Projection onto the exponential cone: a univariate root-finding problem

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Abstract

The exponential function and its logarithmic counterpart are essential corner stones of nonlinear mathematical modeling. In this paper we treat their conic extensions, the exponential cone and the relative entropy cone, in primal, dual and polar form, and show that finding the nearest mapping of a point onto these convex sets all reduce to a single univariate root-finding problem. This leads to a fast algorithm shown numerically robust over a wide range of inputs.

keywords: *projection, exponential cone, relative entropy cone, julia code.*

1 Introduction

In the conic subfield of mathematical optimization—the motivational background for this paper—nonlinear functions are embedded in conic extensions before use. That is, rather than using a convex function $f : \mathbb{R}^n \to \mathbb{R}$ directly, conic optimization researchers and practitioners operate with the closed conic hull of its epigraph, given by

$$\mathrm{cl}\left\{(t, s, r) \in \mathbb{R}^{2+n} \mid \tfrac{t}{s} \geq f\left(\tfrac{r}{s}\right),\ s > 0\right\}, \tag{1}$$

where t, s and r denotes the epigraph, perspective and function variable of the convex cone (1), respectively. From this construction, the reader might recognize that $f(x) = \|x\|_2$ leads to a quadratic cone, whereas $f(x) = \frac{1}{2}\|x\|_2^2$ leads to a rotated quadratic cone, both of which are mainline in proprietary and open-source software for conic optimization.

In case of the exponential function, $f(x) = \exp(x)$, the conic construction (1) leads to the so-called *exponential cone*, which by [4, lemma 4.2.1] can be formulated as the union

$$K_{\exp} := \mathrm{cl}([K_{\exp}]_{++}) = [K_{\exp}]_{++} \cup [K_{\exp}]_0, \tag{2}$$

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joining the two sets

$$\begin{aligned} [K_{\exp}]_{++} &= \left\{(t,s,r) \in \mathbb{R}^3 \mid s > 0,\ t \geq s \exp\left(\tfrac{r}{s}\right)\right\}, \\ \text{and} \qquad [K_{\exp}]_0 &= \left\{(t,s,r) \in \mathbb{R}^3 \mid s = 0,\ t \geq 0,\ r \leq 0\right\}. \end{aligned} \tag{3}$$

We refer to $[K_{\exp}]_{++}$ and $[K_{\exp}]_0$ as the perspective interior and perspective boundary of the exponential cone, referring in both name and subscript to the represented subdomain of s.

In case of the concave logarithm function, using $f(x) = -\log(x) = \log(x^{-1})$ in (1) leads the so-called *relative entropy cone*. This cone, $K_{\log}$, is nevertheless nothing but an orthogonal transformation of the exponential cone, realized from definition by the relationship $(t,s,r) \in K_{\log} \Leftrightarrow (r,s,-t) \in K_{\exp}$.

We shall not dwell on the general advantages of working with conic extensions, referring instead to [7], but will nevertheless try to explain why the exponential cone (of all possible conic extensions) is an interesting subject of study.

From a mathematical modeling perspective, the expressive abilities of the exponential cone is vast. For instance, given the simple relationship between $K_{\exp}$ and $K_{\log}$ above, it should come as no surprise that both the exponential epigraph and logarithmic hypograph sets can be represented using the exponential cone, namely,

$$t \geq \exp(r) \Leftrightarrow (t,1,r) \in K_{\exp}, \quad \text{and} \quad t \leq \log(r) \Leftrightarrow (r,1,t) \in K_{\exp}.$$

More broadly, it can be used to represent convex compositions of exponentials, logarithms, entropy functions, product logarithms (such as Lambert W), softmax and softplus known from neural networks, and generalized posynomials known from geometric programming [4, 12]. These constructions are usually made in conjunction with quadratic cones, but note that this is purely for the sake of simplicity and computational performance. In principle, the exponential cone is powerful enough to represent semidefiniteness of a 2x2 symmetric matrix variable (which is an orthogonal transformation of a 3-dimensional quadratic cone) [3], and thus single-handily represent all the above.

From a computational perspective, the exponential cone has a numerically stable 3-self-concordant barrier function [13], and is both nonempty, closed and convex by construction, making it compatible with most duality, facial reduction and interior-point properties desired for conic optimization [16]. It is not a symmetric cone, however, as is traditionally needed to ensure fast and reliable convergence, but the implementations in ECOS [18], SCS [14] and MOSEK [11, 5] have shown that this property is not necessary for large-scale computational viability. In particular, Dahl and Andersen claims in [5] that they achieve *"good numerical performance, on level with standard symmetric cone (e.g., quadratic cone) algorithms"*.

The projection problem addressed in this paper asks for the nearest mapping of a point $v_0 \in \mathbb{R}^3$ onto the exponential cone, that is,

$$d(v_0) = \min\{\|v - v_0\|_2 \mid v \in K_{\exp}\}. \tag{4}$$

This problem was studied in support of the research leading to [5], in order to establish sound grounds for error analysis of the proposed implementation. In particular, while forward error for the set membership constraint, $v_0 \in K_{\text{exp}}$, could easily be defined in any number of ways, e.g.,

$$\begin{cases} \left| \left[t - s \exp(\frac{r}{s}) \right]_- \right| & \text{if } s > 0, \\ \|([t]_-,\ |s|,\ [r]_+)\| & \text{otherwise}, \end{cases} \tag{5}$$

where $[x]_+ = \max(x, 0)$ and $[x]_- = \min(x, 0)$, it can only ever provide answers to whether the constraint is satisfied or not in the chosen computational precision. Specifically, due to the condition number of the exponential function, it is hard to put meaningful tolerances on acceptable levels of forward error as the function value may vary widely with small changes to input.

In contrast, the projection distance, $d(v_0)$ from (4), provides a measure of backward error for the set membership constraint, $v_0 \in K_{\text{exp}}$, which depends solely on the geometric properties of the exponential cone and not on its internal algebraic representation. Moreover, as desired, small changes to input gives small changes to output, bounded according to $d(v_0 + \Delta v_0) \leq d(v_0) + \|\Delta v_0\|_2$ by the triangle inequality.

To a broader audience, the applications of a fast and stable solution method for the projection problem (4) extends well beyond judging membership constraints and reporting errors. The solution itself, the projection of v_0 onto K_{exp} denoted as $[v_0]_{K_{\text{exp}}}$, is useful for first-order feasibility seeking methods and heuristics [2, 14]. Moreover, this solution gives rise to a supporting hyperplane for the exponential cone with maximal separation of the given point and finds use, e.g., in incremental constructions of outer approximations [9].

Previous work on the projection problem (4) is given by the analysis of [15], printed in 2013 with an implementation attached to its online material, and an undocumented function in SCS [14] tracing back to 2013 also.

The work of [15] propose closed-form solutions for a subset of inputs to the projection problem (4)—which are all restated with proof in Section 3—as well as necessary optimality conditions for the remaining set of points to be solved by Newton's method. These conditions are not sufficient, however, due to a lack of nonnegativity on their dual multiplier, λ, allowing solutions such as

$$\begin{pmatrix} t_0 \\ s_0 \\ r_0 \end{pmatrix} = \begin{pmatrix} \exp(3)+1 \\ 2\exp(3)+1 \\ 3-\exp(3) \end{pmatrix}, \quad \begin{pmatrix} t \\ s \\ r \end{pmatrix} = \begin{pmatrix} \exp(3) \\ 1 \\ 3 \end{pmatrix}, \quad \lambda = -1,$$

which suggest a projection distance of $\approx$45. In comparison, the heuristic solution $(s_0 \exp(r_0/s_0), s_0, r_0)$ is only at distance $\approx$6. Whether this relaxation can be justified algorithmically is outside the scope of this paper, but it does seem as if the starting point they use, found in online material, is always in the convergence region of the correct solution with $\lambda > 0$. The uncertainty of correctness, combined with the stated intention of code to be pedagogical, not performant, nevertheless strongly suggest improvements to be possible.

The undocumented work of [14] is harder to assess, but can be observed to solve the projection problem (4) by iteratively applying Newton's method to solve a converging series of univariate root-finding subproblems. In particular, this series of subproblems are parameterized by a single argument, and the solution of each subproblem is used to determine if the given argument was too big or too small. This allows bisection strategies to locate the correct argument value and, in turn, obtain the final solution.

In the remainder of this paper, the projection problem (4) is simplified to a single univariate root-finding problem, in no obvious relation to the series of problems solved in [14], and solved by numerical methods. The basic theory is covered in Section 2, along with results for how to solve other variants (namely, orthogonal transformations) of the projection problem. Subsequent analysis leads to closed-form solutions in Section 3 and the univariate root-finding problem in Section 4. Finally, the ingredients for a fast and numerically stable algorithm is discussed in Section 5, leading to a proof-of-concept implementation for which the computational experiences are reported in Section 6.

2 Optimality conditions

The Moreau decomposition theorem [10] elegantly states that if a point is written as a sum of two orthogonal components belonging to a primal-polar pair of non-empty closed convex cones, then these components are the respectively projections of that point onto the pair of cones. Applied to the exponential cone, this means that the projection problem (4) has optimal solution $v^\star = v_p$, with objective value $d(v_0) = \|v_d\|_2$, in terms of the unique feasible solution to the Moreau system

$$v_0 = v_p + v_d, \quad v_p \in K_{\exp}, \quad v_d \in K_{\exp}^\circ, \quad v_p^T v_d = 0, \tag{6}$$

where $K_{\exp}^\circ$ is the polar form of the exponential cone (2). Using [4, Theorem 4.3.3], this polar cone can be formulated as the union

$$K_{\exp}^\circ := \mathrm{cl}([K_{\exp}^\circ]_{++}) = [K_{\exp}^\circ]_{++} \cup [K_{\exp}^\circ]_0, \tag{7}$$

joining the two sets

$$\begin{aligned} [K_{\exp}^\circ]_{++} &= \left\{(t,s,r) \in \mathbb{R}^3 \mid r > 0,\ (-\mathbf{e})t \ge r \exp\left(\tfrac{s}{r}\right)\right\}, \\ \text{and} \qquad [K_{\exp}^\circ]_0 &= \left\{(t,s,r) \in \mathbb{R}^3 \mid r = 0,\ (-\mathbf{e})t \ge 0,\ s \le 0\right\}, \end{aligned} \tag{8}$$

where $\mathbf{e} = \exp(1)$. To the curious reader, note that the Moreau conditions (6) are merely an easily understood and elegant reformulation of the first-order KKT conditions for the projection problem (4), resembling Lagrangian stationarity, primal feasibility, dual feasibility and complementarity.

As previously stated, the Moreau conditions (6) solves not only the projection problem onto the exponential cone, but also the projection problem onto its polar cone. Moreover, due to invariance of the Moreau system to orthogonal transformations $H \in \mathbb{R}^{3x3}$, we see that any solution triplet (v_0, v_p, v_d) projecting onto $K_{\exp}$ and $K_{\exp}^{\circ}$, can be transformed to a solution triplet (Hv_0, Hv_p, Hv_d) projecting onto $HK_{\exp}$ and $(HK_{\exp})^{\circ} = HK_{\exp}^{\circ}$. This property empowers any algorithm solving the projection problem (4), to also solve a host of related projection problems. To be concrete, consider a solution to (6) such as

$$\begin{pmatrix} t_0 \\ s_0 \\ r_0 \end{pmatrix} = \begin{pmatrix} \exp(1)-1 \\ 1 \\ \exp(1)+1 \end{pmatrix}, \quad \begin{pmatrix} t_p \\ s_p \\ r_p \end{pmatrix} = \begin{pmatrix} \exp(1) \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} t_d \\ s_d \\ r_d \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ \exp(1) \end{pmatrix}, \tag{9}$$

from which we know that v_p is the projection of v_0 onto the primal cone, $K_{\exp}$, and v_d is the projection of v_0 onto the polar cone, $K_{\exp}^{\circ}$. In other words,

$$\left[\begin{pmatrix} \exp(1)-1 \\ 1 \\ \exp(1)+1 \end{pmatrix}\right]_{K_{\exp}} = \begin{pmatrix} \exp(1) \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \left[\begin{pmatrix} \exp(1)-1 \\ 1 \\ \exp(1)+1 \end{pmatrix}\right]_{K_{\exp}^{\circ}} = \begin{pmatrix} -1 \\ 0 \\ \exp(1) \end{pmatrix}.$$

Since dual cones can be defined from polar cones by the transformation $H = -\mathrm{I}$, i.e., $K_{\exp}^{*} = HK_{\exp}^{\circ}$, we can use this result to derive that

$$\left[-\begin{pmatrix} \exp(1)-1 \\ 1 \\ \exp(1)+1 \end{pmatrix}\right]_{K_{\exp}^{*}} = -\begin{pmatrix} -1 \\ 0 \\ \exp(1) \end{pmatrix}.$$

Furthermore, since the relative entropy cone is given by $K_{\log} = HK_{\exp}$, where $H = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, we may also derive that

$$\left[\begin{pmatrix} -\exp(1)-1 \\ 1 \\ \exp(1)-1 \end{pmatrix}\right]_{K_{\log}} = \begin{pmatrix} -1 \\ 1 \\ \exp(1) \end{pmatrix}.$$

In general, one can compute the projection of v_0 onto any primal-polar pair, $HK_{\exp}$ and $HK_{\exp}^{\circ}$, by first projecting $H^T v_0$ onto $K_{\exp}$ and $K_{\exp}^{\circ}$, and then returning Hv_p and Hv_d as the solution.

3 Closed-form solutions

A simple trick to force out closed-form solutions to the Moreau system (6), is to strengthen the complementarity condition, $v_p^T v_d = 0$, to the elementwise zero condition $t_p \cdot t_d = s_p \cdot s_d = r_p \cdot r_d = 0$. Given the readily available sign relations in the exponential cone (2) and its polar (7), namely

$$\begin{aligned} &t_p \geq 0,\ r_p \text{ free},\ s_p \geq 0, \text{ and } [t_p = 0] \Rightarrow [s_p = 0] \Rightarrow [r_p \leq 0], \\ &t_d \leq 0,\ r_d \geq 0,\ s_d \text{ free}, \text{ and } [t_d = 0] \Rightarrow [r_d = 0] \Rightarrow [s_d \leq 0], \end{aligned}$$

this is achieved under the condition

$$[t_p = 0] \text{ or } [t_d = 0] \text{ or } [s_p = 0] \text{ or } [r_d = 0] \text{ or } [s_d \leq 0 \text{ and } r_p \leq 0]. \tag{10}$$

Formally, as is left for the reader to verify, the relation is given by

$$[v_p^T v_d = 0] \text{ and } (10) \iff [t_p \cdot t_d = s_p \cdot s_d = r_p \cdot r_d = 0]. \qquad (11)$$

With the elementwise zero condition in place, the Moreau system (6) can now be shown to imply the projection rules stated in [15].

Theorem 3.1. *The Moreau system* (6) *is solved in satisfaction of condition* (10) *if and only if it is found by one of the following projection rules.*

1. *If* $v_0 \in K_{\exp}$*, then* $v_p = v_0$ *and* $v_d = 0$.
2. *If* $v_0 \in K_{\exp}^{\circ}$*, then* $v_p = 0$ *and* $v_d = v_0$.
3. *If* $r_0 \leq 0$ *and* $s_0 \leq 0$*, then* $v_p = \big([t_0]_+,\, 0,\, r_0\big)$ *and* $v_d = \big([t_0]_-,\, s_0,\, 0\big)$.

Proof. All solutions above satisfy conditions (6) and (10). Hence, to complete the proof, assume (10) to hold such that $t_p^T t_d = s_p^T s_d = r_p^T r_d = 0$ by (11).

- Case $t_d = 0$ (implies $r_d = 0$) and $s_d = 0$: The Moreau system reduces to $(t_0, s_0, r_0) = (t_p, s_p, r_p) \in K_{\exp}$ covered by rule 1.
- Case $t_d = 0$ (implies $r_d = 0$) and $s_p = 0$: The Moreau system reduces to $t_0 = t_p \geq 0$, $s_0 = s_d \leq 0$ and $r_0 = r_p \leq 0$ covered by rule 3.
- Case $t_p = 0$ (implies $s_p = 0$) and $r_d = 0$: The Moreau system reduces to $t_0 = t_d \leq 0$, $s_0 = s_d \leq 0$ and $r_0 = r_p \leq 0$ covered by rule 3.
- Case $t_p = 0$ (implies $s_p = 0$) and $r_p = 0$: The Moreau system reduces to $(t_0, s_0, r_0) = (t_d, s_d, r_d) \in K_{\exp}^{\circ}$ covered by rule 2.

□

Note that the projection rules are not disjoint in the sense that membership $v_0 \in [K_{\exp}]_0$ covered by rule 1 (resp. $v_0 \in [K_{\exp}^{\circ}]_0$ covered by rule 2), are also covered by the third projection rule.

4 The univarite root-finding problem

Let $v_0 \in \mathcal{F}$ denote the set of points not covered by the projection rules of Theorem 3.1. That is, points for which the solution to the Moreau system (6) violates condition (10), and thus instead must satisfy

$$[t_p > 0] \text{ and } [t_d < 0] \text{ and } [s_p > 0] \text{ and } [r_d > 0] \text{ and } [s_d > 0 \text{ or } r_p > 0]. \qquad (12)$$

This allows one to simplify the Moreau system (6) in a long series of steps, omitted here due to the end-result being independently provable. The essence of the derivation is based on the fact that, in general, the Moreau system (6) allows for the dual feasibility and complementary constraints to be combined as a single normal cone constraint. In our case, this is given by

$$v_d \in K_{\exp}^{\circ} \cap v_p^{\perp} = N_{K_{\exp}}(v_p),$$

where $N_{K_{\exp}}(v_p)$ is the cone of normal vectors to $K_{\exp}$ at the point v_p. Given the simplifying nature of (12), and the characterization of normal cones in [17, page 283] for sets of the form $\{x \in \mathbb{R}^n \mid g(x) \leq 0\}$ generated by proper convex functions, the following results can be obtained.

Lemma 4.1. *The Moreau system* (6) *is satisfied by all solutions to the following reduced system,*

$$t_0 = t_p + t_d, \quad s_p > 0, \quad r_d > 0,$$

using substitutions

$$\begin{aligned} v_p &= (t_p, s_p, r_p) = (\exp(\rho), 1, \rho)\, s_p, & s_p &= \tfrac{(\rho-1)r_0+s_0}{\rho^2-\rho+1}, \\ v_d &= (t_d, s_d, r_d) = (-\exp(-\rho), 1-\rho, 1)\, r_d, & r_d &= \tfrac{r_0-\rho s_0}{\rho^2-\rho+1}, \end{aligned}$$

depending solely on the primal ratio, $\rho = \frac{r_p}{s_p} = 1 - \frac{s_d}{r_d}$.

Proof. By inspection. For instance, $v_p^T v_d = -s_p r_d + s_p(1-\rho) r_d + \rho s_p r_d = 0$ verifies complementarity in the Moreau system (6).

□

Theorem 4.2. *Assuming $v_0 \in \mathcal{F}$, solving the Moreau system* (6) *is equivalent to finding the unique root of the function*

$$h(\rho) = \frac{(\rho-1)r_0 + s_0}{\rho^2 - \rho + 1} \exp(\rho) - \frac{r_0 - \rho s_0}{\rho^2 - \rho + 1} \exp(-\rho) - t_0,$$

on the nonempty strict domain, $l < \rho < u$, given by

$$l = \begin{cases} 1 - s_0/r_0 & \textit{if } r_0 > 0, \\ -\infty & \textit{otherwise}, \end{cases} \quad \textit{and} \quad u = \begin{cases} r_0/s_0 & \textit{if } s_0 > 0, \\ \infty & \textit{otherwise}, \end{cases}$$

and the transformations to/from the root ρ (i.e., the primal ratio) and the pair of projections (v_p, v_d) are as stated in Lemma 4.1.

Proof. Compared to the reduced system of Lemma 4.1, note that a root satisfies $h(\rho) = t_p + t_d - t_0 = 0$, as well as the bounds $[s_p > 0] \Leftrightarrow [(\rho-1)r_0 + s_0 > 0]$ and $[r_d > 0] \Leftrightarrow [r_0 - \rho s_0 > 0]$ expanding, respectively, to

$$\begin{cases} \rho > 1 - s_0/r_0 & \text{if } r_0 > 0, \\ \rho < 1 - s_0/r_0 & \text{if } r_0 < 0, \\ s_0 > 0 & \text{if } r_0 = 0, \end{cases} \quad \text{and} \quad \begin{cases} \rho < r_0/s_0 & \text{if } s_0 > 0, \\ \rho > r_0/s_0 & \text{if } s_0 < 0, \\ r_0 > 0 & \text{if } s_0 = 0, \end{cases}$$

where the last two cases of each expansion can be ignored. Specifically, the last is a tautology, $[r_0 = 0] \Rightarrow [s_0 > 0]$ and $[s_0 = 0] \Rightarrow [r_0 > 0]$ on $v_0 \in \mathcal{F}$, and the second is dominated; $[r_0 < 0] \Rightarrow [s_0 > 0]$ whereby $r_0/s_0 < 1 - s_0/r_0$ holds for the two upper bounds, and $[s_0 < 0] \Rightarrow [r_0 > 0]$ whereby $1 - s_0/r_0 > r_0/s_0$ holds for the two lower bounds. By Lemma 4.1, a root of $h(\rho)$ is thus a solution to the Moreau system (6). By uniqueness of projection, it hence suffices to show that

$h(\rho)$ always has a unique root on $l < \rho < u$ for all $v_0 \in \mathcal{F}$. This is a consequence of $h(\rho)$ being smooth, strictly increasing and changing sign on this particular domain as shown next in Theorem 4.5. □

The next theorem, following two minor analytic results, characterize important properties of $h(\rho)$ that are both essential to the proof of Theorem 4.2 and of great benefit to the development of root-finding algorithms for this function.

Lemma 4.3. *If $s_0 > 0$, then $[h(u) \leq 0] \Leftrightarrow [v_0 \in K_{\exp}]$ for $u = r_0/s_0$.*

Proof. Shown by $h(u) = s_0 \exp(u) - t_0$ whereby $[h(u) = 0] \Leftrightarrow [v_0 \in \mathbf{bd}(K_{\exp})]$, using both $((u-1)r_0 + s_0)/(u^2 - u + 1) = s_0$ and $(r_0 - us_0) = 0$ to simplify. This characterizes all roots of $h(u)$ as a function of v_0. The claim hence follows from decreasing t_0 which exits $K_{\exp}$ and increases $h(u)$. □

Lemma 4.4. *If $r_0 > 0$, then $[h(l) \geq 0] \Leftrightarrow [v_0 \in K_{\exp}^{\circ}]$ for $l = 1 - s_0/r_0$.*

Proof. Shown by $h(l) = -r_0 \exp(-l) - t_0$ whereby $[h(l) = 0] \Leftrightarrow [v_0 \in \mathbf{bd}(K_{\exp}^{\circ})]$, using both $((l-1)r_0 + s_0) = 0$ and $(r_0 - ls_0)/(l^2 - l + 1) = r_0$ to simplify. This characterizes all roots of $h(l)$ as a function of v_0. The claim hence follows from increasing t_0 which exits $K_{\exp}^{\circ}$ and decreases $h(l)$. □

Theorem 4.5. *Assuming $v_0 \in \mathcal{F}$, the function $h(\rho)$ is smooth, changing sign and strictly increasing on the domain, $l < \rho < u$, specified by Theorem 4.2.*

Proof. The function $h(\rho)$ is

(i) *smooth*, because it can be stated as a quotient of smooth functions plus a constant, $h = \frac{f}{g} - t_0$, with positive denominator $g = \rho^2 - \rho + 1 \geq 3/4$ for all $\rho \in \mathbb{R}$. It follows that the derivate, $h' = (f'g + fg')/g^2$, is a quotient of two smooth functions with positive denominator and the argument repeats.

(ii) *changing sign*, because $h(\rho) \to \infty$ for $\rho \to \infty$ by dominant term $\exp(\rho)$ with positive coefficient, and $h(\rho) \to -\infty$ for $\rho \to -\infty$ by dominant term $\exp(-\rho)$ with negative coefficient. If $s_0 > 0$, then $h(\rho)$ is strictly positive already at $\rho = u$ by Lemma 4.3. Likewise, if $r_0 > 0$, then $h(\rho)$ is strictly negative already at $\rho = l$ by Lemma 4.4.

(iii) *strictly increasing*, because strict decrease and repeating function values can both be ruled out. The former by results shown in (ii), the latter since $h(\rho_1) = h(\rho_2) = \tilde{h}$, for a pair of arguments, $l < \rho_1 < \rho_2 < u$, implies $h(\rho_1) = h(\rho_2) = 0$ for the perturbed parameter $\tilde{v}_0 = (t_0 + \tilde{h}, s_0, r_0)$. Under this perturbation the reduced system of Lemma 4.1 would thus be satisfied by two distinct values of ρ, yielding two distinct solutions to the Moreau system (6), which contradicts uniqueness in the projection of $\tilde{v}_0$. □

5 Solution techniques

If a given instance of the projection problem (4) is not solved in presolve via the projection rules of Theorem 3.1, it can be solved by finding the unique root of the function $h(\rho)$ as demonstrated in Theorem 4.2. Although root-finding can be hard in general, Theorem 4.5 showed us that $h(\rho)$ is both smooth and strictly increasing within the bracket of interest, $l < \rho < u$, making solutions methods much easier to craft. Moreover, as will be shown in Section 5.2, this bracket can always be strengthen to a finite range by simple preprocessing steps, further expanding the design space of solutions methods. This paper makes no effort to compare the multitude of root-finding approaches, nor argue for one over the other. Instead, the richness of the subject is underlined.

To start out, a solution method can be based on a wide variety of search techniques including bisection, regula falsi [6], and Newton's method [1]. These three techniques request an increasing level of information from left to right, stepping from function value sign attributes, to function values, and up to function derivatives. Higher-order information has the potential to give higher convergence rates near the root, but must be balanced against the increased computational cost of each iteration and the added sensitivity to computational errors affecting the quality of this information.

One step further, methods can be combined in a ramp-up strategy where one starts from a stable method (e.g., bisection) and only switch to more aggressive methods if improved convergence rates are guaranteed (e.g., via the Kantorovich theorem or similar convergence tests for the Newton method [8]). A ramp-down strategy is similarly possible, in which aggressive methods are attempted until iteration limits or numerical criteria triggers a switch to methods that are less sensitive to computational errors. It is not cut in stone either that $h(\rho)$ is the best function to work on as, e.g., $f(\rho) = h(\rho)g(\rho)$, given $g(\rho) = \rho^2 - \rho + 1 \geq 3/4$ for all $\rho \in \mathbb{R}$, allow all higher-order derivatives to be computed at the same cost as the function itself.

For the purpose of this paper, it suffices to show two results of significance to the success of said solution methods. In Section 5.1 we present a short list of heuristics solutions and show show their effectiveness in handling roots outside the numerical range of the exponential function. Then, in Section 5.2, we show that the weak bracket, $l < \rho < u$, can be strengthened to a finite (and often tight) range around the root in a simple preprocessing step. This is instrumental to the convergence of bisection methods and to the construction of initial guesses used, e.g., in Newton's method.

5.1 Heuristic solutions

For any point not covered by the projection rules of Theorem 3.1, i.e., $v_0 \in \mathcal{F}$, the Moreau system (6) can still be satisfied with limited error—solely affecting the stationarity condition—by perturbing it to a neighbor, $v_0 + \Delta v$, where the closed-form solution is known. This technique leads directly to the following heuristic solution pairs.

Lemma 5.1. *Let $v_0 \in \mathcal{F}$ and consider the projection rules of Theorem 3.1.*

1. *If $s_0 > 0$, one may increase t_0 until the first rule applies. This gives the heuristic solution pair*

$$\tilde{v}_p = \big(\max(t_0, s_0 \exp(r_0/s_0)),\ s_0,\ r_0\big) \in K_{\exp}, \quad \tilde{v}_d = 0 \in K_{\exp}^{\circ}.$$

2. *If $r_0 > 0$, one may decrease t_0 until the second rule applies. This gives the heuristic solution pair*

$$\tilde{v}_p = 0 \in K_{\exp}, \quad \tilde{v}_d = \big(\min(t_0, -r_0 \exp(s_0/r_0 - 1)),\ s_0,\ r_0\big) \in K_{\exp}^{\circ}.$$

3. *One may always decrease s_0 and r_0 until the third rule applies. This gives the heuristic solution pair*

$$\tilde{v}_p = ([t_0]_+,\ 0,\ [r_0]_-) \in K_{\exp}, \quad \tilde{v}_d = ([t_0]_-,\ [s_0]_-,\ 0) \in K_{\exp}^{\circ}.$$

□

The list of heuristic solutions above is not complete by any means, and can be extended in a number of ways.

First of, while the third rule is based on the actual projection onto the set $\mathbb{R} \times \mathbb{R}^2_-$ of points that it cover, rule 1 and rule 2 are not. In particular, they just follow an arbitrary interior point direction given, respectively, as $(1,0,0) \in \mathbf{int}(K_{\exp})$ and $(-1,0,0) \in \mathbf{int}(K_{\exp}^{\circ})$, until the set of points they cover is reached. Any other interior point could be used instead.

Secondly, by scaling invariance in the Moreau system (6), each and every symbolic solution for some v_0, such as (9), can be turned into a projection rule covering all points in the conic hull of this v_0. Projection onto these rays further adds to the list of heuristic solution pairs.

What can be said about the particular selection of heuristics in Lemma 5.1 is that they appear to work well in practice to overcome numerical challenges. Taking $v_0 = (8, -8, 0.01) \in \mathcal{F}$ as example, the function $h(\rho)$ exhibits an large root bounded from below by $\rho > l = 1 - s_0/r_0 = 801$. To be clear, this is way beyond the finite range of the exponential function $\exp(\rho)$, used in $h(\rho)$, when evaluated in standard IEEE 64-bit floating-point arithmetic. Fortunately, by choice of $\tilde{v}_p = (t_0, 0, [r_0]_-) \in K_{\exp}$ and $\tilde{v}_d = (-r_0 \exp(s_0/r_0 - 1), s_0, r_0) \in K_{\exp}^{\circ}$, the Moreau system errors for this pair turn out to be

$$\|\tilde{v}_p + \tilde{v}_d - v_0\|_2 = |\tilde{t}_d| \approx 10^{-350}, \text{ and } |\tilde{v}_p^T \tilde{v}_d| = |t_0 t_d| \approx 10^{-349}, \tag{13}$$

which solves the problem for all practical purposes. Interestingly, all pairs $(\tilde{v}_p, \tilde{v}_d)$ as they are listed in Lemma 5.1 yield much larger errors, and it is only by mixing the argmin of $\|\tilde{v}_p - v_0\|_2$ with the argmin of $\|\tilde{v}_d - v_0\|_2$ over all candidates, that a practical solution is produced.

While the formal reasoning behind the seemingly general effectiveness of these heuristics to solve numerically challenging instances is not yet established, the intuitive answer is clear. When the lower bracket is high, the rate of change in $h(\rho)$—powered by $\exp(\rho)$—is enormous and the constant offset, t_0, is easily covered by extremely tiny steplengths from l. It is thus reasonable to believe that the observation is an effect of the fact that v_d tends to the above applied heuristic solution $v_d \to \tilde{v}_d = (-r_0 \exp(s_0/r_0 - 1), s_0, r_0)$ for $\rho \to l$. Analogous arguments can be applied to the equivalent example, $v_0 = (-8, 0.01, -8) \in \mathcal{F}$, acting on the opposite end of the spectrum where $\rho < u = r_0/s_0 = -800$.

5.2 Bracket strengthening

The search bracket $l < \rho < u$ given by Theorem 4.2 may be infinitely wide which can be problematic to convergence analysis and the design of effective search methods. Fortunately, this issue can be solved.

The "just get it working" approach to finite brackets is exponential jump search which is so simple it can be explained by example. If one were to find an upper bound to the root of a monotonic increasing functions such as $h(\rho)$, start at any value and iteratively double the stepsize, $h(\tilde{\rho} + 2^p)$ for $p = \{0, 1, \ldots\}$, until the function becomes positive. At this point the argument overstepped the root and therefore bound it from above.

A more analytical approach would require a thorough investigation of $h(\rho)$, e.g., using under- and overestimators. In this section, however, we shall make a small detour and exploit that the terms of the function actually has meaning, i.e., $h(\rho) = t_p + t_d - t_0$ as described by Lemma 4.1, and bound the terms t_p and t_d directly using the underlying projection problems. Bounds for ρ are then derived in subsequent steps.

Lemma 5.2. *The epigraph variables of a solution to the Moreau system* (6) *are bounded according to*

$$\begin{array}{rcccl} [t_0]_+ & \leq & t_p & \leq & \min\left(\Delta_d,\ \Delta_p + t_0\right), \\ -[t_0]_- & \leq & -t_d & \leq & \min\left(\Delta_p,\ \Delta_d - t_0\right), \end{array}$$

where

$$\Delta_p = \sqrt{\|\tilde{v}_p - v_0\|_2^2 - [s_0]_-^2}, \quad \Delta_d = \sqrt{\|\tilde{v}_d - v_0\|_2^2 - [r_0]_-^2},$$

for any choice of $\tilde{v}_p \in K_{\exp}$ *and* $\tilde{v}_d \in K_{\exp}^{\circ}$.

Proof. The bounds $\Delta_p + t_0$ and $\Delta_d - t_0$ are found by the pair of relations

$$\begin{array}{rcccl} \sqrt{(t_p - t_0)^2 + [s_0]_-^2} & \leq & \|v_p - v_0\|_2 & \leq & \|\tilde{v}_p - v_0\|_2, \\ \sqrt{(t_0 - t_d)^2 + [r_0]_-^2} & \leq & \|v_d - v_0\|_2 & \leq & \|\tilde{v}_d - v_0\|_2, \end{array}$$

where $[s_0]_-$ is included to tighten the former, using $s_p \geq 0$ on $K_{\exp}$, as is $[r_0]_-$ in the latter using $r_d \geq 0$ on $K_{\exp}^{\circ}$. The rest is found by bound propagation on $t_0 = t_p + t_d$, where $t_p > 0$ and $-t_d > 0$ by cone definitions (2) and (7). □

With this information at hand, we now derive a set of under- and overestimators, e.g., $\underline{t_p}(\rho) \leq t_p(\rho) \leq \overline{t_p}(\rho)$ for all $l < \rho < u$, that are simple enough to provide us with closed-form bound expressions to strengthen the bracket. The first pair of estimators preserve the exponential growth, by instead replacing its linear-over-quadratic coefficient by a constant.

Corollar 5.3. *Assuming $v_0 \in \mathcal{F}$, an overestimator of t_p is given by*

$$\overline{t_p} = \frac{(\psi_p - 1)r_0 + s_0}{\psi_p^2 - \psi_p + 1}\exp(\rho), \quad \psi_p = \begin{cases} 1/2 & \text{if } r_0 = 0, \\ \frac{r_0 - s_0 + \sqrt{r_0^2 + s_0^2 - r_0 s_0}}{r_0} & \text{otherwise.} \end{cases}$$

Proof. Given $\exp(\rho) > 0$, the constant ψ_p is simply chosen as the unconstrained argmax of $\frac{(\rho-1)r_0+s_0}{\rho^2-\rho+1}$. Its value can be improved by also considering the bracket, $l < \rho < u$, but we forego this optimization for simplicity of proofs. □

Corollar 5.4. *Assuming $v_0 \in \mathcal{F}$, an underestimator of t_d is given by*

$$\underline{t_d} = -\frac{r_0 - \psi_d s_0}{\psi_d^2 - \psi_d + 1}\exp(-\rho), \quad \psi_d = \begin{cases} 1/2 & \text{if } s_0 = 0, \\ \frac{r_0 - \sqrt{r_0^2 + s_0^2 - r_0 s_0}}{s_0} & \text{otherwise.} \end{cases}$$

Proof. Given $-\exp(-\rho) < 0$, the constant ψ_d is simply chosen as the unconstrained argmax of $\frac{r_0 - \rho s_0}{\rho^2-\rho+1}$. Its value can be improved by also considering the bracket, $l < \rho < u$, but we forego this optimization for simplicity of proofs. □

The second pair of estimators preserves the position of the root (that is, $t_p(l) = 0$ if $r_0 > 0$, respective $t_d(u) = 0$ if $s_0 > 0$), by keeping the linear expression and instead replace its exponential-over-quadratic coefficient by a constant.

Corollar 5.5. *Assuming $v_0 \in \mathcal{F}$, any choice of $\alpha \in \mathbb{R}$ leads to an underestimator of t_p on the possibly reduced domain, $l \leq \alpha < \rho < u$, given by*

$$\underline{t_p} = \underline{\omega_p}(\alpha)((\rho - 1)r_0 + s_0), \quad \underline{\omega_p}(\alpha) = \begin{cases} \omega_p(\alpha), & \text{if } \alpha \geq 2 \\ \min(\omega_p(\alpha), \omega_p(2)) & \text{otherwise,} \end{cases}$$

where $\omega_p(\rho) = \frac{\exp(\rho)}{\rho^2-\rho+1} > 0$.

Proof. Given $(\rho - 1)r_0 + s_0 > 0$ by definition of the bracket in Theorem 4.2, we simply need $\underline{\omega_p}(\alpha)$ to be an underestimator of $\omega_p(\rho)$ on $l \leq \alpha < \rho < u$. This follows from asymptotic increase, $\omega_p(\rho) \to \infty$ for $\rho \to \infty$, combined with the fact that $\omega_p(\rho)$ has exactly one local minimizer; the point $\rho = 2$. Numerical evaluation shows the maximum relative error to be $\max_{\rho \in \mathbb{R}} |\frac{\omega_p(\rho) - \underline{\omega_p}(\rho)}{\omega_p(\rho)}| \approx 9\%$, so the underestimator is fairly tight near α in general. □

Corollar 5.6. *Assuming $v_0 \in \mathcal{F}$, any choice of $\alpha \in \mathbb{R}$ leads to an overestimator of t_d on the possibly reduced domain, $l < \rho < \alpha \leq u$, given by*

$$\overline{t_d} = \overline{\omega_d}(\alpha)(r_0 - \rho s_0), \quad \overline{\omega_d}(\alpha) = \begin{cases} \omega_d(\alpha), & \text{if } \alpha \leq -1, \\ \max(\omega_d(\alpha), \omega_d(-1)), & \text{otherwise,} \end{cases}$$

where $\omega_d(\rho) = \frac{-\exp(-\rho)}{\rho^2-\rho+1} < 0$.

Proof. Given $r_0 - \rho s_0 > 0$ by definition of the bracket in Theorem 4.2, we simply need $\overline{\omega_d}(\alpha)$ to be an overestimator of $\omega_d(\rho)$ on $l < \rho < \alpha \leq u$. This follows from asymptotic decrease, $\omega_p(\rho) \to -\infty$ for $\rho \to -\infty$, combined with the fact that $\omega_p(\rho)$ has exactly one local maximizer; the point $\rho = -1$. Numerical evaluation shows the maximum relative error to be $\max_{\rho \in \mathbb{R}} |\frac{\omega_d(\rho) - \overline{\omega_d}(\rho)}{\omega_d(\rho)}| \approx 2\%$, so the overestimator is fairly tight near α in general. □

Finally we are able to show that bounds from Lemma 5.2 are sufficient to provide us with finite bounds on ρ using the estimators above. Note that Corollar 5.3 and Corollar 5.4 are not needed to establish this fact, but we keep them here because their contribution is only limited by the bounds of Lemma 5.2.

Theorem 5.7. *Assuming $v_0 \in \mathcal{F}$, any choice of $\tilde{v}_p \in K_{\exp}$ and $\tilde{v}_d \in K^{\circ}_{\exp}$ (preferably close to v_0) leads to a finite bracket around ρ. In particular,*

(i) $[t_0 > 0] \Leftrightarrow$ *A finite lower bound is found by Corollar 5.3.*

(ii) $[t_0 < 0] \Leftrightarrow$ *A finite upper bound is found by Corollar 5.4.*

(iii) $[r_0 > 0] \Leftrightarrow$ *A finite lower bound is found by Theorem 4.2. A finite upper bound is found by Corollar 5.5.*

(iv) $[s_0 > 0] \Leftrightarrow$ *A finite lower bound is found by Corollar 5.6. A finite upper bound is found by Theorem 4.2.*

Proof. Let $t_p^l \leq t_p \leq t_p^u$ and $t_d^l \leq t_d \leq t_d^u$ denote the finite bounds of Lemma 5.2. The statements are proven in order to show that a finite bracket can indeed be constructed for all points $v_0 \in \mathcal{F}$.

(i) Corollar 5.3 gives $\overline{t_p} = c\exp(\rho)$ for a constant $c > 0$. A finite lower bound can thus be derived, $[\ \overline{t_p} \geq t_p^l\ ] \Leftrightarrow [\ -\infty < \log(c^{-1}t_p^l) \leq \rho\ ]$, under the assumption $t_p^l = [t_0]_+ > 0$.

(ii) Corollar 5.4 gives $\underline{t_d} = -c\exp(-\rho)$ for a constant $c > 0$. A finite upper bound can thus be derived, $[\ \underline{t_d} \leq t_d^u\ ] \Leftrightarrow [\ \rho \leq -\log(-c^{-1}t_d^u) < \infty\ ]$, under the assumption $t_d^u = [t_0]_- < 0$.

(iii) Corollar 5.5 gives $\underline{t_p} = c((\rho-1)r_0 + s_0)$ for a constant $c > 0$. A finite upper bound can thus be derived, $[\ \underline{t_p} \leq t_p^u\ ] \Leftrightarrow [\ \rho \leq 1 + r_0^{-1}(c^{-1}t_p^u - s_0) < \infty\ ]$, under the assumption $r_0 > 0$.

(iv) Corollar 5.6 gives $\overline{t_d} = c(r_0 - \rho s_0)$ for a constant $c < 0$. A finite lower bound can thus be derived, $[\ \overline{t_d} \geq t_d^l\ ] \Leftrightarrow [\ -\infty < s_0^{-1}(r_0 - c^{-1}t_d^l) \leq \rho\ ]$, under the assumption $s_0 > 0$.

□

6 Implementation

A proof-of-concept implementation in the Julia programming language has been made available online at `github.com/HFriberg/projection`. The code works as described in Algorithm 1, and is generic in the floating-point type so users can easily switch from the standard IEEE 64-bit type to any software emulated higher-precision type available in Julia.

Algorithm 1: Projection of v_0 onto the exponential cone

Data: $v_0 \in \mathbb{R}^3$.
Result: Projections $v_p \in K_{\exp}$ and $v_d \in K_{\exp}^{\circ}$.

$v_p \leftarrow \arg\min \|v_0 - \tilde{v}_p\|_2$ for $\tilde{v}_p$ in the primal heuristics of Lemma 5.1.
$v_d \leftarrow \arg\min \|v_0 - \tilde{v}_d\|_2$ for $\tilde{v}_d$ in the polar heuristics of Lemma 5.1.

`/* see if the projection rules of Theorem 3.1 are satisfied. */`
if ***not*** $\left[\|v_0 - v_p\|_2 \approx 0\right.$ ***or*** $\|v_0 - v_d\|_2 \approx 0$ ***or*** $\left.[r_0 \leq 0 \text{ \textbf{and} } s_0 \leq 0]\right]$ **then**

> `/* see if the Moreau system (6) is nearly satisfied to counter numerical difficult examples such as in (13). */`
> **if** ***not*** $[\ \|v_p + v_d - v_0\| \approx 0$ ***and*** $v_p^T v_d \approx 0\ ]$ **then**
>
> > $(\rho^l, \rho^u) \leftarrow$ finite bracket from bounds of Theorem 5.7.
> > `// computed using` $\alpha = l$ `in Corollar 5.5 and` $\alpha = u$ `in Corollar 5.6.`
> >
> > $\rho \leftarrow$ root of the function $h(\rho)$ from Theorem 4.2 on $\rho^l \leq \rho \leq \rho^u$.
> > `// computed using Newton's method with dampened boundary steps, switching to binary search if not converged in 20 iterations.`
> >
> > $v_p \leftarrow \arg\min \|v_0 - \tilde{v}_p\|_2$ for $\tilde{v}_p \in \{v_p,\ v_p(\rho)$ from Lemma 4.1$\}$.
> > $v_d \leftarrow \arg\min \|v_0 - \tilde{v}_d\|_2$ for $\tilde{v}_d \in \{v_d,\ v_d(\rho)$ from Lemma 4.1$\}$.

To test the performance and reliability of the proposed implementation, a benchmark consisting of 85^3 points is assembled to span the expected numerical range of inputs from 1e-9 to 1e9. That is,

$$v_0 = (t_0, s_0, r_0) \in (-I \cup \{0\} \cup I)^3,$$
$$\text{for } I = \{\exp(x) \mid x \in \{-20, -19 \ldots, 20, 21\}\}$$

On an ordinary office laptop from 2019—equipped with an Intel Core i5-8265U CPU @ 1.60GHz—running the Generic Linux distribution of Julia v.1.5.3 in single-threaded mode, it performs around 175'000 projections per second (around 5.7e-6 seconds each) on average over this benchmark set. In terms of quality, the Moreau system (6) was satisfied with maximum relative errors on stationarity and complementarity given by

$$\frac{\|v_p + v_d - v_0\|_2}{\max(1, \|v_0\|_2)} \approx 1.1\text{e-}8, \text{ and } \frac{|v_p^T v_d|}{\max(1, \|v_0\|_2)} \approx 1.5\text{e-}7,$$

which is considered acceptable. In comparison, switching to the `Float128` type from `Quadmath.jl`, which is a software emulated 128-bit floating-point type based on the GCC Quad-Precision Math Library API, this improves to

$$\frac{\|v_p + v_d - v_0\|_2}{\max(1, \|v_0\|_2)} \approx 4.2\text{e-}14, \text{ and } \frac{|v_p^T v_d\|_2}{\max(1, \|v_0\|_2)} \approx 1.2\text{e-}19,$$

at one order of magnitude increase in time. In both cases, the set membership conditions of the Moreau system (6) are satisfied to working precision by design, as can be verified by how points are constructed.

7 Conclusion

The proposed implementation is good enough for most practical purposes in terms of speed and quality, and its precision can be made arbitrarily high by switching to other floating-point types in the Julia language. Yet, there are still many opportunities left to improve the algorithm.

The bracket strengthening techniques of Section 5.2 are chosen for simplicity over performance and can obviously be improved in a number of ways, but the real question is whether we can do so without making the overall projection algorithm slower? In particular, what level of complexity in bound computations strike the right balance for a fast implementation?

Moreover, in preparation of the proposed implementation, the author only tried one solution method for the root-finding problem—a basic implementation of Newton's method with fallback to binary search—and so it is likely that tweaks (or entirely different approaches) exists to improve performance. Hopefully, further work will reveal such optimizations.

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