On Recognizing Staircase Compatibility

Andreas Bärmann[1], Patrick Gemander[2], Alexander Martin[1] and Maximilian Merkert[3]

[1] Andreas.Baermann@fau.de
Alexander.Martin@fau.de
Lehrstuhl für Wirtschaftsmathematik,
Department Mathematik,
Friedrich-Alexander-Universität Erlangen-Nürnberg,
Cauerstraße 11, 91058 Erlangen, Germany

[2] Patrick.Gemander@fau.de
Gruppe Optimization
Fraunhofer-Arbeitsgruppe für Supply Chain Services SCS,
Fraunhofer-Institut für Integrierte Schaltungen IIS,
Nordostpark 93, 90411 Nürnberg, Germany

[3] Maximilian.Merkert@ovgu.de
Institut für Mathematische Optimierung,
Fakultät für Mathematik,
Otto-von-Guericke-Universität Magdeburg,
Universitätsplatz 2, 39106 Magdeburg, Germany

Abstract

For the problem to find an m-clique in an m-partite graph, staircase compatibility has recently been introduced as a polynomial-time solvable special case. It is a property of a graph together with an m-partition of the vertex set and total orders on each subset of the partition. In optimization problems involving m-cliques in m-partite graphs as a subproblem, it allows for totally unimodular linear programming formulations which have shown to efficiently solve problems from different applications. In this work, we address questions concerning the recognizability of this property in the case where the m-partition of the graph is given, but suitable total orders are to be determined. While finding these total orders is NP-hard in general, we give several conditions under which it can be done in polynomial time. For bipartite graphs, we present a polynomial-time algorithm to recognize staircase compatibility, and show that staircase total orders are unique up to a small set of reordering operations. On m-partite graphs, where the recognition problem is NP-complete in the general case, we identify a polynomially solvable subcase and also provide a corresponding algorithm to compute the total orders. Finally, we evaluate the performance of our ordering algorithm for m-partite graphs on a set of artificial instances as well as real-world instances from a railway timetabling application. It turns out that applying the ordering algorithm to the real-world instances and subsequently solving the problem via the aforementioned totally unimodular reformulations indeed outperforms a generic formulation which does not exploit staircase compatibility.

Keywords: Staircase Structure, Clique Problem, Multiple-Choice Constraints, Scheduling

Mathematics Subject Classification: 90C27 - 90C35 - 90C57 - 90C90

1 Introduction

The clique problem is among the most prominent combinatorial optimization problems studied in the literature. The interest in this problem is closely related to its importance for the modelling of applied optimization tasks, such as e.g. scheduling problems. In order to solve practical problems involving cliques efficiently, it is usually important to exploit the structure of the underlying graph as far as possible. Being able to recognize special cases of the problem from the incidence structure of the graph can beneficially inform both the choice of the optimization model used and the derivation of fast solution algorithms.

In this work, we focus on recognizing a polynomially solvable subcase of the problem to find an m-clique in an m-partite graph. This special case is called *staircase compatibility* and allows for totally unimodular linear programming formulations of polynomial size, see [BGMS18]. In particular, the formulation based on a dual network flow problem introduced there has shown to be very efficient for solving problems arising in different applications, such as railway timetabling ([BMS20]) and flow problems with piecewise-linear arc costs ([LM16]). In [JM18], the authors present a new formulation for the vertex colouring problem based on the same dual-flow formulation and show that it is computationally superior on many instances. Although their instances do not have the staircase property, a reformulation trick allows them to use this formulation anyway. From these successes, there arises a natural interest to analyse the problem of recognizing the staircase property on a given graph $G = (V, E)$ and to generalize the applicability of the aforementioned dual-flow formulation for the clique problem.

At its core, staircase recognition reduces to finding suitable total orders on the subsets of a partition of the vertex set of the given graph. By way of incidence matrices, staircase compatibility gives an alternate approach to so-called staircase matrices. Namely, a bipartite graph has the staircase property if its adjacency matrix can be re-sorted into *completely dense* staircase form, i.e. the non-zeros in each row form a single block which is not interrupted by zero-elements. This way, staircase compatibility is closely linked to staircase matrices, which are an important structure in applied mathematics, cf. [Fou84]. These matrices are the natural target form to be achieved when performing Gauss-Jordan elimination and allow for efficient algorithms to solve linear programs (see [Fou82a, Fou82b]), to name only two examples.

Constructing conflict graphs from constraint matrices is one possible way to find clique structures in mixed-integer programming problems, see e.g. [BS19, ABG$^+$20]. Apart from staircase compatibility, m-cliques in an m-partite graph can also be found in polynomial time if the so-called dependency graph is a forest; see [BGM20b], where the problem is studied under the name *clique problem with multiple-choice constraints*. For polyhedral properties of the general clique problem we refer to [Bor98, BBPP99].

Contribution We study the problem to recognize the staircase property in a graph with a given m-partition of the vertex set. For bipartite graphs, we can check the staircase property in time $\mathcal{O}(|V| + |E|)$, and the algorithm can easily be extended to the case of m-partite graphs. Furthermore, we can give a full characterization of staircase orderings on bipartite graphs by showing that they are unique up to a few trivial operations on the two total orders they consist of. We will see that on bipartite graphs, the computation of staircase orderings is possible in polynomial time as well. For this task, we derive an algorithm which runs in $\mathcal{O}(|V_1|(|V| + |E|))$, where V_1 can be chosen to be the smaller of the two bipartite subsets of vertices. This scheme can be extended to a polynomial-time algorithm for computing staircase orderings on m-partite graphs where all bipartite subgraphs G_{ij} induced by two subsets V_i and V_j of the partition are connected. For general m-partite graphs, however, we show that deciding existence of a staircase ordering is NP-complete. Finally, if a given ordering is not staircase on an m-partite graph, it is possible to establish the staircase property by adding new edges to the graph. The problem of finding an ordering with the minimal number of edges to be added to the graph

such that the staircase property is fulfilled is NP-complete even in the case that all bipartite subgraphs G_{ij} are connected.

These results are particularly interesting because in many cases they allow for efficiently checking practical instances of m-clique problems on m-partite graphs for the staircase property of their underlying graph. Such staircase-sortable instances are then solvable in polynomial time as a linear program (LP), as shown in [BGMS18]. In a computational study, we will demonstrate that our algorithms can efficiently check for the staircase property. Furthermore, for instances from a real-world timetabling problem we will see that first resorting them to be staircase and then solving them via the formulation from [BGMS18] is significantly faster than using a generic model for the problem.

Structure The remainder of the article is organized as follows. Section 2 introduces the required notation and definitions to describe the problem. We then focus on bipartite graphs in Section 3, where we consider the recognition and uniqueness of staircase orderings and present a polynomial-time algorithm to compute staircase orderings. In Section 4, we extend our results to m-partite graphs. We first show that recognizing staircase partitions is NP-complete in general. Then we present an algorithm to solve the problem in a polynomially solvable special case. Finally, in Section 5 we present the results of our computational results on artificial instances as well as real-world instances from a railway timetabling application.

2 Notation

The following notation is used throughout the article. We denote the *neighbourhood* of a vertex $u \in V$ in an undirected graph $G = (V, E)$ by $N_G(u) := \{v \in V \mid \{u, v\} \in E\}$, and for subsets $U \subseteq V$ we analogously write $N_G(U) := \bigcup_{u \in U} N_G(u)$. If it is clear which graph is referred to, we may simply write $N(u)$. The following definition introduces some fundamental graph structures related to the clique problem on m-partite graphs.

Definition 2.1. (*Subgraphs G_{ij} and Dependency Graph $\mathcal{G}$*) *Let $G = (V, E)$ be an m-partite graph together with a partition $\mathcal{V} = \{V_1, \dots, V_m\}$ of V. For two subsets $V_i, V_j \in \mathcal{V}$ with $i \neq j$, we write*

$$G_{ij} := (V_i \cup V_j, E_{ij})$$

for the subgraph of G induced by $V_i \cup V_j$, where E_{ij} is the corresponding edge set. Note that all subgraphs G_{ij} are bipartite.

Further, we call $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with

$$\mathcal{E} := \left\{ \{V_i, V_j\} \subseteq \mathcal{V} \mid i \neq j \wedge \exists u \in V_i \colon N_{G_{ij}}(u) \neq V_j \right\}$$

the dependency graph *of G. Note that $\{V_i, V_j\} \in \mathcal{E}$ is equivalent to G_{ij} not being a complete bipartite graph.*

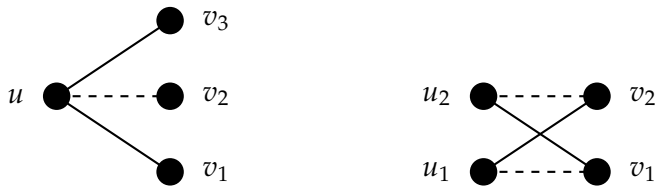
The main focus of our work is the so-called *staircase property* of a graph together with an m-partition of its vertex set and total orders on each subset. This property establishes local structures on all bipartite subgraphs G_{ij} and can be used to efficiently solve certain clique problems on the graph. It is defined as follows.

Definition 2.2. (*Ordering, Staircase Ordering*) *Let $G = (V, E)$ be an m-partite graph, together with a partition $\mathcal{V} = \{V_1, \dots, V_m\}$ of V and total orders $>_i$ on the subsets $V_i \in \mathcal{V}$. For simplicity, we refer to the set $\{>_1, \dots, >_m\}$ of total orders as an* ordering *on G if $\mathcal{V}$ is clear from the context. An ordering on G is* staircase *if for all subgraphs G_{ij} of G the folllowing two conditions hold:*

$$u \in V_i \wedge v_1 >_j v_2 >_j v_3 \in V_j \wedge \{u, v_1\}, \{u, v_3\} \in E_{ij} \Rightarrow \{u, v_2\} \in E_{ij} \tag{SC1}$$

$$u_1 >_i u_2 \in V_i \wedge v_1 >_j v_2 \in V_j \wedge \{u_1, v_2\}, \{u_2, v_1\} \in E_{ij} \Rightarrow \{u_1, v_1\}, \{u_2, v_2\} \in E_{ij}. \tag{SC2}$$

Condition (SC1) ensures that the neighbourhoods of all vertices are continuous with respect to the total order, whereas (SC2) yields a kind of monotonicity on the edge set E_{ij}. The two conditions are illustrated in Fig. 1.



(a) Illustration of (SC1) (b) Illustration of (SC2)

Figure 1: Illustration of the two staircase conditions. If the solid edges are contained in the graph, the dashed ones must be contained as well.

Note that both (SC1) and (SC2) are trivially satisfied for complete bipartite subgraphs G_{ij}. Therefore, it is sufficient for the ordering on G to be staircase if they hold for those G_{ij} with $\{V_i, V_j\} \in \mathcal{E}$. From the definition it also immediately follows that removing vertices from the graph does not destroy the staircase property. Moreover, it is easy to show that (SC1) is implied by (SC2) if G_{ij} does not contain vertices with degree zero. In the next definition, we canonically extend the notion of a staircase ordering onto a partition $\mathcal{V}$ on V.

Definition 2.3. (*Staircase Partition*) *Let $G = (V, E)$ be an m-partite graph with vertex set partition $\mathcal{V} = \{V_1, \dots, V_m\}$. We call $\mathcal{V}$* staircase *if there exists a staircase ordering on $\mathcal{V}$.*

We close this section with some notation used to more concisely describe our algorithms.

Definition 2.4. (*Minimal and Maximal Neighbour Index*) *Let $G = (V_1 \cup V_2, E)$ be a bipartite graph with total orders $>_1, >_2$ on V_1 and V_2, respectively, and $u \in V_1$ with $\deg_G(u) \geq 1$. We write*

$$\underline{\lambda}_{V_2}(u) := |\{v \in V_2 \mid w >_2 v \,\forall w \in N_G(u)\}| + 1$$
$$\overline{\lambda}_{V_2}(u) := |V_2| - |\{v \in V_2 \mid v >_2 w \,\forall w \in N_G(u)\}|$$

for the minimal and maximal index of any neighbour of u in V_2 in the total order $>_2$.

3 Staircase Orderings on Bipartite Graphs

We start by presenting our results for staircase orderings on bipartite graphs. First, we state an algorithm to determine within polynomial time whether a given ordering on a bipartite graph is staircase. In Section 3.2, we then show that a staircase ordering on a bipartite graph is unique up to some easily recognizable situations where the order of vertices may be changed. Finally, an algorithm to compute a staircase ordering on a given bipartite graph within polynomial time is given in Section 3.3.

3.1 Recognizing Staircase Orderings

In this section, we will give a polynomial-time algorithm to verify whether a given ordering on a bipartite graph is staircase. Let $G = (V_1 \cup V_2, E)$ with $V_1 = \{u_1, \dots, u_{|V_1|}\}$ and $V_2 = \{v_1, \dots, v_{|V_2|}\}$ be a bipartite graph and consider the *canonical ordering* on G, given by the total orders

$$(\forall u_i, u_j \in V_1)\, u_i >_1 u_j :\Leftrightarrow i > j \text{ and}$$
$$(\forall v_i, v_j \in V_2)\, v_i >_2 v_j :\Leftrightarrow i > j.$$

We are able to show that it can be checked in polynomial time whether the canonical ordering on G is staircase.

Lemma 3.1. *It can be checked in time $O(|V| + |E|)$ whether the canonical ordering $\{>_1, >_2\}$ on a bipartite graph $G = (V_1 \cup V_2, E)$ with $V_1 = \{u_1, \ldots, u_{|V_1|}\}$ and $V_2 = \{v_1, \ldots, v_{|V_2|}\}$ is staircase.*

Proof. Note that for any triplet $u_i, u_j, u_k \in V_1$, $i > j > k$ where u_i and u_k are in the same connected component C of G, Condition (SC1) ensures that u_j is in C as well. In other words, all vertices belonging to C appear consecutively in their respective total orders $>_1$ and $>_2$, which can be verified in time $O(|V| + |E|)$ using breadth-first search. Apart from this requirement, the staircase conditions can be checked on each connected component of G separately. For the remainder of the proof, we therefore assume w.l.o.g. that G is connected. The following algorithm then solves the problem.

Algorithm 1 Recognizing a staircase ordering on a bipartite graph

Input: A connected bipartite graph $G = (V_1 \cup V_2, E)$ where $V_1 = \{u_1, \ldots, u_{|V_1|}\}$ and $V_2 = \{v_1, \ldots, v_{|V_2|}\}$

Output: Returns whether the canonical ordering on G is staircase

1: **if** $\overline{\lambda}_{V_2}(u_1) - \underline{\lambda}_{V_2}(u_1) + 1 > |N_G(u_1)|$ **then**
2: **return** "The canonical ordering violates (SC1)"
3: **end if**
4: **for** $i \in \{2, \ldots, |V_1|\}$ **do**
5: **if** $\underline{\lambda}_{V_2}(u_i) < \underline{\lambda}_{V_2}(u_{i-1})$ **or** $\overline{\lambda}_{V_2}(u_i) < \overline{\lambda}_{V_2}(u_{i-1})$ **then**
6: **return** "The canonical ordering violates (SC2)"
7: **end if**
8: **if** $\overline{\lambda}_{V_2}(u_i) - \underline{\lambda}_{V_2}(u_i) + 1 > |N_G(u_i)|$ **then**
9: **return** "The canonical ordering violates (SC1)"
10: **end if**
11: **end for**
12: **return** "The canonical ordering on G is staircase"

In Line 1, we directly check (SC1) for the neighbourhood of u_1. Afterwards, we check (SC1) for all remaining $u_i \in V_1$ and (SC2) for certain key vertices of G. The following case distinction shows that these checks indeed suffice to decide whether the canonical ordering on G is staircase:

- Case 1 – The canonical ordering on G fulfils (SC1) and (SC2):
  It is clear that the checks in Lines 1 and 8 do not fail as they are equivalent to (SC1) with the vertex $u \in V_1$. Suppose there exists some $i \in \{2, \ldots, |V_1|\}$ such that $k := \underline{\lambda}_{V_2}(u_i) < j := \underline{\lambda}_{V_2}(u_{i-1})$, i.e. we fulfil the first condition of the check in Line 5. Then we have $u_i >_1 u_{i-1}$ as well as $v_j >_2 v_k$. Moreover, we have $\{u_i, v_k\}, \{u_{i-1}, v_j\} \in E$ and $\{u_{i-1}, v_k\} \notin E$ by definition of $\underline{\lambda}_{V_2}(u_{i-1})$. However, this is a contradiction to (SC2). Therefore, $\underline{\lambda}_{V_2}(u_i) \geq \underline{\lambda}_{V_2}(u_{i-1})$ must hold for all $i \in \{2, \ldots, |V_1|\}$. Analogously, we get that $\overline{\lambda}_{V_2}(u_i) \geq \overline{\lambda}_{V_2}(u_{i-1})$ holds for all $i \in \{2, \ldots, |V_1|\}$. Hence, Algorithm 1 works correctly if the canonical ordering is staircase.

- Case 2a – The canonical ordering violates (SC2):
  In this case, we can find $u_{i_1}, u_{i_2} \in V_1, v_{j_1}, v_{j_2} \in V_2$ with $i_1 < i_2, j_1 < j_2$ and edges $\{u_{i_1}, v_{j_2}\}, \{u_{i_2}, v_{j_1}\} \in E$ such that $\{u_{i_1}, v_{j_1}\} \notin E$ or $\{u_{i_2}, v_{j_2}\} \notin E$ holds. Assume we have $\{u_{i_2}, v_{j_2}\} \notin E$. If $N_G(u_{i_2})$ contains a vertex v_{j_3} with $j_2 < j_3$, the check in Line 8 will fail for $i = i_2$. Therefore, we can assume w.l.o.g. $j_1 \leq \overline{\lambda}_{V_2}(u_{i_2}) < j_2$. However, this immediately

yields $j_1 \leq \overline{\lambda}_{V_2}(u_{i_2}) < j_2 \leq \overline{\lambda}_{V_2}(u_{i_1})$. Hence, for some $i \in \{i_1+1, ..., i_2\}$ the check in Line 5 fails. Analogously, either of the checks in Line 5 or 8 must fail if $\{u_{i_1}, v_{j_1}\} \notin E$.

- Case 2b – The canonical ordering violates (SC1):
  If we find $u \in V_1, v_1, v_2, v_3 \in V_2$ such that (SC1) is violated, the check in Line 1 or the one in Line 8 fails, depending on whether $u = u_1$ or $u = u_i, i \in \{2, \ldots, |V_2|\}$. Otherwise, we can find $u_1, u_2, u_3 \in V_1, v \in V_2$ such that (SC2) is violated, i.e. we have $\{u_1, v\}, \{u_3, v\} \in E$ while $\{u_2, v\} \notin E$. However, the vertex u_2 is by assumption not isolated and we can therefore find some vertex $v' \in V_2$ with $\{u_2, v'\} \in E$. As Fig. 2 illustrates, the quadruple u_1, u_2, v', v violates (SC2) if $v >_2 v'$ and the quadruple u_2, u_3, v, v' violates (SC2) if $v' >_2 v$. In both cases, we can apply the arguments used in Case 2b.

To summarize, Algorithm 1 correctly recognizes a canonical ordering which is staircase. For non-staircase canonical orderings, at least one of the checks performed in Lines 1, 5 and 8 fails. The running time of Algorithm 1 is $O(|V| + |E|)$, as the computation of all $\underline{\lambda}_{V_2}(u)$ and $\overline{\lambda}_{V_2}(u)$ as well as the preprocessing step to identify the connected components of G can be done within this amount of time.

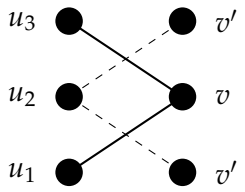


Figure 2: Illustration of reduction of Case 2b to Case 2a

□

3.2 Uniqueness of Staircase Orderings

We have just established that recognizing whether an ordering on a bipartite graph is staircase can be done in polynomial time. Now we will further analyse the structure of staircase orderings on bipartite graphs. More precisely, we will show that it is possible to explicitly state all operations under which a given ordering remains staircase. This gives a full characterization of staircase orderings on bipartite graphs as they are indeed unique up to the following three operations.

First, it is easy to see that w.r.t. an ordering being staircase we can consider each connected component separately. Second, if we have $N_G(u) = N_G(v)$ for a pair of vertices in a given subset V_i, we can remove one of the two, consider the remaining graph and reinsert the removed vertex at the end. We will therefore assume, for a given bipartite graph $G = (V_1 \cup V_2, E)$, the following two conditions w.l.o.g.:

$$G \text{ is connected} \tag{1}$$

$$(\forall u, v \in V, u \neq v)\, N_G(u) \neq N_G(v). \tag{2}$$

Third, any given staircase ordering remains staircase if all total orders are reversed at once, to which we refer as *reversing* the ordering. Note that this also holds for staircase orderings on m-partite graphs with $m > 2$. Theorem 3.3 states that a given staircase ordering on a bipartite graph is in fact unique up to the operations described above. Before proving Theorem 3.3, which is the main result of this section, the following auxiliary lemma gives additional insight on the lengths of shortest paths in bipartite graphs with a given staircase ordering.

Lemma 3.2. *Let G be a connected bipartite graph together with a staircase ordering $\{>_1, >_2\}$. Further, let $u, v, w \in V_1$ such that $u >_1 v >_1 w$, and let P_{uv} and P_{uw} be shortest paths between u and v or w, respectively. Then $|P_{uv}| \leq |P_{uw}|$.*

Proof. We prove the claim by contradiction. Suppose there exist vertices $u, v, w \in V_1$ with $u >_1 v >_1 w$ and shortest paths $P_{uv} = (u, \ldots, v), P_{uw} = (u, \ldots, w', x, w)$ such that $|P_{uv}| > |P_{uw}|$. Note that P_{uw} is of even length greater or equal to two, as G is bipartite. We distinguish two cases:

- Case 1 – $w' >_1 v$:
  If we have $v \in N(x)$, then P_{uv} is no longer a shortest path since we can construct a shorter one by simply choosing v instead of w as the last vertex in P_{uw}. Hence, $v \notin N(x)$ must hold. However, Fig. 3 illustrates that this already violates (SC1).

- Case 2 – $v >_1 w'$:
  In this case, we have $u >_1 v >_1 w'$ and $|P_{uv}| > |P_{uw'}| = |P_{uw}| - 2$. Therefore, we can start from the beginning with w' assuming the role of w. This must eventually yield a contradiction because the path P_{uw} is shortened by two, and once its length is equal to two this second case can no longer occur due to $u = w'$ and $u >_1 v >_1 w$.

□

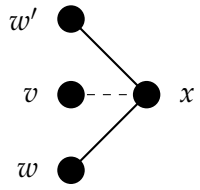


Figure 3: Contradiction for $w' >_1 v$

The condition that G be connected in Lemma 3.2 is not strictly necessary and could be replaced by assuming the paths P_{uv} and P_{uw} to actually exist. Then it would suffice to consider only the induced (connected) subgraph. With this observation in mind, it may seem that the lemma above yields an alternate characterization of the staircase property. However, as the example in Fig. 4 shows, this is not generally the case, even if G is connected. Nonetheless, Lemma 3.2 allows us to prove the main result of this section.

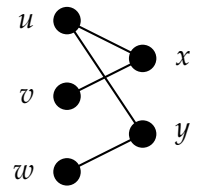
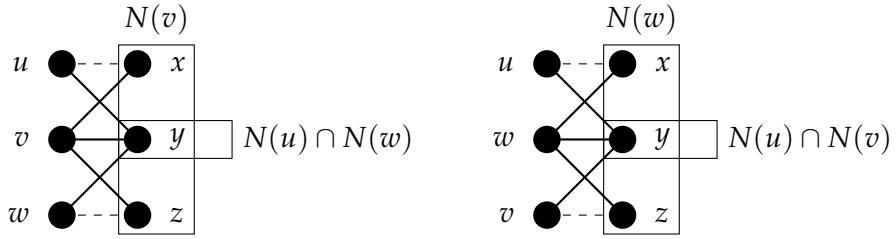


Figure 4: Although the paths P_{uv} and P_{uw} are of equal length, the ordering shown is not staircase since the quadruple u, v, x, y violates (SC2).

Theorem 3.3. *Let $G = (V_1 \cup V_2, E)$ be a bipartite graph which satisfies Conditions (1) and (2). Then a staircase ordering $\{>_1, >_2\}$ on G is unique up to reversal.*

Proof. Suppose there are two different staircase orderings $\{>_1, >_2\}$ and $\{\succ_1, \succ_2\}$ that cannot be transformed into each other by reversing either ordering.

Since the graph G is connected but not complete bipartite, we can find vertices $u, v \in V_1$ and $x, y \in V_2$ that induce the subgraph shown in Fig. 5.



(a) $N(v) \setminus N(w)$ must be empty (b) $N(w) \setminus N(v)$ must be empty

Figure 6: Violations of (SC2)

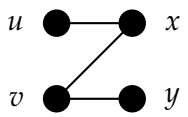


Figure 5: Induced subgraph in G that implies the same order between u, v as between x, y.

In this subgraph, we have

$$u >_1 v \Leftrightarrow x >_2 y \text{ and}$$
$$u \succ_1 v \Leftrightarrow x \succ_2 y$$

due to Condition (SC2). Consequently, it is not possible to retain the staircase conditions by reversing only one total order of a given ordering. In other words, we can assume w.l.o.g. that there exist vertices $u, v \in V_1$ such that $u >_1 v \wedge v \succ_1 u$ holds.

After possibly reversing all four total orders at once as well as possibly switching V_1 and V_2, we can, by the arguments above, find vertices $u, v, w \in V_1$ with $u >_1 v >_1 w$ and $u \succ_1 w \succ_1 v$. Note that in this case, Lemma 3.2 directly implies $|P_{uv}| = |P_{uw}|$.

We now show that we can additionally assume the paths P_{uv} and P_{uw} to be of length two. If P_{uw} is of length greater or equal to four, let $P_{uw} = (u, \ldots, w', x, w)$. Then the path $P_{uw'}$ is shorter than P_{uw}, and with the contraposition of Lemma 3.2 we have $u >_1 w' >_1 v >_1 w$ as well as $\{w', x\}, \{w, x\} \in E$. Hence, the edge $\{v, x\}$ must be contained in E to not violate (SC1). Renaming w' to u now yields the desired assumption.

As the shortest paths P_{uv} and P_{uw} are of length two, we have that $N(u) \cap N(v)$ and $N(u) \cap N(w)$ are both non-empty. From (SC1) and $u >_1 v >_1 w$ it follows that $N(u) \cap N(w) \subseteq N(u) \cap N(v)$. Analogously, from (SC1) and $u \succ_1 w \succ_1 v$ we can also conclude $N(u) \cap N(v) \subseteq N(u) \cap N(w)$ and hence, $N(u) \cap N(v) = N(u) \cap N(w) \neq \emptyset$.

Finally, Figures 6a and 6b illustrate that $N(v) \setminus N(w)$ and $N(w) \setminus N(v)$ must both be empty, as otherwise (SC2) is violated. However, we then have $N(v) = N(w)$, which is a contradiction to Condition 2. This completes the proof. □

This theorem fully characterizes the operations which can be applied to a staircase ordering to retain the staircase property.

3.3 Computing Staircase Orderings

We next introduce a polynomial-time algorithm that computes a staircase ordering on a given bipartite graph if possible, and otherwise states that no such ordering exists. As in the previous section, we may consider each connected component of a bipartite graph separately. The following algorithm then solves the problem.

Algorithm 2 Finding a staircase ordering on a bipartite graph

Input: A connected bipartite graph $G = (V_1 \cup V_2, E)$

Output: Lists L_1, L_2 of the vertices in V_1, V_2 representing a staircase ordering on G or the result that no staircase ordering exists

1: **for** $r \in V_1$ **do**
2: Apply breadth-first search to G starting in r to sort vertices in $V_1 \cup V_2$ into sets $N_i = \{v \in V_1 \cup V_2 \mid \text{shortest path from } r \text{ to } v \text{ is of length } i\}$, and let $i_{\max}$ be the length of the longest shortest path in G starting in r.
3: $L_1 = [r], L_2 = [\,]$
4: **for** $N_i = N_1, N_2, \ldots, N_{i_{\max}}$ **do**
5: Let H be the subgraph of G induced by $N_{i-1} \cup N_i$.
6: Sort vertices $u \in N_i$ in descending order by their degree in H.
7: **if** $i < |\mathcal{N}|$ **then**
8: Let H' the subgraph of G induced by $N_i \cup N_{i+1}$
9: Sort vertices $u \in N_i$ with equal degree in H in ascending order by their degree in H'.
10: **end if**
11: **if** i even **then**
12: Append the (now ordered) elements in N_i to the list L_2
13: **else**
14: Append the (now ordered) elements in N_i to the list L_1
15: **end if**
16: **end for**
17: **if** isStaircase$(L_1 \cup L_2, E)$ **then**
18: **return** L_1, L_2
19: **end if**
20: **end for**
21: **return** "A staircase ordering on G does not exist"

There are no immediately obvious candidates for the uppermost vertex in a staircase ordering. Hence, the outer for-loop iterates over all vertices of V_1. For each choice $r \in V_1$, the inner for-loop constructs an ordering with r as the uppermost vertex. The constructed ordering is then tested, e.g. by Algorithm 1, whether it is staircase. We prove the correctness of Algorithm 2 in Lemma 3.4, and show in Lemma 3.5 that its runtime is $O(|V_1|(|V| + |E|))$.

Lemma 3.4. *Algorithm 2 is correct.*

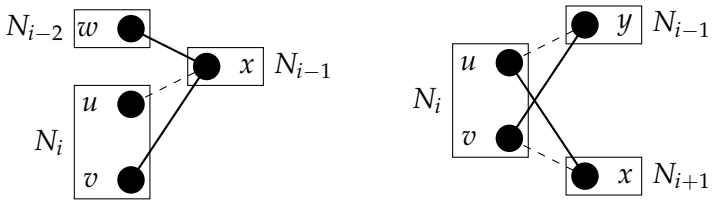
Proof. It is clear that the algorithm terminates in all cases.

If no staircase ordering exists on G, the staircase test in Line 16 will always fail and the algorithm correctly returns that no staircase ordering exists on G.

Otherwise, the outer for-loop will eventually choose a root $r \in V_1$ as the uppermost vertex of the total order $>_1$ such that a staircase ordering can be constructed. It remains to show that the operations within the inner for-loop are either strictly necessary to create a staircase ordering or simply choose one of multiple valid options.

Due to the contraposition of Lemma 3.2, vertices must be ordered by the length of the shortest path to the root vertex r. This is done by the combination of sorting the vertices into subsets N_i during the breadth-first search and then appending them to the lists L_1 and L_2 in the correct order. Second, the exact order of vertices within each N_i is determined by the operations in Lines 6 and 9. Depending on which operation determines the order between two vertices $u, v \in N_i$ we distinguish three cases. For ease of notation, we write $>_*$ to refer to either $>_1$ or $>_2$ depending on whether the vertices are contained in V_1 or V_2.

- $\deg_H(u) < \deg_H(v)$:
  This case can only occur if $|N_i| > 1$ as well as $|N_{i-1}| > 1$ hold. Due to $|N_0| = 1$, we have



(a) Case $\deg_H(u) < \deg_H(v)$ (b) Case $\deg_{H'}(u) > \deg_{H'}(v)$

Figure 7: Subcases for proof of Lemma 3.4

$i \geq 2$. Further, we can find a vertex $x \in N_{i-1}$ with $\{u, x\} \notin E, \{v, x\} \in E$. The vertex x, in turn, must be connected to some vertex $w \in N_{i-2}$. As illustrated in Fig. 7a, $v >_* u$ must hold as otherwise (SC1) is violated.

- $\deg_H(u) = \deg_H(v) \wedge \deg_{H'}(u) > \deg_{H'}(v)$:
  Here, $|N_1| > 1$ must hold and we therefore have $i \geq 1$. Analogously to the previous case, we can find a vertex $x \in N_{i+1}$ with $\{u, x\} \in E$ and $\{v, x\} \notin E$. Fig. 7b illustrates that, in order to not violate (SC2), $v >_* u$ must hold.

- $\deg_H(u) = \deg_H(v) \wedge \deg_{H'}(u) = \deg_{H'}(v)$:
  Suppose there are vertices $x, y \in N_{i-1}$ with $\{u, x\}, \{v, y\} \in E, \{v, x\}, \{u, y\} \notin E$, which can only occur if $i \geq 2$. Then, by the same arguments as in the first case, we get $u >_* v$ as well as $v >_* u$, which is a contradiction. Therefore, we can conclude $N_H(u) = N_H(v)$. Analogously, we can apply the arguments from the second case to conclude $N_{H'}(u) = N_{H'}(v)$. In total, we have $N(u) = N(v)$, and hence, u and v can be ordered arbitrarily.

Altogether, all sorting operations are either strictly necessary for an ordering to be staircase, or simply represent a choice between multiple valid orders. Therefore, the algorithm does find a staircase ordering if and only if one exists. □

Lemma 3.5. *Algorithm 2 runs in time $O(|V_1|(|V| + |E|))$.*

Proof. Breadth-first search has time-complexity $O(|V| + |E|)$. Each set N_i can be sorted in time $O(|N_i| + \max_{v \in N_i} deg_G(v))$ via counting sort. The time complexity of the combined time required for sorting can be bounded by the estimate

$$\sum_{i=1}^{k}(|N_i| + \max_{v \in N_i} deg_G(v)) \leq \sum_{i=1}^{k}|N_i| + \sum_{i=1}^{k}\max_{v \in N_j} deg_G(v) \leq |V| + |E|.$$

As stated by Lemma 3.1, each computed ordering can be checked in time $O(|V| + |E|)$ by Algorithm 1.

In total, an iteration of the outer for-loop takes $O(|V| + |E|)$ time. Since there are at most $|V_1|$ iterations, this completes the proof. □

In Sections 3.1 to 3.3, we have studied staircase orderings on bipartite graphs and concluded that we can determine an ordering on a bipartite graph to be staircase withing polynomial time and also compute one in polynomial time. Further, a staircase ordering on a bipartite graph is unique up to changing the order of connected components, changing the order of vertices with equal neighbourhood and reversing the entire ordering. We will see in the next section that the recognition problem on m-partite graphs with given partition is NP-complete in the general case. However, we will identify a special case which is solvable in polynomial time.

4 Staircase Orderings on m-partite Graphs

We now consider the staircase recognition problem on m-partite graphs. More precisely, the partition $\mathcal{V} = \{V_1, \ldots, V_m\}$ on the vertex set of the graph is assumed to be given and it is to be determined whether $\mathcal{V}$ is a staircase partition, i.e. if there exist corresponding staircase orderings $\{>_1, \ldots, >_m\}$ on its subsets.

It can be shown that recognizing staircase partitions in the general case is NP-complete. In Section 4.1, we will give a proof that uses a reduction from the NP-complete *Betweenness Problem*, which has been presented in the dissertation [Mer17]. In the special case where all bipartite subgraphs G_{ij} of an m-partite graph are connected, we will see in Section 4.2 that Algorithm 2 can be extended to compute staircase orderings within polynomial time.

4.1 Complexity of Recognizing Staircase Partitions

We start by showing that recognizing a staircase partition is NP-complete on general m-partite graphs.

Theorem 4.1. *Let $G = (V, E)$ be an m-partite graph with vertex set partition $\mathcal{V} = \{V_1, \ldots, V_m\}$. The problem to decide whether a staircase ordering on G exists is NP-complete, even if the dependency graph $\mathcal{G}$ is cycle-free.*

Proof. We prove the theorem by giving a polynomial-time reduction from the *Betweenness Problem*. In an instance of Betweenness, a ground set M is given together with a set $\mathcal{M}$ of ordered triples of elements from M. One has to determine whether there exists a total order on M such that the middle item of each given triple is placed somewhere between the other two items. This problem is known to be NP-complete [Opa79].

From an instance of Betweenness, we construct an equivalent instance of the recognition problem for staircase partitions as follows: We have a special subset $V_0 = M$, where each vertex is identified with an item from Betweenness. For the triples $t = (u, v, w) \in \mathcal{M}$, we construct subsets consisting of 3 vertices each that are identified with the items from the respective triple (but of course different from the vertices in V_0). Each vertex in such a subset is solely compatible with the vertex in V_0 corresponding to the same item. Moreover, each subset V_t representing a triple t from $\mathcal{M}$ comes together with an auxiliary subset V_t' consisting of two vertices. This subset V_t' has dependencies only with V_t and these are designed to force v_t, the designated middle element of the triple, to be sorted in between the other two elements in V_t (see Fig. 8). Note that there are exactly two possibilities for choosing orderings on V_t and V_t', which correspond to the two possible orderings of the elements u, v, w from t that put v in the middle. The bipartite subgraphs between subsets or auxiliary subsets corresponding to different triples are complete bipartite graphs such that the dependency graph $\mathcal{G}$ is a tree of depth 2 with root vertex V_0 and leaves V_t', $t \in \mathcal{M}$.

On the one hand, any solution of Betweenness gives us a total order on M that we can use for V_0. This ordering will ensure that the middle element of each V_t for $t \in \mathcal{S}$ is between the other elements of t. Therefore, there will be no crossing of edges between V_t and V_0, and hence no violation of (SC2) if the ordering on V_t and V_t' is oriented correctly. On the other hand, every choice of total orders on the subsets that yield a staircase ordering on G immediately gives us an ordering of V_0 that is a solution to Betweenness. Finally, the transformation of instances is clearly polynomial in the encoding size of the input. □

We would like to remark that for an instance of the recognition problem constructed as in the above proof, no ordering can possibly violate (SC1) without also violating (SC2), which means that finding orderings that just satisfy (SC2) is also NP-complete.

Generally, instances in the proof of Theorem 4.1 feature subgraphs G_{ij} with isolated vertices. Although the general staircase ordering problem is NP-complete, we can show that the problem becomes solvable in polynomial time under an additional connectedness assumption.

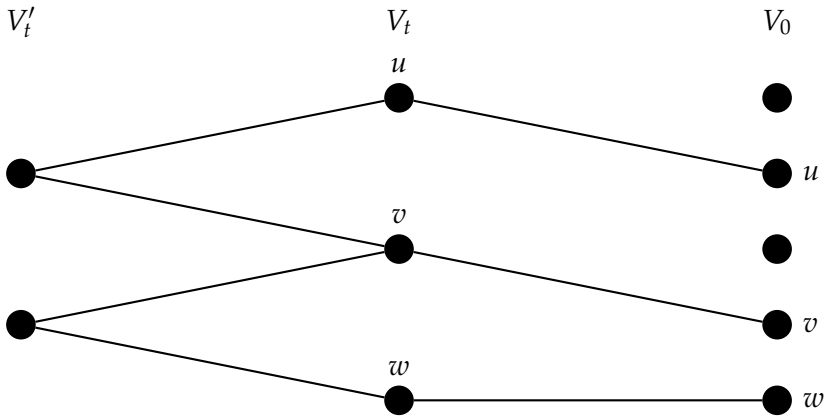


Figure 8: Illustration of the construction from the proof of Theorem 4.1, showing subsets V_0 as well as V_t and V'_t for a triple $t = (u, v, w)$ from the Betweenness problem

4.2 A Polynomial Special Case

In this subsection, we will show that the problem becomes polynomially solvable if all G_{ij} are connected. Before proving this statement, we want to introduce a way to represent all staircase orderings on a given connected subgraph G_{ij} as well as a way to intersect these representations.

As mentioned in Subsection 3.2, there are two ways to modify an ordering on a connected bipartite graph G_{ij} while maintaining the staircase property of the ordering. First, we can simultaneously reverse the ordering and second, we can switch places of vertices u, v with $N_{G_{ij}}(u) = N_{G_{ij}}(v)$.

The first operation can only yield a staircase ordering on G if the total orders of all V_i are reversed at once. Therefore, we can "ground" the ordering by fixing the total order on some V_i and aligning all other total orders with the fixed one.

In contrast, the second operation can change the total order on a given V_i in a way that cannot be uniformly extended to all other subsets V_j. To represent this operation, we use *preorders*, i.e. reflexive, transitive binary relations. Let G_{ij} be some bipartite subgraph of G and $>_i, >_j$ the total orders on V_i, V_j found by Algorithm 2. Then all valid partial reorderings on V_i and V_j are represented by the total preorders

$$(\forall k \in \{i, j\})\, u \geq_k v :\Leftrightarrow u >_k v \vee (v >_k u \wedge N_{G_{ij}}(u) = N_{G_{ij}}(v)).$$

In other words, if $\{\succ_i, \succ_j\}$ is another staircase ordering on G_{ij}, applying this definition to the ordering itself or its reverse will yield $\geq_i$ and $\geq_j$. The next two lemmas show under which circumstances we can merge these preorders.

Lemma 4.2. *Let G_{ij}, G_{jk} be bipartite connected graphs and $\geq_i, \geq_j, \succeq_j, \succeq_k$ the total preorders representing all staircase orderings on G_{ij} and G_{jk} respectively. If, even after possibly reversing $\succeq_j$ and $\succeq_k$, there exists no vertex $u \in V_j$ with $u \geq_j v, u \succeq_j v$ for all $v \in V_j$, there cannot be total orders $>_i, >_j, >_k$ such that the orderings $\{>_i, >_j\}$ and $\{>_j, >_k\}$ are staircase on G_{ij} and G_{jk} respectively.*

Proof. Suppose there are total orders $>_i, >_j, >_k$ such that the orderings $\{>_i, >_j\}$ and $\{>_j, >_k\}$ are staircase on G_{ij} and G_{jk} respectively. Then there exists a vertex $u \in V_j$ with $u >_j v\, \forall v \in V_j$. However, $>_j$ must be represented by $\geq_j$ as well as $\succeq_j$ and therefore we have $u \geq_j v, u \succeq_j v\, \forall v \in V_j$, which is a contradiction. □

Lemma 4.3. *Let G_{ij}, G_{jk} be bipartite connected graphs and $\geq_i, \geq_j, \succeq_j, \succeq_k$ the total preorders representing all staircase orderings on G_{ij} and G_{jk} respectively. Further, assume there exists some vertex $u \in V_j$*

with $u \geq_j v, u \succeq_j v \; \forall v \in V_j$. Then there exist staircase orderings $\{>_i, >_j\}$ and $\{>_j, >_k\}$ on G_{ij} and G_{jk} respectively if and only if the composed preorder

$$u \trianglerighteq_j v :\Leftrightarrow u \geq_j v \wedge u \succeq_j v$$

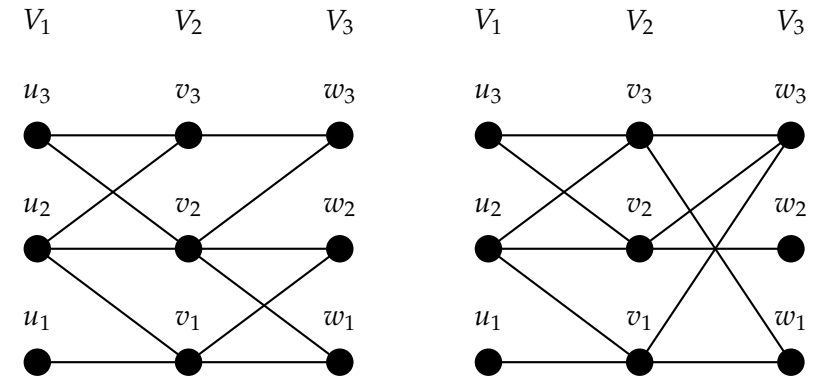
is total. Further, the total preorders $\geq_i, \trianglerighteq_j, \succeq_k$ represent exactly the set of total orders $>_i, >_j, >_k$ such that the orderings $\{>_i, >_j\}$ and $\{>_j, >_k\}$ are staircase on G_{ij} and G_{jk} respectively.

Proof. Let the composed preorder $\trianglerighteq_j$ be total and $>_j$ some total order represented by $\trianglerighteq_j$. Then $>_j$ is also represented by $\geq_j$ and $\succeq_j$ as $\trianglerighteq_j$ is the composition of these two preorders. Hence, we can find total orders $>_i, >_k$ such that the orderings $\{>_i, >_j\}$ and $\{>_j, >_k\}$ are staircase on G_{ij} and G_{jk}, respectively. In fact, any triple $>_i, >_j, >_k$ of total orders represented by $\geq_i, \trianglerighteq_j, \succeq_k$ gives us staircase orderings on G_{ij} and G_{jk}. In summary, we can construct staircase orderings on G_{ij} and G_{jk} from the total preorder $\trianglerighteq_j$, and each combination of total orders $>_i, >_j, >_k$ represented by $\geq_i, \trianglerighteq_j, \succeq_k$ yields staircase orderings on G_{ij} and G_{jk}.

Conversely, if total orders $>_i, >_j, >_k$ exist such that the orderings $\{>_i, >_j\}$ and $\{>_j, >_k\}$ are staircase on G_{ij} and G_{jk}, respectively, we have $u >_j v$ or $v >_j u$ for all pairs $u, v \in V_j$. If $u >_j v$, both $u \geq_j v, u \succeq_j v$ must hold, and by definition we have $u \trianglerighteq_j v$. Analogously, we get $v \trianglerighteq_j u$ if $v >_j u$. As u, v were chosen arbitrarily, $\trianglerighteq_j$ is a total preorder. Further, $>_i, >_j, >_k$ are represented by $\geq_i, \trianglerighteq_j, \succeq_k$. Altogether, if total orders $>_i, >_j, >_k$ exist such that the orderings $\{>_i, >_j\}$ and $\{>_j, >_k\}$ are staircase on G_{ij} and G_{jk} respectively, then $\trianglerighteq_j$ must be total and the total orders are represented by $\geq_i, \trianglerighteq_j, \succeq_k$. This completes the proof. □

Example 4.4. *Consider the graph G shown in Fig. 9a, which can be ordered to be staircase. In the subgraph G_{12}, we can arbitrarily arrange v_3 and v_2, and in the subgraph G_{23} we can arbitrarily arrange v_2 and v_1. We write this as $v_3 =_2 v_2$ and $v_2 \approx_2 v_1$, respectively. As a result, we have $v_3 =_2 v_2 \geq_2 v_1$ and $v_3 \succeq_2 v_2 \approx_2 v_1$. The composed preorder, which is indeed total, is then given by $v_3 \trianglerighteq_2 v_2 \trianglerighteq_2 v_1$.*

In contrast, the bipartite subgraphs H_{12} and H_{23} of H shown in Fig. 9b can be ordered staircase with the preorders $v_3 =_2 v_2 \geq_2 v_1$ and $v_3 \approx_2 v_1 \succeq_2 v_2$, respectively. However, in the composed preorder $\trianglerighteq_2$ we have neither $v_2 \trianglerighteq_2 v_1$ nor $v_1 \trianglerighteq_2 v_2$, and there indeed exists no staircase ordering on H.



(a) Graph G with total composed preorder

(b) Graph H with non-total composed preorder

Figure 9: An example where the composed preorder is not total and no staircase ordering on G exists

We proceed to prove the main theorem of this section, which states that, if all subgraphs G_{ij} are connected, the problem to decide whether a staircase ordering exists on a given graph with given partition is solvable in polynomial time. Our proof will be constructive, yielding an algorithm to even compute a staircase ordering if it exists, and we will investigate its performance in Section 5.

Theorem 4.5. *Let G be an m-partite graph with a partition $\mathcal{V} = \{V_1, \dots, V_m\}$ of the vertex set such that all G_{ij} are connected. Then it can be decided within polynomial time whether a staircase ordering on G exists.*

Proof. We prove this theorem by describing an algorithm that solves the problem in polynomial time. The idea is to perform a breadth-first search on the dependency graph and to use Algorithm 2 on G_{ij} whenever an edge $\{V_i, V_j\}$ is traversed. If necessary, the computed total orders on V_i and V_j will be merged with already existing orders on V_i and V_j. In the case where a staircase ordering exists, the breadth-first search traverses all edges and the last merging step yields a staircase ordering on G. Otherwise, the process stops as one of the merging step fails or Algorithm 2 already recognizes that no staircase ordering exists for some subgraph G_{ij}. If the dependency graph $\mathcal{G}$ consists of more than one connected component, we can simply apply the above procedure to each connected component. Therefore, we can assume w.l.o.g. that $\mathcal{G}$ is connected as well.

It is clear that a staircase ordering on G cannot exist if Algorithm 2 determines that for some G_{ij} no staircase ordering can be found. On the other hand, if a staircase ordering on G does exist, Algorithm 2 does find a staircase ordering on G_{ij}, which yields new total preorders $\succeq_i, \succeq_j$ on V_i and V_j respectively. Suppose there already exists some total preorder $\geq_i$ on V_i. If, even after possibly reversing $\succeq_i$ and $\succeq_j$, we cannot find a vertex $u \in V_i$ with $u \geq_i v, u \succeq_i v \; \forall v \in V_i$, which can be checked in polynomial time, then by Lemma 4.2 there exists no staircase ordering on G. Otherwise consider the composed preorder

$$u \trianglerighteq_i v \Leftrightarrow u \geq_i v \wedge u \succeq_i v.$$

If the composed preorder is not total, then by Lemma 4.3 there exists no staircase ordering on G. We want to emphasize that applying Algorithm 2 to some G_{ij} yields two total preorders $\geq_i, \geq_j$ that need to be merged and we must either reverse both new total preorders or none as reversing only one of them would yield an ordering that is not staircase on G_{ij}. This includes the case where, for example, $\geq_j$ is the first total preorder on V_j. If the breadth-first search finishes and the last merging step succeeds, we have total preorders on all $V_i \in \mathcal{V}$. Deriving any combination of total orders then yields a staircase ordering on G. The resulting algorithm can then be summarized as follows.

Algorithm 3 Finding a staircase ordering on m-partite graphs with connected subgraphs G_{ij}

Input: A graph $G = (V, E)$ together with a partition $\mathcal{V}$ of its vertex set V and the property that all subgraphs G_{ij} are connected

Output: Returns a staircase ordering or the message that no staircase ordering exists

```
Initialize total preorders on all V_i ∈ 𝒱 as u ≥ v ∀u, v ∈ V_i Select root vertex R ∈ 𝒱
for edge {i, j} being traversed by the breadth-first search starting in R do
    Sort G_ij via Algorithm 2
    if No staircase ordering on G_ij exists then
        return "A staircase ordering on G does not exist"
    end if
    Compute corresponding preorders on V_i and V_j
    Unify the new preorders with existing preorders on V_i and V_j
    if Unification failed then
        return "A staircase ordering on G does not exist"
    end if
end for
return The computed staircase ordering on G
```

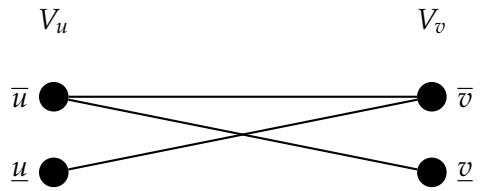


Figure 10: Construction of the compatibility graph for the proof of Theorem 4.6, showing subsets V_u and V_v for $u, v \in E(H)$

In short, the breadth-first-search approach can indeed decide whether a staircase ordering exists on the graph G. Further, its running time is polynomial in the input size, as we traverse each edge in the dependency graph $\mathcal{G}$ at most once, Algorithm 2 runs in polynomial time, and finally, each merging step can be done in polynomial time as well. □

It is worth mentioning that, since the instances in the proof of Theorem 4.1 may feature subgraphs G_{ij} with isolated vertices, it does not state that deciding whether staircase sorting is hard under the assumption of nonempty neighbourhoods in each G_{ij}, which is a slightly weaker assumption than the one made in Theorem 4.5.

Before presenting the results of our computational experiments, we want to state an interesting observation. Any given ordering on any given graph can be made staircase by adding additional edges to the graph until the staircase conditions are fulfilled. In fact, only adding edges of violated staircase conditions until the ordering becomes staircase, yields, in terms of number of additional edges, the optimal *staircase relaxation*. If the partition $\mathcal{V}$ is not staircase, it may then be relevant to find an ordering with minimal staircase relaxation. The following theorem, which is a slight modification of [Mer17, Theorem 5.20], extending it to graphs satisfying our connectedness assumption from Theorem 4.5, yields a surprising result. Namely, it shows that finding such an ordering remains NP-complete, even if all bipartite subgraphs G_{ij} are connected and each $V_i \in \mathcal{V}$ containts at most two elements.

Theorem 4.6. *Let G be an m-partite graph with a partition $\mathcal{V} = \{V_1, \dots, V_m\}$ of the vertex set such that all G_{ij} are connected. Then the optimization problem to find an ordering on G such that its staircase relaxation is minimal is NP-complete, even when the problem is restricted to at most two elements per partition.*

Proof. We show the theorem by a polynomial-time reduction from the famous MaxCut problem. For a given graph H, it asks for partitioning the vertices into two sets such that the number of arcs between both partitions is maximized. The weighted version of the corresponding decision problem was one of Karp's 21 NP-complete problems [Kar72], and also the unweighted case is well known to be NP-hard [GJ79].

From an instance of MaxCut on a graph H, we construct an equivalent instance of the optimization problem from Theorem 4.6 with only two elements per partition: For each $v \in V(H)$, construct a partition $V_v = \{\underline{v}, \overline{v}\}$ consisting of the two elements $\underline{v}$, $\overline{v}$. For $\{u, v\} \in E(H)$ we then add the edges $\{\underline{u}, \overline{v}\}, \{\overline{u}, \overline{v}\}, \{\overline{u}, \underline{v}\}$ (see Fig. 10). For $u, v \in V(H), \{u, v\} \notin E(H)$ we add edges such that G_{uv} is complete bipartite, i.e. $\{V_u, V_v\} \notin \mathcal{G}$. Hence, all resulting bipartite subgraphs G_{ij} are connected.

Therefore, sorting $\underline{u} < \overline{u}$ and $\underline{v} < \overline{v}$ would contribute exactly one missing edge to the objective, while flipping one of the orderings (but not both) satisfies (SC1) and (SC2) for the two partitions V_u and V_v. As a consequence, any cut in H containing l edges with vertex subsets S, T corresponds to a choice for total orderings on the subsets V_v, $v \in V(H)$ with $|E(H)| - l$ violations, namely $\underline{v} < \overline{v}\ \forall v \in S$ together with $\underline{v} > \overline{v}\ \forall v \in T$, and vice versa. □

5 Computational Experiments

In the following, we will empirically evaluate Algorithm 3 to decide whether a staircase ordering on a given m-partite graph exists, and to compute such an ordering if it does. First, we consider two sets of randomly generated instances. On a set of small instances, we benchmark the performance of Algorithm 3 against the mixed-integer program (MIP) for finding staircase orderings suggested in [Mer17]. Several sets of larger instances are then used to test the limits of Algorithm 3. Second, we demonstrate that our algorithm can staircase-sort a set of real-world railway timetabling instances. It will turn out that using the sorted version, which allows for using the totally unimodular dual-flow formulation from [BGMS18], is much more efficient than using a generic model for the problem.

We have implemented Algorithm 3 in Python 3.7.9, using the Python library NetworkX 2.5 ([HSSC08]) for our graph operations. For the MIP approach, we used the Python API of Gurobi 9.1.0 ([GO20]). All experiments have been run on machines with Xeon E3-1240 v6 CPUs ("Kaby Lake", 4 cores, HT disabled, 3.7 GHz base frequency) and 32 GB RAM.

5.1 Computational Experiments on Random Instances

The artificial instances in our first benchmark set were generated as follows. First, we created a dependency graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with m vertices and $\mathcal{E} = \emptyset$. For each pair of vertices $V_i, V_j \in \mathcal{V}$, $i \neq j$ in $\mathcal{G}$, we then added the edge $\{V_i, V_j\}$ with probability $p \in [0, 1]$. To ensure that the generated instances do not decompose into multiple smaller instances, we randomly chose two vertices V_i and V_j from two different randomly chosen connected components of $\mathcal{G}$ and iteratively added the corresponding edge $\{V_i, V_j\}$ until the entire dependency graph was connected. If the so-obtained $\mathcal{G}$ was cycle-free, we randomly added one more edge such that it contained at least one cycle. This later allowed us to create a non-staircase version of the instance without simply adding a non-staircase subgraph G_{ij} (see below). From this dependency graph $\mathcal{G}$, we then generated G as follows. For each $V_i \in \mathcal{V}$, we randomly drew a subset size $n_i \in [\underline{n}, \overline{n}], \underline{n}, \overline{n} \in N^+$ and added the vertices $v_{i,1}, \ldots, v_{i,n_i}$ to subset V_i. Finally, for each edge $\{V_i, V_j\} \in \mathcal{E}$, we use Algorithm 4 as a generator for the (connected) subgraph G_{ij}.

After initializing the vertex sets V_i and V_k, the edge set E as well as the auxiliary variables $minIdx, maxIdx$ in Lines 1 to 5, Algorithm 4 iterates over the vertex indices in V_i in Line 6. For each $v_{i,k}$, its neighbourhood is generated as follows. First, we draw an integer which will represent the smallest index of any neighbour of $v_{i,k}$ in Lines 9 and 10. The exponential function is chosen in order to prevent the generated graphs from being sparse at one end of the ordering while being very dense at the other end of the ordering. We then check if the drawn value indeed represents the index of a neighbour of $v_{i,k}$ (Line 11), and, if necessary, adjust it in Lines 12 to 16 such that the generated graph remains connected. A similar procedure follows for finding a new maximal neighbour index. When valid indices are found, the new edges are added in Line 29. Note, that the minimal and maximal neighbourhood indices increase monotonously in each iteration, and therefore the canonical ordering is indeed staircase for G_{ij}.

To test the performance of Algorithm 3 on non-staircase instances, we added a non-staircase duplicate of each instance to our test set. Since Algorithm 2 already detects if no staircase ordering exists on a given G_{ij}, which therewith holds for the entire graph G, we broke the staircase property in a way that forces Algorithm 3 to also take interdependencies between subgraphs G_{ij} into account. In particular, we created a set of all edges in $\mathcal{E}$ that are contained in at least one cycle in $\mathcal{G}$ and chose one of these edges uniformly at random. In the corresponding subgraph $G_{ij} = (V_i \cup V_j, E_{ij})$, we then flipped V_j, i.e. we changed the edge set to

$$\tilde{E}_{ij} = \{\{v_{i,n}, v_{j,m'}\} \mid \{v_{i,n}, v_{j,m}\} \in E_{ij}, m' = |V_j| - m + 1\}.$$

Algorithm 4 Generator for bipartite graphs with staircase canonical ordering

Input: Subset sizes $n_i, n_j \in \mathbb{N}^+$
Output: A bipartite graph G_{ij} such that the canonical ordering is staircase

$V_i = \{v_{i,1}, \ldots, v_{i,n_i}\}$
$V_j = \{v_{j,1}, \ldots, v_{v,n_j}\}$
$E_{ij} = \emptyset$
$minIdx = 0$
$maxIdx = 0$
for $k = 1, 2, \ldots, n_i$ **do**
  foundNewMin =False
  while not foundNewMin **do**
    Choose $x \geq 0$ with probability density function $f(x) = \lambda e^{-\lambda x}$ where $\lambda = \frac{n_i}{n_j}$
    $newMinIdx = minIdx + \lfloor (x + 0.5) \rfloor$ {round to nearest integer}
    if $newMinIdx < n_j$ **then**
      if $newMinIdx > maxIdx$ **then**
        $minIdx = maxIdx$
      else
        $minIdx = newMinIdx$
      end if
      foundNewMin =True
    end if
  end while
  foundNewMax =False
  while not foundNewMax **do**
    Choose $x \geq 0$ with probability density function $f(x) = \lambda e^{-\lambda x}$ where $\lambda = \frac{n_i}{n_j}$
    $newMaxIdx = maxIdx + \lfloor (x + 0.5) \rfloor$ {round to nearest integer}
    if $minIdx \leq newMaxIdx < n_j$ **then**
      $maxIdx = newMaxIdx$
      foundNewMax =True
    end if
  end while
  $E_{ij} = E_{ij} \cup \{\{v_{i,k}, v_{j,l}\} \mid l \in \{minIdx, minIdx + 1, \ldots, maxIdx\}\}$
end for
return $G_{ij} = (V_i \cup V_j, E_{ij})$

To further improve the accuracy of our computations, we created five random instances for each set of parameters $m, p, \underline{n}, \overline{n}$ used. Our result tables use the parameter naming scheme

$$m\text{-}p\text{-}\underline{n}\text{-}\overline{n}, \tag{3}$$

with the postfix "*-nonSC*" to indicate the non-staircase duplicates.

Finally, we prepend each vertex in $\mathcal{V}$ and V with four random lowercase letters and rebuild the graphs $\mathcal{G}$ and G by adding vertices and edges in the resulting random lexicographic order to destroy any orders that NetworkX might derive from the vertex names or the order in which they have originally been added to the graphs.

An MIP formulation based on the linear ordering problem which is able to decide whether

a staircase ordering exists, can be stated as follows:

$$\begin{aligned}
&\text{find} && z && && && (4a)\\
&\text{s.t.} && z_{uv} + z_{vu} &= 1 && \forall V_i \in \mathcal{V}\ \forall\, u \neq v \in V_i && (4b)\\
& && z_{uv} + z_{vw} + z_{wu} &\leq 2 && \forall V_i \in \mathcal{V}\ \forall\, u \neq v \neq w && (4c)\\
& && z_{uv} - z_{v'u'} &= 0 && \forall u >_i v, u' >_j v' : \text{(SC2) is violated} && (4d)\\
& && z_{uv} &\in \{0,1\} && \forall (u,v) \in \textstyle\bigcup_{i=1}^{m} (V_i \times V_i). && (4e)
\end{aligned}$$

Variables $z_{uv} \in F$ stand for the decision to order vertex $u \in V$ before vertex $v \in V$ for each u and v in the same subset of $\mathcal{V}$. Constraint (4a) ensures that exactly one of two possible orders between two vertices is chosen, while Constraint (4b) requires the order within a subset to be acyclic. Finally, Constraint (4d) demands that the orders fulfil Condition (SC2). Note that we do not need to explicitly model (SC1), as it is implied by (SC2) if G_{ij} is connected. Cf. [Mer17] for this MIP approach.

Table 1 shows our results for the comparison between building and solving the MIP and using Algorithm 3. In the columns, we state the parameter set (see (3)), the dimensions of the graphs $\mathcal{G}$ and G as well as the time required to build the MIP model, solve the MIP model, or solve the problem with Algorithm 3. For increased readability, each row represents the averages of the five instances with the given parameter set, where the resulting graph dimensions are rounded to integer values.

In terms of performance, we make two important observations. On the one hand, when comparing the pure solution times, the MIP model outperforms our Python-based implementation of Algorithm 3 on the given benchmark set. Second, however, the MIP model takes a very long time to build, as its number of constraints is double-quadratic in the number of vertices per subset of the partition. For example, the MIP model for the first instance with parameters 10-0.5-10-10 consists of only 9,182 rows, 900 columns and 20,764 nonzeros, whereas the MIP model of the first instance with parameters 10-0.5-50-50 already consists of 1,391,617 rows, 24,500 columns and 3,175,234 nonzeros. As a result, increasing the subset size from 10 to 50 already led to a 1000-fold increase in MIP build time in our test.

Although the picture remains largely the same between staircase and non-staircase instances, we can observe that solution times tend to be lower for the non-staircase instances, as either method can stop once it has found the first conflict. The largest variance has been found for the five instances of 150-0.5-10-10, ranging from 57.8 seconds to 65.0 seconds, whereas solution times for the five non-staircase instances show much higher variance and range from 6.6 seconds to 56.6 seconds.

Finally, we conclude that although the model construction time of the MIP scales roughly linearly with $|\mathcal{E}|$, it quickly becomes prohibitively long for practical application as instance size increases. We therefore neglect this approach for the larger instances we present in the following.

The results on the impact of varying the subset size are given in Table 2. The experiments with the first five parameter sets show that computation time quickly increases with subset size, as Algorithm 2, which is used to sort the bipartite subgraphs G_{ij}, is of quadratic complexity in the number of vertices per subset. The experiments with the next five parameter sets show that increasing the variance in subset size while leaving the expected number of vertices per subset at 250 vertices also slightly increases computation time. As before, we find that even with just a single conflict prohibiting a staircase ordering from existing, computation times are significantly lower for non-staircase instances.

The next set of random instances features a gradually increasing number of vertices in the dependency graph. We consider it to study the impact of an increasing number of bipartite subgraphs G_{ij}. The results can be seen in Table 3. Here we see that the computation time of

Table 1 Performance of sorting algorithm and linear-ordering MIP on small random instances

					Time (s)		
Instance	$\|\mathcal{V}\|$	$\|\mathcal{E}\|$	$\|V\|$	$\|E\|$	MIP Building	MIP Solution	Algorithm
10-0.5-10-10	10	21	100	643	4.0	0.0	0.1
10-0.5-20-20	10	21	200	1846	81.0	0.1	0.2
10-0.5-30-30	10	19	300	2675	390.4	0.3	0.6
10-0.5-40-40	10	22	400	4968	1604.0	1.2	1.5
10-0.5-50-50	10	21	500	7053	4335.6	2.4	2.5
50-0.5-10-10	50	619	500	19894	329.4	0.3	3.1
75-0.5-10-10	75	1385	750	44454	972.5	0.9	8.5
100-0.5-10-10	100	2467	1000	79517	2236.0	2.4	18.8
125-0.5-10-10	125	3897	1250	125680	4305.4	3.5	35.3
150-0.5-10-10	150	5599	1500	179797	7300.8	6.3	60.6
10-0.5-10-10-nonSC	10	21	100	643	4.0	0.0	0.0
10-0.5-20-20-nonSC	10	21	200	1846	81.5	0.1	0.2
10-0.5-30-30-nonSC	10	19	300	2675	381.2	0.3	0.3
10-0.5-40-40-nonSC	10	22	400	4968	1626.2	1.0	1.1
10-0.5-50-50-nonSC	10	21	500	7053	4267.8	2.5	1.3
50-0.5-10-10-nonSC	50	619	500	19894	316.6	0.3	1.2
75-0.5-10-10-nonSC	75	1385	750	44454	964.5	0.7	2.0
100-0.5-10-10-nonSC	100	2467	1000	79517	2193.1	1.2	11.8
125-0.5-10-10-nonSC	125	3897	1250	125680	4283.4	1.8	26.4
150-0.5-10-10-nonSC	150	5599	1500	179797	7230.8	4.1	31.3

Algorithm 3 scales linearly with $|\mathcal{E}|$. Again, computation times for non-staircase instances are significantly lower, but show higher variance.

Table 3 Performance of sorting algorithm on random instances with varying dependency graph size

Instance	$\|\mathcal{V}\|$	$\|\mathcal{E}\|$	$\|V\|$	$\|E\|$	Algorithm Time (s)
100-0.1-50-50	100	511	5000	165292	72.0
200-0.1-50-50	200	2034	10000	655546	374.1
300-0.1-50-50	300	4476	15000	1440756	994.2
400-0.1-50-50	400	7958	20000	2579813	2093.1
500-0.1-50-50	500	12518	25000	4033818	3694.5
100-0.1-50-50-nonSC	100	511	5000	165292	25.9
200-0.1-50-50-nonSC	200	2034	10000	655546	217.2
300-0.1-50-50-nonSC	300	4476	15000	1440756	293.3
400-0.1-50-50-nonSC	400	7958	20000	2579813	988.0
500-0.1-50-50-nonSC	500	12518	25000	4033818	1893.2

Finally, we created a testset that, in terms of dependency graph size and density as well as subset size, approximates the structure of the real-world railway timetabling application studied in the next section. Table 4 states the results. First, we want to emphasize that the smaller subset size allows us to solve the problem for much larger graphs. Further, the results confirm that the computation time grows linearly with the number of bipartite subgraphs and is generally lower for non-staircase instances. On the non-staircase instances of this testset we could once again observe significant variance of solution times. The greatest outliers arose for 250000-0.00002-10-10-nonSC, where solution time ranged from 463.5 seconds to 1573.7 seconds, an almost four-fold increase between the fastest and slowest recognition.

Table 2 Performance of sorting algorithm on random instances with varying subset size

Instance	$\lvert\mathcal{V}\rvert$	$\lvert\mathcal{E}\rvert$	$\lvert V\rvert$	$\lvert E\rvert$	Algorithm Time (s)
100-0.1-50-50	100	511	5000	165292	72.0
100-0.1-100-100	100	507	10000	443142	486.8
100-0.1-150-150	100	506	15000	808059	1442.6
100-0.1-200-200	100	481	20000	1192116	3111.3
100-0.1-250-250	100	490	25000	1670222	5991.0
100-0.1-200-300	100	472	24961	1588812	5521.0
100-0.1-150-350	100	494	25027	1632905	6025.6
100-0.1-100-400	100	486	25794	1783508	6816.1
100-0.1-50-450	100	499	25669	1729277	6843.8
100-0.1-1-500	100	494	25788	1655008	6770.1
100-0.1-50-50-nonSC	100	511	5000	165292	25.9
100-0.1-100-100-nonSC	100	507	10000	443142	206.1
100-0.1-150-150-nonSC	100	506	15000	808059	548.0
100-0.1-200-200-nonSC	100	481	20000	1192116	1543.5
100-0.1-250-250-nonSC	100	490	25000	1670222	3721.1
100-0.1-200-300-nonSC	100	472	24961	1588812	3321.3
100-0.1-150-350-nonSC	100	494	25027	1632905	1953.6
100-0.1-100-400-nonSC	100	486	25794	1783508	4817.9
100-0.1-50-450-nonSC	100	499	25669	1729277	1791.3
100-0.1-1-500-nonSC	100	494	25788	1655008	3069.2

Table 4 Performance of sorting algorithm on random instances with large sparse dependency graph

Instance	$\lvert\mathcal{V}\rvert$	$\lvert\mathcal{E}\rvert$	$\lvert V\rvert$	$\lvert E\rvert$	Algorithm Time (s)
100000-0.00002-10-10	100000	116292	1000000	3745465	273.5
150000-0.00002-10-10	150000	232991	1500000	7499847	580.6
200000-0.00002-10-10	200000	403666	2000000	12994694	1042.3
250000-0.00002-10-10	250000	626913	2500000	20180041	1696.4
300000-0.00002-10-10	300000	901312	3000000	29000083	2564.7
100000-0.00002-10-10-nonSC	100000	116292	1000000	3745465	107.3
150000-0.00002-10-10-nonSC	150000	232991	1500000	7499847	404.6
200000-0.00002-10-10-nonSC	200000	403666	2000000	12994694	561.2
250000-0.00002-10-10-nonSC	250000	626913	2500000	20180041	939.5
300000-0.00002-10-10-nonSC	300000	901312	3000000	29000083	1420.5

To summarize, the results show that Algorithm 3 clearly outperforms the MIP if model build time is considered. If model build time is disregarded then, at least on our testset consisting of very small instances, the MIP outperforms our Python-based implementation of Algorithm 3. Further, computation times scale, in accordance with our theoretical results, linearly with number of bipartite subgraphs G_{ij} and quadratically with subset size. Finally, computation times tend the be shorter on non-staircase instances as the algorithm can terminate on finding the first conflict.

5.2 Computational Experiments on Timetabling Instances

Now, we evaluate the performance of Algorithm 3 on a set of real-world railway timetabling instances. In the application, we are given a timetable draft, and the task is to slightly adjust the departure times to reduce peak power consumption, which typically accounts for 20–25% of the train operator company's electricity bill. For full details of the application and different solution approaches, we refer to [BMS17], [BGMS18] and [BMS20], as well as [BGM+20a] and [BGM20b] for an extension of the approach onto underground timetabling. In [BGMS18], it has been shown that explicit knowledge of a staircase ordering allows the usage of a compact formulation to efficiently solve the timetabling problem and drastically reduce solution times. In the case of the largest instance, containing all nationwide passenger traffic of Germany's

largest railway company over a whole day, the computation time has been reduced from about three hours to about three minutes.

In our test, we simulated the lack of explicit knowledge of the staircase ordering by randomizing the vertex names as described for the random instances in the previous section and also tested five randomizations for each instance. As not all bipartite subgraphs G_{ij} were connected in these instances, some additional presolve was required. In particular, a vertex with degree zero in any G_{ij} can be removed entirely as it cannot be part of a feasible solution, i.e. a valid timetable. After removing all isolated vertices, the condition that all G_{ij} are connected is fulfilled and we can directly apply Algorithm 3.

We have repeated the computations in [BGMS18, Table 1] with our up-to-date version of the instances and the soft- and hardware specified at the beginning of this section. The column $\Delta(s)$ in Table 5 states the time saved if the problem is solved by first applying the aforementioned presolve method (Presol.), then sorting the graphs using Algorithm 3 (Alg.) and finally using the dual-flow formulation (DF) to optimize the schedule, instead of directly solving the problem via the naive formulation (NA). In this test, we even used an improved version of the naive formulation, see [BGM20b, Formulation (1)], which performs slightly better than the version used in [BGMS18]. On most instances in the test set, our three-step approach clearly outperforms solving the problem via the (improved) naive formulation. Most notably, we are able to solve the instance *Würzburg Hbf* in less than 30 minutes, whereas the naive formulation runs into the time limit of 10 hours.

Table 5 Comparison of optimization performance on railway instances

					Time (s)				
Instance	$\lvert\mathcal{V}\rvert$	$\lvert\mathcal{E}\rvert$	$\lvert V\rvert$	$\lvert E\rvert$	NA	DF	Presol.	Alg.	Δ(s)
Bayreuth Hbf	327	306	2287	8668	0.8	0.2	0.1	0.5	0.0
Zeil	762	822	5329	23382	9.7	0.4	0.3	1.2	7.8
Passau	1040	1114	7273	31673	162.4	8.6	0.4	1.6	151.8
Jena Paradies	1102	1229	7712	35372	89.0	4.2	0.5	1.8	82.5
Lichtenfels	1650	1944	11546	56012	1914.3	14.6	0.8	3.0	1895.9
Kiel Hbf	2130	2582	14885	75783	60.9	2.6	1.0	4.0	53.3
Erlangen	2969	4056	20769	0.12M	1769.9	48.9	2.1	5.8	1713.1
Aschaffenburg	3463	4467	24227	0.13M	198.2	2.4	1.8	7.0	187.0
Bamberg	3644	4481	25495	0.13M	843.1	15.8	1.8	6.9	818.6
Würzburg Hbf	4456	6041	31165	0.18M	36003.6	1741.5	2.7	10.0	34249.4
Ulm Hbf	5729	7608	40077	0.22M	122.9	6.6	3.3	12.5	100.5
Leipzig Hbf (tief)	6810	18592	47631	0.56M	3.6	0.7	20.9	1.7	-19.7
Dresden	6936	11853	48522	0.35M	2610.0	70.1	9.2	12.4	2518.3
Frankfurt (Main)	8626	13343	60332	0.40M	361.0	15.3	9.5	16.0	320.2
Stuttgart Hbf	11594	46184	81112	1.40M	331.3	8.0	51.2	23.7	248.4
Nürnberg	12189	20995	85258	0.63M	179.8	5.7	20.1	19.9	134.1
Hamb.-Altona(S)	12373	60939	86556	1.93M	528.5	6.3	95.5	16.7	410.0
Berlin Hbf(S)	16114	0.10M	0.11M	3.15M	33.4	5.2	126.6	0.9	-99.3
Regio Nord	13379	20670	93582	0.61M	204.6	8.1	15.0	24.4	157.1
Regio Nordost	16496	24760	0.12M	0.74M	846.8	10.2	17.4	33.6	785.6
S-Bahn Hamburg	17533	89859	0.12M	2.85M	749.8	12.5	154.3	24.7	558.3
Regio Südwest	24191	36125	0.17M	1.07M	244.5	6.5	28.0	34.3	175.7
Regio Hessen	25092	80859	0.18M	2.48M	42.2	12.2	116.4	9.5	-95.9
Regio BW	30172	74513	0.21M	2.24M	2225.5	16.7	82.5	57.8	2068.5
Regio Südost	31917	55003	0.22M	1.64M	508.4	14.8	48.1	37.4	408.1
Regio NRW	47026	0.11M	0.33M	3.48M	608.2	24.3	169.7	36.9	377.3
Regio Bayern	49262	0.15M	0.34M	4.66M	379.6	21.9	236.0	38.3	83.4
S-Bahn Berlin	53353	0.35M	0.37M	10.99M	541.1	47.0	474.4	4.0	15.7
Fernverkehr	7053	11455	49323	0.35M	58.0	17.7	6.6	17.0	16.7
Regionalverkehr	0.31M	1.01M	21.58M	31.25M	5824.6	1965.3	1467.9	200.7	2190.7
Deutschland	0.32M	1.04M	22.07M	32.43M	2350.4	1558.6	1634.1	173.7	-1016.0

We would like to note that in [BGM20b] each scheduled departure could also choose from a discrete set of travel times, which changes the structure of the problem such that a large portion of the subgraphs G_{ij} no longer fulfil the staircase property. We ran preliminary tests on these

instances and our algorithm determined all instances to be non-staircase within less than one second.

6 Conclusions

In this work, we have we discussed theoretical aspects of the recognizability of staircase compatibility and derived polynomial-time algorithms to solve this recognition problem. Furthermore, we demonstrated the practical applicability of the latter. For the special case of bipartite graphs we first showed that a given ordering can be verified to be staircase in time $O(|V| + |E|)$. Moreover, staircase orderings on bipartite graphs are unique up to three obvious operations on the ordering and can be computed in time $O(|V_1|(|V| + |E|))$. We then generalized the recognition problem onto m-partite graphs, where it becomes NP-hard in the general case. Further, we presented the polynomial subcase, where all bipartite subgraphs G_{ij} are connected and stated an algorithm that solves the decision problem in polynomial time by constructing a staircase ordering if one exists. Finally, we benchmarked the algorithm against solving the sorting problem via the MIP formulation presented in [Mer17] on small randomly generated instances, tested the behaviour of our algorithm on a set of larger randomly generated instances and eventually used it to find a staircase ordering on a set of real-world timetabling instances. From our tests on small artificial instances, the MIP approach turned out to be impractical due to the long time the models take to construct, while Algorithm 3 does not suffer from such weaknesses. On the real-world timetabling instances, our results show that applying Algorithm 3, and then solving the problem with a reformulation exploiting the staircase ordering, is significantly faster than solving the problem with the formulations available when the staircase ordering is not known.

In the future, it would be interesting to study whether the dual-flow formulations perform well on instances where no staircase ordering exists, but adding a low number of edges fulfils the staircase property. As Theorem 4.6 shows that finding an ordering with minimal staircase relaxation remains NP-complete even if all subgraphs G_{ij} are connected, it is especially interesting to study whether there exist polynomially solvable subcases of this problem.

References

[ABG+20] Janniele A. S. Araujo, Haroldo G. Santos Bernard, Gendron, Sanjay Dominik Jena, Samuel S. Brito, and Danilo S. Souza. Strong bounds for resource constrained project scheduling: Preprocessing and cutting planes. *Computers & Operations Research*, 113, 2020.

[BBPP99] Immanuel M. Bomze, Marco Budinich, Panos M. Pardalos, and Marcello Pelillo. *Handbook of Combinatorial Optimization: Supplement Volume A*, chapter The Maximum Clique Problem, pages 1–74. Springer, 1999.

[BGM+20a] Andreas Bärmann, Patrick Gemander, Alexander Martin, Maximilian Merkert, and Frederik Nöth. *Success Stories of Industrial Mathematics in Germany*, chapter Energy-Efficient Timetabling in a German Underground System. Springer, 2020. To appear, available under: `http://www.optimization-online.org/DB_FILE/2020/04/7728.pdf`.

[BGM20b] Andreas Bärmann, Patrick Gemander, and Maximilian Merkert. The clique problem with multiple-choice constraints under a cycle-free dependency graph. *Discrete Applied Mathematics*, 283:59–77, 2020.

[BGMS18] Andreas Bärmann, Thorsten Gellermann, Maximilian Merkert, and Oskar Schneider. Staircase compatibility and its applications in scheduling and piecewise linearization. *Discrete Optimization*, 29:111–132, 2018.

[BMS17] Andreas Bärmann, Alexander Martin, and Oskar Schneider. A comparison of performance metrics for balancing the power consumption of trains in a railway network by slight timetable adaptation. *Public Transport*, 9(1-2):95–113, 2017.

[BMS20] Andreas Bärmann, Alexander Martin, and Oskar Schneider. Efficient formulations and decomposition approaches for power peak reduction in railway traffic via timetabling. *Transportation Science*, 2020. To appear, available under `www.optimization-online.org/DB_HTML/2020/09/7996.html`.

[Bor98] Ralf Borndörfer. *Aspects of Set Packing, Partitioning, and Covering*. PhD thesis, TU Berlin, 1998.

[BS19] Samuel Souza Brito and Haroldo Gambini Santos. Preprocessing and cutting planes with conflict graphs. `https://arxiv.org/pdf/1909.07780.pdf`, 2019.

[Fou82a] Robert Fourer. Solving staircase linear programs by the simplex method, 1: Inversion. *Mathematical Programming*, 1982.

[Fou82b] Robert Fourer. Solving staircase linear programs by the simplex method, 2: Pricing. *Mathematical Programming*, 1982.

[Fou84] Robert Fourer. Staircase matrices and systems. *SIAM Review*, 26(1):1–70, 1984.

[GJ79] Michael R. Garey and David S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W.H. Freeman and Company, New York, 1979.

[GO20] LLC Gurobi Optimization. Gurobi optimizer reference manual, 2020.

[HSSC08] Aric Hagberg, Pieter Swart, and Daniel S Chult. Exploring network structure, dynamics, and function using networkx. Technical report, Los Alamos National Lab.(LANL), Los Alamos, NM (United States), 2008.

[JM18] Adalat Jabrayilov and Petra Mutzel. New integer linear programming models for the vertex coloring problem. In *Latin American Symposium on Theoretical Informatics*, pages 640–652. Springer, 2018.

[Kar72] Richard M. Karp. Reducibility among combinatorial problems. In Raymond E. Miller, James W. Thatcher, and Jean D. Bohlinger, editors, *Complexity of Computer Computations: Proceedings of a symposium on the Complexity of Computer Computations, held March 20–22, 1972, at the IBM Thomas J. Watson Research Center, Yorktown Heights, New York*, pages 85–103. Springer US, Boston, MA, 1972.

[LM16] Frauke Liers and Maximilian Merkert. Structural investigation of piecewise linearized network flow problems. *SIAM Journal on Optimization*, 26(4):2863–2886, 2016.

[Mer17] Maximilian Merkert. *Solving Mixed-Integer Linear and Nonlinear Network Optimization Problems by Local Reformulations and Relaxations*. PhD thesis, Friedrich-Alexander-Universität Erlangen-Nürnberg (FAU), 2017.

[Opa79] Jaroslav Opatrny. Total ordering problem. *SIAM Journal on Computing*, 8(1):111–114, 1979.

Acknowledgements

We acknowledge financial support by the Bavarian Ministry of Economic Affairs, Regional Development and Energy through the Center for Analytics – Data – Applications (ADA-Center) within the framework of BAYERN DIGITAL II. Moreover, we thank the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) for their support within GRK 2297 MathCoRe as well as Project Z01 in CRC TRR 154.