation laws.” In: *Journal de Mathématiques Pures et Appliquées* 107.1 (2017), pp. 1–40. doi: [10.1016/j.matpur.2016.04.005](https://doi.org/10.1016/j.matpur.2016.04.005).
- [21] J.-L. Lions, Y. Maday, and G. Turinici. “Résolution d’EDP par un schéma en temps “pararéel”.” In: *Comptes Rendus de l’Académie des Sciences - Series I - Mathematics* 332.7 (2001), pp. 661–668. doi: [10.1016/S0764-4442\(00\)01793-6](https://doi.org/10.1016/S0764-4442(00)01793-6).
- [22] P.-L. Lions. “On the Schwarz alternating method. III. A variant for nonoverlapping subdomains.” In: *Third International Symposium on Domain Decomposition Methods for Partial Differential Equations (Houston, TX, 1989)*. SIAM, Philadelphia, PA, 1990, pp. 202–223.
- [23] Y. Maday, M.-K. Riahi, and J. Salomon. “Parareal in time intermediate targets methods for optimal control problems.” In: *Control and optimization with PDE constraints*. Vol. 164. Internat. Ser. Numer. Math. Birkhäuser/Springer Basel AG, Basel, 2013, pp. 79–92.
- [24] Y. Maday, J. Salomon, and G. Turinici. “Monotonic time-discretized schemes in quantum control.” In: *Numerische Mathematik* 103.2 (2006), pp. 323–338. doi: [10.1007/s00211-006-0678-x](https://doi.org/10.1007/s00211-006-0678-x).
- [25] Y. Maday and G. Turinici. “A parareal in time procedure for the control of partial differential equations.” In: *Comptes Rendus Mathématique. Académie des Sciences. Paris* 335.4 (2002), pp. 387–392. doi: [10.1016/S1631-073X\(02\)02467-6](https://doi.org/10.1016/S1631-073X(02)02467-6).
- [26] H. Schaefer. “Über die Methode sukzessiver Approximationen.” In: *Jahresbericht der Deutschen Mathematiker-Vereinigung* 59 (1957), pp. 131–140.
- [27] W. Schmaedeke. “Mathematical theory of optimal control for semilinear hyperbolic systems in two independent variables.” In: *SIAM Journal on Control* 5 (1967), pp. 138–152. doi: [10.1137/0305008](https://doi.org/10.1137/0305008).
- [28] S. Ulbrich. “Generalized SQP methods with “parareal” time-domain decomposition for time-dependent PDE-constrained optimization.” In: *Real-time PDE-constrained optimization*. Vol. 3. Computational Science & Engineering. SIAM, 2007, pp. 145–168. doi: [10.1137/1.9780898718935.ch7](https://doi.org/10.1137/1.9780898718935.ch7).
- [29] S.-L. Wu and T.-Z. Huang. “A fast second-order parareal solver for fractional optimal control problems.” In: *Journal of Vibration and Control* 24.15 (2018), pp. 3418–3433. doi: [10.1177/1077546317705557](https://doi.org/10.1177/1077546317705557).
- [30] S.-L. Wu and J. Liu. “A parallel-in-time block-circulant preconditioner for optimal control of wave equations.” In: *SIAM Journal on Scientific Computing* 42.3 (2020), A1510–A1540. doi: [10.1137/19M1289613](https://doi.org/10.1137/19M1289613).

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