On Hölder Calmness of Minimizing Sets

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December 16, 2020, Revised March 17, 2021

Abstract. We present conditions for Hölder calmness and upper Hölder continuity of optimal solution sets to perturbed optimization problems in finite dimensions. Studies on Hölder type stability were a popular subject in variational analysis already in the 1980ies and 1990ies, and have become a revived interest in the last decade. In this paper, we focus on conditions for Hölder calmness of the argmin mapping in the case of non-isolated minima. We recall known ideas and results in this context for general as well as special parametric programs, refine them and discuss particular settings, including nonlinear programs and convex semi-infinite optimization problems.

Keywords. Perturbed optimization problems, minimizing sets, stability conditions, Hölder calmness, upper Hölder continuity.

2010 Mathematics Subject Classification. 49J53, 90C31, 90C25, 90C34.

1   Introduction

Given a metric space $(T, d(\cdot, \cdot))$, a function $f : \mathbb{R}^n \times T \to \mathbb{R}$, a multifunction $M : T \rightrightarrows \mathbb{R}^n$ and a reference point $\bar{t} \in T$, we study in this paper the parametric optimization problem

$$P(t): \quad f(x,t) \to \min_x \quad \text{s.t.} \quad x \in M(t), \quad t \text{ varies near } \bar{t}, \tag{1.1}$$
$$\text{where } f \text{ is locally Lipschitzian and } M \text{ is closed,}$$

and special realizations of this *basic model*. Recall that the *graph* and the *domain* of a multifunction $\Phi : T \rightrightarrows \mathbb{R}^n$ are defined by $\operatorname{gph}\Phi := \{(t,x) \in T \times \mathbb{R}^n \mid x \in \Phi(t)\}$ and $\operatorname{dom}\Phi := \{t \in T \mid \Phi(t) \neq \emptyset\}$, respectively, and Φ is closed if $\operatorname{gph}\Phi$ is a closed set.

We are going to present sufficient conditions for (q-order) Hölder calmness of the *argmin mapping* (also called *optimal set mapping*) of the problem (1.1),

$$t \mapsto \Psi(t) := \operatorname*{argmin}_x \{f(x,t) \mid x \in M(t)\}.$$

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Recall (cf. [33, 32]) that for $0 < q \leq 1$, a multifunction $\Phi : T \rightrightarrows \mathbb{R}^n$ is called *q-order calm* at $(\bar{t}, \bar{x}) \in \mathrm{gph}\,\Phi$ if for some $\varepsilon, \delta, \varrho > 0$,

$$\Phi(t) \cap B(\bar{x}, \varepsilon) \subset \Phi(\bar{t}) + \varrho\,(d(t,\bar{t}))^q\,\mathbb{B} \quad \forall t \in B(\bar{t}, \delta), \tag{1.2}$$

where $\Phi(t) \cap B(\bar{x}, \varepsilon) = \emptyset$ for $t \neq \bar{t}$ is possible. Here $B(\bar{x}, \varepsilon)$ and $B(\bar{t}, \delta)$ denote the closed ε- and δ-neighborhood of $\bar{x}$ and $\bar{t}$, respectively, and $\mathbb{B}$ is the closed unit ball in $\mathbb{R}^n$. In the case $q = 1$ we say that Φ *is calm* or *proper calm* at $(\bar{t}, \bar{x})$. If in (1.2) the restriction of $\Phi(t)$ to $B(\bar{x}, \varepsilon)$ is omitted, Φ is called *upper Hölder continuous of order q* at $\bar{t}$. Though our approaches and results will include the case $q = 1$, we are mainly interested in the q-order for $0 < q < 1$.

The study of Hölder calmness and upper Hölder continuity of minimizing sets was rather popular in the 1980ies and 1990ies, see, e.g., [1, 4, 25, 26, 44, 45, 23, 8, 9, 10]. Sufficient conditions were given by assuming constraint qualifications (or related regularity assumptions), combined with the second-order analysis of the data, including 2nd- or higher-order growth conditions. Under assumptions ensuring that the initial solution $\bar{x} \in \Psi(\bar{t})$ is locally isolated, a complete theory was developed at that time, see the book by Bonnans and Shapiro [10]. For the general case $\Psi(\bar{t}) \neq \{\bar{x}\}$, we refer e.g. to Sect. 4.4 and Sect. 4.9 in [10] and to the papers [26, 23, 8, 9].

In the last decade, there is an revived interest in studying Hölder calmness and closely related properties (q-order metric subregularity, q-order local error bounds) in applications to optimization, we refer e.g. to the papers [33, 18, 27, 35, 31, 21, 32].

However, Hölder calmness of the argmin mapping has been recovered only in a few papers. In [21, Thm. 4], weak constraint qualifications and a (refined) strong quadratic growth condition were assumed to obtain $\frac{1}{2}$-order upper Lipschitz behavior of stationary solutions and minimizing sets for a class of parametric problems which include perturbed mathematical programs with equilibrium or vanishing constraints (*MPECs* or *MPVCs*). This result is not covered by classical ones, though, the assumptions imply that the initial stationary solution to $P(\bar{t})$ is locally isolated. The authors of [32, Sect. 4] use an approach via Hölder error bounds to get q-order calmness of the argmin mapping of a parametric convex semi-infinite program and of an associated inequality system.

In our paper, we present two types of sufficient conditions for Hölder calmness of the argmin mapping of the parametric problem (1.1), where we focus on the general case that $\Psi(\bar{t})$ is not a singleton. By the first approach, we refine a result obtained in Theorem 2.2 of [26]. In contrast to a stronger regularity requirement in [26], we assume now only calmness and Lipschitz lower semicontinuity at the initial pair $(\bar{t}, \bar{x}) \in \mathrm{gph}\,\Psi$, and the applied growth condition is supposed to hold only locally. The second approach is borrowed from the idea in [32] to use q-order calmness of an auxiliary multifunction (which is in special cases the solution set mapping of an inequality system) as a sufficient condition for the q-calmness of the argmin mapping, see also [12, 30] for the case $q = 1$. This allows the application of conditions for the Hölder calmness of level set mappings, which are given e.g. in [27, 32, 33, 47].

The paper is organized as follows. After presenting some notation and prerequisites in Section 2, we show in Section 3 that relatively mild regularity assumptions and a growth condition of order $\kappa \geq 1$ ensure calmness of order κ^{-1} for the argmin

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mapping of (1.1). The statement and its assumptions are discussed for special settings and results known from the literature. In Section 4, we prove that q-order calmness of minimizing sets can be checked by known conditions for q-order calmness of an related auxiliary mapping.

2 Notation and Preliminaries

To define further continuity concepts besides that of q-order calmness (1.2), let $\Phi : T \rightrightarrows \mathbb{R}^n$ be a multifunction with some given $(\bar{t}, \bar{x}) \in \mathrm{gph}\,\Phi$. Φ is said to have the *Aubin property* at $(\bar{t}, \bar{x})$ [1] iff there are $\varepsilon, \delta, \varrho > 0$ such that

$$\Phi(t) \cap B(\bar{x}, \varepsilon) \subset \Phi(t') + \varrho\, d(t, t')\mathbb{B} \quad \forall t, t' \in B(\bar{t}, \delta). \tag{2.1}$$

Defining $\mathrm{dist}(z, X) := \inf_x \{\|x - z\| \mid x \in X\}$ ($z \in \mathbb{R}^n$, $X \subset \mathbb{R}^n$), Φ is called *Lipschitz lower semicontinuous (Lipschitz l.s.c.)* at $(\bar{t}, \bar{x})$ [2] iff there are $\delta, \varrho > 0$ such that

$$\mathrm{dist}(\bar{x}, \Phi(t)) \leq \varrho\, d(t, \bar{t}) \quad \forall t \in B(\bar{t}, \delta). \tag{2.2}$$

Obviously, the Aubin property implies both calmness and Lipschitz lower semicontinuity. Note that the opposite direction fails, let e.g. $\Phi : \mathbb{R} \rightrightarrows \mathbb{R}$ with $\Phi(t) = \{0\}$ for $t \neq 0$, $\Phi(0) = \mathbb{R}$ and $\bar{x} = \bar{t} = 0$. If $\Phi(\bar{t}) \cap B(\bar{x}, \varepsilon) = \{\bar{x}\}$ holds in the definition (1.2), we speak of *isolated q-order calmness*.

We say that Φ is *lower semicontinuous (l.s.c.)* at $(\bar{t}, \bar{x})$ iff

$$\mathrm{dist}(\bar{x}, \Phi(t)) \to 0 \quad \text{for } t \to \bar{t}. \tag{2.3}$$

Note that (2.3) (and hence (2.2)) implies for each $\varepsilon > 0$, $\Phi(t) \cap B(\bar{x}, \varepsilon) \neq \emptyset$ if t is sufficiently close to $\bar{t}$.

Φ is called *upper semicontinuous in Berge's sense (B-u.s.c.)* at $\bar{t}$ iff for any open set $Q \supset \Phi(\bar{t})$ there is some neighborhood $\mathcal{N}$ of $\bar{t}$ such that $\Phi(t') \subset Q$ for all $t' \in \mathcal{N}$.

Given the parametric program (1.1), M will be called its *feasible set mapping*, while

$$\varphi(t) := \inf_x \{f(x, t) \mid x \in M(t)\}, \, t \in T,$$

denotes its *infimum value function*. Given $t \in T$ and a closed set $\emptyset \neq V \subset \mathbb{R}^n$, we define
$$\begin{aligned} M_V(t) &:= M(t) \cap V, \\ \Psi_V(t) &:= \mathrm{argmin}_x \{f(x, t) \mid x \in M_V(t)\}, \\ \varphi_V(t) &:= \inf_x \{f(x, t) \mid x \in M_V(t)\}. \end{aligned} \tag{2.4}$$

Lemma 1. *Consider the parametric optimization problem (1.1) under the assumptions imposed there. Given $(\bar{t}, \bar{x}) \in \mathrm{gph}\,\Psi$ and $\varepsilon > 0$, let $V = B(\bar{x}, \varepsilon)$ and suppose that M is l.s.c. at $(\bar{t}, \bar{x})$. Then Ψ_V is B-u.s.c. at $\bar{t}$.*

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[1] This concept was first studied in [2, 3] and originally designated *pseudo-Lipschitz continuity*; the name *Aubin property* goes back to [43]. Further, in [36], the term *Lipschitz-like* was introduced for this property.

[2] The terminology *Lipschitz l.s.c.* is taken from the authors' book [28]. In the more recent literature, one finds also names like *recession with linear rate* [24] or *inner calm* [6] for this concept.

Proof. As assumed in (1.1), M is closed at $\bar{t}$, and so M_V is closed at $\bar{t}$, too. Since V is compact, M_V is B-u.s.c. at $\bar{t}$, by [5, Lemma 2.2.3]. Further, M is l.s.c. at $(\bar{t}, \bar{x})$, i.e., $\mathrm{dist}(\bar{x}, M(t)) \to 0$ as $t \to \bar{t}$. Hence, there is some $\delta > 0$ such that $M(t) \cap V \neq \emptyset$ for all $t \in B(\bar{t}, \delta)$, and so $\mathrm{dist}(\bar{x}, M_V(t)) \to 0$ as $t \to \bar{t}$, i.e., M_V is l.s.c. at $(\bar{t}, \bar{x})$. Since f is continuous at $(\bar{t}, \bar{x})$ and for every $t \in B(\bar{t}, \delta)$, $\Psi_V(t)$ is - by the Weierstrass theorem - a nonempty subset of the compact set $B(\bar{x}, \varepsilon)$, then both Theorem 4.2.2(1') and Theorem 4.2.1(4) in [5] apply, and ensure that Ψ_V is B-u.s.c. at $\bar{t}$. $\qquad\square$

We conclude the section by some notation. $\|\cdot\|$ denotes any norm in $\mathbb{R}^n$, and $\mathbb{B}^\circ$ is the open unit ball in this norm. Given a compact subset I of a metric space, $C(I, \mathbb{R})$ denotes the linear space of continuous functions $i \in I \mapsto b_i \in \mathbb{R}$ equipped with the norm $\|b\| = \max_{i \in I} b_i$. By $h \in C^k$ (or $h \in C^{1,1}$) we abbreviate the property that h is a k-times continuously differentiable function (or a C^1 function with locally Lipschitzian derivative Dh). Let $\mathbb{R}_+$ be the set of nonnegative real numbers, and $\mathbb{R}_\infty = \mathbb{R} \cup \{\infty\}$. For brevity, we write $[1, m] := \{1, \ldots, m\}$.

3 Hölder Calmness under a Growth Condition

In this section, we present sufficient conditions for Hölder calmness of minimizing sets under certain regularity properties for the constraints and a growth condition for the objective function. Consider the parametric model (1.1).

As usual in the literature (see e.g. [10, Def. 3.1, Def. 3.141]), we say that for a given nonempty closed set $S \subset M(\bar{t})$ and a constant $\kappa \geq 1$, $f(\cdot, \bar{t})$ *satisfies the local κ-order growth condition at $\bar{x} \in S$ with respect to S* if for some open neighborhood $\mathcal{O}$ of $\bar{x}$ and a constant $c > 0$,

$$GC(\kappa): \qquad f(x, \bar{t}) \geq f(s, \bar{t}) + c\,\mathrm{dist}^\kappa(x, S)\ \forall s \in S\ \forall x \in M(\bar{t}) \cap \mathcal{O} \qquad (3.1)$$

holds true. Obviously, $f(s, \bar{t}) = const$ for all $s \in S \cap \mathcal{O}$, and $S \cap \mathcal{O}$ is a set of local minimizers of $P(\bar{t})$. If (3.1) even holds for some open set $\mathcal{O} \supset S$, we say that $f(\cdot, \bar{t})$ satisfies $GC(\kappa)$ at S. In this case, S is called a *set of weak sharp minima of order κ*, see [11]. If $f(\cdot, \bar{t})$ satisfies $GC(\kappa)$ at a compact set S, then S is a *complete local minimizing set* (*CLM set*) in the sense of Robinson [41], i.e., $S = \Psi_{\mathrm{cl}\,Q}(\bar{t})$ holds for some open set $Q \supset S$. Without loss of generality we will work with $S = \Psi(\bar{t})$, the analysis could be similarly done for CLM sets.

Theorem 1. *Consider the parametric model (1.1). Let $\bar{x}$ be an optimal solution of $P(\bar{t})$. Suppose that M is calm and Lipschitz l.s.c. at $(\bar{t}, \bar{x})$, and $f(\cdot, \bar{t})$ satisfies the local κ-order growth condition (3.1) at $\bar{x}$ with respect to $S = \Psi(\bar{t})$. Then the argmin mapping Ψ is κ^{-1}-order calm at $(\bar{t}, \bar{x})$.*

Proof. First we make some preparations. By the assumptions, $S = \Psi(\bar{t})$ is a closed set. Further, M is calm and Lipschitz l.s.c. at $(\bar{t}, \bar{x})$, hence there are $\beta_M, \varepsilon_M, \delta_M > 0$ such that

$$
\begin{aligned}
M(t) \cap B(\bar{x}, \varepsilon_M) &\subset M(\bar{t}) + \beta_M d(t, \bar{t}) \mathbb{B} &&\forall t \in B(\bar{t}, \delta_M), &&(3.2)\\
\mathrm{dist}(\bar{x}, M(t)) &\leq \beta_M d(t, \bar{t}) &&\forall t \in B(\bar{t}, \delta_M). &&(3.3)
\end{aligned}
$$

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Note that we have taken the same constant β_M both in (3.2) and (3.3) without losing generality, similarly for δ_M. Moreover, the Lipschitz property of f particularly means that with some positive constants ε_f, δ_f and β_f, the condition

$$|f(x,t) - f(y,\bar{t})| \le \beta_f(\|x-y\| + d(t,\bar{t})) \quad \forall x,y \in B(\bar{x},\varepsilon_f) \quad \forall t \in B(\bar{t},\delta_f) \qquad (3.4)$$

is fulfilled. As assumed, there exists a neighborhood $\mathcal{O}$ of $\bar{x}$ such that, with some $c > 0$, the growth condition (3.1) holds true. Now, choose $\varepsilon > 0$ so that

$$B(\bar{x},2\varepsilon) \subset \mathcal{O}, \quad \text{and} \quad V := B(\bar{x},\varepsilon) \ \text{ satisfies } \ 0 < \varepsilon \le \min\left\{\frac{\varepsilon_f}{2}, \varepsilon_M\right\}. \qquad (3.5)$$

Let $M_V(t)$ and $\Psi_V(t)$, $t \in T$, be defined according to (2.4).

By Lemma 1, Ψ_V is B-u.s.c. at $\bar{t}$. Hence, for some $\delta_0 > 0$ one has

$$\Psi_V(t) \subset \Psi_V(\bar{t}) + \frac{\varepsilon}{2}\mathbb{B}^\circ \quad \forall t \in B(\bar{t},\delta_0). \qquad (3.6)$$

Suppose that δ is a constant satisfying

$$0 < \delta \le \min\left\{\delta_f, \delta_M, \delta_0, \frac{\varepsilon}{2\beta_M}\right\}, \qquad (3.7)$$

which finishes our preparations. Now we are going to prove that for some $\varrho > 0$,

$$\text{dist}^\kappa(x,S) \le \varrho \, d(t,\bar{t}) \quad \forall x \in \Psi_V(t) \ \forall t \in B(\bar{t},\delta). \qquad (3.8)$$

This immediately implies that Ψ is κ^{-1}-order calm at $(\bar{t},\bar{x})$, since for $t \in T$,

$$\Psi_V(t) = \Psi(t) \cap V \ \text{ provided that } \ \Psi(t) \cap V \ne \emptyset. \qquad (3.9)$$

Indeed, if $\Psi(t) \cap V \ne \emptyset$ then $\varphi(t) = f(z,t) \le f(\xi,t)$ holds for some $z \in \Psi(t) \cap V$ and (especially) for all $\xi \in M(t) \cap V$, so $\varphi(t) = \varphi_V(t)$ and $\Psi_V(t) = M(t) \cap V \cap \{x \mid f(x,t) = \varphi(t)\} = \Psi(t) \cap V$.

To show (3.8), consider any $t \in B(\bar{t},\delta)$ and $x \in \Psi_V(t)$, where ε, δ and V are given according to (3.5) and (3.7), respectively. Next we apply the existence of auxiliary points $x^C \in M(\bar{t})$ and $x^L \in M(t)$ such that (by calmness and Lipschitz l.s.c.) Lipschitz estimates for $\|x - x^C\|$ and $\|\bar{x} - x^L\|$ hold in terms of $d(t,\bar{t})$. Along with the growth condition, the latter will be used for estimates in terms of the locally Lipschitzian function f.

Using $\|x - \bar{x}\| \le \varepsilon$ (by $x \in V$), (3.2) and (3.7), we get

$$\|x - x^C\| \le \beta_M d(t,\bar{t}) \le \frac{\varepsilon}{2} \ \text{ for some } x^C \in M(\bar{t}), \qquad (3.10)$$

where $\|x^C - \bar{x}\| \le \|x^C - x\| + \|x - \bar{x}\| \le \frac{3\varepsilon}{2}$, and so, according to (3.1) and $B(\bar{x},2\varepsilon) \subset \mathcal{O}$,

$$f(x^C,\bar{t}) \ge f(\bar{x},\bar{t}) + c\,\text{dist}^\kappa(x^C,S) \ \text{ for some } c > 0. \qquad (3.11)$$

Because of (3.3) and (3.7), we have

$$\|\bar{x} - x^L\| \le \beta_M d(t,\bar{t}) \le \frac{\varepsilon}{2} \ \text{ for some } x^L \in M(t). \qquad (3.12)$$

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Note that $t \in B(\bar{t}, \delta_f)$, $x, x^L \in B(\bar{x}, \varepsilon) \subset B(\bar{x}, \varepsilon_f)$ and $\|x^C - \bar{x}\| \leq \frac{3\varepsilon}{2} \leq \varepsilon_f$. By defining
$$e(t) := \beta_f(\beta_M + 1)d(t, \bar{t}),$$
(3.4), (3.10) and (3.12) then give
$$
\begin{aligned}
|f(x^C, \bar{t}) - f(x, t)| &\leq \beta_f(\|x^C - x\| + d(t, \bar{t})) \leq e(t), \\
|f(x^L, t) - f(\bar{x}, \bar{t})| &\leq \beta_f(\|x^L - \bar{x}\| + d(t, \bar{t})) \leq e(t).
\end{aligned}
$$

Using these estimates, one has
$$
\begin{aligned}
f(x^C, \bar{t}) &\leq f(x, t) + |f(x^C, \bar{t}) - f(x, t)| \leq f(x, t) + e(t), \\
f(x^L, t) - e(t) &\leq f(x^L, t) - |f(x^L, t) - f(\bar{x}, \bar{t})| \leq f(\bar{x}, \bar{t}).
\end{aligned}
$$

Therefore, together with (3.11),
$$
\begin{aligned}
f(x^L, t) - e(t) \leq f(\bar{x}, \bar{t}) &\leq f(x^C, \bar{t}) - c\,\mathrm{dist}^\kappa(x^C, S) \\
&\leq f(x, t) + e(t) - c\,\mathrm{dist}^\kappa(x^C, S),
\end{aligned}
$$

which entails
$$f(x^L, t) - f(x, t) \leq 2e(t) - c\,\mathrm{dist}^\kappa(x^C, S). \tag{3.13}$$

To finish the proof, let now
$$\|x - s\| = \mathrm{dist}(x, S) \text{ and } \|x^C - \hat{s}\| = \mathrm{dist}(x^C, S) \text{ for some } s, \hat{s} \in S.$$

Using (3.6) and (3.10), we then have
$$
\begin{aligned}
\|x - s\| \leq \mathrm{dist}(x, \Psi_V(\bar{t})) &< \frac{\varepsilon}{2} \\
\|x^C - \hat{s}\| \leq \|x^C - s\| \leq \|x^C - x\| + \|x - s\| &< \varepsilon.
\end{aligned}
$$

Hence, again with (3.10),
$$\|x - \hat{s}\| \leq \|x - x^C\| + \|x^C - \hat{s}\| < 2\varepsilon,$$

and so, by applying the mean value theorem to the function $\tau \in [0, 2\varepsilon) \mapsto \tau^\kappa$, for $\kappa \geq 1$,
$$
\begin{aligned}
\|x - \hat{s}\|^\kappa &\leq (\|x^C - \hat{s}\| + \|x - x^C\|)^\kappa \\
&\leq \|x^C - \hat{s}\|^\kappa + \kappa(2\varepsilon)^{\kappa-1}\|x - x^C\| \\
\text{(3.10)} \quad &\leq \mathrm{dist}^\kappa(x^C, S) + \kappa(2\varepsilon)^{\kappa-1}\beta_M d(t, \bar{t}).
\end{aligned}
$$

This implies
$$\mathrm{dist}^\kappa(x, S) \leq \|x - \hat{s}\|^\kappa \leq \mathrm{dist}^\kappa(x^C, S) + \kappa(2\varepsilon)^{\kappa-1}\beta_M d(t, \bar{t}), \tag{3.14}$$

while we have from (3.13), $x \in \Psi_V(t)$ and $x^L \in M_V(t)$ (3.12), by definition of $e(t)$,
$$c\,\mathrm{dist}^\kappa(x^C, S) \leq 2e(t) + f(x, t) - f(x^L, t) \leq 2\beta_f(\beta_M + 1)d(t, \bar{t}).$$

Combining this with (3.14), we thus obtain

$$
\begin{aligned}
\mathrm{dist}^\kappa(x, S) &\leq \mathrm{dist}^\kappa(x^C, S) + \kappa(2\varepsilon)^{\kappa-1}\beta_M d(t, \bar{t}) \\
&\leq c^{-1}(2\beta_f(\beta_M + 1) + c\kappa(2\varepsilon)^{\kappa-1}\beta_M)d(t, \bar{t}).
\end{aligned}
$$

By setting $\varrho := c^{-1}\left(2\beta_f(\beta_M + 1) + c\kappa(2\varepsilon)^{\kappa-1}\beta_M\right)$, (3.8) is shown. This completes the proof. $\qquad\square$

Note. According to (3.9), $\Psi_V(t) = \Psi(t) \cap V$ only holds if $\Psi(t) \cap V \neq \emptyset$. Under the assumptions of the theorem, $\Psi(t) \cap V$ is not automatically nonempty for t near $\bar{t}$. Consider for $\kappa \geq 1$ the example of minimizing the function $x \in \mathbb{R} \mapsto \min\{|x|^\kappa, e^x\}$ (or some smoothed modification) subject to $x \leq t$ at $(\bar{x}, \bar{t}) = (0, 0)$. $\qquad\diamond$

Corollary 1. *Consider the parametric model (1.1). Suppose that for some $\bar{t} \in T$, $S = \Psi(\bar{t})$ is nonempty and compact, M is calm and Lipschitz l.s.c. at $(\bar{t}, x)$ for each $x \in S$, and for some $\kappa \geq 1$, $f(\cdot, \bar{t})$ satisfies $GC(\kappa)$ at S. Then there exist some open bounded set $Q \supset S$ and some constant $\varrho > 0$ such that for t near $\bar{t}$,*

$$
\mathrm{dist}^\kappa(\xi, \Psi(\bar{t})) \leq \varrho\, d(t, \bar{t}) \quad \forall \xi \in \Psi(t) \cap Q. \tag{3.15}
$$

Proof. The assumptions here ensure that the assumptions of Theorem 1 are fulfilled for each $x \in \Psi(\bar{t})$ (x now replaces $\bar{x}$). By this theorem, we find that for each $x \in S = \Psi(\bar{t})$, there are positive constants $\varrho(x)$, $\varepsilon(x)$ and $\delta(x)$ such that if $t \in B(\bar{t}, \delta(x))$, then

$$
\mathrm{dist}^\kappa(\xi, S) \leq \varrho(x)\, d(t, \bar{t}) \quad \forall \xi \in \Psi(t) \cap B(x, \varepsilon(x)). \tag{3.16}
$$

The sets $B^\circ(x, \varepsilon(x)) := \{\xi \in \mathbb{R}^n \mid \|\xi - x\| < \varepsilon(x)\}$, $x \in S$, define a covering of S by open balls. Now, standard compactness arguments apply. Indeed, since S is compact, there are finitely many points $x^1, \ldots, x^r \in S$ such that $S \subset Q := \bigcup_{i=1}^r B^\circ(x^i, \varepsilon(x^i))$, where Q is open. Putting $\varrho := \max\{\varrho(x^1), \ldots, \varrho(x^r)\}$ and $\delta := \min\{\delta(x^1), \ldots, \delta(x^r)\}$, we see that (3.16), applied to $x \in \{x^i \mid i \in [1, r]\}$, implies

$$
\mathrm{dist}^\kappa(\xi, S) \leq \varrho\, d(t, \bar{t}) \quad \forall \xi \in \Psi(t) \cap Q = \bigcup_{i=1}^r \left(\Psi(t) \cap B^\circ(x^i, \varepsilon(x^i))\right),
$$

provided that $t \in B(\bar{t}, \delta)$. This completes the proof. $\qquad\square$

Let us now discuss specializations and the assumptions of the preceding theorem and its corollary, where we focus on the case of non-isolated initial solutions (i.e., $\Psi(\bar{t})$ is not a singleton) and q-order calmness for $0 < q = \kappa^{-1} < 1$.

Note that Hölder stability for isolated solutions of $P(\bar{t})$ is almost completely handled for nonlinear programs and further classes of optimization problems in the Chapters 4 and 5 of [10]. Further, note that the assertion of Corollary 1 can be strengthened for parametric nonlinear programs with C^2 or $C^{1,1}$ data. For such problems even calmness (i.e. $\kappa = 1$ in (3.15)) holds provided that the quadratic growth condition $GC(2)$ at $S = \Psi(\bar{t})$ and the *linear independence constraint qualification* are satisfied, see e.g. [26, Thm. 3.3] or similar results in [44] and [10, §4.9.3]. Calmness of minimizing sets also holds for certain convex parametric programs with fixed constraint set under a so-called strong quadratic growth condition, see [8, 23].

Theorem 1 and its corollary modify and refine the result of Theorem 2.2 in [26]. In particular, they are given in the up-to-date language of Hölder calmness and under weaker regularity properties of the constraints. Note that some ideas in the above proof go already back to [1] where a standard nonlinear program with smooth constraints for $S = \{\bar{x}\}$ was studied.

In the literature on Hölder stability of isolated minimizers or minimizing sets, the constraint set mapping M is usually supposed to have the Aubin property at $(\bar{t}, \bar{x})$ (cf. e.g. [1, 44, 10, 32]) which implies (3.2) and (3.3), or the Lipschitz l.s.c. assumption (3.3) at $(\bar{t}, \bar{x})$ is replaced by the stronger requirement

$$M(\bar{t}) \cap B(\bar{x}, \varepsilon_M) \subset M(t) + \beta_M d(t, \bar{t})\mathbb{B} \text{ for } t \text{ near } \bar{t},^3 \qquad (3.17)$$

cf. [25, 26].

Example 1. [13, Example 1] *Property (3.17) and hence the Aubin property are already violated in the very simple example $M = \Gamma$,*

$$\Gamma(t) := \{x \in \mathbb{R} \mid tx \leq 0\}, \ t \in \mathbb{R}, \ (\bar{t}, \bar{x}) = (0, 0).$$

On the other hand, Γ is calm and Lipschitz l.s.c. at $(0, 0)$.

A typical realization of Corollary 1 concerns the parameterized optimization problem

$$\widetilde{P}(t) : \qquad \min_x \ f(x, t) \text{ s.t. } G(x, t) \in K, \ t \text{ varies near } \bar{t}, \qquad (3.18)$$

where $\bar{t} \in T$, T is a Banach space, $f : \mathbb{R}^n \times T \to \mathbb{R}$ is locally Lipschitz, $G : \mathbb{R}^n \times T \to \mathbb{R}^m$ is a C^1 function, and $K \subset \mathbb{R}^m$ is a closed convex set. Let M and Ψ be the feasible and optimal set mapping, respectively, of (3.18). *Robinson's constraint qualification (RCQ)* [38] holds at $\bar{x} \in M(\bar{t})$ if

$$0 \in \text{int}\{G(\bar{x}, \bar{t}) + D_x G(\bar{x}, \bar{t})\mathbb{R}^n - K\},$$

which becomes the *Mangasarian-Fromovitz constraint qualification (MFCQ)* in the case $K = -\mathbb{R}_+^r \times \{0_{m-r}\}$ considered later.

Corollary 2. *Consider the parametric problem (3.18). Suppose that $S = \Psi(\bar{t})$ is nonempty and compact, f satisfies $GC(2)$ at S, and RCQ holds at every point $\bar{x} \in S$. Then there exist some open bounded set $Q \supset S$ and some $\varrho > 0$ such that for t near $\bar{t}$,*

$$\text{dist}^2(\xi, \Psi(\bar{t})) \leq \varrho \, d(t, \bar{t}) \ \ \forall \xi \in \Psi(t) \cap Q.$$

Proof. The result immediately follows from Corollary 1, since M has the Aubin property at $(\bar{t}, \bar{x})$ provided that RCQ holds at $\bar{x} \in M(\bar{t})$, see [38] or [10, Prop. 2.89]. □

If f is assumed to be a C^2 function, the result of the preceding corollary is a special case of Proposition 4.41 in [10] which was given in a Banach space setting for

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³In [25] the name *pseudo-Lipschitzian** was suggested for a multifunction which is calm and satisfies (3.17), see also [26, 42] for its application in stability analysis. If only (3.17) holds, the name *Lipschitz-l.s.c.** was recently introduced in [13].

the functions f and G and the set K, and which, in addition, permits approximate solutions of $\widetilde{P}(t)$.

In Theorem 1 and its corollaries, the growth condition $GC(\kappa)$ is a crucial (but abstract) requirement. Sufficient conditions or characterizations for $GC(\kappa)$ in the case of non-isolated minima are rare. For general $\kappa \geq 1$ one finds such conditions in terms of suitable higher-order directional derivatives in [46] (see also the references therein); however, it is not easy to verify them. This is possible in particular for $\kappa = 1, 2$ under smoothness or convexity assumptions, cf. e.g. [10, 11, 46]. If f is the point-wise maximum of finitely many C^2 functions, characterizations of $GC(2)$ for non-isolated minima can be found in [8, 9]. Concerning quadratic growth in programs of the type (3.18) with C^2 data f and G, we refer to [10, Sect. 3.5] for criteria in terms of the given problem. In particular, the local quadratic growth condition holds under some local 2nd-order sufficient optimality condition, see for example [10, Thm. 3.150] or [7].

Calmness for various types of constraint systems has been studied extensively, see e.g. [16, 19, 22, 29, 33, 34, 37, 40]. As mentioned above, M is calm and Lipschitz l.s.c. at $(\bar{t}, \bar{x})$ provided that M has the Aubin property at this point. Characterizations and applications of the Aubin property are well-studied for many classes of multifunctions and variational problems, we refer here exemplarily to the books [3, 10, 15, 28, 36, 43] for details. Therefore, Theorem 1 has potential applications to different types of models. $M(t)$ can represent not only standard constraint systems, but also, for example, solutions of a complementarity problem, of a generalized equation, or of a (lower level) optimization problem under perturbations.

For special constraint set mappings, the differences between Aubin property, Lipschitz lower semicontinuity and property (3.17) are marginal, but do exist. An interesting study of this phenomenon and of the related moduli in the context of (finite and semi-infinite) linear inequality systems has been recently published in [13]. Further, the mapping in Example 1 is a special polyhedral multifunction which does not satisfy the Aubin property, but is Lipschitz l.s.c. and calm. On the other hand, it is well-known that in some standard situations, the feasible set mapping M of problem (1.1) is calm and Lipschitz l.s.c. if and only if it has the Aubin property. This is recalled in the next remark.

Remark 1. It is known that for a canonically perturbed constraint system

$$\mathcal{F}(t) := \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{l} g_i(x) \leq t_i, \; i \in [1, r], \\ g_j(x) = t_j, \; j \in [r+1, m] \end{array} \right\}, \; t = (t_1, \ldots, t_m) \in \mathbb{R}^m, \quad (3.19)$$

with $g_i : \mathbb{R}^n \to \mathbb{R}$ ($i \in [1, m]$), the implication

$$\mathcal{F} \text{ is Lipschitz l.s.c. at } (0, \bar{x}) \;\Rightarrow\; \mathcal{F} \text{ has the Aubin property at } (0, \bar{x})$$

(and so the equivalence) holds for two standard situations, namely

  (i) if all g_i, $i \in [1, m]$, are C^1 functions,

  (ii) or if $g_1, \ldots, g_r$ are convex and $g_j(x) = \langle a^j, x \rangle - b_j$ ($a^j \in \mathbb{R}^n$, $b_j \in \mathbb{R}$), $j \in [r+1, m]$.

For (i), the implication was given in [29, Lemma 1], by showing that the Lipschitz l.s.c. property implies the MFCQ which is equivalent with the Aubin property [38].

For (ii) one even has: if $\mathcal{F}$ is l.s.c. at $(0, \bar{x})$ then $\mathcal{F}(t) \neq \emptyset$ for small $\|t\|$, which implies the generalized Slater CQ (i.e. $\{g_i(\tilde{x}) < 0, \ i \in [1, r], \ \langle a^j, \tilde{x} \rangle = b_j, \ j \in [r+1, m]\}$ holds for some $\tilde{x}$, and $\{a^{r+1}, \ldots, a^m\}$ are linearly independent) and hence (see [39]) the Aubin property. $\diamond$

We recall an example from [17] to illustrate that the *independent* perturbation of all parameters in the system (3.19) is crucial to get the equivalence (i) discussed in Remark 1.

Example 2. [17, Example 2.5] *Define the multifunction $M = \Omega$,*

$$\Omega(t) := \{x \in \mathbb{R}^2 \mid x_2(x_2 - x_1^2) \geq 0, \ x_2 = t\}, \ t \in \mathbb{R}.$$

Obviously,

$$\Omega(t) = \begin{cases} \{(x_1, t) \mid |x_1| \leq \sqrt{t}\} & \text{if } t > 0, \\ \{(x_1, x_2) \mid x_2 = t\} & \text{if } t \leq 0. \end{cases}$$

Hence, $\operatorname{dist}(x, \Omega(0)) = |t|$ *for all* $(t, x) \in \operatorname{gph} \Omega$ *and* $\operatorname{dist}(0, \Omega(t)) = |t|$ *for all t, and so Ω is in particular calm and Lipschitz l.s.c. at $(0, 0)$. On the other hand, Ω has not the Aubin property, since for any $t > 0$, one has $(\sqrt{t}, t) \in \Omega(t)$ and (for the Euclidean norm $\| \cdot \|_2$)*

$$\operatorname{dist}((\sqrt{t}, t), \Omega(t/4)) = \|(\sqrt{t}, t) - (\sqrt{t}/2, t/4)\|_2 = \sqrt{t/4 + 9t^2/16}.$$

It is easy to see that MFCQ does not hold at $0 \in \Omega(0)$. $\diamond$

Let us continue with a brief reference to conditions in [21], which guarantee calmness and Lipschitz lower semicontinuity for a solution mapping defined by disjunctive constraints, but do not imply that the Aubin property is satisfied.

Remark 2. Consider the constraint set mapping M of the parametric problem (3.18), but assume that K is the union of finitely many convex polyhedra, and G is a C^2 function. Let $\bar{x} \in M(\bar{t})$. In [21], the so-called *first order sufficient condition for metric subregularity (FOSCMS)* of the mapping $\widetilde{M}(x) := G(\bar{t}, x) - K$ at $(\bar{x}, 0)$ (i.e., its inverse is calm at $(0, \bar{x})$) is used as constraint qualification. FOSCMS is based on Gfrerer's concept [20] of a *limiting normal cone to a subset K in direction u* and guarantees that M is Lipschitz l.s.c. and calm, see [21, Prop. 1]. In special settings, this constraint qualification can be rewritten in terms of the problem data. This has been done e.g. for MPECs, see [21, Sect. 5].

It is worth noting that FOSCMS, in general, does not imply metric regularity of $\widetilde{M}$, which equivalently means that the Aubin property for $p \mapsto \{x \mid p \in G(\bar{t}, x) - K\}$ does not follow automatically. The latter is demonstrated by [21, Example 3] for a special MPEC. This is in contrast to standard nonlinear programs (see Remark 1). $\diamond$

We conclude this section by a remark concerning the case that an argmin mapping acts as the constraint set mapping M in the problem (1.1). Then it may happen that the Lipschitz l.s.c. assumption on M leads to a special (and less interesting) setting.

Remark 3. [28, Sect. 4.2] Consider the optimization problem

$$\min f(x) - \langle t, x \rangle \ \text{ s.t. } \ x \in X,$$

where X is a nonempty subset of a real Hilbert space T, f is any function from X to $\mathbb{R}$, and t varies near $0 \in T$. Let

$$\mathcal{M}(t) := \underset{x}{\operatorname{argmin}}\{f(x) - \langle t, x \rangle \mid x \in X\}, \; t \in T.$$

Then there holds, by Theorem 4.8 in [28]: *If* $(0, \bar{x}) \in \operatorname{gph} \mathcal{M}$ *and* $\mathcal{M}$ *is Lipschitz l.s.c. at* $(0, \bar{x})$, *then* $\mathcal{M}(0) = \{\bar{x}\}$ *and* $\mathcal{M}$ *is isolated calm at* $(0, \bar{x})$.

Thus, if $M := \mathcal{M}$ is Lipschitz l.s.c., then both M and Ψ in (1.1) are isolated calm at $(0, \bar{x})$. Note that $\mathcal{M}(\cdot)$ is even locally single-valued near $(0, \bar{x})$ if $\mathcal{M}$ has the Aubin property at this point [28, Cor. 4.7]. $\diamond$

4 Hölder Calmness via Associated Inequalities

Following the ideas in [12, 30], we show that q-order calmness of minimizing sets for the basic model (1.1) is implied by the corresponding calmness property for the solution set of some associated inequality system. This extends partially Theorem 4.7 in [32] which was proved in the context of convex semi-infinite programs under canonical perturbations.

First we recall an auxiliary result on (proper) calmness of optimal values.

Lemma 2. [30, Lemma 3.1] *Consider the problem (1.1). Given* $(\bar{t}, \bar{x}) \in \operatorname{gph} \Psi$, *suppose that* M *is calm and Lipschitz l.s.c. at* $(\bar{t}, \bar{x})$. *Then there exists some* $\varepsilon > 0$ *such that, with* $V = B(\bar{x}, \varepsilon)$, *the function* φ_V *is calm at* $\bar{t}$, *i.e.,* $|\varphi_V(t) - \varphi_V(\bar{t})| \leq \varrho_\varphi \, d(t, \bar{t})$ *holds for some neighborhood* U *of* $\bar{t}$, *some constant* $\varrho_\varphi > 0$ *and all* $t \in U \cap \operatorname{dom} \varphi_V$.

The q-order calmness of the optimal set mapping Ψ of (1.1) will be related to the multifunction

$$L(t, \mu) \; := \; \{x \in M(t) \mid f(x, \bar{t}) \leq \mu\}, \quad t \in T, \; \mu \in \mathbb{R}. \tag{4.1}$$

To prove the next theorem, one can argue the same way as in the case $q = 1$, which is handled in [30, Thm. 3.1]. For completeness, we give the simple proof.

Theorem 2. *Consider the problem (1.1). Suppose that for the reference point* $(\bar{t}, \bar{x}) \in \operatorname{gph} \Psi$,

  (i) the feasible set mapping M *is calm and Lipschitz l.s.c. at* $(\bar{t}, \bar{x})$ *and*

  (ii) for $q \in (0, 1]$, *the multifunction* $L = L(t, \mu)$ *in (4.1) is* q-*order calm at* $((\bar{t}, \varphi(\bar{t})), \bar{x})$.

Then the argmin mapping Ψ *is* q-*order calm at* $(\bar{t}, \bar{x})$.

Proof. The assumptions of the theorem imply that for some constants $\varepsilon, \delta, \varrho > 0$, with $V = B(\bar{x}, \varepsilon)$, $U = B(\bar{t}, \delta)$ and $\bar{\mu} = \varphi(\bar{t})$,

$$L(t, \mu) \cap V \; \subset \; L(\bar{t}, \bar{\mu}) + \varrho \, (d(t, \bar{t}) + |\mu - \bar{\mu}|)^q \, \mathbb{B} \quad \forall t \in U \; \forall \mu \in B(\bar{\mu}, \delta), \tag{4.2}$$

$$|f(x, t) - f(y, \bar{t})| \; \leq \; \varrho \, (\|x - y\| + d(t, \bar{t})) \quad \forall x, y \in V \quad \forall t \in U. \tag{4.3}$$

Since the assumptions of Lemma 2 are satisfied, we conclude that for some modulus $\varrho_\varphi > 0$ and some neighborhood $\tilde{U} \subset U$ of $\bar{t}$,

$$|\varphi_V(t) - \varphi_V(\bar{t})| \le \varrho_\varphi \, d(t, \bar{t}) \qquad \forall t \in \tilde{U} \cap \operatorname{dom} \Psi_V. \tag{4.4}$$

Now let $U' \subset \tilde{U}$ be a neighborhood of $\bar{t}$ such that for all $t \in U'$ and all $x \in V$ both

$$\varrho_\varphi \, d(t, \bar{t}) \le \frac{\delta}{2} \quad \text{and} \quad |f(x, t) - f(x, \bar{t})| \le \varrho \, d(t, \bar{t}) \le \frac{\delta}{2} \tag{4.5}$$

hold true, where (4.3) was used. By definition and (3.9), one has $L(\bar{t}, \varphi(\bar{t})) = \Psi(\bar{t})$, $\Psi_V(\bar{t}) = \Psi(\bar{t}) \cap V$, $\varphi_V(\bar{t}) = \varphi(\bar{t})$ and

$$x \in \Psi(t) \quad \Leftrightarrow \quad x \in L(t, \mu(x, t)) \text{ where } \mu(x, t) := \varphi(t) + f(x, \bar{t}) - f(x, t).$$

Consider any $t \in U'$ and suppose $\Psi(t) \cap V \ne \emptyset$, otherwise q-order calmness is trivially satisfied. Hence, we obtain $\Psi_V(t) = \Psi(t) \cap V$ and $\varphi(t) = \varphi_V(t)$ from (3.9), and by (4.4) and (4.5),

$$\begin{aligned}
|\mu(x, t) - \varphi(\bar{t})| &= |\varphi_V(t) + f(x, \bar{t}) - f(x, t) - \varphi_V(\bar{t})| \\
&\le |\varphi_V(t) - \varphi_V(\bar{t})| + |f(x, \bar{t}) - f(x, t)| \le \delta.
\end{aligned}$$

So, we can apply (4.2) (recall $\bar{\mu} = \varphi(\bar{t})$ and $L(\bar{t}, \bar{\mu}) = \Psi(\bar{t})$), and it follows

$$\Psi(t) \cap V = L(t, \mu(x, t)) \cap V \subset \Psi(\bar{t}) + \varrho \, (d(t, \bar{t}) + |\mu(x, t) - \bar{\mu}|)^q \, \mathbb{B}.$$

Then $|\mu(x, t) - \bar{\mu}| \le |\varphi_V(t) - \varphi_V(\bar{t})| + |f(x, \bar{t}) - f(x, t)| \le (\varrho_\varphi + \varrho) d(t, \bar{t})$ yields

$$\Psi(t) \cap V \subset \Psi(\bar{t}) + \varrho(1 + \varrho_\varphi + \varrho)^q \, (d(t, \bar{t}))^q \, \mathbb{B}.$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Let us continue with a discussion of the above result for semi-infinite or finite parametric optimization problems of the type

$$P(c, b): \qquad \min \, h(x) + \langle c, x \rangle \ \text{ s.t. } \ g_i(x) \le b_i, \, i \in I, \tag{4.6}$$

where I is any compact set in a metric space, $h, g_i : \mathbb{R}^n \to \mathbb{R}$ $(i \in I)$ are locally Lipschitz, $(i, x) \mapsto g_i(x)$ and $i \mapsto b_i$ are continuous, and (c, b) varies in the parameter space $T = \mathbb{R}^n \times C(I, \mathbb{R})$ near some given $\bar{t} = (\bar{c}, \bar{b}) \in T$. If $I = [1, m]$, this reduces to a standard nonlinear program.

In the analysis of the model (4.6), let $M(t) = M(b)$, $\Psi(t) = \Psi(c, b)$ and $\varphi(t) = \varphi(c, b)$ symbolize the constraint set mapping, the optimal set mapping and the optimal value function. The auxiliary mapping L in (4.1) now becomes

$$L(b, \mu) = \{x \mid h(x) + \langle \bar{c}, x \rangle \le \mu, \ g_i(x) \le b_i, \, i \in I\}, \tag{4.7}$$

where b varies near $\bar{b}$ and μ varies near $\varphi(\bar{c}, \bar{b})$. Given $((\bar{c}, \bar{b}), \bar{x}) \in \operatorname{gph} \Psi$ and the optimal value $\bar{\mu} = \varphi(\bar{t})$, we define the auxiliary function,

$$H(x) := \max\{ \max_{i \in I} \, (g_i(x) - \bar{b}_i) \, , \, h(x) + \langle \bar{c}, x \rangle - \bar{\mu} \, \}.$$

Corollary 3. *Consider the problem (4.6). Suppose that the feasible set mapping M is calm and Lipschitz l.s.c. at $((\bar{c}, \bar{b}), \bar{x}) \in \mathrm{gph}\,\Psi$. Then the argmin mapping Ψ is q-order calm at $((\bar{c}, \bar{b}), \bar{x})$ with $0 < q \leq 1$ if the multifunction L (4.7) is q-order calm at $((\bar{b}, \bar{\mu}), \bar{x})$, or, equivalently, if the level set map*

$$L_H(\nu) := \{x \mid H(x) \leq \nu\}, \; \nu \text{ varies near } 0, \tag{4.8}$$

is q-order calm at $(0, \bar{x})$.

Proof. Obvious from Theorem 2 and the definitions of L (4.7) and H. □

For level set mappings, the following characterization of q-order calmness has been given by the second author in [33, Prop. 3.4] (see also [27, Cor. 4.3]):

Let X be a complete metric space, $F : X \to \mathbb{R}_\infty$ be lower semicontinuous and $F(\bar{x}) = 0$. The level set map $x \in X \mapsto L_F(\nu) := \{x \mid F(x) \leq \nu\}$ is q-order calm at $(0, \bar{x})$ if and only if there are $\delta, \varepsilon, \lambda > 0$ such that

$$\begin{array}{c} \text{for every } x \in B(\bar{x}, \varepsilon) \text{ with } 0 < F(x) \leq \delta \text{ there is} \\ \text{some } x' \text{ satisfying } \max\{0, F(x')\}^q - F(x)^q < -\lambda d(x', x). \end{array} \tag{4.9}$$

This can be reformulated in certain procedures to check q-order calmness, see [33] for details. An application to standard nonlinear programs in the case $q = \frac{1}{2}$ is discussed below.

Consider now the model (4.6) under the assumptions

$$h, g_i \; (i \in I) \text{ are convex functions.} \tag{4.10}$$

In this case, Corollary 3 covers the implications (iii) $\Rightarrow$ (i) $\Rightarrow$ (ii) of Theorem 4.7 in [32], which was given under the assumption that $M(\bar{b})$ satisfies the Slater CQ (i.e., $\exists \tilde{x} \; \forall i \in I : \; g_i(\tilde{x}) < 0$). It is known (cf. [39]) that the Slater CQ implies the Aubin property, and vice versa, hence M is Lipschitz l.s.c. and calm.

Remark 4. Theorem 4.7 in [32] is proved by showing and utilizing that q-order calmness of L is equivalent to the property for H to have a q-order error bound. The latter holds if and only if

$$\liminf_{x \to \bar{x}, \; H(x) \downarrow 0} H(x)^{q-1} \mathrm{dist}(0, \partial H(x)) > 0,$$

see [32, Prop. 4.3], where $\partial H(x)$ means the subdifferential of convex analysis.

An important additional result of [32, Thm. 4.7] is that under Slater CQ the opposite direction of Corollary 3 holds if h and g_i are linear. ◇

For a standard nonlinear program with C^2 data, a sufficient condition for the $\frac{1}{2}$-order Hölder calmness of inequality systems, given in [33], can be immediately used. Consider the model (4.6) and assume that

$$h, g_i, \; i \in I = [1, m], \text{ are } C^2 \text{ functions.} \tag{4.11}$$

Then Theorem 4.11 in [33] gives that L_H (4.8) is $\frac{1}{2}$-order calm at $(0, \bar{x})$ if $0 \notin \partial^C H(\bar{x})$ or if (otherwise) there exists some $\gamma > 0$ such that

$$\min\{\|x^*\| \mid x^* \in \partial^C H(x)\} \geq \gamma \|x - \bar{x}\| \text{ for } x \text{ near } \bar{x}, \tag{4.12}$$

where $\partial^C H$ denotes Clarke' subdifferential [14]. Note that (4.12) is equivalent to some injectivity condition on the contingent derivative of $\partial^C H$ at $(\bar{x}, 0)$, for details of this characterization we refer to [33, Sect. 4.2].

Finally, we recall from [30, Prop. 4.2] a refined version of Theorem 2 for the parametric problem (1.1) in the case of *proper* calmness (i.e. $q = 1$). There it was shown that the multifunction L (4.1) is calm at $(\bar{t}, \bar{x})$ provided that

(i) M is calm and Lipschitz l.s.c. at $(\bar{t}, \bar{x})$,

(ii) the level set map $L_0(\mu) := \{x \mid f(x, \bar{t}) \le \mu\}$ is calm at $(\varphi(\bar{t}), \bar{x})$, and

(iii) the restricted level set map $\mu \mapsto L_0(\mu) \cap M(\bar{t})$ is calm at $(\varphi(\bar{t}), \bar{x})$.

In the proof of this statement, an intersection theorem for calm multifunctions is applied (cf. [28, Thm. 2.5]), and it is used that for locally Lipschitzian f the inverse L_0^{-1} of the level set mapping L_0 has the Aubin property at $(\bar{x}, \varphi(\bar{t}))$.

For a detailed discussion of further specializations to the settings (4.10) and (4.11) in the case $q = 1$, we refer to [30, Sect. 4].

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