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Commodity Prioritized Maximum Dynamic Multi-Commodity Flow Problem

Tanka Nath Dhamala · Durga Prasad Khanal · Urmila Pyakurel

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Abstract Due to different disasters, natural or men made, world is facing the problem of massive damage of life and property. To save the life of maximum number of evacuees, an efficient evacuation planning is essential. Prioritization is the process of deciding the relative importance or urgency of things or objects. It helps to focus on the objectives and goals in an efficient way. Prioritized flow problems are applicable for large scale disaster management problems. Prioritization of evacuees on the basis of case sensitive or risk of life is one of the best and reliable way of evacuation planning. In this paper, we introduce the maximum evacuation planning problem with commodity (evacuee) priority ordering and develop multi-commodity flow model for priority based evacuation planning problem. We present polynomial time algorithms to solve the problem in static as well as dynamic flow models.

Keywords Multi-commodity · maximum flow · commodity priority · evacuation planning

1 Introduction

Network flows are widely used to solve real life problems. Dynamic flow consists of transmission of commodities from the supply zone to the demand zone

Tanka Nath Dhamala
Central Department of Mathematics, Tribhuvan University, Kathmandu, Nepal
E-mail: amb.dhamala@daadindia.org

Durga Prasad Khanal
Tribhuvan University, Saraswati Multiple Campus, Nepal
E-mail: durgapsdkhanal@gmail.com

Urmila Pyakurel
Central Department of Mathematics, Tribhuvan University, Kathmandu, Nepal
E-mail: urmilapyakurel@gmail.com
Corresponding author

incorporating with time factor necessary to travel through. Mostly considered applications of this problem are modeled for communication network, highway and railway networks, supply and demand chains and massage routine. Ford and Fulkerson [7] introduced the maximum static as well as dynamic flow problems in which maximum amount of flow is shifted from the source node to the sink node. The detailed illustrations of maximum flow can be found in the books [1,8] and survey papers [3,17].

Minieka [19] introduced the lexicographic maximum static flow problem and presented polynomial time algorithm to solve it. Megiddo [18] studied this problem in single source and multi sink network. Hamacher and Tufekci [11] presented evacuation problem as a lexicographic min cost flow problem and presented an algorithm based on time expanded network. They divided the building in to different prioritized zone and evacuate with successive higher priority zones as quickly as possible. Hoppe and Tardos [13] modeled evacuation problem as a flow problem in dynamic network. The first polynomial time algorithm to solve lexicographic maximum dynamic flow and quickest transshipment problems are presented by the same authors in [14] and PhD thesis of Hoppe [12].

Multi-commodity flow problem concerns with the transshipment of more than one commodities from respective sources to the corresponding sinks without violating capacity constraints on the arcs. The maximum multi-commodity flow problem deals with the transshipment of maximum number of objects in given time horizon. Fleischer and Skutella [6] generalized the dynamic flow problem introduced by Ford and Fulkerson [8] to the case of a multi-commodity network. The static multi-commodity flow problem is solved in polynomial-time by using the ellipsoid or interior point method. However, the multi-commodity flow over time problem is $\mathcal{NP}$-hard, [10]. Approximate solutions of the quickest multi-commodity flow problem are presented in [6] by using T-length bounded approximation and condensed time-expanded network.

Contraflow is one of the most useful techniques in evacuation planning problem where, flipping of the orientation of arcs towards the destination node is used to increase the outbound capacity of arcs and reduce the time horizon. Different authors in different time use heuristic, analytic and simulation techniques using contraflow configuration. Kim et al. [16] presented macroscopic models by incorporating multiple sources, road capacity constraints, congestion and scalability. Various problems with contraflow can be found in [2,21, 23–25]. Different algorithmic approaches and their applications can be found in the survey paper of Dhamala et al. [5]. Pyakurel and Dempe [20] introduced the maximum dynamic contraflow problem with intermediate storage and presented polynomial time algorithm to solve the problem. By reverting only necessary arcs, Dhamala et al. [4] presented algorithms to solve quickest multi-commodity flow problem. Authors in [9] extended this result in case of continuous time setting. Recently, maximum multi-commodity flow over time problem with partial contraflow is presented by Pyakurel et al. [22].

In day to day life, every individual has the list of work to be completed within certain time period. The optimal use of limited time is only possible by efficient planning of works with priority. Prioritization is a vital skill that is used to manage the variety of tasks by deciding the relative importance or the urgency of things. Though priority may not be uniform for each individual, it helps to fulfill the objectives, achieve the goals and improve the efficiency of the work.

In this paper, we introduce the model for the prioritized evacuation planning problem in which evacuees are categorized according to their case sensitive and evacuate by using multi-commodity flow model. Considering different category of evacuees as commodities, we use the model of multi-commodity flow problem. Evacuees are sent from respective sources to corresponding sinks so that prioritization of sources automatically prioritize the sinks in same order. In each arc, evacuees are sent in some priority order without violating capacity constraint for each time steps. We present polynomial time algorithms to solve maximum evacuation planning problems for static and dynamic flow models. This paper is an extension of extended abstract [15] that was presented on ISAHP 2020.

We organize the paper as follows. Section 2 provides the mathematical formulations of flow models. In Section 3, we present static as well as dynamic prioritized algorithms to solve the maximum evacuation planning problems with commodity priority. The paper is concluded in Section 4.

2 Mathematical Formulation of Flow Models

The multi-commodity evacuation planning problem with priority order concerns with the transshipment of distinctly categorized evacuees from their respective sources to corresponding sinks without violating the capacity constraints through a given network topology so that the total number of evacuees of each category is shifted efficiently. After the disasters, every individual may not be hurt equally. So evacuation of the group of people on the basis of case sensitive or the risk of life is essential to save their lives. We represent each differently categorized evacuees as multiple commodity groups. We set necessary mathematical notations for the formulation of flow models as follows.

2.1 Notations

Let $\mathcal{N} = (V, A, K, u, \tau, d_i, S, D, T)$ be a dynamic network topology with node set V, arc set $A \subseteq V \times V$, and the set of commodities $K = \{1, 2, ..., k\}$. The non-negative capacity function $u : A \to \mathcal{R}^+$ limits the flow of commodities on the arc and a non-negative transit time function $\tau : A \to \mathcal{R}^+$ measures the time to transship the flow associated with each arc $e = (v, w)$. We use n and m to denote the number of nodes and arcs in $\mathcal{N}$. The demand for each commodity $i \in K$, denoted by d_i, is routed through a unique source-sink pair

(s_i, t_i), where $s_i \in S \subset V$ and $t_i \in D \subset V$ with $S \cap D = \emptyset$. The time period T is denoted by $\mathcal{T} = \{0, 1, ..., T\}$ in discrete time settings.

The flow rate function is defined by $\Phi : A \times \mathcal{T} \to \mathcal{R}^+$, where $\Phi_e^i(\theta)$ represents the flow rate of commodity $i \in K$ on arc e at time $\theta \in \mathcal{T}$. In case of static flow, this function is defined by $x : A \to \mathcal{R}^+$, where x_e^i represents the flow of commodity i on arc e. The time parameters T and τ are absent in this case.

2.2 Priority ordering on evacuation planning.

At the time of evacuation, each and every evacuees are equally important to be transshipped from danger zone to the safe place. But the people who are critically injured, need quick treatment to safe their lives. Similarly, minor injured, old aged people and pregnant women or with babies, normal evacuees may be successively lower in priority order while evacuation process. Critically injured people are to be transshipped to the well equipped hospitals, whereas minor injured evacuees in the health centers. Similarly, old aged people and pregnant women or with babies are to be sent in the care centers and the normal people in some safe shelters. The collection of evacuees are made according to their priority order (case sensitive) on different collection centers. Our assumption is that the evacuees within the group (collection center) are of homogeneous character and between the groups are of heterogeneous character. Thus, evacuation problem becomes a multi-commodity flow problem with commodity priority where commodities are transshipped from respective sources (collection centers s_i) to corresponding sinks (destinations t_i) without violating capacity constraints on the arcs.

We define the commodity priority ordering function $\mathbb{P} : K \to \mathcal{Z}^+$ such that $\mathbb{P}(i) \prec \mathbb{P}(i+1) \;\; \forall i \in K$, where $\mathbb{P}(i)$ represent the commodities with i^{th} priority that are to be transshipped from s_i to t_i. The symbol $\mathbb{P}(i) \prec \mathbb{P}(i+1)$ represents that $\mathbb{P}(i)$ is higher in priority than $\mathbb{P}(i+1)$. At any instance of time θ, if more than one commodities are to be entering on an arc, then commodity of first priority $\mathbb{P}(1)$ enters at first and if flow of first commodity is strictly less than arc capacity, then commodity of second priority $\mathbb{P}(2)$ is to be entered and so on with successive order.

The priority order in term of path flow is as follows. Let $\mathbf{P}_i$ be a set of paths for commodity i (i.e. evacuees of priority group i). The bottleneck capacity of path $P \in \mathbf{P}_i$ is $\nu_P^i = \min\{u_e \mid e \in P\}$. The dynamic prioritized flow function on the arc e at time θ is

$$\Phi_e^i(\theta) = \begin{cases} \nu_P^1(\theta) & \text{if } i = 1 \\ \min\{u_e - \sum_{m=1}^{i-1} \Phi_e^m(\theta),\;\; \nu_P^i(\theta)\} & \text{if } i \geq 2 \end{cases} \quad \forall P \in \mathbf{P}_i. \tag{2.1}$$

Similarly, the static prioritized flow on the arc e with priority order is

$$x_e^i = \begin{cases} \nu_P^1 & \text{if } i = 1 \\ \min\{u_e - \sum_{m=1}^{i-1} x_e^m,\;\; \nu_P^i\} & \text{if } i \geq 2 \end{cases} \quad \forall P \in \mathbf{P}_i. \tag{2.2}$$

2.3 Static prioritized flow model.

A static multi-commodity flow x for the given static network $\mathcal{N}$ without time dimension is the sum of all nonnegative static flows x^i defined by the functions $x^i : A \to \mathcal{R}^+$ for each commodity i satisfying the prioritized flow function (2.2) and constraints (2.3 - 2.4).

$$\mathcal{G}_v = \begin{cases} d_i & \text{if } v = s_i \\ -d_i & \text{if } v = t_i \quad \forall i \in K \\ 0 & \text{otherwise} \end{cases} \tag{2.3}$$

$$0 \leq x_e = \sum_{i \in K} x_e^i \leq u_e \;\; \forall e \in A \tag{2.4}$$

where, the net flow at node v is

$$\mathcal{G}_v = \sum_{e \in A_v^{out}} x_e^i - \sum_{e \in A_v^{in}} x_e^i$$

The sets $A_v^{in} = \{(w, v) \mid w \in V\}$ and $A_v^{out} = \{(v, w) \mid w \in V\}$ denote the set of incoming arcs to node v and outgoing arcs from node v, respectively, such that $A_D^{out} = \emptyset$ and $A_S^{in} = \emptyset$ in the case of without arc reversals. Third condition of the constraints in (2.3) are flow conservation constraints for each commodity at intermediate nodes and the net flow is presented in the remaining conditions. The constraints in (2.4) are bundle constraints bounded by the arc capacities.

2.4 Dynamic prioritized flow model.

For a given dynamic network $\mathcal{N}$ with constant transit times on arcs, a multi-commodity flow over times Φ in the network with k commodities is the sum of flows defined by functions $\Phi_e^i : A \times \mathcal{T} \to \mathcal{R}^+$ satisfying the prioritized flow function (2.1) and constraints (2.5 - 2.7).

$$\mathcal{G}_{v,i}^+(T) - \mathcal{G}_{v,i}^-(T) = \begin{cases} d_i & \text{if } v = s_i \\ -d_i & \text{if } v = t_i \quad \forall i \in K, \\ 0 & \text{otherwise} \end{cases} \tag{2.5}$$

$$\mathcal{G}_{v,i}^-(\theta) - \mathcal{G}_{v,i}^+(\theta) \geq 0, \;\; \forall \;\; \theta \in \mathcal{T}, \; v \notin \{s_i, t_i\}, \text{ and } \; i \in K. \tag{2.6}$$

$$0 \leq \Phi_e(\theta) = \sum_{i=1}^{k} \Phi_e^i(\theta) \leq u_e, \; \forall \; e \in A, \;\; \theta \in \mathcal{T} \tag{2.7}$$

where,

$$\mathcal{G}_{v,i}^-(\theta) := \sum_{e \in A_v^{in}} \sum_{\delta = \tau_e}^{\theta} \Phi_e^i(\delta - \tau_e) \quad \text{and} \quad \mathcal{G}_{v,i}^+(\theta) := \sum_{e \in A_v^{out}} \sum_{\delta = 0}^{\theta} \Phi_e^i(\delta).$$

Here, the third condition of the constraints in (2.5) are flow conservation constraints at intermediate nodes in time horizon T, whereas the constraints in (2.6) represent non-conservation of flow at intermediate time points $\theta \in \mathcal{T}$. Similarly, the bundle constraints in (2.7) are bounded above by the capacities on the arcs. The goal is to transship the maximum amount of flow d_i in given time horizon T from s_i to t_i which is stated in first two conditions of (2.5).

3 Commodity Prioritized Maximum Flow

In this section, we introduce static as well as dynamic flow problems for commodity prioritized maximum flow with multi-commodity flow model and present polynomial algorithms to solve them. At first, we define the terminologies that are used hereafter.

Definition 1 Maximal flow: A maximal flow in the network $\mathcal{N}$ is a flow that maximizes the net flow out from the sources (into the sinks). We denotes the maximal flow that can leave $s_i \in S$ by $M(s_i)$.

Definition 2 Lexicographically maximal flow: If $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_k \subseteq S$, then a maximal flow that can delivers $M(S_i)$ units of flow from each subset S_i, for all $i = 1, \ldots, k$ is called a lexicographically maximal flow on the sources.

3.1 Commodity prioritized maximum static multi-commodity flow

Prioritization is sequentially deciding the importance of objects. Minieka [19] has shown the existence of lexicographic flow in static network. For given set of sources $\{s_1, \ldots, s_k\}$, he constructed the set of chain of sources $S_i = \{s_1, \ldots, s_i\}$ $\forall i = 1, \ldots, k$ and proved that amount of flow leaving the sources S_i is simultaneously maximum. Our model is prioritized multi-commodity flow model where each commodity is to be transshipped from respective source to corresponding sink. So prioritization of sources automatically prioritize the sinks in same order. Together with this, our model also fixes the priority of commodities in each bundle arc where more than one commodities are entering the arc at the same time. Figure 1 represents the construction of set inclusion in sources S_i (similarly for sinks) for multi-commodity flow problem with priority order.

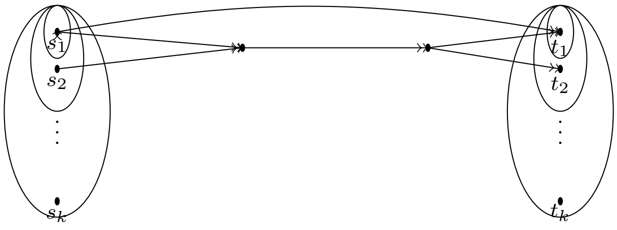

Fig. 1 Multi-commodity flow network with set inclusion.

The prioritized maximum static multi-commodity flow problem with commodity priority is as follows.

Problem 1 Let $\mathcal{N} = (V, A, K, u, d_i, S, D)$ be the given multi-commodity flow network with commodity priority order $\mathbb{P}(i) \prec \mathbb{P}(i+1) \ \ \forall\ i \in K$. Then the prioritized maximum multi-commodity flow problem is to transship the net flow d_i from s_i to t_i by using priority order without violating the capacity constraints on arcs.

To solve this problem, we first prioritize the commodities and set $S_1 = s_1,\ S_2 = s_1 \cup s_2, \ldots, S_k = \cup_{i=1}^{k} s_i$ so that $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_k \subseteq S$. While sending flow on bundle arcs, priority is given to the commodities as described in Equation (2.2) of Section 2.2. We extend the lex-max flow algorithm of Minieka [19] to solve the problem in polynomial time. Here, we present an algorithm to solve the problem.

Algorithm 1: Commodity prioritized maximum static multi-commodity flow algorithm

```
Input  : Given multi-commodity network $\mathcal{N} = (V, A, K, u, d_i, S, D)$.
Output: Commodity prioritized multi-commodity flow on $\mathcal{N}$.
 1. Prioritize the commodities such that $\mathbb{P}(i) \prec \mathbb{P}(i+1)\ \ \forall i \in K$.
 2. Define $S_1 = s_1,\ S_2 = s_1 \cup s_2, \ldots, S_k = \cup_{i=1}^{k} s_i$.
 3. Compute lex-max flow by using algorithm of Minieka [19] and priority ordering
    function (2.2).
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Theorem 1 *For given network $\mathcal{N}$, if $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_k \subseteq S$ then there exists a maximal flow that delivers $M(S_i)$ units from subset S_i, $\forall i = 1, \ldots, k$. Indeed, Algorithm 1 gives lexicographically maximum flow with priority ordering.*

Proof We prove the theorem by induction. Since $S_1 - D_1$ contains single commodity flow from s_1 to t_1, it is single source and single sink flow. So maximum flow algorithm of Ford and Fulkerson [8] provides the maximum number of flow units leaving S_1. That is $M(S_1)$ is maximal flow. We suppose that such maximal flow $M(S_i)$ exists for $S_i,\ i = i, \ldots, m-1,\ \ (m \leq k)$. We prove that maximal flow exists for $S_m \subseteq S$.

Construct an extended network $\mathcal{N}'$ by adding two new nodes s' and t' with additional set of infinite capacitated arcs (s', s_i) for $s_i \in S_m$, (t_i, t') for all $t_i \in D$ and (t', s_i) for $s_i \in S \setminus S_m$ (see Figure 2). Let flow on arcs (s', s_i) for $s_i \in S_m$ is equal to net flow out from sources s_i. Considering s' as source and t' as sink nodes for extended network $\mathcal{N}'$, it is now reduced to single source and single sink network. As our assumption, there exists a maximal flow which sends $M(S_i)$ units of flow for each subset S_i for $i = 1, \ldots, m-1$. Using maximimum flow of [8] in $\mathcal{N}'$ with source node s' and sink node t', it generates the set of $s' - t'$ paths. Some paths in $s' - t'$ may use the arcs of

S_{m-1} but due to commodity prioritized function (2.2), the residual (remaining) capacity from S_{m-1} is used by S_m. So, there is no conflict of use of common arc in priority order which behaves as single commodity flow paths after capacity sharing.

As flow in $S_i,\ i = 1, \ldots, m-1$ are maximal and $M(S_m)$ unit of flow delivered from s' to t' in $\mathcal{N}'$ is maximum flow, S_m sends maximal flow with priority order. Due to set inclusions, this flow is lexicographically maximum flow with commodity priority ordering.

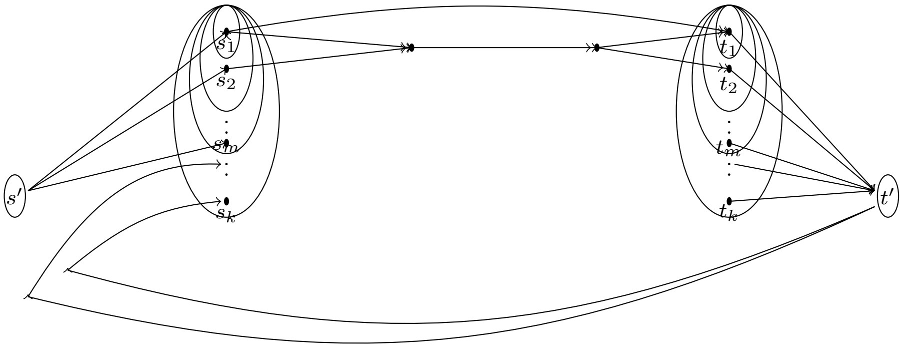
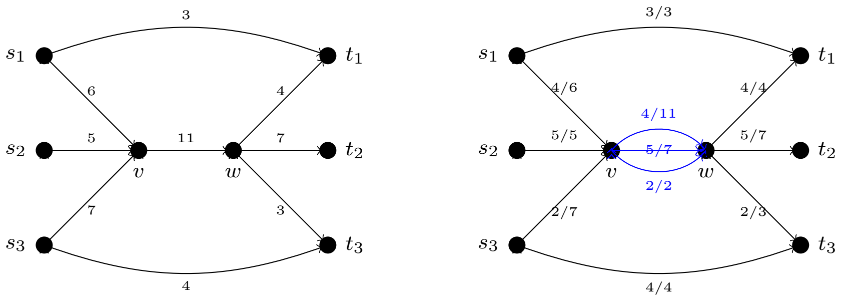


Fig. 2 Extended network $\mathcal{N}'$ of given network $\mathcal{N}$.

Theorem 2 *Algorithm 1 solves commodity prioritized maximum static multi-commodity flow problem efficiently.*

Proof First we prove the feasibility of the theorem. Steps 1 and 2 are feasible which can be obtained in constant time. The priority function (2.2) is applied in each intermediate arcs within $O(m)$ times. As lex-max algorithm of Minieka [19] is polynomial time solvable, Step 3 is also feasible. Next, the optimality of algorithm is dominated by the optimality of Step 3 which provides optimal solution for each $S_k = \cup_{i=1}^{k} s_i$.

Example 1 Consider a multi-commodity network with three commodities that are to be transshipped from s_i to t_i for $i = 1, 2, 3$ as shown in Figure 3. While sending flow, first priority is given to commodity-1 through two paths: $s_1 - t_1$ with flow value 3 units and $s_1 - v - w - t_1$ with bottleneck capacity 4. Path $s_1 - v - w - t_1$ uses arc (v, w) which is bundle arc used by other commodities as well with priority order. The remaining capacity on this arc is $11 - 4 = 7$, so commodity-2 is transshipped through path $s_2 - v - w - t_2$ with flow of 5 units, because the minimum of remaining capacity of arc and bottleneck capacity of path is 5. Similarly, third priority is given to commodity-3 in which we can send 2 units of flow on the path $s_3 - v - w - t_3$ and 4 units of flow is send through path $s_3 - t_3$. Total amount of prioritized flow for commodity-1, commodity-2 and commodity-3 are $d_1 = 7$ units, $d_2 = 5$ units and $d_3 = 6$ units, respectively.



(a) arc capacity

(b) flow/capacity

Fig. 3 (a) Multi-commodity flow network, (b) Solution with priority order.

3.2 Commodity prioritized maximum dynamic multi-commodity flow

Commodity prioritized maximum flow problem is applicable in evacuation planning problem where maximum number of evacuees should be transshipped from respective sources to corresponding destinations according to their case sensitive as quickly and efficiently as possible. In this subsection, we first describe the chain decomposable flow and introduce problem for commodity prioritized maximum dynamic multi-commodity flow. We also present polynomial time algorithm to solve the problem based on Hoppe and Tardos [14].

A chain flow $\gamma^i = (y^i, P)$, $P \in \mathbf{P}_i$ is a static flow of commodity i along path P with flow value y^i and length $\tau(\gamma^i)$. The set $\Gamma^i = \{\gamma_1^i, \dots, \gamma_l^i\}$ is said to be chain decomposition of static flow x^i if $\sum_{p=1}^{l} \gamma_p^i = x^i \quad \forall i \in K$. If all the chain flows of Γ^i use the arcs in same direction as x^i, then it is called standard chain decomposition of x^i. We denote $\Gamma = \cup_{i=1}^{k} \Gamma^i$, the set of chain decomposition of static flow $x = \sum_i x^i$ of all commodities.

The feasible dynamic flow with given time horizon T can be obtained by using standard chain decomposition Γ^i of feasible static flow x^i as follows. Send y^i units of flow through each chain flow $\gamma^i = (y^i, P) \in \Gamma$ at every time step from time zero to time $T - \tau(\gamma^i)$. The sum of these flows γ^i over each chain flow and over all commodities i provides the feasible dynamic flow, called temporally repeated flow. This temporally repeated dynamic flow can be represented as

$$|[\Gamma]^T| = \sum_{\gamma^i \in \Gamma} (T - \tau(\gamma^i) + 1)|\gamma^i| = (T + 1)|x| - \sum_{e \in E} \tau_e x_e. \tag{3.1}$$

Here, $[\Gamma]$ is the resulting dynamic flow induced by chain decomposition Γ. By taking transit time τ_e as cost c_e and creating the arcs $t_i s_i$ with infinite capacity and cost as $-(T+1)$, Equation (3.1) also represents minimum cost circulation.

Although the standard chain decomposition is useful to solve many dynamic flow problems, the temporally repeated solutions may not always be applicable for the lexicographic maximum flow or prioritized maximum flow problems. To solve this difficulty, dynamic flow with non-standard chain decomposition is essential. Non-standard chain decomposition is the chain flow which may use oppositely directed flows on arcs. Unfortunately, Equation (3.1) may not be feasible for non-standard chain decomposition. Let γ be a feasible chain flow that uses an arc $e = (v, w) \in A$ and γ' be another chain flow that uses opposite arc $e' = (w, v)$ of e which cancels the flow γ in arc e. If the portion of γ from source node to the node $v \in V$ is longer than corresponding part of γ', then the resulting dynamic flow is not feasible. This is because, flow γ' reaches to node v and starts canceling the flow earlier than the flow γ reaches. For a large time θ and $\forall\ \gamma' \in \Gamma'$ with $\tau(\gamma') \leq \theta$, the net flow send from source s_i is

$$|[\Gamma]|_{s_i} = \sum_{\gamma \in \Gamma_{s_i}} \theta|\gamma| - \sum_{\gamma' \in \Gamma'_{s_i}} (\theta - \tau(\gamma')|\gamma'|. \tag{3.2}$$

Hoppe and Tardos [13,14] used the term generalized temporally repeated flow to the resulting dynamic flow induced by non-standard chain decomposition.

Now we introduce the evacuation planning problem for prioritized maximum dynamic multi-commodity flow with commodity priority as follows.

Problem 2 Let $\mathcal{N} = (V, A, K, u, \tau, d_i, S, D, T)$ be the given network with commodity (evacuee) priority ordering $\mathbb{P}(i) \prec \mathbb{P}(i+1)\ \ \forall i \in K$. Then the prioritized maximum multi-commodity evacuation planning problem is to evacuate the maximum number of evacuees d_i from s_i to t_i in given time horizon T by using priority order and without violating capacity constraints on the arcs.

To solve this problem, we first assume that evacuees are collected at the different collection centers with priority order. As commodities (evacuees) are prioritized with urgency of treatment for the saving of life, source nodes are prioritized with respect to case sensitive. Also, being the multi-commodity transshipment, sinks are prioritized with same order as their respective sources does. Solution procedure starts by introducing super node s^* and connect $s^* s_i$ and $t_i s^*\ \ \forall i \in K$, where capacities and transit times are allocated as $u_{s^* s_i} = u_{t_i s^*} = \infty,\ \tau_{s^* s_i} = 0,\ \tau_{t_i s^*} = -(T+1)$. We denote this extended network by $\mathcal{N}^0$ and the minimum cost circulation f^0 is calculated in the static network by taking transit time as cost. For each iteration $i = 1, \ldots, k$, edge $s^* s_i$ is deleted from the network $\mathcal{N}^{i-1}$ to create $\mathcal{N}^i$ and then calculate minimum cost maximum flow Φ^i from s^* to s_i in the residual network of the flow f^{i-1} in $\mathcal{N}^i$. The minimum cost flow is now updated by $f^i = f^{i-1} + \Phi^i$.

While sending flow on bundle arcs at any time θ, first priority $\mathbb{P}(1)$ is given to the first commodity ($i = 1$) and if arc is unsaturated then the successive priority $\mathbb{P}(2)$ is given to second commodity, and then $\mathbb{P}(3)$ for third commodity and so on, respectively by using Equation (2.1). We modify the algorithm of Hoppe and Tardos [14] to calculate prioritized dynamic flow. Prioritized maximum dynamic flow algorithm is presented as follows.

Algorithm 2: Commodity prioritized maximum dynamic multi-commodity flow algorithm

```
Input  : Given multi-commodity evacuation network $\mathcal{N} = (V, A, K, u, \tau, d_i, S, D, T)$.
Output: Commodity prioritized maximum dynamic evacuation planning on $\mathcal{N}$.
 1. Prioritize the commodities such that $\mathbb{P}(i) \prec \mathbb{P}(i+1)\ \ \forall i \in K$.
 2. Construct extended network $N^0$ by introducing super node $s^*$ and connecting
    $s^* s_i$ and $t_i s^*\ \ \forall i \in K$, where $u_{s^* s_i} = u_{t_i s^*} = \infty$, $\tau_{s^* s_i} = 0$, $\tau_{t_i s^*} = -(T+1)$.
    • $f^i$ = minimum cost circulation using $\tau$ as edge cost
    • $\mathit{\Gamma}^i$ = standard chain decomposition of $f^i$
 3. Initialize $f^0 = 0$ and $\mathit{\Gamma}^0 = \emptyset$.
 4. For $i = 1, \ldots, k$,
    • delete edge $s^* s_i$ from $N^{i-1}$
    • $\mathit{\Phi}^i$ = minimum cost circulation by using $\tau$ as edge costs with priority ordering
    function (2.1) on arcs
    • $f^i = f^{i-1} + \mathit{\Phi}^i$
    • $\mathit{\Delta}^i$ = standard chain decomposition of $\mathit{\Phi}^i$
    • $\mathit{\Gamma}^i = \mathit{\Gamma}^{i-1} + \mathit{\Delta}^i$
 5. $\mathit{\Gamma}^k$ = prioritized maximum dynamic flow.
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It is to be noted that, after declaring the commodity priority on each arc and sending flow from respective sources to corresponding sinks, our problem becomes like as single source single sink problem because no flow from s_i can send to $D \setminus t_i$. Sharing of the capacity on bundle arc is obtained by priority ordering function (2.1). Also priority ordering of source with respect to the commodity obviously prioritize the sink in same order.

Theorem 3 *Dynamic flow* $[\mathit{\Gamma}^k]$ *is feasible.*

Proof First we show the satisfiability of the capacity constraints by using induction. Since $\mathit{\Gamma}^0$ is the zero flow so it obeys capacity constraint. Again, for iteration $1 \leq i \leq k$, let $[\mathit{\Gamma}^{i-1}]$ is feasible and we want to show $[\mathit{\Gamma}^i]$ is feasible. As $[\mathit{\Gamma}^i] = [\mathit{\Gamma}^{i-1}] + [\mathit{\Delta}^i]$, so for any time step θ and $e = (v, w) \in A$, if $[\mathit{\Delta}^i]_e(\theta) = 0$ then $\mathit{\Gamma}^i = \mathit{\Gamma}^{i-1}\ \forall i \in K$. Thus capacity constraint is satisfied. If $[\mathit{\Delta}^i]_e(\theta) \neq 0$, we define $p^i(v)$ as the minimum cost of path from s^* to any vertex v in the residual network of flow f^i in $\mathcal{N}^i$. By Hoppe [12], $p^i(v) \leq \theta$ and it implies $[\mathit{\Gamma}^i]_e(\theta) = f^i_e(\theta)$. Again $[\mathit{\Delta}^i]$ is a feasible dynamic flow in $\mathcal{N}^i$, it satisfies the capacity constraint.

Next, we show the satisfiability of the flow conservation constraints. Except at the source nodes, conservation constraints hold trivially because the chain decomposition $\mathit{\Gamma}$ induce the chain flows terminating at sources. As no sources have incoming arcs in $\mathcal{N}$, it assures that sources have no incoming dynamic flows.

The symbol $[\Gamma]$ is used to represent chain decomposition of dynamic flow for any large time step θ. As $[\Gamma]_e(\theta) = 0$ for any arc $e \in A$ and any $\theta \geq T + 1$, so $[\Gamma]$ has time horizon T, [14].

Theorem 4 $[\Gamma^k]$ *is lexicographic maximum dynamic flow with priority ordering of commodities.*

Proof To prove the stated theorem, we have to show that the amount of dynamic flow leaving from the sources $S_i = \{s_1, \ldots, s_i\}$ to the sinks $D_i = \{t_1, \ldots, t_i\}$ is maximum for all $i = 1, \ldots, k$. In each iteration, we construct the $S_i - D_i$ cut C_i in the time expanded network, which separates source set $S_i(0)$ from the set of sink nodes $D_i(T)$.

Defined cut $C_i = \{v(\theta) \mid p^i(v) \leq \theta, v \in V\}$. Then, $p^i(s_l) = 0$ for $s_l \in S_i$ (ie, $l \leq i$) and so $s_l(0) \in C_i$. Again, each arc $t_i s^*$ have infinite capacity and transit time $-(T+1)$ for all $i \in K$, so $p^i(t_i) = T + 1$ and so $t_i(\theta) \notin C_i$ for any time $\theta \leq T$. That is, $D_i(\theta) \notin C_i$, $\forall \theta \leq T$. This shows that every flow from S_i must cross cut C_i to reach D_i by time T. As in [13,14], the dynamic flow $[\Gamma^k]$ saturates cut C_i. That is, $[\Gamma^k]$ induces lexicographic maximum dynamic flow.

Again, as our problem is multi-commodity flow problem where different commodities may appear in the bundle arcs at the same time θ. We us priority ordering function defined in Equation (2.1) to give the commodity priority on each bundle arc and without violating the lexicographic order of the sources.

Lemma 1 *Algorithm 2 solves the commodity prioritized maximum evacuation planning problem in polynomial time complexity.*

Proof The complexity of minimum cost flow computation is $O((m \; log \; n)(m + n \; log \; n))$ and complexity of prioritization in the bundle arc is bounded by $O(m)$. Since algorithm iterates k times, so the complexity of the algorithm is bounded by $O(k((m \; log \; n)(m + n \; log \; n))$.

Example 2 Consider a network of Example 1 with capacity and transit time on each arc as shown in Figure 4. The shortest distance of node v from sources are $p^1(v) = 0$, $p^2(v) = 1$ and $p^3(v) = 0$. Since flows leaving from s_1 and s_3 at $\theta = 0$ reaches v at $\theta = 0$, so first priority on bundle arc (v, w) is given to commodity-1 and second priority is for commodity-3 with prioritized flow of value 4 and 3 units, respectively. But from $\theta = 1$ onward, the prioritized flow for commodity-1, commodity-2 and commodity-3 are 4, 5 and 2 units, respectively. Let $T = 10$ be given time horizon. By using standard chain decomposition of flow with priority order, total amount of flow for three commodities are $d_1 = (11-2)3 + (11-3)4 = 59$ units, $d_2 = (11-4)5 = 35$ units and $d_3 = 3 + (10-4)2 + (11-3)4 = 47$ units, respectively.

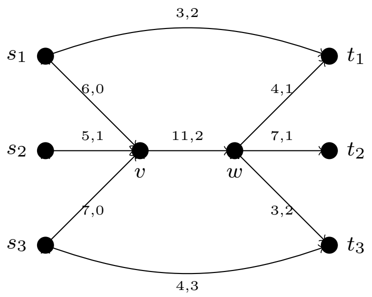

Fig. 4 Dynamic multi-commodity network with capacity and transit time on the arcs.

4 Conclusion

Evacuation planning problem is one of the challenging issues in current complicated situation. Transmission of maximum number of different type of evacuees from respective sources to corresponding sinks without violating capacity constraints in given time horizon is the maximum multi-commodity evacuation planning problem. At the time of evacuation, priority is given to those evacuees who need emergency service (treatment) for saving the lives, is known as prioritized evacuation planning problem. As multi-commodity flow over time problem is $\mathcal{NP}$-hard, we reduce it to the single commodity flow problem by using priority ordering of commodity on each bundle arc.

In this paper, we introduced prioritized maximum evacuation planning problems for static and dynamic flow in which priority is given on the basis of case sensitive. Our models are multi-commodity flow models with priority order. We have presented polynomial time algorithms to solve the problems.

References

1. Ahuja, R.K., Magnanti, T.L. and Orlin, J.B., *Network Flows: Theory, Algorithm and Applications*. Prentice Hall, Englewood Cliffs, (1993).
2. Arulselvan, A., *Network Model for Disaster Management*. PhD thesis, University of Florida USA, (2009).
3. Aronson, J.E., A survey of dynamic network flows. *Annals of Operations Research*, 20, 1-66, (2009).
4. Dhamala, T.N., Gupta, S.P., Khanal, D.P. and Pyakurel, U., Quickest multi-commodity flow over time with partial lane reversals. *Journal of Mathematics and Statistics*, 16(1), 198-211, (2020).
5. Dhamala, T.N., Pyakurel, U. and Dempe, S., A critical survey on the network optimization algorithms for evacuation planning problems. *International Journal of Operations Research*, 15(3), 101-133, (2018).
6. Fleischer, L. and Skutella, M., Quickest flows over time. *SIAM Journal on Computing*, 36(6), 1600-1630, (2007).
7. Ford, L.R. and Fulkerson D.R., A suggested computation for maximal multi-commodity network flows. *Rand corporation*, 3, 27-58, (1958).
8. Ford, L.R. and Fulkerson D.R., *Flows in Networks*. Princeton University Press, Princeton, New jersey, (1962).

9. Gupta, S.P., Khanal, D.P., Pyakurel, U. and Dhamala, T.N., Approximate algorithms for continuous-time quickest multi-commodity contraflow problem. *The Nepali Math. Sc. Report*, accepted, (2020).
10. Hall, A., Hippler, S. and Skutella, M., Multi-commodity flows over time: Efficient and complexity. *Science Direct*, 379, 387-404, (2007).
11. Hamacher, H.W. and Tufekci, S., On the use of lexicographic min cost flow in evacuation modeling. *Naval Research Logistics*, 34, 487-503, (1987).
12. Hoppe, B., *Efficient dynamic network flow algorithms.* PhD thesis, Cornell University, (1995).
13. Hoppe, B. and Tardos, E., Polynomial time algorithms for some evacuation problems. *Proceedings of the* 5^{th} *Annual ACM-SIAM Symposium on Discrete Algorithms*, 433-441, (1994).
14. Hoppe, B. and Tardos, E., The quickest transshipment problem. *Mathematics of Operations Research*, 25, 36-62, (2000).
15. Khanal, D.P., Pyakurel, U. and Dhamala, T.N., Prioritized multi-commodity flow model and algorithm. *Presented on: International Symposium on Analytic Hierarchy Process 2020 (ISAHP 2020)*, (2020).
16. Kim, S., Shekhar, S. and Min, M., Contraflow transportation network reconfiguration for evacuation route planning. *IEEE Transactions on Knowledge and Data Engineering*, 20(8), 1115-1129, (2008).
17. Kotnyek, B., *An annotated overview of dynamic network flows.* Rapport de recherche 4936, INRIA Sophia Antipolis, (2003).
18. Megiddo, N., Optimal flows in networks with multiple sources and sinks. *Mathematical Programming*, 7, 97-107, (1974).
19. Minieka, E., Maximal Lexicographic and Dynamic Network flows. *Operations Research*, 21, 517-527, (1973).
20. Pyakurel, U. and Dempe, S., Network Flow with Intermediate Storage: Models and Algorithms. *SN Operations Research Forum*, doi:10.1007/s43069-020-00033-0, (2020).
21. Pyakurel, U. and Dhamala, T.N., Models and algorithms on contraflow evacuation planning network problems. *International Journal of Operations Research*, 12, 36-46, (2015).
22. Pyakurel, U., Gupta, S.P., Khanal, D.P. and Dhamala, T.N., Continuous time dynamic contraflow models and algorithms. *International Journal of Mathematics and Mathematical Sciences - Hindawi*, https://doi.org/10.1155/2020/2676378, (2020).
23. Pyakurel, U., Nath, H.N., Dempe, S. and Dhamala, T.N., Efficient dynamic flow algorithms for evacuation planning problems with partial lane reversal. Mathematics, 7, 1-29, (2019).
24. Pyakurel, U., Nath, H.N. and Dhamala, T.N., Efficient contraflow algorithms for quickest evacuation planning. *Science China Mathematics*, 61, 2079-2100, (2018).
25. Rebennack, S., Arulselvan, A., Elefteriadou, L. and Pardalos, P.M., Complexity analysis for maximum flow problems with arc reversals. *Journal of Combinatorial Optimization*, 19, 200-216, (2010).