Stability Analysis of Discrete-Time Linear Complementarity Systems

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Abstract

A Discrete-Time Linear Complementarity System (DLCS) is a dynamical system in discrete time whose state evolution is governed by linear dynamics in states and algebraic variables that solve a Linear Complementarity Problem (LCP). The DLCS is the hybrid dynamical system that is the discrete-time counterpart of the well-known Linear Complementarity System (LCS). We derive sufficient conditions for Lyapunov stability of a DLCS when using a quadratic Lyapunov function that depends only on the state variables and a quadratic Lyapunov function that depends both on the state and the algebraic variables. The sufficient conditions require checking the feasibility of a copositive program over nonconvex cones. Our results only assume that the LCP is solvable and do not require the solutions to be unique. We devise a novel, exact cutting plane algorithm for the verification of stability and the computation of the Lyapunov functions. To the best of our knowledge, our algorithm is the first exact approach for stability verification of DLCS. A number of numerical examples are presented to illustrate the approach. Though our main object of study in this paper is the DLCS, the proposed algorithm can be readily applied to the stability verification of LCS. In this context, we show the equivalence between the stability of a LCS and the DLCS resulting from a time-stepping procedure applied to the LCS for all sufficiently small time steps.

1   Introduction

A Discrete-Time Linear Complementarity System (DLCS) is linear dynamical system in a discrete time where the state evolution is governed by *complementarity* constraints. Mathematically, a DLCS is written as

$$x_{k+1} = Ax_k + C\lambda_{k+1} \tag{1a}$$

$$0 \leq \lambda_{k+1} \perp Dx_k + F\lambda_{k+1} \geq 0, \tag{1b}$$

where $x_k \in \mathbb{R}^{n_x}$ is the state of the system at the time-step k and *complementarity variables* $\lambda_k \in \mathbb{R}^{n_c}$ are the algebraic variables satisfying complementarity constraints in (1b). The dimensions of the matrices in (1) are $A \in \mathbb{R}^{n_x \times n_x}$, $C \in \mathbb{R}^{n_x \times n_c}$, $D \in \mathbb{R}^{n_c \times n_x}$, and $F \in \mathbb{R}^{n_c \times n_c}$. The complementarity constraints (1b) distinguish the DLCS from a standard linear dynamical system. A DLCS can be naturally derived by time sampling a Linear Complementarity System (LCS) [18, 19, 8]. A LCS is a continuous-time linear dynamical systems governed by *complementarity* constraints and can be written mathematically as follows:

$$\frac{dx}{dt} = \widetilde{A}x(t) + \widetilde{C}\lambda(t) \tag{2a}$$

$$0 \leq \lambda(t) \perp \widetilde{D}x(t) + \widetilde{F}\lambda(t) \geq 0, \tag{2b}$$

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where the matrices $\widetilde{A}, \widetilde{C}, \widetilde{D}, \widetilde{F}$ have conformal dimensions. An appropriate time-stepping scheme such as *explicit* or *implicit* Euler [31] can be shown to yield the DLCS (1). For example, Pang and Stewart [31, §3] propose simulating a LCS using time-stepping formulations of the form

$$x_{k+1} = x_k + \Delta t(\widetilde{A}x_{k+\theta} + \widetilde{C}\lambda_{k+1}) \tag{3a}$$

$$0 \le \lambda_{k+1} \perp \widetilde{D}x_{k+1} + \widetilde{F}\lambda_{k+1} \ge 0, \tag{3b}$$

where Δt is the time-step, $\theta \in [0,1]$ and $x_{k+\theta} = (1-\theta)x_k + \theta x_{k+1}$. The dynamics of the time-stepping formulations can be transformed to a DLCS by rearranging (3a) as

$$x_{k+1} = (I_{n_x} + \Delta t\widetilde{A}_\theta^{-1}\widetilde{A})x_k + \Delta t\widetilde{A}_\theta^{-1}\widetilde{C}\lambda_{k+1} \tag{4a}$$

$$0 \le \lambda_{k+1} \perp \widetilde{D}x_{k+1} + \widetilde{F}\lambda_{k+1} \ge 0 \tag{4b}$$

where $\widetilde{A}_\theta = (I_{n_x} - \theta\Delta t\widetilde{A})^{-1}$. Specifically, substituting for x_{k+1} in (4b) using (4a), we obtain the DLCS (1). The use of $\theta = 0, \frac{1}{2}, 1$ results in the explicit Euler, trapezoidal, or implicit Euler time-stepping schemes, respectively. Convergence of the solutions of the time-stepping formulations (3) as $\Delta t \to 0$ to a solution of the LCS (2) are provided in [31].

In dynamical systems, the discrete-time sampled variant plays an important role in implementing control algorithms in practice. For example, discrete-time control algorithms, such as model predictive control (MPC) [26], rely on the time-stepping formulations of continuous-time linear dynamical systems. In an analogous manner, the DLCS naturally arises in applications that can be modeled with LCS. The modeling and study of dynamical systems with complementarity conditions has been steadily increasing, with application in electrical circuit simulation, robotics, nonsmooth mechanics, economics, bioengineering [1, 5, 9, 10, 29, 30, 35, 36, 39, 40, 41]. Another important recent application of interest is robotics, where complementarity is required to model the friction forces that arise in contact-based manipulation tasks [12, 25, 27, 32, 34, 33].

Stability is a fundamental issue in dynamical systems, and stability verification by computing stability certificates is a key to guaranteeing performance and safety in real-world systems. Lyapunov stability is a widely used concept for analyzing the stability of dynamical systems [23], and its extension to hybrid and switched systems has been considered by several authors [4, 15, 20, 21, 22, 24, 38]. The papers extending stability to hybrid systems assume that a Lyapunov-like function exists for each mode's vector field and for the entire state space. In many hybrid and switched systems, however, each mode is active only over a subset of the state space, especially for those systems whose switchings are triggered by state evolution, such as in the LCS and DLCS. Hence, the above results are rather restrictive, even for linear switched systems. To rectify this, Çamlibel and Schumacher [11] proposed copositive Lyapunov functions for conewise linear systems in which the feasible region of each mode is a polyhedral cone. Bundfuss and Dür [6] present necessary and sufficient conditions for the existence of such Lyapunov functions when the cone is polyhedral. Çamlibel, Pang and Shen [8] derived sufficient conditions for the stability of LCS (2). Extending standard approaches that employ piecewise-quadratic Lyapunov functions depending only on state $x(t)$, the authors in [8] proposed an extended quadratic Lyapunov function that depends on both $x(t)$ and $\lambda(t)$. However, the authors did not provide an algorithm for computing such a Lyapunov function. Recently, Aydinoglu, Preciado, and Posa [2] extended the sufficient conditions of [8] under the assumption of an existence of a feedback controller that depends on both $x(t)$ and $\lambda(t)$. The authors also proposed an algorithm based on bilinear matrix inequalities for jointly determining the feedback controller and the Lyapunov function. However, we are not aware of similar results for the DLCS.

In this paper, we focus on the stability analysis of the DLCS and the development of algorithms for computing a Lyapunov function certifying stability. The paper makes three main contributions.

First, in §3 we derive sufficient conditions for the stability of a DLCS using two quadratic Lyapunov functions. The first Lyapunov function is a quadratic function that depends only on the states of the DLCS (1). Such a Lyapunov function is referred to as a *Common Quadratic Lyapunov Function* (CQLF) [24, 38]. The second Lyapunov function is a quadratic function that depends on both the states and complementarity variables of the DLCS (1). This is the discrete-time analogue

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of the *Extended Quadratic Lyapunov Function* (EQLF) introduced in [8]. The stability conditions are provided under the assumption that F is a Q-matrix and a R_0-matrix [14]. The Q-matrix is the largest class of matrices for which the complementarity constraints in (1b) have a solution for all x_k. Our assumptions on the matrix F are weaker than those used in [2, 8].

Second, in §4 we present exact algorithms for computing the CQLF and EQLF for a DLCS. The computation of Lyapunov functions requires finding a feasible solution for a copositive program that is defined over a nonconvex cone—in particular, the union of convex cones. It is well-known that checking whether a given matrix is copositive is a co-NP-complete problem [28]. We pose the problem of computing the Lyapunov functions as semi-infinite programming problem and devise a cutting-plane algorithm. The cutting plane algorithm alternates a linear programming master problem and a separation problem formulated as a Mixed Integer Quadratically Constrained Program (MIQCP). Numerical experiments demonstrating the utility of the algorithm are presented in §5.

Third, in §6 we consider the case where the DLCS is obtained by a time-stepping scheme applied to a LCS. An important question as yet unanswered in the literature is whether stability of the LCS is equivalent to stability of the DLCS when the time-step Δt is sufficiently small. We show equivalence for both the case of CQLF and EQLF under the assumption that the matrix $\widetilde{F}$ in the LCS (2) is a P-matrix.

Notation. The set of reals, nonnegative reals, integers and nonnegative integers is denoted by $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$ and $\mathbb{Z}_+$ respectively. The 1-norm ball is denoted as $\mathbb{B}_1^n = \{w \in \mathbb{R}^n \,|\, \|w\|_1 = 1\}$. For a vector $v \in \mathbb{R}^n$ $[v]_i$ denotes the i-th component of the vector. For $v \in \mathbb{R}^n$ the notation $[v]_\alpha$ for $\alpha \subset \{1, \ldots, n\}$ denotes the subvector obtained by removing the elements $[v]_i$ for $i \notin \alpha$. The notation (x, y) of two column vector x, y is also used to represent a vertical stacking of the vectors $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$. For a matrix $M \in \mathbb{R}^{n \times m}$ $[M]_{ij}$ denotes the (i, j)-th entry in the matrix. For a matrix M and index sets $\alpha \subseteq \{1, \ldots, n\}$, $\beta \subseteq \{1, \ldots, m\}$ the notation $[M]_{\alpha\beta}$ denotes the submatrix of M formed by removing rows not in α and removing columns not in β. The notation $[M]_{\alpha\bullet}$ and $[M]_{\bullet\alpha}$ refers to the submatrix of M obtained by retaining only the rows and columns in α respectively. The notation I_n represents the $n \times n$ identity matrix. The set $\mathbb{S}^n$ denotes the $n \times n$ real, symmetric matrices and $\mathbb{S}_+^n, \mathbb{S}_{++}^n$ denote the set of symmetric positive semidefinite and positive definite matrices. A symmetric positive (semi)definite matrix $N \in \mathbb{S}^n$ is denoted as $N \succ (\succeq)0$.

2 Background

In this section, we present relevant results and notation that are used throughout the paper.

2.1 Linear Complementarity Problem (LCP)

The Linear Complementarity Problem (LCP), denoted as LCP(q,M), is to find a solution $\lambda \in \mathbb{R}^{n_c}$ of
$$0 \le \lambda \perp M\lambda + q \ge 0, \tag{5}$$
where $q \in \mathbb{R}^{n_c}, M \in \mathbb{R}^{n_c \times n_c}$. The solution set of the LCP (5) is denoted as SOL(q,M) $= \{\lambda \,|\, \lambda \text{ satisfies } (5)\}$. If $q = Nx$ for some matrix N, the graph of SOL(Nx, M) is denoted as Gr SOL(Nx, M) $= \{(x, \lambda) \,|\, \lambda \in$ SOL(Nx,M)$\}$. The monograph [14] provide an in-depth treatment describing conditions for the existence and uniqueness of solutions to (LCP). We provide a brief summary of definitions and results that will be useful for subsequent developments.

Given a solution $\lambda \in$ SOL(q,M), we define index sets $\alpha(\lambda)$, $\beta(\lambda)$, $\gamma(\lambda)$ partitioning $\{1, \ldots, n_c\}$ as

$$\alpha(\lambda) := \{i \,|\, [\lambda]_i > 0 = [M\lambda + q]_i\} \tag{6a}$$
$$\beta(\lambda) := \{i \,|\, [\lambda]_i = 0 = [M\lambda + q]_i\} \tag{6b}$$
$$\gamma(\lambda) := \{i \,|\, [\lambda]_i = 0 < [M\lambda + q]_i\}. \tag{6c}$$

If $\beta(\lambda) = \emptyset$, then λ is called a *strict complementarity solution* of LCP(q,M). Otherwise, it is called a *nonstrict complementarity solution*. We will suppress λ when the dependence is clear from the context.

A matrix M is a Q-matrix if LCP(q,M) has a solution for all $q \in \mathbb{R}^{n_c}$. A matrix M is an R_0-matrix if SOL(0,M) = $\{0\}$. A matrix M is said to be a P-matrix if the principal minors are all positive, i.e. $\det([M]_{JJ}) > 0$ for all $J \subseteq \{1, \ldots, n_c\}$. It is known that a P-matrix is both an R_0-matrix and a Q-matrix [14, Theorem 3.9.22]. The following lemma stating properties of solutions to LCP(q,M) will be useful in the remainder of the paper.

Lemma 1 ([14]). *Consider the LCP(q,M) where the matrix M is a P-matrix. Then*

(a) *LCP(q,M) has an unique solution $\lambda(q)$ for every q.*

(b) *$\lambda(q)$ is a piecewise linear function of q, is globally Lipschitz continuous, and is directionally differentiable.*

(c) *There exists a constant $c_M > 0$ such that $\|\lambda(q)\| \leq c_M \|q\|$ for all $q \in \mathbb{R}^{n_c}$.*

2.2 Lyapunov Stability

Consider the discrete-time nonlinear dynamical system

$$x_{k+1} = f(x_k); \quad x_0 = \hat{x}, \tag{7}$$

where $f : \mathbb{R}^{n_x} \to 2^{\mathbb{R}^{n_x}}$ is a mapping into subsets of $\mathbb{R}^{n_x}$. Let x^e be an equilibrium point of the system (7); i.e. $f(x^e) = 0$, and let $x_k(\hat{x})$, $k \in \mathbb{Z}_+$, be a trajectory of the system initialized at $\hat{x}$. We do not assume the trajectory is unique.

Definition 1. *The equilibrium point x^e is*

(i) *stable (in the sense of Lyapunov) if for each $\epsilon > 0$, there is $\delta_\epsilon > 0$ such that*

$$\|\hat{x} - x^e\| < \delta_\epsilon \implies \|x_k(\hat{x}) - x^e\| < \epsilon \quad \forall k \in \mathbb{Z}_+$$

for all trajectories $x_k(\hat{x})$;

(ii) *asymptotically stable if it is stable and there exists $\delta > 0$ such that*

$$\|\hat{x} - x^e\| < \delta \implies \lim_{k \to \infty} x_k(\hat{x}) = x^e \quad \forall k \in \mathbb{Z}_+$$

for all trajectories $x_k(\hat{x})$;

(iii) *exponentially stable if there exist scalars $\delta > 0$, $c > 0$ and $\mu > 0$ such that*

$$\|\hat{x} - x^e\| < \delta \implies \|x_k(\hat{x}) - x^e\| \leq c\mu^k \|\hat{x} - x^e\| \quad \forall k \in \mathbb{Z}_+$$

for all trajectories $x_k(\hat{x})$.

If the function $f(x)$ is positively homogeneous in x, i.e. $f(\tau x) = \tau f(x)$ then the stability results hold *globally* for all initial conditions $\hat{x} \in \mathbb{R}^{n_x}$. Note that the functions defining the dynamics in both LCS and DLCS are positively homogeneous. We discuss how stability analysis for the homogeneous case can be extended to some inhomogeneous systems in § 3.3.

2.3 Semidefiniteness and Matrix Copositivity

Semidefiniteness with respect to a closed, not necessarily convex, cone $\mathcal{K}$ is called copositivity with respect to $\mathcal{K}$. Specifically, a matrix $M \in \mathbb{R}^{n \times n}$ is said to be *copositive* with respect to a cone $\mathcal{K} \subseteq \mathbb{R}^n$ if $u^T M u \geq 0$ for all $u \in \mathcal{K}$. In this case, we say M is a $\mathcal{K}$-copositive matrix, and denote this fact by $M \succeq_{\mathcal{K}} 0$. Similarly, if $u^T M u > 0$ for all $u \in \mathcal{K}$ then M is a strict $\mathcal{K}$-copositive matrix, which we denote by $M \succ_{\mathcal{K}} 0$. Note that the standard semidefinite and copositivity condtions are obtained by choosing $\mathcal{K} = \mathbb{R}^n$ and $\mathcal{K} = \mathbb{R}^n_+$ respectively. For additional details, the reader is referred to [7, 16].

3 Stability of DLCS

In this section we present stability conditions for the DLCS at $x^e = 0$. We assume that $(x^e, \lambda^e) = (0, 0)$ is the unique equilibrium point of DLCS, i.e. (x^e, λ^e) is the only solution of

$$0 = Ax + C\lambda \tag{8a}$$

$$0 \le \lambda \perp Dx + F\lambda \ge 0. \tag{8b}$$

The uniqueness of the equilibrium is guaranteed if F is a R_0-matrix i.e. $\mathrm{LCP}(0,F) = \{0\}$. We first show sufficient conditions for a CQLF in §3.1 and extend these to sufficient conditions for an EQLF in §3.2. A discussion on the applicability of stability results to inhomogeneous DLCS is presented in §3.3 under the assumption that F is a P-matrix.

3.1 Common Quadratic Lyapunov Function (CQLF)

We will first present sufficient conditions for stability of the equilibrium point using a quadratic Lyapunov function that depends only on the states, or a Common Quadratic Lyapunov function (CQLF) [24, 38].

Theorem 1. *Consider the DLCS in (1). Assume that F is a Q-matrix and R_0-matrix. Let $\psi :$ $\mathbb{R}^{n_x + n_c} \to \mathbb{R}$ denote a quadratic function*

$$\psi(x, \lambda) = \begin{pmatrix} x \\ \lambda \end{pmatrix}^T M(P_{xx}) \begin{pmatrix} x \\ \lambda \end{pmatrix}, \quad \text{where} \tag{9}$$

$$M(P_{xx}) = \begin{pmatrix} A^T P_{xx} A - P_{xx} & A^T P_{xx} C \\ C^T P_{xx} A & C^T P_{xx} C \end{pmatrix}, \tag{10}$$

and $P_{xx} \in \mathbb{S}^{n_x}$. If there exists $P_{xx} \succ 0$, then $x^e = 0$ is

(i) stable if $\psi(x, \lambda) \le 0$ for all $(x, \lambda) \in Gr\,SOL(Dx, F)$,

(ii) asymptotically stable if $\psi(x, \lambda) \le 0$ for all $(x, \lambda) \in Gr\,SOL(Dx, F)$, $\psi(x, \lambda) = 0 \implies x = 0$, and

(iii) exponentially stable if $\psi(x, \lambda) < 0$ for all $(x, \lambda) \in Gr\,SOL(Dx, F)$.

Proof. Let x_0 be an arbitrary initial condition for the DLCS and let (x_k, λ_k) for $k \ge 1$ be a trajectory from such an initial state with $\lambda_{k+1} \in \mathrm{Gr\,SOL}(Dx_k, F)$. Consider the function $V(x_k) = x_k^T P_{xx} x_k$. By the positive definiteness of P_{xx}, $V(x) \ge 0$ for all $x \in \mathbb{R}^{n_x}$ and $V(x) = 0$ if and only if $x = x^e$. Thus, $V(x)$ is a Lyapunov function. The Lyapunov function at x_{k+1} is

$$V(x_{k+1}) = \begin{pmatrix} x_k \\ \lambda_{k+1} \end{pmatrix}^T \begin{pmatrix} A^T P_{xx} A & A^T P_{xx} C \\ C^T P_{xx} A & C^T P_{xx} C \end{pmatrix} \begin{pmatrix} x_k \\ \lambda_{k+1} \end{pmatrix}.$$

It can be verified that $\psi(x_k, \lambda_{k+1}) = V(x_{k+1}) - V(x_k)$. If $\psi(x, \lambda) \le 0$ for all $(x, \lambda) \in \mathrm{Gr\,SOL}(Dx, F)$ then $V(x_{k+1}) \le V(x_k)$ for all $x_k \in \mathbb{R}^{n_x}$. In other words, along all trajectories of DLCS from x_0

$$V(x_{k+1}) \le V(x_k) \implies V(x_k) \le V(x_0). \tag{11}$$

Since $P_{xx} \succ 0$ there exists $\omega > 0$ such that $V(x_k) \ge \mu \|x_k\|^2$. Further, $V(x) \le \|P_{xx}\| \|x\|^2$. Combining the two observations in (11) yields $\|x_k\| \le (\|P_{xx}\|/\mu) \|x_0\|^2$ for all k and all trajectories from x_0. The claim on stability in (i) follows. If the condition in (ii) holds then the decrease $\psi(x, \lambda) = 0$ only if $x = 0$, so the the sequence $\{V(x_k)\}$ is strictly decreasing unless $x = 0$. Since $V(x)$ is lower bounded by zero and $V(x) \ge \mu \|x\|^2$ we have that the claim in (ii) holds. If $\psi(x, \lambda) < 0$ then there exists a $\gamma > 0$ such that $\psi(x, \lambda) \le -\gamma \|(x, \lambda)\|^2$ for all $(x, \lambda) \in \mathrm{Gr\,SOL}(Dx, F)$. Then along all trajectories of DLCS

$$V(x_{k+1}) - V(x_k) \le -\gamma \|(x_k, \lambda_{k+1})\|^2 \le -\gamma \|x_k\|^2 \le -(\gamma/\|P_{xx}\|) V(x_k)$$

where the second inequality follows from $\|x_k\|^2 \leq \|(x_k, \lambda_{k+1})\|^2$ and the third from $V(x_k) \leq \|P_{xx}\|\|x_k\|^2$. Hence, $V(x_k) \leq (1 - \gamma/\|P_{xx}\|)^k V(x_0)$ along all trajectories of DLCS from x_0. Combining this with $V(x) \geq \omega\|x\|^2$ yields $\|x_k\|^2 \leq (\|P_{xx}\|/\omega)(1 - \gamma/\|P_{xx}\|)^k\|x_0\|^2$ and the claim in (iii) holds. $\qquad\square$

Theorem 1 shows that the sufficient conditions for exponential stability using a CQLF are to

$$\text{Find} \ \ P_{xx} \in \mathbb{S}^{n_x} \tag{12a}$$

$$\text{such that} \ \ P_{xx} \succ 0, \tag{12b}$$

$$- M(P_{xx}) \succ_{\mathcal{K}} 0, \tag{12c}$$

where $M(P_{xx})$ is defined in (10), and the set $\mathcal{K} \subseteq \mathbb{R}^{n_x+n_c}$ is defined as

$$\mathcal{K} = \text{Gr} \, \text{SOL}(Dx, F) \tag{13}$$

3.2 Extended Quadratic Lyapunov Function (EQLF)

The use of a quadratic function in the states alone can limit the set of systems that can be stabilized. Specifically, there exist linear switching systems that can be stabilized, but for which the CQLF does not exist [24, §2.1.5]. This has motivated the use of unique quadratic Lyapunov functions for each mode of the switching system with continuity imposed across the switching boundaries [4, 15, 20, 22, 24, 38]. As noted in the introduction, this construction can also be restrictive since the Lyapunov decrease condition for a mode is enforced over the entire state-space rather than over the subset of the state-space corresponding to the mode. In a significant departure from prior literature, Çamlibel, Pang and Shen [8] proposed a quadratic Lyapunov function that depends on $(x(t), \lambda(t))$ for the LCS (2). This Lyapunov function is piecewise quadratic since $\lambda(t)$ is a piecewise linear function (see Lemma 1). We follow a similar path for the DLCS and derive sufficient conditions for stability of $(x^e, \lambda^e) = 0$.

Theorem 2. *Consider the DLCS in (1). Assume that F is a Q-matrix and an R_0-matrix. Let $\widehat{\psi} : \mathbb{R}^{n_x+2n_c} \to \mathbb{R}$ denote a quadratic function*

$$\widehat{\psi}(x, \lambda, \widehat{\lambda}) = \begin{pmatrix} x \\ \lambda \\ \widehat{\lambda} \end{pmatrix}^T \widehat{M}(P) \begin{pmatrix} x \\ \lambda \\ \widehat{\lambda} \end{pmatrix}, \tag{14}$$

$$\text{with } \widehat{M}(P) = \begin{pmatrix} A^T P_{xx} A - P_{xx} & A^T P_{xx} C - P_{x\lambda} & A^T P_{x\lambda} \\ C^T P_{xx} A - P_{x\lambda}^T & C^T P_{xx} C - P_{\lambda\lambda} & C^T P_{x\lambda} \\ P_{x\lambda}^T A & P_{x\lambda}^T C & P_{\lambda\lambda} \end{pmatrix}, \tag{15}$$

where $P = \begin{pmatrix} P_{xx} & P_{x\lambda} \\ P_{x\lambda}^T & P_{\lambda\lambda} \end{pmatrix}$, $P_{xx} \in \mathbb{S}^{n_x}$, $P_{\lambda\lambda} \in \mathbb{S}^{n_c}$ and $P_{x\lambda} \in \mathbb{R}^{n_x \times n_c}$. If there exists $P_{xx}, P_{x\lambda}, P_{\lambda\lambda}$ such that the function $\widehat{V}(x, \lambda)$

$$\widehat{V}(x, \lambda) = \begin{pmatrix} x \\ \lambda \end{pmatrix}^T P \begin{pmatrix} x \\ \lambda \end{pmatrix} \tag{16}$$

satisfies $\widehat{V}(x, \lambda) \geq 0$ for all $(x, \lambda) \in \text{Gr} \, \text{SOL}(Dx, F)$ and $\widehat{V}(x, \lambda) = 0$ if and only if $(x, \lambda) = (x^e, \lambda^e)$, then $(x^e, \lambda^e) = 0$ is

(i) *stable if $\widehat{\psi}(x, \lambda, \widehat{\lambda}) \leq 0$ for all $(x, \lambda) \in \text{Gr} \, \text{SOL}(Dx, F)$ and $\widehat{\lambda} \in \text{SOL}(D(Ax + C\lambda), F)$,*

(ii) *asymptotically stable if $\widehat{\psi}(x, \lambda, \widehat{\lambda}) \leq 0$ for all $(x, \lambda) \in \text{Gr} \, \text{SOL}(Dx, F)$, $\widehat{\lambda} \in \text{SOL}(D(Ax + C\lambda), F)$, $\widehat{\psi}(x, \lambda, \widehat{\lambda}) = 0 \implies x = 0$*

(iii) *exponentially stable if $\widehat{\psi}(x, \lambda, \widehat{\lambda}) < 0$ for all $(x, \lambda) \in \text{Gr} \, \text{SOL}(Dx, F)$ and $\widehat{\lambda} \in \text{SOL}(D(Ax + C\lambda), F)$.*

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Proof. Let x_0 be an arbitrary initial condition for the DLCS and let (x_k, λ_k) for $k \geq 1$ be a trajectory from such an initial state with $\lambda_{k+1} \in \mathrm{Gr\,SOL}(Dx_k, F)$. Then

$$
\begin{aligned}
\widehat{V}(x_{k+1}, \lambda_{k+2}) &= \begin{pmatrix} x_{k+1} \\ \lambda_{k+2} \end{pmatrix}^T \begin{pmatrix} P_{xx} & P_{x\lambda} \\ P_{x\lambda}^T & P_{\lambda\lambda} \end{pmatrix} \begin{pmatrix} x_{k+1} \\ \lambda_{k+2} \end{pmatrix} \\
&= \begin{pmatrix} Ax_k + C\lambda_{k+1} \\ \lambda_{k+2} \end{pmatrix}^T \begin{pmatrix} P_{xx} & P_{x\lambda} \\ P_{x\lambda}^T & P_{\lambda\lambda} \end{pmatrix} \begin{pmatrix} Ax_k + C\lambda_{k+1} \\ \lambda_{k+2} \end{pmatrix} \\
&= \begin{pmatrix} x_k \\ \lambda_{k+1} \\ \lambda_{k+2} \end{pmatrix}^T \begin{pmatrix} A^T P_{xx} A & A^T P_{xx} C & A^T P_{x\lambda} \\ C^T P_{xx} A & C^T P_{xx} C & C^T P_{x\lambda} \\ P_{x\lambda}^T A & P_{x\lambda}^T C & P_{\lambda\lambda} \end{pmatrix} \begin{pmatrix} x_k \\ \lambda_{k+1} \\ \lambda_{k+2} \end{pmatrix}.
\end{aligned}
$$

It can be verified that $\widehat{V}(x_{k+1}, \lambda_{k+2}) - \widehat{V}(x_k, \lambda_{k+1}) = \widehat{\psi}(x_k, \lambda_{k+1}, \lambda_{k+2})$. The assumptions on $\widehat{V}$ ensure that $\widehat{V}(x_k, \lambda_{k+1}) \geq 0$ and $\widehat{V}(x_{k+1}, \lambda_{k+2}) \geq 0$. If the condition in (i) holds then

$$
\widehat{V}(x_{k+1}, \lambda_{k+2}) \leq \widehat{V}(x_k, \lambda_{k+1}) \implies \widehat{V}(x_k, \lambda_{k+1}) \leq \widehat{V}(x_0, \lambda_1) \tag{17}
$$

for all k. Further, since $\widehat{V}(x_k, \lambda_{k+1}) > 0$ for all $(x_k, \lambda_{k+1}) \in \mathrm{GrSOL}(Dx_k, F)$ we have that $\widehat{V}(x_k, \lambda_{k+1}) \geq \omega \|(x_k, \lambda_{k+1})\|^2$ for some $\omega > 0$. Further, $V(x_0, \lambda_1) \leq \|P\| \|(x_0, \lambda_1)\|^2$. Applying these two in (17) yields $\|(x_k, \lambda_{k+1})\|^2 \leq (\|P\|/\omega) \|(x_0, \lambda_1)\|^2$ and the claim on stability in (i) follows. If condition in (ii) holds then $\{V(x_k, \lambda_{k+1})$ is strictly decreasing along the trajectories unless $x_k = 0$ and the claim in (ii) follows. If condition in (iii) holds then there exists $\gamma > 0$ such that $\widehat{\psi}(x_k, \lambda_{k+1}, \lambda_{k+2}) \leq -\gamma \|(x_k, \lambda_{k+1}, \lambda_{k+2})\|^2$. Then

$$
\begin{aligned}
\widehat{V}(x_{k+1}, \lambda_{k+2}) - \widehat{V}(x_k, \lambda_{k+1}) &\leq -\gamma \|(x_k, \lambda_{k+1}, \lambda_{k+2})\|^2 \leq -\gamma \|(x_k, \lambda_{k+1})\|^2 \\
&\leq -(\gamma/\|P\|) \widehat{V}(x_k, \lambda_{k+1})
\end{aligned}
$$

where the second inequality follows from $\|(x_k, \lambda_{k+1})\|^2 \leq \|(x_k, \lambda_{k+1}, \lambda_{k+2})\|^2$ and the third inequality from $\widehat{V}(x_k, \lambda_{k+1}) \leq \|P\| \|(x_k, \lambda_{k+1})\|^2$. Thus, $\widehat{V}(x_k, \lambda_{k+1}) \leq (1 - \gamma/\|P\|)^k \widehat{V}(x_0, \lambda_1)$ along all trajectories of DLCS from the initial point x_0. Combining with $\widehat{V}(x_k, \lambda_{k+1}) \geq \omega \|(x_k, \lambda_{k+1}\|^2$ yields

$$
\|(x_k, \lambda_{k+1})\|^2 \leq (\|P\|/\omega)(1 - \gamma/\|P\|)^k \|(x_0, \lambda_1)\|^2
$$

and the claim in (iii) follows. $\qquad\square$

Note that setting $P_{x\lambda} = 0$ and $P_{\lambda\lambda} = 0$ in (15), we obtain the sufficient conditions for stability with a CQLF. Theorem 2 shows that sufficient conditions for exponential stability using an EQLF are to

$$
\text{Find } P \in \mathbb{S}^{n_x + n_c} \tag{18a}
$$
$$
\text{such that } P \succ_{\mathcal{K}} 0, \tag{18b}
$$
$$
-\widehat{M}(P) \succ_{\widehat{\mathcal{K}}} 0, \tag{18c}
$$

where $\widehat{M}(P)$ is defined in (15), and the set $\widehat{\mathcal{K}} \subseteq \mathbb{R}^{n_x + 2n_c}$ is defined as

$$
\widehat{\mathcal{K}} = \{(x, \lambda, \widehat{\lambda}) \,|\, (x, \lambda) \in \mathrm{Gr\,SOL}(Dx, F), \, \widehat{\lambda} \in \mathrm{SOL}(D(Ax + C\lambda), F)\}. \tag{19}
$$

3.3   Extension to Inhomogeneous DLCS

In this section, we consider a generalization of the DLCS, the autonomous inhomogeneous DLCS which is given as

$$
x_{k+1} = Ax_k + C\lambda_{k+1} + g \tag{20a}
$$
$$
0 \leq \lambda_{k+1} \perp Dx_k + F\lambda_{k+1} + h \geq 0, \tag{20b}
$$

where $g \in \mathbb{R}^{n_x}$ and $h \in \mathbb{R}^{n_c}$.

In this section, we show how the stability results in §3.1 and §3.2 are applicable to an inhomogeneous DLCS (20). For the results in this section, we must make the stronger assumption that F is a P-matrix.

A point (x^e, λ^e) is said to be an equilibrium of (20) if

$$x^e = Ax^e + C\lambda^e + g \tag{21a}$$

$$0 \le \lambda^e \perp Dx^e + F\lambda^e + h \ge 0. \tag{21b}$$

Let $(\alpha^e, \beta^e, \gamma^e)$ denote the index partition according to (6) at the equilibrium x^e. Suppose that F is a P-matrix, and let $\lambda(x)$ be the unique solution of the LCP$(Dx + h, F)$. From the global Lipschitz continuity of $\lambda(x)$ (Lemma 1(b)), we have that there exists a neighborhood $\mathcal{N}(x^e)$ around x^e such that for all $x \in \mathcal{N}(x^e)$, $[\lambda(x)]_i > 0$ for all $i \in \alpha^e$ and $[Dx + F\lambda(x)]_i > 0$ for all $i \in \gamma^e$. Hence, $\alpha(\lambda(x)) \supseteq \alpha^e$ and $\gamma(\lambda(x)) \supset \gamma^e$ holds for all $x \in \mathcal{N}(x^e)$.

Thus, for all $x \in \mathcal{N}(x^e)$ the LCP(Dx, F) can be restated as

$$0 = [Dx + F\lambda(x) + h]_{\alpha^e} \tag{22a}$$

$$0 \le [\lambda(x)]_{\beta^e} \perp [Dx + F\lambda(x) + h]_{\beta^e} \ge 0 \tag{22b}$$

$$[\lambda(x)]_{\gamma^e} = 0. \tag{22c}$$

Introducing $\delta x_{k+1} = x_{k+1} - x^e$, $\delta\lambda_{k+1} = \lambda_{k+1} - \lambda^e$, and using the observation in (22), the DLCS in (20) can be written for all $x \in \mathcal{N}(x^e)$ as

$$\delta x_{k+1} = A\delta x_k + C\delta\lambda_{k+1} \tag{23a}$$

$$0 = [D\delta x_k + F\delta\lambda_{k+1}]_{\alpha^e} \tag{23b}$$

$$0 \le [\delta\lambda_{k+1}]_{\beta^e} \perp [D\delta x_k + F\delta\lambda_{k+1}]_{\beta^e} \ge 0 \tag{23c}$$

$$0 = [\delta\lambda_{k+1}]_{\gamma^e}. \tag{23d}$$

The system (23) can be further simplified to the homogeneous DLCS

$$\delta x_{k+1} = \widehat{A}\delta x_k + \widehat{C}[\delta\lambda_{k+1}]_{\beta^e} \tag{24a}$$

$$0 \le [\delta\lambda_{k+1}]_{\beta^e} \perp \widehat{D}\delta x_k + \widehat{F}[\delta\lambda_{k+1}]_{\beta^e} \ge 0, \tag{24b}$$

where $\widehat{A} = (A - [C]_{\bullet\alpha^e}([F]_{\alpha^e\alpha^e})^{-1}[D]_{\alpha^e\bullet}$, $\widehat{C} = [C]_{\bullet\beta^e} - [C]_{\bullet\alpha^e}([F]_{\alpha^e\alpha^e})^{-1}[F]_{\alpha^e\beta^e}$, $\widehat{D} = [D]_{\beta^e\bullet} - [F]_{\beta^e\alpha^e}([F]_{\alpha^e\alpha^e})^{-1}[D]_{\alpha^e\bullet}$, and $\widehat{F} = [F]_{\beta^e\beta^e} - [F]_{\beta^e\alpha^e}([F]_{\alpha^e\alpha^e})^{-1}[F]_{\alpha^e\beta^e}$. Thus, the stability analysis of the previous sections can be applied to the homogenous DLCS in (24) to verify local stability of x^e.

4 A Cutting-Plane Algorithm for Computing the Lyapunov Function

The sufficient conditions (12) and (18) for exponential stability using a CQLF or an EQLF can be written in a general fashion as

$$\text{Find } \mathbf{P} \in \mathbb{S}^n \tag{25a}$$

$$\text{such that } \mathbf{P} \succ_{\mathcal{K}_1} 0, \tag{25b}$$

$$-\mathbf{M}(\mathbf{P}) \succ_{\mathcal{K}_2} 0 \tag{25c}$$

where $\mathcal{K}_1$ and $\mathcal{K}_2$ are closed (not necessarily convex) cones and the matrix $\mathbf{M} \in \mathbb{S}^m$ depends on the matrix $\mathbf{P}$. The feasibility conditions require that $\mathbf{P}$ is strict $\mathcal{K}_1$-copositive and $-\mathbf{M}$ is strict $\mathcal{K}_2$-copositive.

A typical approach to handling the strict $\mathcal{K}_1$-copositivity and strict $\mathcal{K}_2$-copositivity is to relax the conditions into linear matrix inequalities using the S-Lemma [2, 3]. The transformation to such a semidefinite formulation for the CQLF and EQLF is provided in Appendix A. Our intent is to provide an exact algorithm which we describe next.

4.1 Cutting Plane Algorithm

We state by recalling a result from [7] that bounds the set of test vectors for copositivity with respect to a cone.

Lemma 2. *[7, Lemma 1] Let $\| \cdot \|_1$ be the 1-norm on $\mathbb{R}^n$ and $\mathcal{C} \subset \mathbb{R}^n$ be a closed cone. Then*

$$N \succeq_{\mathcal{C}} 0 \iff z^T N z \geq 0 \text{ for all } z \in \mathcal{C}, \|z\|_1 = 1.$$

An analogous results holds for $N \succ_{\mathcal{C}} 0$.

Thus, the feasibility problem (25) can be equivalently cast as the semi-infinite optimization problem

$$\max_{\mu, \mathbf{P} \in \mathbb{S}^n} \mu \tag{26a}$$

$$\text{s.t. } u^T \mathbf{P} u \geq \mu \, \forall \, u \in \mathcal{U} \tag{26b}$$

$$- v^T \mathbf{M}(\mathbf{P}) v \geq \mu \, \forall \, v \in \mathcal{V} \tag{26c}$$

where $\mathcal{U} = \mathcal{K}_1 \cap \mathbb{B}_1^n$ and $\mathcal{V} = \mathcal{K}_2 \cap \mathbb{B}_1^m$. If the optimal objective value of (26) is positive then the origin is exponentially stable for (1). The presence of infinite number of constraints poses a difficulty for computation. We propose to solve (26) using a cutting plane algorithm which alternates between a *master problem* and two *separation problems*. At an iteration l of the cutting plane algorithm the master problem attempts to find a matrix $\mathbf{P}^l$ for which the inequalities (26b)-(26c) hold over finite sets $\mathcal{U}^l, \mathcal{V}^l$. The separation problem verifies if the computed $\mathbf{P}^l$ satisfies the inequalities in (26b)-(26c) for all vectors in $\mathcal{U}$ and $\mathcal{V}$. If satisfied, then $\mathbf{P}^l$ is a matrix for which (25) holds. If not, then the separation problem identifies points u^l, v^l for which the inequalities (25b)-(25c) fail to hold. These points are appended to the sets $\mathcal{U}^l, \mathcal{V}^l$, and the algorithm continues.

The *master problem* associated with (26) is

$$\max_{\mu, \mathbf{P} \in \mathbb{S}^n} \mu \tag{27a}$$

$$\text{s.t. } u^T \mathbf{P} u \geq \mu \, \forall \, u \in \mathcal{U}^l \tag{27b}$$

$$- v^T \mathbf{M}(\mathbf{P}) v \geq \mu \, \forall \, v \in \mathcal{V}^l \tag{27c}$$

$$- 1 \leq [\mathbf{P}]_{ij} \leq 1 \, \forall \, i, j = 1, \ldots, n. \tag{27d}$$

The constraint (27d) is a normalization constraint and does not affect the feasible region due to the homogeneity of (25). Specifically, if $\hat{\mathbf{P}}$ satisfies (25) then $\hat{\mathbf{P}}/(\max_{i,j} \left[\hat{\mathbf{P}}\right]_{ij})$ also satisfies (25). From a computational standpoint (27d) serves to bound the feasible region and ensures that an optimal solution to the linear program (27) always exists as long as $\mathcal{U}^l \neq \emptyset$. Note that from (27) we have that

$$\mu = \max_{\mathbf{P}: |[\mathbf{P}]_{ij}| \leq 1} \min \left\{ \min_{u \in \mathcal{U}^l} u^T \mathbf{P} u, \min_{v \in \mathcal{V}^l} -v^T \mathbf{M}(\mathbf{P}) v \right\}. \tag{28}$$

Thus, if $\mathcal{U}^l \neq \emptyset$, an optimal solution to (27) must have optimal value at most n, since

$$\mu \leq \max_{\mathbf{P}: |[\mathbf{P}]_{ij}| \leq 1} \max_{u \in \mathcal{U}^l} u^T \mathbf{P} u \leq \max_{\mathbf{P}: |[\mathbf{P}]_{ij}| \leq 1, u \in \mathbb{B}_1^n} u^T \mathbf{P} u \leq \max_{u \in \mathbb{B}_1^n} \sum_{ij} u_i u_j \leq n.$$

Since $\mathbf{P} = 0, \mu = 0$ is a feasible solution to (27), the master linear program must have an optimal solution, which we denote as $(\mu^l, \mathbf{P}^l)$.

The *separation problem* associated with (27) for the cone $\mathcal{K}_1$ is

$$\min_{\nu_1 \in \mathbb{R}, u \in \mathbb{R}^n} \nu_1 \tag{29a}$$

$$\text{s.t. } u^T \mathbf{P}^l u \leq \nu_1 \tag{29b}$$

$$u \in \mathcal{K}_1 \cap \mathbb{B}_1^n. \tag{29c}$$

The *separation problem* associated with (27) for the cone $\mathcal{K}_2$ is

$$\min_{\nu_2 \in \mathbb{R}, v \in \mathbb{R}^m} \nu_2 \tag{30a}$$

$$\text{s.t. } -v^T \mathbf{M}(\mathbf{P}^l)v \leq \nu_2 \tag{30b}$$

$$v \in \mathcal{K}_2 \cap \mathbb{B}_1^m. \tag{30c}$$

In the separation problems in (29) and (30) we have included the $\mathcal{K}_1 \cap \mathbb{B}_1^n$ and $\mathcal{K}_2 \cap \mathbb{B}_1^m$ constraint without specifying a formulation. We provide a mixed integer programming formulation for the CQLF conditions (12) in § 4.2 and for the EQLF conditions (18) in § 4.3.

Algorithm 1 summarizes the steps in the cutting plane algorithm. The steps of solving the master and separation problems at each iteration of the algorithm are described in Lines 5-18. The algorithm aims to find $\mathbf{P}$, $u \in \mathbb{B}_1^n$, $v \in \mathbb{B}_1^m$ satisfying

$$u^T \mathbf{P} u \geq \epsilon \; \forall u \in \mathcal{K}_1, \quad -v^T \mathbf{M}(\mathbf{P})v \geq \epsilon \; \forall v \in \mathcal{K}_2, \; (27d). \tag{31}$$

By appropriate scaling, this is equivalent to

$$\mathbf{P} \succ_{\mathcal{K}_1} 0, \; -\mathbf{M}(\mathbf{P}) \succ_{\mathcal{K}_2} 0, \; (27d). \tag{32}$$

Algorithm 1 A cutting plane algorithm to find $\mathbf{P}$ satisfying (25)

1: **Input**: $\epsilon > 0$, MaxIter.
2: **Output**: $\mathbf{P}$ satisfying (25) if one exists.
3: Set $l = 0$. Initialize the sets $\mathcal{U}^0 \subset \mathcal{U}$ and $\mathcal{V}^0 \subset \mathcal{V}$.
4: **while** $l < $ MaxIter **do**
5:     Solve (27) to obtain optimal solution $(\mu^l, \mathbf{P}^l)$.
6:     **if** $\mu^l < \epsilon$ **then**
7:         No solution exists satisfying user-specified tolerances. Terminate.
8:     **end if**
9:     Solve (29) and (30) using $\mathbf{P}^l$ to obtain optimal solutions (ν_1^l, u^l) and (ν_2^l, v^l) respectively.
10:    **if** $\min(\nu_1^l, \nu_2^l) \geq \epsilon$ **then**
11:       Return $\mathbf{P}^l$ as a feasible solution. Terminate.
12:    **end if**
13:    **if** $\nu_1^l < \epsilon$ **then**
14:       $\mathcal{U}^{l+1} \leftarrow \mathcal{U}^l \cup \{u^l\}$
15:    **end if**
16:    **if** $\nu_2^l < \epsilon$ **then**
17:       $\mathcal{V}^{l+1} \leftarrow \mathcal{V}^l \cup \{v^l\}$
18:    **end if**
19:    Set $l \leftarrow l + 1$.
20: **end while**

It is easy to see that if on solving (27) (in Line 5) $\mu^l < \epsilon$ then (26) is infeasible.

Lemma 3. *Suppose $\mu^l < \epsilon$ at some iteration l of Algorithm 1. Then there exists no $\mathbf{P}$ in* (27d) *satisfying* (31).

Proof. If $\mu^l < \epsilon$ then by (28)

$$\max_{\mathbf{P}:|[\mathbf{P}]_{ij}| \leq 1} \min \left\{ \min_{u \in \mathcal{U}^l} u^T \mathbf{P} u, \min_{v \in \mathcal{V}^l} -v^T \mathbf{M}(\mathbf{P})v \right\} < \epsilon$$

$$\implies \max_{\mathbf{P}:|[\mathbf{P}]_{ij}| \leq 1} \min \left\{ \min_{u \in \mathcal{U}} u^T \mathbf{P} u, \min_{v \in \mathcal{V}} -v^T \mathbf{M}(\mathbf{P}) \right\} < \epsilon$$

where the second inequality follows by noting that the minimization is over larger sets $\mathcal{U}^l \subseteq \mathcal{U}$ and $\mathcal{V}^l \subseteq \mathcal{V}$. Hence, there exists no $\mathbf{P}$ satisfying (31), which proves the claim. $\qquad\square$

Thus, if the Algorithm 1 terminates, it either terminates with a $\mathbf{P} \in \mathbb{S}^n$ such that $|[\mathbf{P}]_{ij}| \leq 1$, $\mathbf{P} \succcurlyeq_{\mathcal{K}_1} \epsilon$ and $-\mathbf{M}(\mathbf{P}) \succcurlyeq_{\mathcal{K}_2} \epsilon$, or a proof that no such matrix $\mathbf{P}$ exists.

4.2   Common Quadratic Lyapunov Function

To state the sufficiency conditions (12) required for a CQLF in the general form (25), we set $n = n_x$, $\mathbf{P} = P_{xx}$, $\mathcal{K}_1 = \mathbb{R}^{n_x}$, $m = (n_x + n_c)$, $\mathbf{M} = M(P_{xx})$ in (10), and $\mathcal{K}_2 = \mathcal{K} = \operatorname{Gr} \operatorname{SOL}(Dx, F) \subseteq \mathbb{R}^{n_x + n_c}$. In the separation problem for a CQLF, we are given a matrix P_{xx}^l, and we wish to verify if the conditions (12) hold for this matrix. Thus, we seek a vector $w = (x, \lambda) \in \mathbb{B}_1^{n_x + n_c}$ such that either $w^T P_{xx}^l w \leq 0$ or $-w^T M(P_{xx}^l) w \leq 0$ for some $(x, \lambda) \in \operatorname{Gr} \operatorname{SOL}(Dx, F)$. If no such vector exits, then we have verified that P_{xx}^l satisfies conditions (12), else the associated vector may be added to the appropriate set $\mathcal{U}^l$ or $\mathcal{V}^l$ in the master problem (27).

The separation problem for the constraint $P_{xx} \succ 0$ can be posed as the following mixed integer quadratic program, which minimizes the value of the quadratic form $x^T P_{xx}^l x$ over a 1-norm ball:

$$\min_{x, y, x^+, x^-, \nu} \nu \tag{33a}$$

$$\text{s.t} \quad x^T P_{xx}^l x \leq \nu \tag{33b}$$

$$x = x^+ - x^- \tag{33c}$$

$$\mathbf{1}^T (x^+ + x^-) = 1 \tag{33d}$$

$$0 \leq x^+ \leq y \tag{33e}$$

$$0 \leq x^- \leq \mathbf{1} - y \tag{33f}$$

$$y \in \{0, 1\}^{n_x}, \tag{33g}$$

where $\mathbf{1}$ is a vector of all ones.

The separation problem for the constraint $-M(P_{xx}^l) \succcurlyeq_{\mathcal{K}} 0$ can be posed as the following mixed integer quadratic program in variables $w = (x, \lambda) \in \mathbb{R}^{n_x + n_c}$, binary variables $y \in \{0, 1\}^{n_x}$, $z \in \{0, 1\}^{n_c}$ and auxiliary variables $x^+, x^- \in \mathbb{R}^{n_x}$:

$$\min_{x, \lambda, y, x^+, x^-, \nu} \nu \tag{34a}$$

$$\text{s.t} \quad -(x, \lambda)^T M(P_{xx}^l)(x, \lambda) \leq \nu \tag{34b}$$

$$x = x^+ - x^- \tag{34c}$$

$$\mathbf{1}^T (x^+ + x^- + \lambda) = 1 \tag{34d}$$

$$0 \leq x^+ \leq y \tag{34e}$$

$$0 \leq x^- \leq \mathbf{1} - y \tag{34f}$$

$$y \in \{0, 1\}^{n_x} \tag{34g}$$

$$0 \leq \lambda \leq z \tag{34h}$$

$$0 \leq \sum_{j=1}^{n_x} [D]_{ij} [x]_j + \sum_{j=1}^{n_c} [F]_{ij} [\lambda]_j \leq \theta_i (1 - [z]_i) \quad \forall i = 1, \ldots, n_c \tag{34i}$$

$$z \in \{0, 1\}^{n_c}, \tag{34j}$$

where $\mathbf{1}$ is a vector of all ones of conformal dimension, and the constant

$$\theta_i = \max \left(\max_j \left| [D]_{ij} \right|, \max_k ([F]_{ik}) \right)$$

provides an upper bound on the value of the left hand side of constraint (34i) for any $(x, \lambda) \in \mathbb{B}_1^{n_x + n_c}$. The objective (34a) and constraint (34b) seek to minimize the value of the (possibly nonconvex) quadratic function $(x, \lambda)^T M(P_{xx}^l)(x, \lambda)$. Constraints (34c)—(34g) ensure that $(x, \lambda) \in \mathbb{B}_1^{n_x + n_c}$, and constraints (34h)—(34j) enforce the conditions that $(x, \lambda) \in \operatorname{Gr} \operatorname{SOL}(Dx, F)$.

4.3 Extended Quadratic Lyapunov Function

To state the sufficiency conditions for the EQLF in Theorem 2 in terms of the general setting of (25), we can set $n = (n_x + n_c)$, $\mathbf{P} = P$, $\mathcal{K}_1 = \mathcal{K}$, $m = (n_x + 2n_c)$, $\mathbf{M} = \widehat{M}(P)$, and $\mathcal{K}_2 = \widehat{\mathcal{K}}$ where $\widehat{\mathcal{K}} = \{(x, \lambda, \widehat{\lambda}) \,|\, (x, \lambda) \in \mathrm{GrSOL}(Dx, F), \widehat{\lambda} \in \mathrm{SOL}(D(Ax + C\lambda), F)\}$. In the separation problem for an EQLF, we are given a matrix P^l, and we seek a vector $v = (x, \lambda, \hat{\lambda}) \in \mathbb{B}_1^{n_x + 2n_c}$ such that either

$$(i) \quad (x, \lambda)^T P^l (x, \lambda) < 0 \text{ for some } (x, \lambda) \in \mathrm{Gr\,SOL}(Dx, F); \text{ or}$$

$$(ii) - (x, \lambda, \hat{\lambda})^T \widehat{M}(P^l)(x, \lambda, \hat{\lambda}) < 0 \text{ for some } (x, \lambda) \in \mathrm{Gr\,SOL}(Dx, F)$$
$$\text{and } \widehat{\lambda} \in \mathrm{SOL}(D(Ax + C\lambda), F).$$

The separation problem for the condition (i) ($P \succ_\mathcal{K} 0$) is very similar to (34) and thus not repeated here. The separation problem for condition (ii) ($\widehat{M}(P) \succ_{\widehat{\mathcal{K}}} 0$) can be posed as the following quadratic mixed integer program:

$$\min_{x, \lambda, \hat{\lambda}, y, z, w, x^+, x^-, \nu} \nu \tag{35a}$$

$$\text{s.t} \quad (x, \lambda, \hat{\lambda})^T \widehat{M}(P^l)(x, \lambda, \hat{\lambda}) \leq \nu \tag{35b}$$

$$x = x^+ - x^- \tag{35c}$$

$$\mathbf{1}^T(x^+ + x^- + \lambda + \hat{\lambda}) = 1 \tag{35d}$$

$$0 \leq x^+ \leq y \tag{35e}$$

$$0 \leq x^- \leq \mathbf{1} - y \tag{35f}$$

$$y \in \{0, 1\}^{n_x} \tag{35g}$$

$$0 \leq \lambda \leq z \tag{35h}$$

$$0 \leq \sum_{j=1}^{n_x} [D]_{ij}[x]_j + \sum_{j=1}^{n_c} [F]_{ij}[\lambda]_j \leq \theta_i(1 - [z]_i) \quad \forall i = 1, \ldots, n_c \tag{35i}$$

$$z \in \{0, 1\}^{n_c} \tag{35j}$$

$$0 \leq \hat{\lambda} \leq w \tag{35k}$$

$$0 \leq \sum_{j=1}^{n_x} [DA]_{ij}[x]_j + \sum_{j=1}^{n_c} [DC]_{ij}[\lambda]_j + \sum_{j=1}^{n_c} [F]_{ij}\left[\hat{\lambda}\right]_j \leq \Theta_i(1 - [w]_i) \quad \forall i = 1, \ldots, n_c \tag{35l}$$

$$w \in \{0, 1\}^{n_c}, \tag{35m}$$

where $\Theta_i = \max\left(\max_j \left|[DA]_{ij}\right|, \max_j \left([F]_{ij}\right), \max_j \left([DC]_{ij}\right)\right)$ provides an upper bound on the value of the left-hand-side of (35l) for any feasible solution to the problem. The objective (35a) and constraint (35b) minimize the nonconvex quadratic function $(x, \lambda, \hat{\lambda})^T \widehat{M}(P^l)(x, \lambda, \hat{\lambda})$. Constraints (35c)—(35g) ensure that $(x, \lambda, \hat{\lambda} \in \mathbb{B}_1^{n_x + 2n_c}$. The constraints (35h)—(35j) enforce that $(x, \lambda) \in \mathrm{Gr\,SOL}(Dx, F)$, and the constraints (35k)— (35m) force that $\widehat{\lambda} \in \mathrm{SOL}(D(Ax + C\lambda), F)$.

5 Numerical Experiments

We implemented Algorithm 1 for the computation of CQLF and EQLF in Python and solved the master problems and separation problemss using Gurobi 9.0.2 [17]. In our experiments, we chose separation tolerance $\epsilon = 10^{-6}$ in Algorithm 1. In our implementation, we do not always solve the separation problems for the CQLF and EQLF to optimality. We terminate the separation early, if the solver finds a feasible solution whose objective value is less than or equal to zero. In this case, the incumbent solution is a vector that can be used to separate the current master problem solution and can be added to the set of vectors ($\mathcal{U}^l$ or $\mathcal{V}^l$) in the master problem (27).

5.1 DLCS derived from LCS with $\widetilde{F}$ a P-matrix

We consider the 3 LCS examples from [8] (Examples 3.1-3.3) and another example from Heemels, Schumacher and Weiland [19, §2].

- `cam31` is Example 3.1 [8] with data matrices: $\widetilde{A} = (\,1.0\,)$, $\widetilde{C} = (\,2 \ {-2}\,)$, $\widetilde{D} = \left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right)$, $\widetilde{F} = \left(\begin{smallmatrix} 1 & 3 \\ 0 & 1 \end{smallmatrix}\right)$.

- `cam32` is Example 3.2 [8] with data matrices: $\widetilde{A} = (\,{-1.0}\,)$, $\widetilde{C} = (\,0 \ 1\,)$, $\widetilde{D} = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$, $\widetilde{F} = \left(\begin{smallmatrix} 1 & 3 \\ 0 & 1 \end{smallmatrix}\right)$.

- `cam33` is Example 3.3 [8] with data matrices: $\widetilde{A} = \left(\begin{smallmatrix} -5 & -4 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$, $\widetilde{C} = \left(\begin{smallmatrix} -3 & 0 & 0 \\ -21 & 0 & 0 \\ 0 & 2 & -2 \end{smallmatrix}\right)$, $\widetilde{D} = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{smallmatrix}\right)$, $\widetilde{F} = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{smallmatrix}\right)$.

- `hem2` is Example in §2 [19] with data matrices: $\widetilde{A} = \left(\begin{smallmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{smallmatrix}\right)$, $\widetilde{C} = \left(\begin{smallmatrix} 0 \\ 0 \\ 1 \\ 0 \end{smallmatrix}\right)$, $\widetilde{D} = (\,1 \ 0 \ 0 \ 0\,)$, $\widetilde{F} = (\,1\,)$.

Note that in all the examples above the $\widetilde{F}$ matrix is a P-matrix. In the following we consider DLCS that are obtained from explicit ($\theta = 0$ in (3)) and implicit Euler ($\theta = 1$ in (3)) time-stepping schemes. We considered two possible time-steps of $\Delta t = 0.1$ and 0.05. For these choices of time-stepping schemes and Δt, it can be verified that F is also a P-matrix.

Table 1 summarizes the results for the CQLF computation while Table 2 provides the results for the EQLF. Table 1 shows that the algorithm is successful in identifying a CQLF such that $\mu \geq \epsilon$ for `cam31` and `cam32` for both time-steps. The algorithm shows that a CQLF does not exist such that $\mu \geq \epsilon$ for `cam33` and `hem2` when using the explicit Euler time-stepping scheme and either time-step choices. When using the implicit Euler scheme `cam31`, `cam32` and `cam33` permit identifying a CQLF for both time-step choices. As for `hem2`, a CQLF does not exist for either time-step choices. Table 2 shows that an EQLF with $\mu \geq 10^{-6}$ can be computed for all the examples except for `hem2` for either time-steps when using the explicit Euler. When using the implicit Euler an EQLF is computed for all systems.

In all cases that are stabilized using CQLF (EQLF), we also simulated the system from 100 random initial and verified that the CQLF (EQLF) is indeed decreasing over the trajectories as shown in Theorems 1 and 2.

From the tables it can be seen that the best μ for different time-steps (whenever feasible) is in proportion to the time-steps. The next section shows that this is indeed true for the CQLF (see Lemma 5).

	Name	$\Delta t = 0.1$			$\Delta t = 0.05$		
		Best μ	# iters.	Time(s)	Best μ	# iters.	Time(s)
Explicit	cam31	$4.348 \cdot 10^{-2}$	1	0.006	$2.326 \cdot 10^{-2}$	1	0.006
	cam32	$1.0 \cdot 10^{-1}$	1	0.003	$5.0 \cdot 10^{-2}$	1	0.003
	cam33	$< 10^{-6}$	3	0.171	$< 10^{-6}$	10	0.477
	hem2	$< 10^{-6}$	7	0.157	$< 10^{-6}$	6	0.118
Implicit	cam31	$4.762 \cdot 10^{-2}$	1	0.003	$2.439 \cdot 10^{-2}$	1	0.010
	cam32	$9.091 \cdot 10^{-2}$	1	0.004	$4.762 \cdot 10^{-2}$	1	0.003
	cam33	$3.941 \cdot 10^{-3}$	9	0.624	$4.373 \cdot 10^{-4}$	15	1.106
	hem2	$< 10^{-6}$	22	0.720	$< 10^{-6}$	16	0.320

Table 1: Summary of results from applying Algorithm 1 for the CQLF computation with $\epsilon = 10^{-6}$. Best μ - refers to the best μ obtained at termination; # iters. - number of times the while-loop. in Algorithm 1 is executed; Tot. CPU time (s) - total time for algorithm.

5.2 DLCS with F a Q-matrix and R_0-matrix

Stability results in the literature assume that F is a P-matrix [2, 8], which ensures that the state trajectory of the system is unique. The sufficient conditions for stability of DLCS given in Section 3

		$\Delta t = 0.1$			$\Delta t = 0.05$		
	Name	Best μ	# iters.	Time(s)	Best μ	# iters.	Time(s)
Explicit	cam31	$4.188 \cdot 10^{-2}$	2	0.048	$4.133 \cdot 10^{-2}$	1	0.025
	cam32	$1.766 \cdot 10^{-1}$	1	0.014	$4.870 \cdot 10^{-2}$	1	0.012
	cam33	$4.859 \cdot 10^{-3}$	9	1.630	$1.112 \cdot 10^{-2}$	13	2.500
	hem2	$< 10^{-6}$	8	0.365	$< 10^{-6}$	6	0.321
Implicit	cam31	$8.453 \cdot 10^{-2}$	1	0.033	$4.334 \cdot 10^{-2}$	1	0.029
	cam32	$1.607 \cdot 10^{-1}$	1	0.014	$8.453 \cdot 10^{-2}$	1	0.017
	cam33	$1.2 \cdot 10^{-2}$	4	0.990	$6.723 \cdot 10^{-3}$	8	1.501
	hem2	$9.629 \cdot 10^{-5}$	35	7.658	$2.831 \cdot 10^{-5}$	33	23.152

Table 2: Summary of results from applying Algorithm 1 for the EQLF computation with $\epsilon = 10^{-6}$. Best μ - refers to the best μ obtained at termination; # iters. - number of times the while-loop. in Algorithm 1 is executed; Tot. CPU time (s) - total time for algorithm.

assume the weaker condition that F is a Q-matrix and an R_0 matrix. Here, we demonstrate that the cutting-plane method can find stability conditions for these more general systems by considering the DLCS

$$A = \begin{pmatrix} 0.5 & 0.25 \\ -0.25 & 0.5 \end{pmatrix}, \ C = \begin{pmatrix} 3 \\ 5 \end{pmatrix}, \ D = \begin{pmatrix} 1.0 & 0.0 \\ 0.0 & 0.0 \end{pmatrix}, \ F = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}. \tag{36}$$

Every principal minor of the matrix F is nonnegative, and hence F is a P_0-matrix. Further, F is a R_0-matrix since $SOL(0, F) = \{0\}$. Thus, F is also a Q-matrix. Running our cutting-plane algorithm proves that the system cannot be stabilized using a CQLF. However, the system in (36) can be stabilized using the EQLF with $\mu = 4.057 \cdot 10^{-3}$ and

$$P = \begin{pmatrix} 5.33238468 \cdot 10^{-1} & -1.79873472 \cdot 10^{-1} & -1.0 & -4.13688777 \cdot 10^{-2} \\ -1.79873472 \cdot 10^{-1} & 1.70818637 \cdot 10^{-1} & 2.78264033 \cdot 10^{-1} & -8.65180851 \cdot 10^{-4} \\ -1.0 & 2.78264033 \cdot 10^{-1} & 1.0 & -1.0 \\ -4.13688777 \cdot 10^{-2} & -8.65180851 \cdot 10^{-4} & -1.0 & 7.53616763 \cdot 10^{-2} \end{pmatrix}.$$

6 LCS and DLCS Stability

Çamlibel, Pang and Shen [8, Theorem 3.1] present sufficient conditions for stability using an EQLF for the LCS (2). In this section, we consider the DLCS (1) obtained from the discretization of the LCS (2) for some $\theta \in [0, 1]$ in (4). Recall that by defining the matrix $\widetilde{A}_\theta = (I_{n_x} - \theta \Delta t \widetilde{A})^{-1}$, the time-stepping formulation of the LCS (4) can be identified with the DLCS as

$$\begin{aligned} A &= I_{n_x} + \Delta t \widetilde{A}_\theta^{-1} \widetilde{A}, & C &= \Delta t \widetilde{A}_\theta^{-1} \widetilde{C}, \\ D &= \widetilde{D}(I_{n_x} + \Delta t \widetilde{A}_\theta^{-1} \widetilde{A}), & F &= \widetilde{F} + \Delta t \widetilde{D} \widetilde{A}_\theta^{-1} \widetilde{C}. \end{aligned} \tag{37}$$

The stability results in [8, Theorem 3.1] are obtained under the assumption that $\widetilde{F}$ is a P-matrix. Accordingly, we assume in the rest of the section that $\widetilde{F}$ is a P-matrix. The main result presented in this section is to show that the LCS is exponentially stable using a CQLF (EQLF) if and only if the DLCS is exponentially stable using a CQLF (EQLF) for all Δt sufficiently small and any $\theta \in [0, 1]$.

6.1 CQLF Equivalence

We will begin by stating the conditions for exponential stability of LCS using CQLF. Note that [8, Theorem 3.1] states the asymptotic stability of LCS using EQLF. Just as in the DLCS (see remarks following Theorem 2) we can obtain the stability results for the CQLF after setting certain matrices in the Lyapunov function to zero.

Definition 2. *[8, Theorem 3.1] The LCS* (2) *is asymptotically stable using the CQLF if there exists* $\widetilde{P}_{xx} \in \mathbb{S}^{n_x}$ *such that*

$$\widetilde{P}_{xx} \succ 0, \ and \tag{38a}$$

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$$-\underbrace{\begin{pmatrix} \widetilde{A}^T \widetilde{P}_{xx} + \widetilde{P}_{xx}\widetilde{A} & \widetilde{P}_{xx}\widetilde{C} \\ \widetilde{C}^T \widetilde{P}_{xx} & 0 \end{pmatrix}}_{=:Q(\widetilde{P}_{xx})} \succ_{\mathcal{C}} 0, \tag{38b}$$

where the set $\mathcal{C} \subseteq \mathbb{R}^{n_x+n_c}$ is defined as

$$\mathcal{C} = Gr\,SOL(\widetilde{D}x, \widetilde{F}). \tag{38c}$$

In the following we assume that Δt is chosen such that

$$\Delta t \cdot \theta \cdot \|\widetilde{A}\| < 1, \ \Delta t \cdot \|\widetilde{A}_\theta^{-1}\widetilde{A}\| < 1, \text{ and } F \text{ is a Q-matrix and R}_0\text{-matrix.} \tag{39}$$

Note that the first inequality in (39) ensures that $\widetilde{A}_\theta^{-1}$ exists while the second inequality ensures that A is invertible. Both of these conditions can be ensured for all Δt sufficiently small. Finally since we assume that $\widetilde{F}$ is a P-matrix then there must exist Δt sufficiently small so that $F = \widetilde{F}+\Delta t\widetilde{D}\widetilde{A}_\theta^{-1}\widetilde{C}$ is also a P-matrix, so it is also both a Q-matrix and an R$_0$-matrix. For Δt satisfying (39) we have by (37) that

$$A^{-1} = (I_{n_x} + \Delta t \cdot \widetilde{A}_\theta^{-1}\widetilde{A})^{-1} = I_{n_x} - \Delta t \cdot \widetilde{A} + \Delta t^2 \cdot \widetilde{A}_r \tag{40}$$

where $\widetilde{A}_r$ is a bounded, residual term.

We will begin the analysis by relating the cones $\mathcal{K}$ and $\mathcal{C}$.

Lemma 4. *Suppose $\widetilde{F}$ is a P-matrix, let Δt be such that (39) holds, and let the matrix N be defined as*

$$N = \begin{pmatrix} A & C \\ 0 & I_{n_c} \end{pmatrix}. \tag{41}$$

Then the following hold:

(a) If $(x, \lambda) \in \mathcal{K}$ then $(Ax + C\lambda, \lambda) \in \mathcal{C}$.

(b) If $(x, \lambda) \in \mathcal{C}$ then $(\widetilde{x}, \lambda) \in \mathcal{K}$ where $\widetilde{x} = A^{-1}x - A^{-1}C\lambda$.

(c) There exist constants $c_1, c_2 > 0$ such that $c_1\|(\widetilde{x}, \lambda)\| \leq \|(x, \lambda)\| \leq c_2\|(\widetilde{x}, \lambda)\|$.

Proof. Let $(x, \lambda) \in \mathbb{R}^{n_x+n_c}$ and define $\widetilde{x} = Ax + C\lambda$. From the definition of (41), we have that

$$\begin{pmatrix} \widetilde{x} \\ \lambda \end{pmatrix} = N \begin{pmatrix} x \\ \lambda \end{pmatrix}. \tag{42}$$

The matrix N is invertible with $N^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}C \\ 0 & I_{n_c} \end{pmatrix}$. The claims in (a) and (b) follow immediately. From the upper triangular representation of N, we have that the eigenvalues of N are the eigenvalues of A and I_{n_c}. Hence, $\|N\| \leq \max(\|A\|, 1)$ and $\|N^{-1}\| \leq \max(\|A^{-1}\|, 1)$. Combining this with (42) yields the claim in (c). $\qquad\square$

We now present an alternative representation of the matrix $M(P_{xx})$ (10) in terms of $Q(\widetilde{P}_{xx})$ in (38b).

Lemma 5. *Suppose Δt is sufficiently small so that (39) holds. Let $P_{xx} \in \mathbb{S}^{n_x}$ be given. Then*

$$M(P_{xx}) = N^T \left(\Delta t \cdot Q(P_{xx}) + \Delta t^2 \cdot Q_r(P_{xx})\right) N, \tag{43}$$

where $Q_r(P_{xx})$ is a bounded matrix and $N = \begin{pmatrix} A & C \\ 0 & I_{n_c} \end{pmatrix}$.

Proof. Consider the definition of the matrix M in (10)

$$\begin{aligned} M(P_{xx}) &= N^T N^{-T} \begin{pmatrix} A^T P_{xx} A - P_{xx} & A^T P_{xx} C \\ C^T P_{xx} A & C^T P_{xx} C \end{pmatrix} N^{-1} N \\ &= N^T \begin{pmatrix} P_{xx} - A^{-T} P_{xx} A^{-1} & A^{-T} P_{xx} A^{-1} C \\ C^T A^{-T} P_{xx} A^{-1} & -C^T A^{-T} P_{xx} A^{-1} C \end{pmatrix} N, \end{aligned} \tag{44}$$

where the final equality is obtained by substituting for N^{-1}, multiplying through, and simplifying. Substituting (37) into the blocks of the matrix in (44) and simplifying, we obtain for all Δt satisfying (39) that

$$P_{xx} - A^{-T} P_{xx} A^{-1} = \Delta t(\widetilde{A}^T P_{xx} + P_{xx} \widetilde{A}) + \Delta t^2 \cdot Q_{r,11}(P_{xx}) \tag{45a}$$

$$A^{-T} P_{xx} A^{-1} C = \Delta t P_{xx} \widetilde{C} + \Delta t^2 \cdot Q_{r,12}(P_{xx}) \tag{45b}$$

$$-C^T A^{-T} P_{xx} A^{-1} C = \Delta t^2 \cdot Q_{r,22}(P_{xx}) \tag{45c}$$

where $Q_{r,11}(P_{xx})$, $Q_{r,12}(P_{xx})$, and $Q_{r,22}(P_{xx})$ are bounded residual matrices. Define the matrix $Q_r(P_{xx}) = \begin{pmatrix} Q_{r,11}(P_{xx}) & Q_{r,12}(P_{xx}) \\ Q_{r,12}(P_{xx})^T & Q_{r,22}(P_{xx}) \end{pmatrix}$. Plugging (45) into the expressions in (44) obtains (43), which proves the claim. $\qquad\square$

Combining the Lemmas (4) and (5) yields the main result that there exists a CQLF for the LCS if and only if there exists a CQLF for DLCS.

Theorem 3. *Suppose $\widetilde{F}$ is a P-matrix and Δt is such that (39) holds. Then the LCS is exponentially stable using a CQLF if and only if the DLCS obtained by the time-stepping formulation (4) is exponentially stable using a CQLF.*

Proof. Let $P_{xx} \in \mathbb{S}^{n_x}$ be given. Pre-multiply and post-multiply the equation in (43) by $(x, \lambda) \in \mathcal{K}$ to obtain

$$\begin{pmatrix} x \\ \lambda \end{pmatrix}^T M(P_{xx}) \begin{pmatrix} x \\ \lambda \end{pmatrix} = \begin{pmatrix} \widetilde{x} \\ \lambda \end{pmatrix}^T \left(\Delta t \cdot Q(P_{xx}) + \Delta t^2 \cdot Q_r(P_{xx}) \right) \begin{pmatrix} \widetilde{x} \\ \lambda \end{pmatrix}, \tag{46}$$

where $\widetilde{x} = Ax + C\lambda$. Further, by Lemma 4, we have that $(\widetilde{x}, \lambda) \in \mathcal{C}$.

Consider the only if part of the claim. Suppose the LCS is exponentially stable using a CQLF. Since $-Q(P_{xx}) \succ_{\mathcal{C}} 0$ we have that $-(\widetilde{x}, \lambda)^T Q(P_{xx})(\widetilde{x}, \lambda) \geq \gamma \|(\widetilde{x}, \lambda)\|^2$ for some $\gamma > 0$. Choose Δt smaller than dictated by (39) so that $\Delta t \cdot \|Q_r(P_{xx})\| \leq \gamma/2$. Then we have for such Δt from (46) that $-(x, \lambda)^T M(P_{xx})(x, \lambda) \geq (\gamma \Delta t)/2 \|(\widetilde{x}, \lambda)\|^2$. Combining this with Lemma 4(c) proves the claim. The if part of the claim can be shown using similar arguments. This completes the proof. $\qquad\square$

6.2 EQLF Equivalence

We now consider the the EQLF and derive similar results between the LCS and DLCS. The condition for exponential stability derived in [8] is stated next.

Definition 3. *[8, Theorem 3.1] The LCS (2) is exponentially stable if there exists $\widetilde{P}_{xx} \in \mathbb{S}^{n_x}$, $\widetilde{P}_{x\lambda} \in \mathbb{R}^{n_x \times n_c}$, $\widetilde{P}_{\lambda\lambda} \in \mathbb{S}^{n_c}$ such that*

$$\underbrace{\begin{pmatrix} \widetilde{P}_{xx} & \widetilde{P}_{x\lambda} \\ \widetilde{P}_{x\lambda}^T & \widetilde{P}_{\lambda\lambda} \end{pmatrix}}_{=:\widetilde{P}} \succ_{\mathcal{K}} 0 \tag{47a}$$

$$-\underbrace{\begin{pmatrix} \widetilde{A}^T \widetilde{P}_{xx} + \widetilde{P}_{xx} \widetilde{A} & \widetilde{P}_{xx} \widetilde{C} + \widetilde{A}^T \widetilde{P}_{x\lambda} & \widetilde{P}_{x\lambda} \\ \widetilde{C}^T \widetilde{P}_{xx} + \widetilde{P}_{x\lambda}^T \widetilde{A} & \widetilde{P}_{x\lambda}^T \widetilde{C} + \widetilde{C}^T \widetilde{P}_{x\lambda} & \widetilde{P}_{\lambda\lambda} \\ \widetilde{P}_{x\lambda}^T & \widetilde{P}_{\lambda\lambda} & 0 \end{pmatrix}}_{=:\widehat{Q}(\tilde{P})} \succ_{\mathcal{C}} 0 \tag{47b}$$

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where the set $\widehat{\mathcal{C}} \subseteq \mathbb{R}^{n_x + 2n_c}$ is defined as

$$\widehat{\mathcal{C}} = \left\{ (\widetilde{x}, \lambda, \lambda') \in \mathbb{R}^{n_x + 2n_c} \left| \begin{array}{l} (\widetilde{x}, \lambda) \in Gr\,SOL(\widetilde{D}\widetilde{x}, \widetilde{F}) \\ \left[\widetilde{D}(\widetilde{A}\widetilde{x} + \widetilde{C}\lambda) + \widetilde{F}\lambda' \right]_{\alpha(\lambda)} = 0 \\ 0 \leq [\lambda']_{\beta(\lambda)} \perp \left[\widetilde{D}(\widetilde{A}\widetilde{x} + \widetilde{C}\lambda) + \widetilde{F}\lambda' \right]_{\beta(\lambda)} \geq 0 \\ [\lambda']_{\gamma(\lambda)} = 0 \end{array} \right. \right\} \tag{47c}$$

with $\alpha(\lambda), \beta(\lambda), \gamma(\lambda)$ defined according to (6) for $SOL(\widetilde{D}\widetilde{x}, \widetilde{F})$.

Note that λ' in (47c) is the directional derivative of $\lambda(\widetilde{x})$ along the direction $(\widetilde{A}\widetilde{x} + \widetilde{C}\lambda)$.

Suppose $(x, \lambda, \widehat{\lambda}) \in \widehat{\mathcal{K}}$ with $\widehat{\lambda} \in SOL(D(Ax + C\lambda), F)$. By Lemma 4 we have that $\lambda = SOL(\widetilde{D}\widetilde{x}, \widetilde{F})$. In other words, $\widehat{\lambda}$ is the one-time step evolution of the complementarity variables from the initial state $\widetilde{x}$ using the time-stepping DLCS (3). The next result relates $\widehat{\lambda}$ with λ and λ'. The proof relies on results from Shen and Pang [37] and Chen and Wang [13] which we state next.

Lemma 6. *([37, Lemma 14]) Suppose $\widetilde{F}$ is a P-matrix. Let $\widetilde{D}\widetilde{x} \neq 0$ for all $\widetilde{x} \neq 0$. Let $\lambda = SOL(\widetilde{D}\widetilde{x}, \widetilde{F})$ and $\lambda(\Delta t)$ denote the complementarity variable obtained by evolving the LCS (2) over Δt from initial state $\widetilde{x}$. Then*

$$\lambda(\Delta t) = \lambda + \Delta t \cdot \lambda' + h_1(\widetilde{x}) \cdot o(\Delta t) \tag{48}$$

where the function $h(\widetilde{x})$ satisfies $\|h_1\| = O(\|\widetilde{x}\|)$.

Proof. The claim follows from Lemma 14 in [37]. $\qquad\square$

Lemma 7. *([13, Theorem 3.1]) Suppose $\widetilde{F}$ is a P-matrix. Let $\lambda(\Delta t)$ denote the complementarity variable resulting from evolving the LCS (2) from the initial state $\widetilde{x}$ over a time duration of Δt and $\lambda = SOL(\widetilde{D}\widetilde{x}, \widetilde{F})$. Let $\widehat{\lambda}$ be the one time-step evolution of (3) from the same initial state $\widetilde{x}$. Then there exists a $\overline{\Delta t} > 0$ such that for all $\Delta t < \overline{\Delta t}$*

$$\|\widehat{\lambda} - \lambda(\Delta t)\| \leq \Delta t^2 \cdot O(\|\widetilde{x}\|). \tag{49}$$

Proof. Under the assumptions of the lemma it is readily verified that Theorem 3.1 in [13] holds. Hence, $\|\widehat{\lambda} - \lambda(\Delta t)\|_\infty \leq \Delta t^2 L \|\widetilde{A}\widetilde{x} + \widetilde{C}\lambda\|_\infty$ for some constant $L > 0$ independent of Δt, x, λ. Using Lemma 1 we know that $\|\lambda\|$ can be bounded using $\|\widetilde{x}\|$. Combining this with the above and using the equivalence of norms yields (49). $\qquad\square$

Combining Lemma 6 and 7 gives a relation between $\widehat{\lambda}$ and the pair λ, λ'.

Lemma 8. *Suppose the assumptions of Lemmas 6 and 7 hold. Then*

$$\widehat{\lambda} = \lambda + \Delta t \cdot \lambda' + o(\Delta t) \cdot h_2(\widetilde{x}) \tag{50}$$

where $\|h_2(\widetilde{x})\| \leq O(\|\widetilde{x}\|)$.

Proof. From Lemma 7 we have that $\widehat{\lambda} = \lambda(\Delta t) + \Delta t^2 \cdot h_2(\widetilde{x})$ where $\|h_2(\widetilde{x})\| = O(\|\widetilde{x}\|)$. Substituting for $\lambda(\Delta t)$ from (48) yields (50). $\qquad\square$

We next must relate the set $\widehat{\mathcal{K}}$ in (19) with $\widehat{\mathcal{C}}$.

Lemma 9. *Suppose $\widetilde{F}$ is a P-matrix and $\widetilde{D}\widetilde{x} \neq 0$ for all $\widetilde{x} \neq 0$. Let the matrix $\widehat{N}$ and vector $\widehat{h}$ be defined as*

$$\widehat{N} = \begin{pmatrix} A & C & 0 \\ 0 & I_{n_c} & 0 \\ 0 & -\frac{1}{\Delta t} I_{n_c} & \frac{1}{\Delta t} I_{n_c} \end{pmatrix} \quad and \quad \widehat{h}(\widetilde{x}) = \begin{pmatrix} 0 \\ 0 \\ h_2(\widetilde{x}) \end{pmatrix}. \tag{51a}$$

Then there exists a $\overline{\Delta t} > 0$ such that for all $\Delta t \in (0, \overline{\Delta t})$ the following hold:

(a) If $(\widetilde{x}, \lambda, \lambda') \in \widehat{\mathcal{C}}$ then $\widehat{N}^{-1}\begin{pmatrix} \widetilde{x} \\ \lambda \\ \lambda' \end{pmatrix} + o(\Delta t)\widehat{h}(\widetilde{x}) \in \widehat{\mathcal{K}}$.

(b) If $(x, \lambda, \widehat{\lambda}) \in \widehat{\mathcal{K}}$ then $\widehat{N}\begin{pmatrix} x \\ \lambda \\ \widehat{\lambda} \end{pmatrix} - \frac{o(\Delta t)}{\Delta t}\widehat{h}(Ax + C\lambda) \in \widehat{\mathcal{C}}$.

(c) There exist constants $c_3, c_4 > 0$ such that $c_3 \Delta t \|(\widetilde{x}, \lambda, \lambda')\| \leq \|(x, \lambda, \widehat{\lambda})\| \leq c_4 \|(\widetilde{x}, \lambda, \lambda')\|$.

Proof. From Lemma 4 we have that $(A^{-1}\widetilde{x} - A^{-1}C\lambda, \lambda) \in \mathcal{K}$. Combining this with Lemma 8 and noting that $\widehat{N}^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}C & 0 \\ 0 & I_{n_c} & 0 \\ 0 & I_{n_c} & \Delta t \cdot I_{n_c} \end{pmatrix}$ yields

$$\begin{pmatrix} x \\ \lambda \\ \widehat{\lambda} \end{pmatrix} = \widehat{N}^{-1}\begin{pmatrix} \widetilde{x} \\ \lambda \\ \lambda' \end{pmatrix} + o(\Delta t)\widehat{h}(\widetilde{x}) \tag{52a}$$

$$\begin{pmatrix} \widetilde{x} \\ \lambda \\ \lambda' \end{pmatrix} = \widehat{N}\begin{pmatrix} x \\ \lambda \\ \widehat{\lambda} \end{pmatrix} - \frac{o(\Delta t)}{\Delta t}\widehat{h}(\widetilde{x}). \tag{52b}$$

where the second equality is obtained by left multiplying (52a) with $\widehat{N}$. The claims in (a) and (b) follow from (52a) and (52b) respectively. To prove (c), note that the first and second rows and columns of $\widehat{N}$ can be swapped to produce a lower triangular matrix $\begin{pmatrix} I_{n_c} & 0 & 0 \\ C & A & 0 \\ -\frac{1}{\Delta t}I_{n_c} & 0 & \frac{1}{\Delta t}I_{n_c} \end{pmatrix}$. Based on the lower triangular representation, the eigenvalues of $\widehat{N}$ can be obtained as the eigenvalues of A, I_{n_c}, and $(1/\Delta t)I_{n_c}$. Hence, $\|\widehat{N}\| \leq 1/(\Delta t)$ and $\|\widehat{N}^{-1}\| \leq \max(\|A^{-1}\|, 1)$. By choosing Δt to be smaller if necessary we have from (52a) and (52b) that $\|(x, \lambda, \widehat{\lambda})\| \leq 2\|\widehat{N}^{-1}\|\|(\widetilde{x}, \lambda, \lambda')\|$ and $\|(\widetilde{x}, \lambda, \lambda')\| \leq 2\|\widehat{N}\| \cdot \|(x, \lambda, \widehat{\lambda})\|$. Substituting for $\|\widehat{N}\|$ and $\|\widehat{N}^{-1}\|$ yields the claim in (c). This completes the proof. $\square$

The matrices $\widehat{Q}$ and $\widehat{M}$ are related next.

Lemma 10. *Suppose Δt is chosen to be sufficiently small such that (39) holds. Let $P_{xx} \in \mathbb{S}^{n_x}$, $P_{x\lambda} \in \mathbb{R}^{n_x \times n_c}$, and $P_{\lambda\lambda} \in \mathbb{S}^{n_c}$ be given. Then*

$$\widehat{M}(P) = \widehat{N}^T\left(\Delta t \cdot \widehat{Q}(P) + \Delta t^2 \cdot \widehat{Q}_r(P)\right)\widehat{N} \tag{53}$$

where $\widehat{Q}_r(P)$ is a bounded matrix and $\widehat{N}$ is as defined in (51a).

Proof. Consider the definition of the matrix $\widehat{M}(P)$ in (15)

$$\widehat{M}(P) = \widehat{N}^T\widehat{N}^{-T}\begin{pmatrix} A^T P_{xx} A - P_{xx} & A^T P_{xx} C - P_{x\lambda} & A^T P_{x\lambda} \\ C^T P_{xx} A - P_{x\lambda}^T & C^T P_{xx} C - P_{\lambda\lambda} & C^T P_{x\lambda} \\ P_{x\lambda}^T A & P_{x\lambda}^T C & P_{\lambda\lambda} \end{pmatrix}\widehat{N}^{-1}\widehat{N}$$

$$= \widehat{N}^T\begin{pmatrix} P_{xx} - A^{-T}P_{xx}A^{-1} & U_{x\lambda} & \Delta t P_{x\lambda} \\ U_{x\lambda}^T & U_{\lambda\lambda} & \Delta t P_{\lambda\lambda} \\ \Delta t P_{x\lambda}^T & \Delta t P_{\lambda\lambda} & 0 \end{pmatrix}\widehat{N}, \tag{54}$$

where $U_{x\lambda} = A^{-T}P_{xx}A^{-1}C - A^{-T}P_{x\lambda} + P_{x\lambda}$ and

$$U_{\lambda\lambda} = -C^T A^{-T}P_{xx}A^{-1}C + P_{x\lambda}^T A^{-1}C + C^T A^{-T}P_{x\lambda}.$$

The second equality is obtained from the first by multiplying through and simplifying. Substituting (37) into the blocks of the matrix in (54) and simplifying, we obtain for all Δt satisfying (39) that

$$P_{xx} - A^{-T}P_{xx}A^{-1} = \Delta t(\widetilde{A}^T P_{xx} + P_{xx}\widetilde{A}) + \Delta t^2 \cdot \widehat{Q}_{r,11}, \tag{55a}$$

$$U_{x\lambda} = \Delta t P_{xx}\widetilde{C} + \Delta t^2 \cdot \widehat{Q}_{r,12}, \text{ and} \tag{55b}$$

$$U_{\lambda\lambda} = \Delta t \left(P_{x\lambda}^T D + D^T P_{x\lambda} \right) + \Delta t^2 \cdot \widehat{Q}_{r,22}, \tag{55c}$$

where $\overline{M}_{r,11}$, $\overline{M}_{r,12}$, and $\overline{M}_{r,22}$ are bounded residual matrices. Define the matrix $\overline{M}_r = \begin{pmatrix} \overline{M}_{r,11} & \overline{M}_{r,12} & 0 \\ \overline{M}_{r,12}^T & \overline{M}_{r,22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$.
Plugging (55) into the expressions in (54) yields (53), which proves the claim. $\square$

We are now ready to show the equivalence between the EQLF conditions for the LCS (47) and the DLCS (18).

Theorem 4. *Suppose $\widetilde{F}$ is a P-matrix. Let $\widetilde{D}\widetilde{x} \neq 0$ for all $\widetilde{x} \neq 0$. Then for all Δt sufficiently small, the LCS is exponentially stable using a EQLF if and only if the DLCS obtained by the time-stepping formulation (3) is exponentially stable using a EQLF.*

Proof. Pre-multiply and post-multiply the equation in (53) by $(x, \lambda, \widehat{\lambda}) \in \widehat{\mathcal{K}}$ to obtain

$$\begin{pmatrix} x \\ \lambda \\ \widehat{\lambda} \end{pmatrix}^T \widehat{M} \begin{pmatrix} x \\ \lambda \\ \widehat{\lambda} \end{pmatrix}$$

$$= \left(\begin{pmatrix} \widetilde{x} \\ \lambda \\ \lambda' \end{pmatrix} + \frac{o(\Delta t)}{\Delta t} \widehat{h}(\widetilde{x}) \right)^T \left(\Delta t \cdot \widehat{Q} + \Delta t^2 \cdot \widehat{Q}_r \right) \left(\begin{pmatrix} \widetilde{x} \\ \lambda \\ \lambda' \end{pmatrix} + \frac{o(\Delta t)}{\Delta t} \widehat{h}(\widetilde{x}) \right)$$

$$= \Delta t \begin{pmatrix} \widetilde{x} \\ \lambda \\ \lambda' \end{pmatrix}^T \widehat{Q} \begin{pmatrix} \widetilde{x} \\ \lambda \\ \lambda' \end{pmatrix} + o(\Delta t) \begin{pmatrix} \widetilde{x} \\ \lambda \\ \lambda' \end{pmatrix}^T \widehat{Q}\widehat{h}(\widetilde{x}) + \frac{o(\Delta t)^2}{\Delta t} \widehat{h}(\widetilde{x})^T \widehat{Q}\widehat{h}(\widetilde{x})$$

$$+ \Delta t^2 \left(\begin{pmatrix} \widetilde{x} \\ \lambda \\ \lambda' \end{pmatrix} + \frac{o(\Delta t)}{\Delta t} \widehat{h}(\widetilde{x}) \right)^T \widehat{Q}_r \left(\begin{pmatrix} \widetilde{x} \\ \lambda \\ \lambda' \end{pmatrix} + \frac{o(\Delta t)}{\Delta t} \widehat{h}(\widetilde{x}) \right) \tag{56}$$

where $\widetilde{x} = Ax + C\lambda$. The first equality follows from Lemma 8 for all Δt sufficiently small. Further by Lemma 9, we have that $(\widetilde{x}, \lambda, \lambda') \in \widehat{\mathcal{C}}$.

Consider the only if part of the claim. Suppose LCS is exponentially stable using an EQLF. Since $-\widehat{Q} \succ_{\widehat{\mathcal{C}}} 0$ there exists $\gamma > 0$ such that $-(\widetilde{x}, \lambda, \lambda')^T \widehat{Q}(\widetilde{x}, \lambda, \lambda') \geq \gamma \|(\widetilde{x}, \lambda, \lambda')\|^2$. Then for all Δt sufficiently small the terms other than the leading term in (56) can be made smaller than or equal to $(\gamma \Delta t)/2 \|(\widetilde{x}, \lambda, \lambda')\|^2$. Then for all for such Δt we have from (56) and Lemma 9(c) that

$$-(x, \lambda, \widehat{\lambda})^T \widehat{M}(x, \lambda, \widehat{\lambda}) \geq (\gamma \Delta t)/2 \|(\widetilde{x}, \lambda, \lambda')\|^2 \geq (\gamma \Delta t)/(2c_4) \|(x, \lambda, \widehat{\lambda})\|^2.$$

The proof of the if part of the claim can be obtained in a similar manner. This completes the proof. $\square$

7 Conclusions & Future Work

In this work, we derived sufficient conditions for the Lyapunov stability of a Discrete-Time Linear Complementarity System (DLCS), using both common and extended quadratic Lyapunov functions. We also showed the equivalence between the stability conditions of a Linear Complementarity System and its discrete-time analog for all sufficiently small time-steps. We proposed a cutting plane algorithm to find a Lyapunov function verifying exponential stability by separating points from nonconvex copositive cones, and the algorithm was demonstrated on small DLCS instances. The current investigation opens up a number of avenues for future exploration.

- First, the cutting plane algorithm can be slow on large instances of the DLCS. For large instances, the algorithm often requires performing many rounds of cuts before terminating. Improving the behavior of the cutting plane algorithm is an important future direction of this work.

- Another natural area of improvement for this approach is that a Lyapunov function is only found upon termination of the algorithm. In other words, we do not yet have a heuristic approach for constructing a feasible matrix P_{xx} using information obtained from the cutting plane method. Such an approach can be advantageous since for stability, we are only interested in a feasible solution satisfying the copositivity conditions.

- The incorporation of feedback control and the joint computation of stabilizing controller and Lyapunov function is another avenue for investigation. The inclusion of feedback controls complicates the separation problem since the matrices in the LCP constraints are now dependent on the feedback matrices. This necessitates the development of a different algorithm for the computation of Lyapunov function.

A S-Lemma Formulations for CQLF and EQLF

A.1 Common Quadratic Lyapunov Function

The conditions for exponential stability in Theorem 1(iii) can be written as the feasibility problem (12). Moreover, the inequality in (12c) can be relaxed into a linear matrix inequality using the S-Lemma [3]. Consider the matrix $H \in \mathbb{R}^{(n_x+n_c)}$ defined as

$$H = \begin{pmatrix} D & F \\ 0 & I_{n_c} \end{pmatrix}. \tag{57a}$$

Consider the Linear Matrix Inequality (LMI)

$$M + H^T W H \prec 0 \text{ where } W \in \mathbb{S}^{(n_x+n_c)}, W \geq 0. \tag{57b}$$

Let $\operatorname{conv}(\operatorname{Gr} \operatorname{SOL}(Dx, F))$ denote the convex hull of $\operatorname{Gr} \operatorname{SOL}(Dx, F)$, i.e.

$$\operatorname{conv}(\operatorname{Gr} \operatorname{SOL}(Dx, F)) = \{(x, \lambda) \,|\, \lambda \geq 0, \, Dx + F\lambda \geq 0\}.$$

We show that satisfaction of (57b) implies that $\psi(x, \lambda) < 0$ for all $(x, \lambda) \in \operatorname{conv}(\operatorname{Gr} \operatorname{SOL}(Dx, F))$. Thus, we obtain a relaxation of the copositivity conditions in (12) to a LMI (57b). Multiplying the matrix inequality in (57b) from the left and right by $(x, \lambda) \in \operatorname{conv}(\operatorname{Gr} \operatorname{SOL}(Dx, F))$ we obtain

$$\begin{pmatrix} x \\ \lambda \end{pmatrix}^T \left(M + H^T W H \right) \begin{pmatrix} x \\ \lambda \end{pmatrix} < 0$$

$$\implies \psi(x, \lambda) + \begin{pmatrix} Dx + F\lambda \\ \lambda \end{pmatrix}^T W \begin{pmatrix} Dx + F\lambda \\ \lambda \end{pmatrix} < 0$$

$$\implies \psi(x, \lambda) < 0 \tag{57c}$$

where the second implication follows from $(x, \lambda) \in \operatorname{conv}(\operatorname{Gr} \operatorname{SOL}(Dx, F))$ and $W_{xx}, W_{\lambda\lambda} \geq 0$. Hence, satisfaction of (57b) $\implies$ (12c). The computation of a P_{xx} satisfying (12) can be cast as the LMI system

$$\text{Find } P_{xx} \in \mathbb{S}^{n_x}, W \in \mathbb{S}^{n_x+n_c} \tag{57d}$$

$$\text{such that } P_{xx} \succ 0, W \geq 0, (57b). \tag{57e}$$

A.2 Extended Quadratic Lyapunov Function

Similar to the use of S-Lemma for the CQLF the copositivity conditions in (18) can be turned into a feasibility problem involving LMIs. Consider the matrices $J, J' \in \mathbb{R}^{(n_x+n_c) \times (n_x+2n_c)}$, defined as

$$J = \begin{pmatrix} D & F & 0 \\ 0 & I_{n_c} & 0 \end{pmatrix}, J' = \begin{pmatrix} DA & DC & F \\ 0 & 0 & I_{n_c} \end{pmatrix}. \tag{58a}$$

It can be verified that

$$
\left.\begin{array}{r}
W_1 \in \mathbb{S}^{n_x+n_c}, W_1 \geq 0 \\
P - H^T W_1 H \succ 0
\end{array}\right\} \implies \text{(18b)} \tag{58b}
$$

$$
\left.\begin{array}{r}
W_2, W_3 \in \mathbb{S}^{(n_x+n_c)}, W_2, W_3 \geq 0 \\
\widehat{M} + J^T W_2 J + (J')^T W_3 (J') \prec 0
\end{array}\right\} \implies \text{(18c)}. \tag{58c}
$$

Note that (58b) can be derived in a manner similar to that in (57c). We derive (58c) in the following. Multiply the matrices in the left hand side of (58c) by $(x, \lambda, \widehat{\lambda}) \in \widehat{\mathcal{K}}$ and rearrange to obtain

$$
\begin{pmatrix} x \\ \lambda \\ \widehat{\lambda} \end{pmatrix}^T \left(\widehat{M} + J^T W_2 J + (J')^T W_3 (J') \right) \begin{pmatrix} x \\ \lambda \\ \widehat{\lambda} \end{pmatrix} < 0
$$

$$
\implies \widehat{\psi}(x, \lambda, \widehat{\lambda}) + \begin{pmatrix} Dx + F\lambda \\ \lambda \end{pmatrix}^T W_2 \begin{pmatrix} Dx + F\lambda \\ \lambda \end{pmatrix} +
$$

$$
\begin{pmatrix} DAx + DC\lambda + F\widehat{\lambda} \\ \widehat{\lambda} \end{pmatrix}^T W_3 \begin{pmatrix} DAx + DC\lambda + F\widehat{\lambda} \\ \widehat{\lambda} \end{pmatrix} < 0
$$

$$
\implies \widehat{\psi}(x, \lambda, \widehat{\lambda}) < 0
$$

where the second implication follows from $\lambda \in \mathrm{SOL}(Dx, F))$, $\widehat{\lambda} \in \mathrm{SOL}(DAx + DC\lambda, F)$ and $W_2, W_3 \geq 0$. This prove the implication in (58c). Just as in the case of the CQLF we can show that the LMI in (58c) implies that $\widehat{\psi}(x, \lambda, \widehat{\lambda}) < 0$ for all $(x, \lambda, \widehat{\lambda}) \in \mathrm{conv}\widehat{\mathcal{K}}$.

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