Copositive Duality for Discrete Energy Markets

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Optimization problems with discrete decisions are nonconvex and thus lack strong duality, which limits the usefulness of tools such as shadow prices. It was shown in Burer (2009) that mixed-binary quadratic programs can be written as completely positive programs, which are convex. We apply this perspective by writing unit commitment in power systems as a completely positive program, and then using the dual copositive program and strong duality to design new pricing mechanisms. We show that the mechanisms are revenue-adequate, and, under certain conditions, support a market equilibrium. To facilitate implementation, we also employ a cutting plane algorithm for solving copositive programs exactly, which we further speed up via a second-order cone programming approximation. We provide numerical examples to illustrate the potential benefits of the pricing mechanisms and algorithms.

Key words: Copositive programming, strong duality, unit commitment, electricity market, pricing

1. Introduction

Markets with discrete decisions such as the startup and shutdown of power plants often lack equilibria. A market is in equilibrium if demand equals supply and if its participants have no incentive to change. The absence of a market equilibrium can lead to inefficient allocation of resources and unbalanced supply and demand. A basic difficulty in achieving equilibrium in markets with discreteness is the lack of convexity, which precludes the use of tools like strong duality and the Karush-Kuhn-Tucker (KKT) conditions. Many discrete problems can be written as mixed-integer linear programs (MILPs). Burer (2009) has shown that mixed-binary quadratic programs (MBQPs), a generalization of MILPs, can be written as completely positive programs (CPPs). CPPs and their dual copositive programs (COPs) are convex but NP-hard. While this does not point to a better way of solving MBQPs, it does provide a new notion of duality. In this paper, we use copositive duality to design a new pricing mechanism for nonconvex electricity markets.

This work is accepted at *Management Science*. The link to the published paper is https://pubsonline.informs.org/doi/10.1287/mnsc.2023.00906.

Unit commitment (UC) minimizes the total production cost of generators in electricity markets under inelastic demand, thereby optimizing social welfare. It makes electricity markets nonconvex because the decision to turn a generator on or off is binary. In a convex market, equilibrium prices can be derived from the shadow prices of the social welfare optimization. However, UC is commonly formulated as an MILP, which generally does not have shadow prices, making it difficult to construct efficient electricity prices; see, e.g., (Liberopoulos and Andrianesis 2016). Here, we rewrite the MILP as a CPP, and use the dual COP to design a novel pricing mechanism based on shadow prices. The mechanism has some desirable properties, e.g., it is budget balanced and, under certain conditions, individually rational. It is also straightforward, as we illustrate, to incorporate additional features into this mechanism, such as revenue adequacy of the generators, by adding constraints directly to the dual COP.

An immediate challenge in using copositive duality is the relatively small number of options for solving COPs exactly. Using a copositivity certificate proposed by Anstreicher (2021), we design a cutting plane algorithm that exactly solves COPs when it terminates. The algorithm consists of a sequence of MILPs, which can thus be implemented using standard industrial solvers.

The remainder of the paper is organized as follows. In Section 2 we review the literature on pricing in nonconvex markets and copositive programming, also detail our contributions. In Section 3 we provide the necessary background on CPPs and COPs. In Section 4 we design a COP-based pricing scheme for UC. In Section 5 we present the cutting plane algorithm for solving COPs. In Section 6 we present the results of our computational experiments. We conclude the paper in Section 7.

All proofs are provided in Section B of the e-companion.

2. Literature Review

In this section, we review the relevant literature. Section 2.1 surveys the pricing schemes for nonconvex markets. Section 2.2 reviews the relationship between CPPs, COPs, and integer programs, and solution methods for COPs. Section 2.3 summarizes our contributions.

2.1. Pricing in Discrete Markets

Classical equilibrium theories assume that market participants make decisions by solving linear or convex optimization models. Under this assumption, the market equilibrium prices are related to the shadow prices of the social welfare optimization (Mas-Colell et al. 1995). These prices lead to exchanges that maximize social welfare while respecting constraints.

However, in practice many markets have nonconvexities, e.g., due to binary decisions and indivisible goods. It is difficult to design efficient pricing mechanisms in such markets because of the lack of strong duality. This has now been a subject of research for decades. We refer the reader to Liberopoulos and Andrianesis (2016) for a more thorough review of pricing in nonconvex markets.

A basic source of nonconvexity in electricity markets is the binary startup and shutdown decisions of generators. These decisions are optimized via UC, which is commonly formulated as an MILP (Carrión and Arroyo 2006). The basic idea of most current approaches is to construct approximate shadow prices for the MILP. O'Neill et al. (2005) eliminate the nonconvexity by fixing the binary decisions at their optimal values and obtaining shadow prices from the resulting linear program (LP). This scheme is called the restricted pricing (RP) in the literature. RP and its variants are used by some independent system operators (ISOs) in the US, such as the Pennsylvania-New Jersey-Maryland Interconnect (PJM). However, RP is often too low to cover the costs of generators, in which uplift or make-whole payments are needed to ensure generator profitability, and it can be volatile due to high sensitivity to the load. The convex hull pricing (CHP) of Hogan and Ring (2003) and Gribik et al. (2007) uses the Lagrangian multipliers of the demand constraints as prices, and has been shown to minimize uplift. Common methods for solving the Lagrangian dual problem for CHP include the subgradient and cutting plane algorithms, both of which require some parameter tuning and can suffer slow convergence. A modified version of CHP called extended locational marginal pricing is used by the Midcontinent ISO. Ruiz et al. (2012) propose a primal-dual approach for pricing, which combines the UC problem and the dual of its linear relaxation, as well as revenue adequacy constraints that ensure nonnegative profit for each generator. We note that such revenue adequacy constraints eliminate uplift, but also modify the original UC problem and may therefore result in suboptimal decisions. Recently, Azizan et al. (2020) proposed a parametric pricing function method for markets with nonconvexity, where desirable conditions such as revenue adequacy and individual rationality are directly imposed as constraints and thus are guaranteed to be satisfied. Since the problem is intractable, approximate solutions are obtained via grid search. Milgrom and Watt (2022) propose two linear pricing mechanisms for markets with nonconvexity, and show that they are budget balanced and approximately incentive compatible.

We use the CPP reformulation of MBQP to design a new pricing scheme, which we refer to as copositive duality pricing (CDP). We construct CDP using the dual COP, which benefits from strong duality and the flexibility of an explicit representation. CDP is budget balanced, supports the optimal UC (i.e., the shadow prices incentivize the optimal startups and shutdowns for each individual generator), and is flexible in that it provides direct access to the dual COP. We make use of this feature by adding a revenue adequacy constraint without inducing nonuniform prices or uplift. We refer to the augmented pricing scheme as revenue-adequate CDP (RCDP).

It is desirable for a pricing scheme to ensure individual rationality, i.e., that individual generators have no incentive to deviate from the optimal UC solution. O'Neill et al. (2005) prove that RP is individually rational. CHP satisfies individual rationality in some special cases, but in general it does not support each individual generator's profit-maximizing solution. For example, Gribik et al. (2007) provide sufficient conditions for when CHP satisfies individual rationality. The primal-dual approach does not guarantee individual rationality either. CDP and RCDP do not in general lead to individual rationality, but we provide simple-to-check sufficient conditions under which they do.

Table 1 compares the properties of several pricing schemes, where "✓" and "×" respectively represent true and false, and "∘" indicates the existence of a sufficient condition in the literature for ensuring the property. Column "Uniform only" shows whether the scheme is able to cover total costs with only uniform prices. "Optimal UC" shows whether the prices incentivize the optimal UC solution. "Individual rationality" shows whether each pricing scheme is individually rational.

Table 1 Features of Pricing Schemes

Scheme	Uniform only	Optimal UC	Individual rationality
RP	×	✓	✓
CHP	×	×	∘
Primal-dual	✓	×	×
CDP	×	✓	∘
RCDP	✓	∘	∘

We also mention that there is a related literature stream focusing on markets with indivisible goods. Recent papers include Danilov et al. (2001) and Baldwin and Klemperer (2019), which use discrete convexity to prove the existence of equilibria.

2.2. Copositive Programming

Copositive programming has been shown to generalize a number of NP-hard problems, such as quadratic optimization (Bomze and De Klerk 2002), two-stage adjustable robust optimization (Xu and Burer 2018, Hanasusanto and Kuhn 2018), and MBQPs (Burer 2009). In particular, Burer (2009) shows that an MBQP can be written as a CPP. This reformulation is the basis of our work.

At present, no industrial software can directly solve COPs. Parrilo (2000) constructs a hierarchy of semidefinite programs (SDPs), which is often used to approximate COPs. For example, it is used by De Klerk and Pasechnik (2002) to find the stability number of a graph, and by Hanasusanto and Kuhn (2018) for two-stage distributionally robust linear programs. An exact algorithm for COPs based on simplicial partitions is proposed by Bundfuss and Dür (2009). Bomze et al. (2008) and Bomze et al. (2010) use cutting planes to strengthen the SDP relaxation for COPs, but both are problem specific and can only be used for quadratic programming and clique number problems, respectively. In this paper, we develop a purely MILP-based cutting plane algorithm that exactly solves COPs when it terminates. To speed up convergence, we strengthen the master problem with a second-order cone programming (SOCP) approximation of the COP.

2.3. Our Contributions

We use COP to construct a notion of duality for MILPs and MBQPs. We show that this has several uses, such as (i) providing direct access to dual decisions that support optimal primal decisions (e.g., prices), and this is particularly useful for modeling constraints involving both primal and dual decisions, and (ii) theoretical analysis of problem structure (e.g., using strong duality).

Here we design a new pricing mechanism for UC in power systems, CDP, based on a CPP reformulation of the UC MILP model. We use CPP-COP strong duality to prove that this pricing scheme is revenue neutral, and to obtain sufficient conditions for individual rationality, both desirable features in practice. Using the Shapley-Folkman Lemma, we relax the sufficient conditions and show that individual rationality will hold for most generators in a sufficiently large system. We also derive a bound on the subsidy required for generators that are not individually rational. We design a modified version of the new pricing mechanism, RCDP, which ensures another important property, revenue adequacy, enforced explicitly in the pricing model using the direct access to dual (pricing) variables. We show that RCDP uses only uniform prices, is uplift-free, and, under certain conditions, supports a market equilibrium. Due to its generality, copositive duality can potentially be used to design pricing mechanisms for and analyze equilibria of other nonconvex markets.

To facilitate implementation, we develop a cutting plane algorithm based on a copositivity certificate proposed by Anstreicher (2021), and the algorithm exactly solves COPs when it terminates. This algorithm is easier to implement than that of Bundfuss and Dür (2009), which is to the best of our knowledge the only algorithm in the literature that exactly solves COPs (also when it terminates). Since our algorithm is purely MILP-based, it can handle COPs with both continuous and discrete variable. We speed up convergence by tightening the master problem with an SOCP restriction of the COP. The solutions produced by the SOCP-based cutting plane algorithm are no worse (and typically better) than those obtained from a commonly-used SDP approximation. We apply the algorithm to moderately sized instances of UC.

While our cutting plane algorithm is a useful advance, the technology for solving COPs is not mature. At the same time, there is much debate over which existing pricing mechanism for UC is superior, as some are more tractable, while others have better properties. We therefore view our pricing mechanism as a new conceptual direction, which we hope will lead to significant improvements once there are more efficient tools for solving COPs.

There are numerous discrete and NP-hard problems that can be represented via COP. To date, there have been relatively limited applications of COP to such problems. This is due to the fact that much of the literature on COP is recent, and because at present there are few options for solving COPs. We believe that, via tools like strong duality and the KKT conditions, there are many promising applications of COP. We view our results on discrete energy markets as a significant step in this direction. We also believe the cutting plane algorithm is a useful advance in making COP a tractable modeling framework.

3. Background

In this section, we define CPP and COP and state some basic results.

Throughout the paper, we use bold letters for vectors. We use $\mathrm{Tr}(\cdot)$ for the trace of a matrix, $(\cdot)^\top$ for the transpose of a vector or matrix, $\mathbf{e}_k \in \mathbb{R}^n$ for the k^{th} unit vector, and $\mathrm{opt}(\cdot)$ for the optimal objective value of an optimization problem.

3.1. Preliminaries

Let $\mathcal{S}_n$ be the set of n-dimensional real symmetric matrices. The copositive cone $\mathcal{C}_n$ is defined as:

$$\mathcal{C}_n = \left\{ X \in \mathcal{S}_n \mid \mathbf{y}^\top X \mathbf{y} \geq 0 \text{ for all } \mathbf{y} \in \mathbb{R}^n_+ \right\}. \tag{1}$$

The dual cone of $\mathcal{C}_n$ is the completely positive cone $\mathcal{C}^*_n$:

$$\mathcal{C}^*_n = \left\{ X X^\top \mid X \in \mathbb{R}^{n \times r}_+ \right\}. \tag{2}$$

In COP (CPP), we optimize a linear function of the matrix X subject to linear constraints and $X \in \mathcal{C}_n$ ($X \in \mathcal{C}^*_n$).

Because COP and CPP are convex, strong duality holds if a regularity condition is satisfied, e.g., Slater's condition, which requires the feasible region to have an interior point. It may also hold under certain problem structures (Gao et al. 2019, Cifuentes et al. 2024).

3.2. CPP Reformulation of MBQP

In this section, we state the CPP reformulation of MBQP given in Burer (2009). We also derive its COP dual.

Consider the MBQP:

$$\begin{aligned}
\mathcal{P}^{\text{MBQP}} : \min \quad & \mathbf{x}^\top Q \mathbf{x} + 2\mathbf{c}^\top \mathbf{x} && && \text{(3a)}\\
\text{s.t.} \quad & \mathbf{a}_j^\top \mathbf{x} = b_j && \forall j = 1, ..., m && \text{(3b)}\\
& x_k \in \{0, 1\} && \forall k \in \mathcal{B} && \text{(3c)}\\
& \mathbf{x} \in \mathbb{R}^n_+ && && \text{(3d)}
\end{aligned}$$

where $\mathcal{B} \subseteq \{1, ..., n\}$ is the set of indices of the binary elements of $\mathbf{x}$. Without loss of generality we assume the matrix Q is symmetric.

Burer (2009) gives a CPP reformulation of $\mathcal{P}^{\text{MBQP}}$, which here we refer to as $\mathcal{P}^{\text{CPP}}$, obtained by squaring the linear constraints and substituting lifted variables for the bilinear terms.

$$\mathcal{P}^{\text{CPP}}: \min \quad \text{Tr}(QX) + 2\mathbf{c}^\top \mathbf{x} \tag{4a}$$

$$\text{s.t.} \quad \mathbf{a}_j^\top \mathbf{x} = b_j \qquad \forall j = 1, ..., m \tag{4b}$$

$$\mathbf{a}_j^\top X \mathbf{a}_j = b_j^2 \qquad \forall j = 1, ..., m. \tag{4c}$$

$$x_k = X_{kk} \qquad \forall k \in \mathcal{B} \tag{4d}$$

$$\begin{bmatrix} 1 & \mathbf{x}^\top \\ \mathbf{x} & X \end{bmatrix} \in \mathcal{C}^*_{n+1}. \tag{4e}$$

We now derive the dual of (4). For convenience, define:

$$Y = \begin{bmatrix} 1 & \mathbf{x}^\top \\ \mathbf{x} & X \end{bmatrix}; \; \tilde{Q} = \begin{bmatrix} 0 & \mathbf{0}^{1\times n} \\ \mathbf{0}^{n\times 1} & Q \end{bmatrix}; \; C = \begin{bmatrix} 0 & \mathbf{c}^\top \\ \mathbf{c} & \mathbf{0}^{n\times n} \end{bmatrix}; \; A_j = \begin{bmatrix} 0 & 1/2\mathbf{a}_j^\top \\ 1/2\mathbf{a}_j & \mathbf{0}^{n\times n} \end{bmatrix}, \; \forall j = 1, ..., m. \tag{5}$$

We also define the matrices $\tilde{A}_j = [0, \mathbf{a}_j^\top]^\top [0, \mathbf{a}_j^\top]$, $\forall j = 1, ..., m$, and, for each $k \in \mathcal{B}$, B_k such that

$$(B_k)_{l_1, l_2} = \begin{cases} 1/2 & \text{if } l_1 = k+1,\; l_2 = 1 \text{ or } l_1 = 1,\; l_2 = k+1 \\ -1 & \text{if } l_1 = k+1,\; l_2 = k+1 \\ 0 & \text{otherwise} \end{cases}.$$

Then, $\mathcal{P}^{\text{CPP}}$ can be written as

$$\min \quad \text{Tr}(\tilde{Q}Y) + \text{Tr}(CY) \tag{6a}$$

$$\text{s.t.} \quad \text{Tr}(A_j Y) = b_j \quad \forall j = 1, ..., m \tag{6b}$$

$$\text{Tr}(\tilde{A}_j Y) = b_j^2 \quad \forall j = 1, ..., m \tag{6c}$$

$$\text{Tr}(B_k Y) = 0 \quad \forall k \in \mathcal{B} \tag{6d}$$

$$Y \in \mathcal{C}^*_{n+1}. \tag{6e}$$

Let $\boldsymbol{\gamma}^o$, $\boldsymbol{\beta}^o$, $\boldsymbol{\delta}^o$, and Ω^o be the respective dual variables of constraints (6b) - (6e). The dual of $\mathcal{P}^{\text{CPP}}$ is the following COP:

$$\mathcal{P}^{\text{COP}}: \max_{\boldsymbol{\gamma}^o, \boldsymbol{\beta}^o, \boldsymbol{\delta}^o, \Omega^o} \quad \sum_{j=1}^{m} \left(\gamma_j^o b_j + \beta_j^o b_j^2 \right) \tag{7a}$$

$$\text{s.t.} \quad \tilde{Q} + C - \sum_{j=1}^{m} \gamma_j^o A_j - \sum_{j=1}^{m} \beta_j^o \tilde{A}_j - \sum_{k\in\mathcal{B}} \delta_k^o B_k - \Omega^o = 0 \tag{7b}$$

$$\Omega^o \in \mathcal{C}_{n+1}. \tag{7c}$$

Burer (2009) shows that $\mathcal{P}^{\text{MBQP}}$ and $\mathcal{P}^{\text{CPP}}$ are equivalent, in the sense that (i) $\text{opt}(\mathcal{P}^{\text{MBQP}}) = \text{opt}(\mathcal{P}^{\text{CPP}})$; and (ii) if $(\mathbf{x}^*, X^*)$ is optimal for $\mathcal{P}^{\text{CPP}}$, then $\mathbf{x}^*$ is in the convex hull of optimal solutions of $\mathcal{P}^{\text{MBQP}}$. The second point indicates that $\mathbf{x}^*$ is not necessarily feasible for $\mathcal{P}^{\text{MBQP}}$, i.e., it is possible for some x_k^* with $k \in \mathcal{B}$ to be fractional.

$\mathcal{P}^{\text{CPP}}$ corresponds to (C) in Burer (2009).

Remark 1. If $\mathbf{x}^*$ is an optimal solution of $\mathcal{P}^{\text{MBQP}}$, then $(\mathbf{x}^*, \mathbf{x}^*\mathbf{x}^{*\top})$ is optimal for $\mathcal{P}^{\text{CPP}}$.

Remark 2. Let $(\mathbf{x}^*, X^*)$ be an optimal solution for $\mathcal{P}^{\text{CPP}}$. If $Q \succeq 0$ and $\mathbf{x}^*$ is feasible for $\mathcal{P}^{\text{MBQP}}$, then $\mathbf{x}^*$ is an optimal solution of $\mathcal{P}^{\text{MBQP}}$. Note that the condition $Q \succeq 0$ ensures that the objective value of $\mathcal{P}^{\text{MBQP}}$ is optimal at $\mathbf{x}^*$.

4. Pricing Unit Commitment

In this section we use copositive duality to design a pricing mechanism for UC. UC optimally schedules the startups and shutdowns of the generators in a power system, typically over a 24 hour horizon. The problem is usually formulated as an MILP, in which the startup and shutdown decisions are binary variables. We reformulate the MILP as a CPP in Section 4.1, and use the dual to define prices in Section 4.2, which we modify in Section 4.4 to guarantee revenue adequacy for individual generators.

Let $\mathcal{G}$ be the set of generators and $\mathcal{T}$ the set of time periods. For $t \in \mathcal{T}$, d_t denotes the load, whereas c_g^p, c_g^u, $p_g^{\min}$ and $p_g^{\max}$ are respectively the production cost, startup cost, lower production limit, and upper production limit for $g \in \mathcal{G}$. For $t \in \mathcal{T}$ and $g \in \mathcal{G}$, the decision variable p_{gt} represents the production level, u_{gt} is the binary decision to startup, and z_{gt} is equal to 1 if the generator is online and 0 if offline. A basic MILP model for UC is as follows:

$$
\begin{aligned}
\min \quad & \sum_{g\in\mathcal{G}} \sum_{t\in\mathcal{T}} \left(c_g^p p_{gt} + c_g^u u_{gt}\right) && && \text{(8a)}\\
\text{s.t.} \quad & \sum_{g\in\mathcal{G}} p_{gt} = d_t && \forall t \in \mathcal{T} && \text{(8b)}\\
& u_{gt} \geq z_{gt} - z_{g,t-1} && \forall g \in \mathcal{G}, t \in \mathcal{T} \setminus \{1\} && \text{(8c)}\\
& p_{gt} \geq p_g^{\min} z_{gt} && \forall g \in \mathcal{G}, t \in \mathcal{T} && \text{(8d)}\\
& p_{gt} \leq p_g^{\max} z_{gt} && \forall g \in \mathcal{G}, t \in \mathcal{T} && \text{(8e)}\\
& p_{gt}, u_{gt} \geq 0 && \forall g \in \mathcal{G}, t \in \mathcal{T} && \text{(8f)}\\
& z_{gt} \in \{0,1\} && \forall g \in \mathcal{G}, t \in \mathcal{T}. && \text{(8g)}
\end{aligned}
$$

The objective (8a) minimizes the total cost of power production and generator startup. Constraints (8b) ensure that the total production satisfies the load in each time period. Constraints (8c) link the startup decisions to the online/offline statuses. Constraints (8d) and (8e) set the lower and upper bounds for power output when the generator is on, and ensure the production level is 0 when the generator is off. We impose binary restrictions only on the z_{gt} variables, as the binary constraints on the u_{gt} variables are then implied by (8c) and the objective.

Problem (8) is a simplification of the UC model solved in practice (Carrión and Arroyo 2006, Taylor 2015), which includes additional constraints such as transmission capacity, minimum up/down time, ramping constraints, and energy storage. We omit these constraints to simplify exposition, but could straightforwardly incorporate them in our pricing mechanisms.

To reformulate (8) as a CPP, we first add upper bound constraints (9) to all binary variables:

$$z_{gt} \le 1 \qquad \forall g \in \mathcal{G}, t \in \mathcal{T}. \tag{9}$$

We then add slack variables to all inequality constraints to make them equality constraints. To simplify notation, we write the reformulated model in a more compact form. Let ψ_{jgt} represent the slack variable corresponding to inequality constraint j for generator g at time t. Let $\mathbf{x}^\top = (\mathbf{u}^\top, \mathbf{z}^\top, \mathbf{p}^\top, \boldsymbol{\psi}^\top)$, with bold font denoting vectors. For example, $\mathbf{u}$ denotes the vector of variables u_{gt} for all $g \in \mathcal{G}, t \in \mathcal{T}$. Denote the vector of coefficients for a constraint as $\mathbf{a}_{jgt}$. The MILP (8) can be rewritten as:

$$\begin{aligned}
\mathcal{UC}: \quad \min \quad & \sum_{g \in \mathcal{G}} \sum_{t \in \mathcal{T}} \left(c_g^p p_{gt} + c_g^u u_{gt} \right) && && \text{(10a)}\\
\text{s.t.} \quad & \sum_{g \in \mathcal{G}} p_{gt} = d_t && \forall t \in \mathcal{T} && \text{(10b)}\\
& \mathbf{a}_{jgt}^\top \mathbf{x} = b_{jgt} && \forall j = 1, ..., m, g \in \mathcal{G}, t \in \mathcal{T} && \text{(10c)}\\
& \mathbf{x} \in \mathbb{R}_+^n && && \text{(10d)}\\
& z_{gt} \in \{0, 1\} && \forall g \in \mathcal{G}, t \in \mathcal{T}. && \text{(10e)}
\end{aligned}$$

Constraints (10c) are the individual generator's operational constraints, which include (8c) - (8e) and (9), after the addition of slack variables. Constraints (10c) can also include other operational constraints such as minimum up/down time and ramping constraints.

Thanks to constraints (9) (which is reformulated to (10c)) and (10d), $\mathcal{UC}$ satisfies the key assumption (1) of Burer (2009), which allows us to reformulate $\mathcal{UC}$ as a CPP.

4.1. CPP Reformulation of UC

We reformulate (10) in the form of $\mathcal{P}^{\text{CPP}}$. Let X be the matrix of lifted variables for $\mathbf{x}$ and Y, as defined in (5). To make the correspondence between elements of X and variables in vector $\mathbf{x}$ more explicit, we denote by $X_{k,q}^{vw}$ the element of X corresponding to the row of the v_k variable and the column of the w_q variable. That is, X^{vw} represents the block of X with rows corresponding to the variables v and columns to the variables w. For example, for the UC above, we have

$$X = \begin{bmatrix} X_{11,11}^{uu} & X_{11,12}^{uu} & \dots & X_{11,rl}^{u\psi} \\ X_{12,11}^{uu} & X_{12,12}^{uu} & & \vdots \\ \vdots & & \ddots & \vdots \\ X_{rl,11}^{\psi u} & .. & \dots & X_{rl,rl}^{\psi\psi} \end{bmatrix}.$$

Let $\mathbf{h}_t \in \mathbb{R}^n$ be the coefficient vector for the left-hand side of constraints (10b), i.e., the left-hand side of (10b) can be written as $\mathbf{h}_t^\top \mathbf{x}$, where $\mathbf{h}_t$ is a binary vector with 1's corresponding to p_{gt}, and 0's elsewhere. The CPP reformulation is as follows:

$$
\begin{aligned}
\mathcal{UC}^{\text{CPP}}: \min \quad & \sum_{g\in\mathcal{G}}\sum_{t\in\mathcal{T}} \left(c_g^p p_{gt} + c_g^u u_{gt}\right) && && \text{(11a)}\\
\text{s.t.} \quad & \sum_{g\in\mathcal{G}} p_{gt} = d_t && \forall t\in\mathcal{T} && (\lambda_t) \quad \text{(11b)}\\
& \mathbf{a}_{jgt}^\top \mathbf{x} = b_{jgt} && \forall j=1,...,m, g\in\mathcal{G}, t\in\mathcal{T} && (\phi_{jgt}) \quad \text{(11c)}\\
& \text{Tr}(\mathbf{h}_t\mathbf{h}_t^\top X) = d_t^2 && \forall t\in\mathcal{T} && (\Lambda_t) \quad \text{(11d)}\\
& \text{Tr}(\mathbf{a}_{jgt}\mathbf{a}_{jgt}^\top X) = b_{jgt}^2 && \forall j=1,...,m, g\in\mathcal{G}, t\in\mathcal{T} && (\Phi_{jgt}) \quad \text{(11e)}\\
& z_{gt} = Z_{gt} && \forall g\in\mathcal{G}, t\in\mathcal{T} && (\delta_{gt}) \quad \text{(11f)}\\
& Y \in \mathcal{C}_{n+1}^* && && (\Omega). \quad \text{(11g)}
\end{aligned}
$$

Dual variables are shown to the right of the constraints.

The dual of (11) is:

$$
\begin{aligned}
\mathcal{UC}^{\text{COP}}: \max \quad & \sum_{t\in\mathcal{T}} \left(d_t\lambda_t + d_t^2\Lambda_t + \sum_{j=1}^{m}\sum_{g\in\mathcal{G}} \left(b_{jgt}\phi_{jgt} + b_{jgt}^2\Phi_{jgt}\right)\right) && \text{(12a)}\\
\text{s.t.} \quad & (\boldsymbol{\lambda}, \boldsymbol{\phi}, \boldsymbol{\Lambda}, \boldsymbol{\Phi}, \boldsymbol{\delta}, \Omega) \in \mathcal{F}^{\text{COP}}, && \text{(12b)}
\end{aligned}
$$

where $\mathcal{F}^{\text{COP}}$ denotes the feasible region of the dual problem, which can be written in the form of constraints (7b) - (7c).

Recently, Cifuentes et al. (2024) showed that strong duality holds for the CPP reformulation (4) if the original MBQP has a convex objective function. Since the objective function (10a) is linear and thus convex, strong duality holds for $\mathcal{UC}^{\text{CPP}}$.

4.2. Copositive Duality Pricing

In this section, we describe CDP, a new pricing mechanism for UC. In an electricity market, the system operator (SO) acts as a coordinator, collecting payments from the electricity consumption (commonly referred to as the "load") and distributing them among generators. A pricing mechanism sets the prices for computing these payments. A generator's profit is calculated by subtracting its costs from the revenue it receives from the SO. A generator is *revenue neutral* (or *budget balanced*) if its revenue equals its costs. It is *revenue adequate* if its revenue meets or exceeds its costs, and *revenue deficient* if its revenue falls short of its costs. Similarly, an SO is budget balanced if the payments that it collects and distributes are of an equal amount. A pricing scheme is *individually rational* if the centralized UC decision also maximizes each generator's individual profit, and therefore each market participant has no incentive to deviate from the centrally optimal solution

(Ndrio et al. 2022). As shown later in this section, CDP is budget balanced for both generators and the SO and, under certain conditions, individually rational.

Let $\mathbf{x}^*$ be an optimal solution of $\mathcal{UC}$, and set $X^* = \mathbf{x}^*\mathbf{x}^{*\top}$. According to Remark 1, $(\mathbf{x}^*, X^*)$ is an optimal solution to $\mathcal{UC}^{\text{CPP}}$ (11). The CDP mechanism is defined as follows:

Definition 1 (CDP Mechanism). Let $(\boldsymbol{\lambda}^*, \boldsymbol{\phi}^*, \boldsymbol{\Lambda}^*, \boldsymbol{\Phi}^*)$ be an optimal solution for $\mathcal{UC}^{\text{COP}}$ and $(\mathbf{x}^*, \mathbf{x}^*\mathbf{x}^{*\top})$ be an optimal solution for $\mathcal{UC}^{\text{CPP}}$, where $\mathbf{x}^*$ is an optimal solution for $\mathcal{UC}$. Under the CDP mechanism, at time t the SO:

- collects $\pi_t^{\text{L}} = \lambda_t^* d_t + \Lambda_t^* d_t^2 + \sum_{j=1}^m \sum_{g\in\mathcal{G}} \left(b_{jgt}\phi_{jgt}^* + b_{jgt}^2 \Phi_{jgt}^*\right)$ from the load, and
- pays $\pi_{gt}^{\text{G}} = \lambda_t^* p_{gt}^* + \Lambda_t^* X_{gt,gt}^{pp*} + \sum_{j=1}^m \left(\phi_{jgt}^* \mathbf{a}_{jgt}\mathbf{x}^* + \Phi_{jgt}^* \text{Tr}(\mathbf{a}_{jgt}\mathbf{a}_{jgt}^\top X^*)\right) + \sum_{g'\in -g} f_{gg't}$ to generator g, where $-g = \mathcal{G} \setminus \{g\}$ and $f_{gg't}$ is the share of g's revenue from the cross-term payment $2\Lambda_t^* X_{gt,g't}^{pp*}$. It must satisfy $f_{gg't} + f_{g'gt} = 2\Lambda_t^* X_{gt,g't}^{pp*}$, and if $X_{gt,g't}^{pp*} = 0$, then $f_{gg't} = 0$.

The term π_t^{L} consists of uniform price payments, $d_t\lambda_t^*$ and $d_t^2\Lambda_t^*$, and payments that depend on the shadow prices of operational constraints with non-zero right-hand sides. Note that the quadratic term $d_t^2\Lambda_t^*$ corresponds to the lifted power balance (11d). The shadow price payments for operational constraints are roughly comparable to transmission congestion rents. They could represent, for example, a payment corresponding to a ramping constraint, which, if loosened, would improve the objective. As we later show in Theorem 1, including non-uniform prices ϕ_{jgt}^* and Φ_{jgt}^* in the pricing mechanism is necessary to balance the budget for both the generators and the SO.

The term π_{gt}^{G} depends on generators' optimal production levels and on/off statuses. It is obtained by summing the products of the left-hand sides of constraints (11b) - (11e) with their corresponding dual prices. As in the load payment, $\lambda_t^* p_{gt}^*$ and $\Lambda_t^* X_{gt,gt}^{pp*}$ are volumetric payments for which the prices are uniform for all generators. The term $\Lambda_t^* X_{gt,gt}^{pp*}$ corresponds to the lifted power balance (11d). We also include non-uniform payments in π_{gt}^{G}. In practice, SOs such as New York ISO and PJM provide non-uniform lump-sum payments to generators, as uniform prices alone may not be enough to recover all costs (O'Neill et al. 2005). For π_{gt}^{G}, the terms in the first sum correspond to shadow prices of various operational constraints. The shadow prices are not uniform as they depend on the generator index, g. The second sum $\sum_{g'\in -g} f_{gg't}$ corresponds to the off-diagonal entries in the lifted matrix X^*. It is useful for ensuring that the SO remains budget-balanced, as shown in Theorem 1. This payment also provides flexibility in the pricing mechanism. Since it involves two generators, it is an open question as to how this payment should be divided between those generators. For example, we can assign it to the generator that loses money, or divide it evenly as in Section 4.4. If it is divided evenly, then $f_{gg't} = \Lambda_t^* p_{gt}^* p_{g't}^*$.

We now further examine non-uniform payments resulting from shadow prices. Constraint (9) is the only operational constraint with a non-zero constant on the right-hand-side. The portion of the payment resulting from constraint (9) and the corresponding lifted constraint are included in

π_t^{G}. We refer to the shadow prices of these constraints the *availability prices*, as they signal the availability of the generators at each hour. These are the only non-uniform prices in this example.

Note that in practice, generators follow the optimal decisions set by $\mathcal{UC}$ (10), as mandated by the SO. Therefore, to minimize market distortion, a UC pricing mechanism should incentivize decisions that are as close as possible to those set by $\mathcal{UC}$ (10).

The following theorem shows that CDP is revenue neutral for both the SO and the generators. Note that in general, revenue of the SO does not always equal to the aggregate revenue of the generators. Differences could arise, e.g., when there is transmission congestion, which is a topic for future work. Also, note that the individual generators are in general not revenue neutral under CDP.

THEOREM 1 **(Revenue neutrality of CDP)**. *CDP balances the aggregate revenue and the aggregate cost of the generators. It also balances the budget of the SO.*

4.3. Individual Rationality

Next, we provide a sufficient condition guaranteeing the individual rationality of CDP. The profit-maximization problem of generator g is given by:

$$\begin{aligned}
\pi_g^{\mathrm{Gen}}(\mathbf{p}_{-g}) := \quad \max \quad & \sum_{t\in\mathcal{T}} \Big(\lambda_t^* p_{gt} + \Lambda_t^* p_{gt}^2 + \sum_{j=1}^{m} \left(\phi_{jgt}^* b_{jgt} + \Phi_{jgt}^* b_{jgt}^2 \right) + \sum_{g'\in -g} f_{gg't} \\
& - c_g^p p_{gt} - c_g^u u_{gt} \Big) && \text{(13a)} \\
\text{s.t.} \quad & \mathbf{a}_{jgt}^\top \mathbf{x} = b_{jgt} \qquad \forall t\in\mathcal{T}, j=1,\ldots,m && \text{(13b)} \\
& \mathbf{x} \in \mathbb{R}_+^n && \text{(13c)} \\
& z_{gt} \in \{0,1\} \qquad \forall t\in\mathcal{T}. && \text{(13d)}
\end{aligned}$$

where $\mathbf{p}_{-g}$ denotes the production decision of all generators other than g. The first five terms in the objective represent the total revenue of generator g. The profit $\pi_g^{\mathrm{Gen}}(\mathbf{p}_{-g})$ should not contain entries of X, which are lifted variables that do not have direct physical meanings. As discussed in Remark 1, at optimality we can equivalently use $(p_{gt}^*)^2$ in place of $X_{gt,gt}^{pp*}$ and $p_{gt}^* p_{g't}^*$ in place of $X_{gt,g't}^{pp*}$.

A market mechanism is individually rational if under its prices, no participant would benefit by deviating from its allocated quantity (Milgrom and Watt 2022). For CDP, the prices and allocated quantity are respectively $(\boldsymbol{\lambda}^*, \boldsymbol{\phi}^*, \boldsymbol{\Lambda}^*, \boldsymbol{\Phi}^*)$ and $(\mathbf{x}^*, \mathbf{x}^*\mathbf{x}^{*\top})$. Since $\mathbf{x}^*$ is optimal to $\mathcal{UC}$, CDP is individually rational if and only if the optimal solution of the profit-maximization problem (13) matches the corresponding portion of the optimal solution of $\mathcal{UC}$ for all generators $g \in \mathcal{G}$ (we do not consider the load as it is inelastic). Theorem 2 provides a sufficient condition for when CDP is individually rational.

Theorem 2 **(Individual rationality of CDP)**. *If the price $\boldsymbol{\Lambda}^* = \mathbf{0}$, then CDP is individually rational.*

A market achieves *competitive equilibrium* if the market clears and individual rationality holds. In our pricing scheme, since production equals demand for all time periods due to constraints (11b), the market always clears. If individual rationality holds, then the equilibrium is competitive.

While it is difficult to guarantee when $\boldsymbol{\Lambda}^* = \mathbf{0}$, we observe in our experiments that its entries are often small and sometimes all zero, such as in the Scarf's example in Section 6.1. It may be possible to derive other sufficient conditions for individual rationality, e.g., by decomposing the conic constraint $X \in \mathcal{C}^*$ using matrix completion (Drew and Johnson 1998). This is a topic of future work.

When $\boldsymbol{\Lambda}^* = \mathbf{0}$, constraints (11d) are redundant and can be removed from $\mathcal{UC}^{\text{CPP}}$. In other words, the relaxation of $\mathcal{UC}^{\text{CPP}}$ with constraints (11d) removed, which we denote as $\mathcal{UC}^{\text{CPP}'}$, is an exact reformulation of $\mathcal{UC}$ when $\boldsymbol{\Lambda}^* = 0$.

Note that constraints (11d) and (11e) are a type of reformulation linearization technique (RLT) constraints (Sherali and Adams 2013). RLT constraints are useful for constructing the conic reformulations of MIQP (Fattahi et al. 2017), and it is an open question when a particular RLT constraint in $\mathcal{P}^{\text{CPP}}$ becomes redundant. Studying the redundancy of specific RLT constraints not only could shed light on individual rationality of our pricing mechanism, but also could speed up the computation of polynomial optimization (Bienstock et al. 2020). This is a topic for future research.

However, even if RLT constraints (11d) are not redundant, we can still ensure individual rationality of most generators in a large-scale system by slightly modifying CDP, by paying generators based on prices and primal optimal solutions from $\mathcal{UC}^{\text{CPP}'}$, as defined below:

$$\mathcal{UC}^{\text{CPP}'} := \min_{(\mathbf{x},X)\in\mathcal{X}'} \sum_{g\in\mathcal{G}} \sum_{t\in\mathcal{T}} \left(c_g^p p_{gt} + c_g^u u_{gt}\right), \tag{14}$$

where $\mathcal{X}' := \{(\mathbf{x}, X) | (11b), (11c), (11e), (11f), \begin{bmatrix} 1 & \mathbf{x}_g^\top \\ \mathbf{x}_g & X_g \end{bmatrix} \in \mathcal{C}^*_{n_g+1}, g \in \mathcal{G}\}$, with n_g the dimension of $\mathbf{x}_g$.

$\mathcal{UC}^{\text{CPP}'}$ is obtained by removing (11d) from $\mathcal{UC}^{\text{CPP}}$ and decomposing the completely-positive constraints by g. In Lemma 2 of the e-companion, we show that $\mathcal{UC}^{\text{CPP}'}$ is as tight as a $\mathcal{UC}^{\text{CPP}}$ without the RLT constraints (11d). In Proposition 1, we further demonstrate the tightness of $\mathcal{UC}^{\text{CPP}}$:

Proposition 1. *Assume $|\mathcal{G}| > |\mathcal{T}|+1$. There exists an optimal solution of $\mathcal{UC}^{CPP'}$, which we denote by $(\bar{\mathbf{x}}, \bar{X})$, where at most $|\mathcal{T}|+1$ generators have non-integral solutions for $\bar{z}_{gt}, t \in \mathcal{T}$.*

The proof is based on the Shapley-Folkman Lemma (Bertsekas 2009), and is Lemma 4 in the e-companion. The Shapley-Folkman Lemma is often used in the literature to derive properties of nonconvex but separable problems (Milgrom and Watt 2022, Vujanic et al. 2014). A separable problem is an optimization problem whose Lagrange dual function can be decomposed into smaller subproblems (Bertsekas 2009).

We denote $(\bar{\boldsymbol{\lambda}}, \bar{\boldsymbol{\phi}}, \mathbf{0}, \bar{\boldsymbol{\Phi}})$ as a set of optimal shadow prices corresponding to the primal optimal solution $(\bar{\mathbf{x}}, \bar{X})$, as defined in Proposition 1. We change CDP's payments to generators by using the prices $(\bar{\boldsymbol{\lambda}}, \bar{\boldsymbol{\phi}}, \mathbf{0}, \bar{\boldsymbol{\Phi}})$ and the primal solution $(\bar{\mathbf{x}}, \bar{X})$. Proposition 2 provides a lower bound for the number of generators that are individually rational after this modification.

Proposition 2. *Assume $|\mathcal{G}| > |\mathcal{T}|+1$. Then a CDP that pays generators using prices $(\bar{\boldsymbol{\lambda}}, \bar{\boldsymbol{\phi}}, \mathbf{0}, \bar{\boldsymbol{\Phi}})$ and the primal optimal solution $(\bar{\mathbf{x}}, \bar{X})$ ensures individual rationality of at least $|\mathcal{G}|-|\mathcal{T}|-1$ generators.*

Proposition 2 implies that when there is a large number of generators in the system, the payments based on $(\bar{\boldsymbol{\lambda}}, \bar{\boldsymbol{\phi}}, \mathbf{0}, \bar{\boldsymbol{\Phi}})$ and $(\bar{\mathbf{x}}, \bar{X})$ ensure individual rationality of most generators. For those generators, their optimal primal solution is integral and is feasible to the profit-maximization problem (13), as shown in the proof of Proposition 2. For the remaining generators, the prices $(\bar{\boldsymbol{\lambda}}, \bar{\boldsymbol{\phi}}, \mathbf{0}, \bar{\boldsymbol{\Phi}})$ support non-integral solutions that are not feasible. Consequently, those generators will not supply enough electricity to clear the market. The SO can either ration the demand, which is undesirable, or subsidize those generators to cover the extra cost of production when it is not efficient. Proposition 3 provides bounds on both the subsidy and the over-production caused by the subsidy.

Proposition 3. *Under a CDP that pays generators using prices $(\bar{\boldsymbol{\lambda}}, \bar{\boldsymbol{\phi}}, \mathbf{0}, \bar{\boldsymbol{\Phi}})$ and the primal optimal solution $(\bar{\mathbf{x}}, \bar{X})$, the total subsidy for covering the cost of generators that are not individually rational is $\mathcal{O}((|\mathcal{T}|+1)|\mathcal{T}|)$, and the over-production at each time period is $\mathcal{O}(|\mathcal{T}|+1)$.*

Proposition 2 and 3 motivate the following modification on CDP's payment to generators: At time t, the SO pays $\pi_{gt}^{\mathrm{G}'} = \bar{\lambda}_t \bar{p}_{gt} + \sum_{j=1}^{m} \left(\bar{\phi}_{jgt} \mathbf{a}_{jgt} \bar{\mathbf{x}} + \bar{\Phi}_{jgt} \mathrm{Tr}(\mathbf{a}_{jgt} \mathbf{a}_{jgt}^{\top} \bar{X}) \right)$ to generator g. If the generator is not individually rational, then the SO also subsidizes the uncovered cost due to non-integral solutions.

Since we do not change the load's payments under CDP, the load still pays the prices from $\mathcal{UC}^{\mathrm{CPP}}$. Because $\mathrm{opt}\left(\mathcal{UC}^{\mathrm{CPP}}\right) \geq \mathrm{opt}\left(\mathcal{UC}^{\mathrm{CPP}'}\right)$, the revenue from the load is enough to cover payments to generators after this modification, if not considering the subsidy. The SO can impose an additional charge on the load to fund the subsidy, which can be very small compared with the total payment in a large-scale system with many generators operating.

In summary, if $\boldsymbol{\Lambda}^* = \mathbf{0}$, then CDP ensure individual rationality of all generators. In general, we can modify CDP's payments to generators to ensure individual rationality of at least $|\mathcal{G}|-|\mathcal{T}|-1$

generators (when $|\mathcal{G}| > |\mathcal{T}|+1$). To avoid rationing, a subsidy is needed to cover the costs of additional production. The bounds on the subsidy and the resulting over-production are not affected by the number of generators $|\mathcal{G}|$. Consequently, in a large-scale system with many generators operating, the ratios of subsidy to total payment and over-production to total production can be very small. Note that this modification could lead to inefficient productions that are different from an optimal central dispatch, yet due to bounded subsidy, the cost of such inefficiency is relatively small in a large-scale system.

4.4. Ensuring Individual Revenue Adequacy

CDP does not guarantee each individual generator's revenue adequacy, i.e., nonnegative profit. This is also the case with some other schemes, including RP and CHP. In CDP, we can enforce revenue adequacy by adding constraints directly to the dual of $\mathcal{UC}^{\text{CPP}}$.

Revenue adequacy can be enforced through the non-uniform prices ϕ^*_{jgt} and Φ^*_{jgt}, which can be different for each generator, and/or through the uniform prices λ_t and Λ_t. We use uniform pricing because it is easier to implement in practice. In Section C of the e-companion, we present a different version that uses availability prices as well.

We enforce revenue adequacy by adding a new constraint to the dual problem, which ensures that the payment to each generator is no less than its costs. The resulting augmented dual problem is given by:

$$\max \quad \sum_{t\in\mathcal{T}} \left(d_t\lambda_t + d_t^2\Lambda_t + \sum_{j=1}^{m}\sum_{g\in\mathcal{G}} (b_{jgt}\phi_{jgt} + b_{jgt}^2\Phi_{jgt}) \right) \tag{15a}$$

$$\text{s.t.} \quad \sum_{t\in\mathcal{T}} \left(p^*_{gt}\lambda_t + p^{*2}_{gt}\Lambda_t + \sum_{g'\in -g} p^*_{gt}p^*_{g't}\Lambda_t \right) \geq \sum_{t\in\mathcal{T}} \left(c^p_g p^*_{gt} + c^u_g u^*_{gt} \right) \quad \forall g\in\mathcal{G} \tag{15b}$$

$$(\boldsymbol{\lambda}, \boldsymbol{\phi}, \boldsymbol{\Lambda}, \boldsymbol{\Phi}, \boldsymbol{\delta}, \Omega) \in \mathcal{F}^{\text{COP}}. \tag{15c}$$

The objective (15a) and the constraint (15c) are the same as in the original dual problem, $\mathcal{UC}^{\text{COP}}$. The left-hand side of constraint (15b) is the total revenue of generator g, assuming that the cross-term payments are divided evenly between generators. The right-hand side of (15b) is the total cost of generator g. Constraint (15b) ensures that each generator is revenue adequate, and thus no generator-dependent uplift payment is needed.

The prices are computed by solving (15). The optimal values of primal variables, p^*_{gt} and u^*_{gt}, could be obtained by solving $\mathcal{UC}$. We refer to this pricing mechanism as Revenue-adequate CDP (RCDP). More formally, the RCDP mechanism is defined as follows.

Definition 2 (RCDP Mechanism). Let $(\boldsymbol{\lambda}^*, \boldsymbol{\Lambda}^*)$ be an optimal solution for (15) and $(\mathbf{x}^*, \mathbf{x}^*\mathbf{x}^{*\top})$ be an optimal solution for $\mathcal{UC}^{\text{CPP}}$, where $\mathbf{x}^*$ is an optimal solution for $\mathcal{UC}$. Under the RCDP mechanism, at hour t the SO:

- collects $\tilde{\pi}^{\mathrm{L}}_t = \lambda^*_t d_t + \Lambda^*_t d_t^2$ from the load, and
- pays $\tilde{\pi}^{\mathrm{G}'}_{gt} = \left(\lambda^*_t p^*_{gt} + \Lambda^*_t X^{pp*}_{gt,gt}\right) + \sum_{g' \in -g} p^*_{gt} p^*_{g't} \Lambda^*_t$ to the generator g.

A benefit of RCDP is that the generators pay only uniform prices, which makes it easier to implement in practice. We say a pricing scheme uses only uniform prices if all generators see the same price, i.e., only $\boldsymbol{\lambda}^*$ and $\boldsymbol{\Lambda}^*$ are used as prices. Since $X^{pp*}_{gt,gt} = p^{*2}_{gt}$, the payment $\tilde{\pi}^{\mathrm{G}'}_{gt} = \left(\lambda^*_t + \Lambda^*_t \sum_{g' \in \mathcal{G}} p^*_{g't}\right) p^*_{gt}$. By conservation of power, $\sum_{g' \in \mathcal{G}} p^*_{g't} = d_t$, the load at time t. Hence, the payment simplifies to $(\lambda^*_t + \Lambda^*_t d_t) p^*_{gt}$.

Note that since we use only the uniform prices and ignore the availability prices, the total payments under RCDP no longer equal the total costs. However, RCDP still balances the utility's revenue and payments. We formalize this result in the following proposition:

Proposition 4 **(Revenue neutrality of RCDP)**. *RCDP balances the budget of the SO.*

The optimal solution to the augmented dual problem (15) does not necessarily correspond to the original primal problem $\mathcal{UC}^{\mathrm{CPP}}$. The following proposition provides a straightforward way to check whether RCDP incentivizes the optimal UC solution.

Proposition 5. *RCDP incentivizes the optimal solution of $\mathcal{UC}$ if the optimal value of (15) equals the optimal value of $\mathcal{UC}^{COP}$.*

Proposition 5 says that RCDP supports the optimal solution when the addition of constraints (15b) do not change the optimal value of $\mathcal{UC}^{\mathrm{COP}}$, i.e., if $\mathcal{UC}^{\mathrm{COP}}$ has multiple solutions, then constraints (15b) eliminate solutions that are not revenue adequate. In our experiments, we found that the optimal objective value of (15) satisfied the condition (within a small numerical tolerance) in most instances.

We are also interested in whether individual rationality holds under RCDP. Combining the sufficient conditions in Theorem 2 and Proposition 5 yields following proposition:

Proposition 6 **(Individual rationality of RCDP)**. *If the price $\boldsymbol{\Lambda}^* = \mathbf{0}$, and if the optimal value of (15) equals the optimal value of $\mathcal{UC}^{COP}$, then the market mechanism with RCDP is individually rational.*

We omit the proof from the e-companion because it is similar to that of Theorem 2.

By modifying (15) with (i) using p^*_{gt} and u^*_{gt} from $\mathcal{UC}^{\mathrm{CPP}'}$, (ii) setting $\boldsymbol{\Lambda} = \mathbf{0}$, and (iii) replacing $\mathcal{F}^{\mathrm{COP}}$ with the dual feasible region of $\mathcal{UC}^{\mathrm{CPP}'}$, we can also extend the results of Propositions 2 and 3 to RCDP if the optimal value of the modified problem equals $\mathrm{opt}(\mathcal{UC}^{\mathrm{COP}'})$.

Because the addition of constraints (15b) restricts the original dual problem (12), it is worth checking if the problem is still feasible. We know that if either the primal or dual is feasible, bounded, and has an interior point, then the other is also feasible (Luenberger and Ye 2015). We therefore focus on the dual of (15). Let $q_g \geq 0$ be the dual multiplier of (15b). The dual of (15) is the following:

$$\begin{aligned}
\min \quad & \sum_{g\in\mathcal{G}}\sum_{t\in\mathcal{T}} \left(c_g^p p_{gt} + c_g^u u_{gt} - (c_g^p p_{gt}^* + c_g^u u_{gt}^*) q_g\right) && && \text{(16a)}\\
\text{s.t.} \quad & \sum_{g\in\mathcal{G}} \left(p_{gt} - p_{gt}^* q_g\right) = d_t && \forall t \in \mathcal{T} && \text{(16b)}\\
& \operatorname{Tr}(\mathbf{h}_t \mathbf{h}_t^\top X) - \sum_{g\in\mathcal{G}} \left(p_{gt}^{*2} + \sum_{g'\in -g} p_{gt}^* p_{g't}^*\right) q_g = d_t^2 && \forall t \in \mathcal{T} && \text{(16c)}\\
& \text{(11c)}, \text{(11e)}, \text{(11f)} && && \text{(16d)}\\
& q_g \geq 0 && g \in \mathcal{G} && \text{(16e)}\\
& Y \in \mathcal{C}_{n+1}^*. && && \text{(16f)}
\end{aligned}$$

Observe that this is similar to $\mathcal{UC}^{\text{CPP}}$, but with additional terms multiplying q_g in the objective and constraints. Note that (16) includes both p_{gt}, a variable, and p_{gt}^*, part of the solution to $\mathcal{UC}^{\text{CPP}}$.

The CPP (16) is feasible because we recover $\mathcal{UC}^{\text{CPP}}$ by setting all q_g to zero. It is bounded due to the equality (16b), and the fact that the production level p_{gt} is usually bounded by generator capacity. If the problem has an interior, then its dual, (15), is feasible.

To summarize, when the conditions of Theorem 2 and Proposition 6 are satisfied, CDP and RCDP support optimal UC solutions and market equilibrium. CDP includes both uniform and non-uniform prices, as well as uplift, while RCDP only needs uniform prices, which are easier to implement in practice. In our experiments, RCDP always leads to higher payments from consumers than CDP, so the choice between CDP and RCDP can be viewed as a tradeoff between efficiency and practical simplicity.

Our pricing schemes satisfy many important properties and can be viewed as a natural extension of uniform, equilibrium-supporting shadow prices from convex markets to discrete markets. In particular, RCDP is to the best of our knowledge the first shadow pricing-based scheme for discrete markets that uses only uniform prices, is uplift-free, and under certain conditions supports a market equilibrium.

4.5. Properties of Pricing Mechanisms

The properties of a pricing mechanism depend on: (i) the choice of the base primal UC formulation, (ii) how duality is used, and (iii) the design of the payments. In what follows, we discuss the impact of these factors by comparing CDP/RCDP with RP and CHP with respect to uplift, profit, and load payments.

A generator's revenue can come from both price-based and uplift payments. The uplift payment to a revenue-deficient generator equals the difference between its price-based revenue and costs.

The load's payment equals the total revenue of generators. Since the load is inelastic, a lower payment is desirable.

(1) Payment and Profit Under CDP and RCDP: We denote the prices under CDP as $(\boldsymbol{\lambda}^{\text{CDP}}, \boldsymbol{\phi}^{\text{CDP}}, \boldsymbol{\Lambda}^{\text{CDP}}, \boldsymbol{\Phi}^{\text{CDP}})$. Before uplift, some generators are already profitable while others lose money. The latter receive uplift and end up with zero profit. Since revenue neutrality holds under CDP, the total pre-uplift profits of the profitable generators equals the losses of the others.

Under CDP, the price-based payments equals $\text{opt}(\mathcal{UC})$ because of revenue neutrality. Therefore,

$$\text{load payment under CDP} = \text{opt}(\mathcal{UC}) + \text{uplift payments under CDP}.$$

On the other hand, under RCDP all generators are revenue-adequate, and thus there is no uplift payment. We denote the prices under RCDP as $(\boldsymbol{\lambda}^{\text{RCDP}}, \boldsymbol{\Lambda}^{\text{RCDP}})$.

If the condition in Proposition 5 holds, then there exists a set of prices $(\boldsymbol{\lambda}^{\text{CDP}}, \boldsymbol{\phi}^{\text{CDP}}, \boldsymbol{\Lambda}^{\text{CDP}}, \boldsymbol{\Phi}^{\text{CDP}})$ under CDP such that $\boldsymbol{\lambda}^{\text{CDP}} = \boldsymbol{\lambda}^{\text{RCDP}}$ and $\boldsymbol{\Lambda}^{\text{CDP}} = \boldsymbol{\Lambda}^{\text{RCDP}}$.

(2) Comparison with RP: RP is obtained by solving the following linear program:

$$\begin{aligned}
\mathcal{UC}^{\text{RP}}: \quad \min \quad & \sum_{g\in\mathcal{G}}\sum_{t\in\mathcal{T}} \left(c_g^p p_{gt} + c_g^u u_{gt}\right) && && \text{(17a)}\\
\text{s.t.} \quad & \sum_{g\in\mathcal{G}} p_{gt} = d_t && \forall t\in\mathcal{T} && (\lambda_t) \quad \text{(17b)}\\
& \mathbf{a}_{jgt}^\top \mathbf{x} = b_{jgt} && \forall j=1,\dots,m, g\in\mathcal{G}, t\in\mathcal{T} && (\phi_{jgt}) \quad \text{(17c)}\\
& z_{gt} = z_{gt}^* && \forall g\in\mathcal{G}, t\in\mathcal{T} && (\iota_{gt}) \quad \text{(17d)}\\
& \mathbf{x}\in\mathbb{R}_+^n, && && \text{(17e)}
\end{aligned}$$

which is the same as $\mathcal{UC}$ except that the binary variables are fixed to their optimal solutions (17d). RP is obtained from the optimal dual variables of constraints (17b) and (17d), which we denote $(\boldsymbol{\lambda}^{\text{RP}}, \boldsymbol{\iota}^{\text{RP}})$. The SO collects $\sum_{t\in\mathcal{T}}\left(\lambda_t^{\text{RP}} d_t + \sum_{g\in\mathcal{G}} \iota_{gt}^{\text{RP}} z_{gt}^*\right)$ from the load, and pays $\sum_{t\in\mathcal{T}}\left(\lambda_t^{\text{RP}} p_{gt}^* + \iota_{gt}^{\text{RP}} z_{gt}^*\right)$ to generator g. It is straightforward to check that RP balances the revenue and payment of the SO.

Due to strong duality of LP, we have

$$\sum_{g\in\mathcal{G}}\sum_{t\in\mathcal{T}} \left(c_g^p p_{gt} + c_g^u u_{gt}\right) = \sum_{t\in\mathcal{T}}\left(\lambda_t^{\text{RP}} d_t + \sum_{g\in\mathcal{G}}\left(\iota_{gt}^{\text{RP}} z_{gt}^* + \sum_{j=1}^m \phi_{jgt}^{\text{RP}} b_{jgt}\right)\right).$$

If all $b_{jgt} = 0$, as for $\mathcal{UC}^{\text{RP}}$ if we ignore the redundant upper bounds on z_{gt}, then RP balances the aggregated revenue and cost of generators. It is mentioned in the literature that under RP every generator's net profit is zero (Azizan et al. 2020). If we add constraints with nonzero b_{jgt}, such as ramping constraints, to (8) and they are binding, then individual revenue neutrality may no longer hold.

When every generator's net profit is zero, there is no uplift payment under RP, and we have

$$\text{load payment under RP} = \text{opt}(\mathcal{UC}).$$

This implies that the load payment under RP is no more than that after uplift under CDP, and under RCDP.

(3) Comparison with CHP: CHP can be obtained via the following Lagrangian dual problem (Wang et al. 2013):

$$\mathcal{UC}^{\text{CHP}}: \quad \max_{\boldsymbol{\lambda}} \quad \min_{\mathbf{x}} \quad \sum_{g\in\mathcal{G}}\sum_{t\in\mathcal{T}}\left(c_g^p p_{gt} + c_g^u u_{gt}\right) + \sum_{t\in\mathcal{T}}\lambda_t(d_t - \sum_{g\in\mathcal{G}} p_{gt}) \tag{18a}$$

$$\text{s.t.} \quad \mathbf{a}_{jgt}^\top \mathbf{x} = b_{jgt} \qquad \forall j = 1,...,m, g\in\mathcal{G}, t\in\mathcal{T} \tag{18b}$$

$$\mathbf{x}\in\mathbb{R}_+^n \tag{18c}$$

$$z_{gt}\in\{0,1\} \qquad \forall g\in\mathcal{G}, t\in\mathcal{T}. \tag{18d}$$

We denote the vector of prices under CHP as $\boldsymbol{\lambda}^{\text{CHP}}$, which corresponds to optimal values of the Lagrangian multipliers $\lambda_t, t\in\mathcal{T}$. Due to nonconvexity, generally strong duality does not hold for the Lagrange dual problem.

CHP does not guarantee revenue adequacy for each generator, and thus may have nonzero uplift payments. Therefore,

$$\text{load payment under CHP} = \text{payments from } \boldsymbol{\lambda}^{\text{CHP}} + \text{uplift payments under CHP}.$$

Because prices under CHP do not correspond to the optimal UC solution, it is difficult to analytically compare the load payment under CHP with that under CDP and RCDP. In our experiments, the load payment under CHP is usually higher than CDP, and slightly lower than RCDP.

We now discuss the pricing implications of tightening the base primal UC formulation. One can tighten a given UC formulation with the addition of valid inequalities. These are typically linear or convex constraints added to an MILP, which shrink the search space of the corresponding linear relaxations and reduce the relaxation gap. Adding valid inequalities can make the prices obtained from linear relaxations more efficient. However, it has been observed that for RP, strengthening linear relaxations of UC with valid inequalities can lead to shadow prices with peculiar properties (O'Neill et al. 2005).

Because $\mathcal{UC}^{\text{CPP}}$ is an exact convex reformulation of UC, adding valid inequalities will not change the optimal objective value, or the fact that CDP is efficient. However, addition of constraints to $\mathcal{UC}^{\text{CPP}}$ could potentially be useful for selecting prices with desirable properties, e.g., fairness.

One can obtain an alternative CPP reformulation by lifting a different MILP formulation, e.g., an MILP with additional valid inequalities. Pricing mechanisms like CDP and RCDP may then be constructed in a similar manner. However, modifying the CPP in this way could complicate the resulting pricing mechanism; for instance, new RLT and conic constraints could add more couplings between generators, further compromising separability, and making it harder to interpret the corresponding dual variables.

We also mention that one could in principal construct a pricing mechanism using the SDP relaxation of UC by Quarm and Madani (2021), which is also conic. In the future, we intend to characterize how a pricing mechanism is affected by adding valid inequalities for linear and conic relaxations of UC.

5. Cutting Plane Algorithm

At present, no industrial solver can handle COPs. In the literature they are often approximately solved via SDPs (see Section A of the e-companion). To solve COPs exactly, we propose a cutting plane algorithm, which returns the optimal solution when it terminates. To the best of our knowledge, this is the first use case and computational assessment of the copositivity certificate by Anstreicher (2021). We also speed up the algorithm by tightening the master problem with an SOCP restriction of the COP.

5.1. The Algorithm

The cutting plane algorithm is applicable for the following general type of COPs with linear constraints over a copositive cone:

$$\max \quad \mathbf{q}^\top \boldsymbol{\lambda} + \mathrm{Tr}(H^\top \Omega) \tag{19a}$$

$$\text{s.t.} \quad \mathbf{d}^\top \boldsymbol{\lambda} + \mathrm{Tr}(D_i^\top \Omega) = g_i \quad \forall i = 1, \ldots, m \tag{19b}$$

$$\boldsymbol{\lambda} \geq \mathbf{0} \tag{19c}$$

$$\Omega \in \mathcal{C}_{n_c} \tag{19d}$$

where $\boldsymbol{\lambda}$ is an n_l-dimensional vector, $\mathcal{L}$ is the index set for integer variables in $\boldsymbol{\lambda}$, $\Omega \in \mathbb{R}^{n_c \times n_c}$. The notation $\mathcal{C}_{n_c}$ denotes an n_c-dimensional copositive cone. Note that the COP problems $\mathcal{P}^{\mathrm{COP}}$ (7) is a special case of the COP (19).

The algorithm starts by removing the conic constraint (19d) to obtain the initial master problem, which is iteratively refined by the addition of cuts. At each iteration, we solve the master problem to obtain an optimal solution, denoted by $(\bar{\boldsymbol{\lambda}}, \bar{\Omega})$. To determine whether $\bar{\Omega}$ is copositive, we employ the MILP problem proposed by Anstreicher (2021), which checks copositivity using a recent characterization of copositive matrices given in Dickinson (2019). Before presenting the MILP problem, we first introduce Lemma 1, which restates Theorem 3.3 in (Dickinson 2019).

Lemma 1. *Let β be the index set for a submatrix of $\bar{\Omega}$, and let $|\beta| = n_\beta$. Let $\bar{\Omega}_{\beta\beta} \in \mathbb{R}^{n_\beta \times n_\beta}$ denote the principal submatrix of $\bar{\Omega}$ that consists of rows and columns with indices in β. The matrix $\bar{\Omega}$ is not copositive if and only if there is a submatrix $\bar{\Omega}_{\beta\beta}$ and a vector $\mathbf{z}_\beta \in \mathbb{R}^{n_\beta}_+$ that satisfies $\bar{\Omega}_{\beta\beta}\mathbf{z}_\beta = -\mathbf{1}$.*

The MILP problem, which serves as the separation problem in our cutting plane algorithm, is given by:

$$\mathcal{SP}(\Omega):\ \max\ w \tag{20a}$$

$$\text{s.t.}\ \Omega\mathbf{z} \leq -w\mathbf{1} + \mathbf{m}^\top(\mathbf{1} - \mathbf{u}) \tag{20b}$$

$$\mathbf{1}^\top\mathbf{u} \geq q \tag{20c}$$

$$w \geq 0 \tag{20d}$$

$$\mathbf{0} \leq \mathbf{z} \leq \mathbf{u} \tag{20e}$$

$$\mathbf{u} \in \{0,1\}^{n_c} \tag{20f}$$

where $q = 1$ (or a larger integer, depending on problem structure), $\mathbf{1}$ is a vector of all ones, and $\mathbf{m} \in \mathbb{R}^{n_c}_{++}$ is a vector of large numbers. The MILP identifies non-copositive matrices by finding a submatrix $\bar{\Omega}_{\beta\beta}$ that satisfies Lemma 1. The binary variable u_i equals 1 if the i^{th} row and column are selected for the submatrix. By Theorem 2 of Anstreicher (2021), $\bar{\Omega}$ is copositive if and only if the optimal objective of $\mathcal{SP}(\bar{\Omega})$ is zero.

At any iteration, if the optimal value of the subproblem is zero, then the master problem solution is feasible and optimal for the COP (19), and we can terminate the algorithm. Otherwise, as $\bar{\Omega}$ is not copositive, we add the following cut to the master problem:

$$\bar{\mathbf{z}}^\top\Omega\bar{\mathbf{z}} \geq 0 \tag{21}$$

where $\bar{\mathbf{z}}$ is an optimal solution of $\mathcal{SP}(\bar{\Omega})$.

Proposition 7. *If the optimal value of $\mathcal{SP}(\bar{\Omega})$ is nonzero, then (21) cuts off $\bar{\Omega}$.*

Therefore, if $\bar{\Omega}$ is not copositive, then the MILP (20) produces a vector $\bar{\mathbf{z}}$ that violates the inequality $\mathbf{z}^\top\bar{\Omega}\mathbf{z} \geq 0$.

Note that the cut (21) does not eliminate any feasible solutions from (19). This is because for any $\bar{\mathbf{z}} \in \mathbb{R}^{n_c}_+$, any copositive matrix Ω satisfies $\bar{\mathbf{z}}^\top\Omega\bar{\mathbf{z}} \geq 0$.

Since $\mathcal{SP}(\bar{\Omega})$ is an MILP, we can strengthen its LP relaxation to improve its computational efficiency. Anstreicher (2021) suggests doing so by setting $m_i = 1 + \sum_{j=1,\ j\neq i}^{n_c}\{\bar{\Omega}_{ij} : \bar{\Omega}_{ij} > 0\}$, $i =$

This result is not stated explicitly in Anstreicher (2021), but is useful to show the validity of the cutting planes.

$1,...,n_c$. This is an upper bound for $w + \bar{\Omega}_i^\top \mathbf{z}$, which ensures that the constraint (20b) is redundant when $u_i = 0$. Note that $\bar{\Omega}_{ii}$ is not in the definition of m_i because $u_i = 0$ implies $z_i = 0$. In addition, Lemma 1 indicates that if $\mathrm{diag}(\bar{\Omega}) \geq 0$, then we only need to check $\bar{\Omega}_{\beta\beta}\mathbf{z}_\beta = -\mathbf{1}$ for $n_\beta \geq 2$. Thus, the separation problem can be strengthened by letting $q = 2$. In our algorithm, $\mathrm{diag}(\bar{\Omega}) \geq 0$ is a valid inequality because $\Omega \in \mathcal{C}_{n_c}$. Thus, when initializing the master problem, we can add the constraint $\mathrm{diag}(\Omega) \geq 0$, and set $q = 2$ in $\mathcal{SP}(\bar{\Omega})$.

In some cases the master problem is unbounded at initialization. There are several ways to deal with this. One is to impose a large upper bound on the elements of Ω. This bound can be gradually relaxed and then removed eventually.

It is worth noting that our algorithm can also solve COPs with integer variables, in which case the master problem becomes an MILP. Since our algorithm uses a (mixed-integer) linear master problem and an MILP separation problem, handling discrete variables is much easier compared with using the SDP approximations, which require solving mixed-integer SDPs.

Exploiting our COP's connection to an MBQP, we propose helpful algorithmic enhancements by taking advantage of the problem structure. More specifically, in this paper, we reformulate the MBQP as a CPP, then dualize the CPP to obtain a COP. Usually the original MBQP can be solved with reasonable efficiency, but it is the solution of the dual COP that we are interested in. We can use solutions of the MBQP problem, and the strong duality property between CPP and COP to help the cutting plane algorithm in the following ways:

- If the optimal objective of the MBQP and the cutting plane algorithm are equal, then the algorithm has terminated at an optimal solution.
- Suppose x^* is optimal for the MBQP. Then we can tighten the master problem by adding the complementary slackness constraint $\mathrm{Tr}(x^* x^{*\top}\Omega) = 0$.

In addition, since $\Omega \in \mathcal{C}$, the constraint $\mathrm{Tr}(x^* x^{*\top}\Omega) \geq 0$ is always valid, and this constraint is helpful in speeding up the computation in our experiments.

We were unfortunately unable to prove that the cutting plane algorithm terminates in finitely many iterations. However, note that the objective of the master problem is non-increasing throughout the algorithm. In addition, note that the other exact algorithm for COPs, the simplicial partition method (Bundfuss and Dür 2009), is also shown to be exact only in the limit. We find in our numerical experiments that the cutting plane algorithm usually converges in a reasonable computational time, and when it does not, the approximate solution obtained from the last iteration is often still useful.

5.2. Tightening the Master Problem

We can further improve performance of the cutting plane algorithm, in regards to obtaining exact or approximate solutions, with a tighter master problem initialization. A natural choice for such a master problem would be the use of the SDP approximation of the COP. Let $\mathcal{S}_n^+$ be the n-dimensional positive semidefinite (PSD) cone and $\mathcal{N}_n$ be the cone of n-dimensional entrywise nonnegative matrices. Then $\mathcal{S}_n^+ + \mathcal{N}_n \subseteq \mathcal{C}_n$ (Parrilo 2000). Therefore, introducing the new decision matrices V and N, we can approximate the constraint $\Omega \in \mathcal{C}_n$ with the restriction:

$$V + N = \Omega \tag{22a}$$

$$N \geq 0 \tag{22b}$$

$$V \in \mathcal{S}_n^+. \tag{22c}$$

Unfortunately, solving an SDP master problem at each iteration is a substantial computational burden. Furthermore, the SDP approximation might be very restrictive.

For better tractability and approximation, we relax the SDP constraints to second-order cone (SOC) constraints (Kim and Kojima 2003). Using the fact that any $V \in \mathcal{S}_n^+$ has nonnegative diagonal entries and principal minors, we replace (22c) with the following constraints:

$$V_{ii} \geq 0 \qquad \forall i = 1, \ldots, n \tag{23a}$$

$$V_{ii}V_{jj} \geq V_{ij}^2 \quad \forall i \neq j;\ i,j = 1, \ldots, n, \tag{23b}$$

where (23b) are SOC constraints that require all two-by-two principal minors to be nonnegative. Accordingly, we obtain the following strengthened initial master problem:

$$\max \quad \mathbf{q}^\top \boldsymbol{\lambda} + \mathrm{Tr}(H^\top \Omega) \tag{24a}$$

$$\text{s.t.} \quad \mathbf{d}^\top \boldsymbol{\lambda} + \mathrm{Tr}(D_i^\top \Omega) = g_i \quad \forall i = 1, ..., m \tag{24b}$$

$$\boldsymbol{\lambda} \geq \mathbf{0} \tag{24c}$$

$$V + N = \Omega \tag{24d}$$

$$V_{ii} \geq 0 \qquad \forall i = 1, \ldots, n \tag{24e}$$

$$V_{ii}V_{jj} \geq V_{ij}^2 \qquad \forall i \neq j;\ i,j = 1, \ldots, n \tag{24f}$$

$$N \geq 0. \tag{24g}$$

Enforcing $\Omega \in \mathcal{C}_{n_c}$ with the addition of cutting planes, the algorithm will solve (24) with the additional constraint $\Omega \in \mathcal{C}_{n_c}$, given that this restricted COP is feasible. That is to say, the algorithm will either converge to a feasible (not necessarily optimal) solution of the original problem, the COP (19), or prove infeasibility of the restricted COP due to the inclusion of the constraints (24d)-(24g). Note that in the former case the solution will be no worse than that of the commonly used SDP approximation of the COP. Moreover, in order to obtain a feasible solution in the latter case, and if desired to further improve the obtained feasible solutions (even to converge to an optimal COP solution), we can continue running the cutting plane algorithm by gradually removing constraints (24d)-(24g) from the master problem. In our computational experiments, we found that the algorithm with the SOC-restricted initial master problem converged significantly faster than its exact version and to generally good approximations of prices for CDP and RCDP.

6. Numerical Results

In this section we present our numerical experiments. Section 6.1 implements several pricing schemes on Scarf's example. Section 6.2 compares different pricing schemes for a nonconvex electricity market. Section 6.3 showcases the performance of our cutting plane algorithm and its enhancements on various UC instances. We also include a comparison of our COP cutting plane algorithm with other COP algorithms on a commonly used benchmark problem in e-companion Section D.

All experiments are implemented in Julia using the optimization package JuMP.jl (Dunning et al. 2017). The COP cutting plane algorithm was implemented using CPLEX (CPLEX, IBM ILOG 2022), and the SOCP master problems in the strengthened cutting plane algorithm are solved using Gurobi (Gurobi Optimization, LLC 2023). We use Mosek (MOSEK ApS 2022) to solve SDP approximations.

6.1. Pricing in Scarf's Example

Scarf's example is often used to compare pricing schemes for nonconvex markets. We use the modified version from Hogan and Ring (2003) to compare CDP with RP and CHP, which are currently used by utilities in the U.S.. In the modified Scarf's example, there are three types of generators: smokestack, high technology, and medium technology. Let $\mathcal{G}_i$ be the set of generators of type $i=1,2,3$. We have $|\mathcal{G}_1|=6, |\mathcal{G}_2|=5, |\mathcal{G}_3|=5$. The binary variables $u_{g_i}, g_i \in \mathcal{G}_i, i=1,2,3$, represent startup decisions, and the continuous variables $p_{g_i}, g_i \in \mathcal{G}_i, , i=1,2,3$, represent production decisions. Scarf's example solves the following cost minimization problem:

$$\min \quad \sum_{g_1\in\mathcal{G}_1}(53u_{g_1}+3p_{g_1})+\sum_{g_2\in\mathcal{G}_2}(30u_{g_2}+2p_{g_2})+\sum_{g_3\in\mathcal{G}_3}7p_{g_3} \tag{25a}$$

$$\text{s.t.} \quad \sum_{g_1\in\mathcal{G}_1}p_{g_1}+\sum_{g_2\in\mathcal{G}_2}p_{g_2}+\sum_{g_3\in\mathcal{G}_3}p_{g_3}=d \tag{25b}$$

$$p_{g_3}\geq 2u_{g_3} \qquad \forall g_3\in\mathcal{G}_3 \tag{25c}$$

$$p_{g_1}\leq 16u_{g_1} \qquad \forall g_1\in\mathcal{G}_1 \tag{25d}$$

$$p_{g_2}\leq 7u_{g_2} \qquad \forall g_2\in\mathcal{G}_2 \tag{25e}$$

$$p_{g_3}\leq 6u_{g_3} \qquad \forall g_3\in\mathcal{G}_3 \tag{25f}$$

$$p_g\geq 0, u_g\in\{0,1\} \qquad \forall g\in\mathcal{G}_1,\mathcal{G}_2,\mathcal{G}_3 \tag{25g}$$

where the objective is to minimize the total cost. Constraint (25b) ensures the total production equals the demand. Constraints (25c) set the lower bound for the production of medium technology generators when they are on. Constraints (25d)-(25f) set the capacity of each generator.

We experiment with various demand levels from 5 to 160, with a step length of 5. In Figure 1 we compare the following aspects of RP, CHP, CDP and RCDP:

1. The uniform prices. Notice that for CDP and RCDP, both λ_t^* and Λ_t^* are uniform prices. In all of our experiments Λ_t^* equals zero. Therefore, we report only λ_t^* for those schemes.
2. Sum of generator profits, as calculated by deducting costs from total revenue, which includes both price-based payments and uplift.
3. The payments from non-uniform prices ϕ_{jgt}^* and Φ_{jgt}^*.
4. The make-whole uplift payments. This payment is made when the revenue from electricity prices is not enough to cover the costs. It is equal to the difference between the revenue and costs.

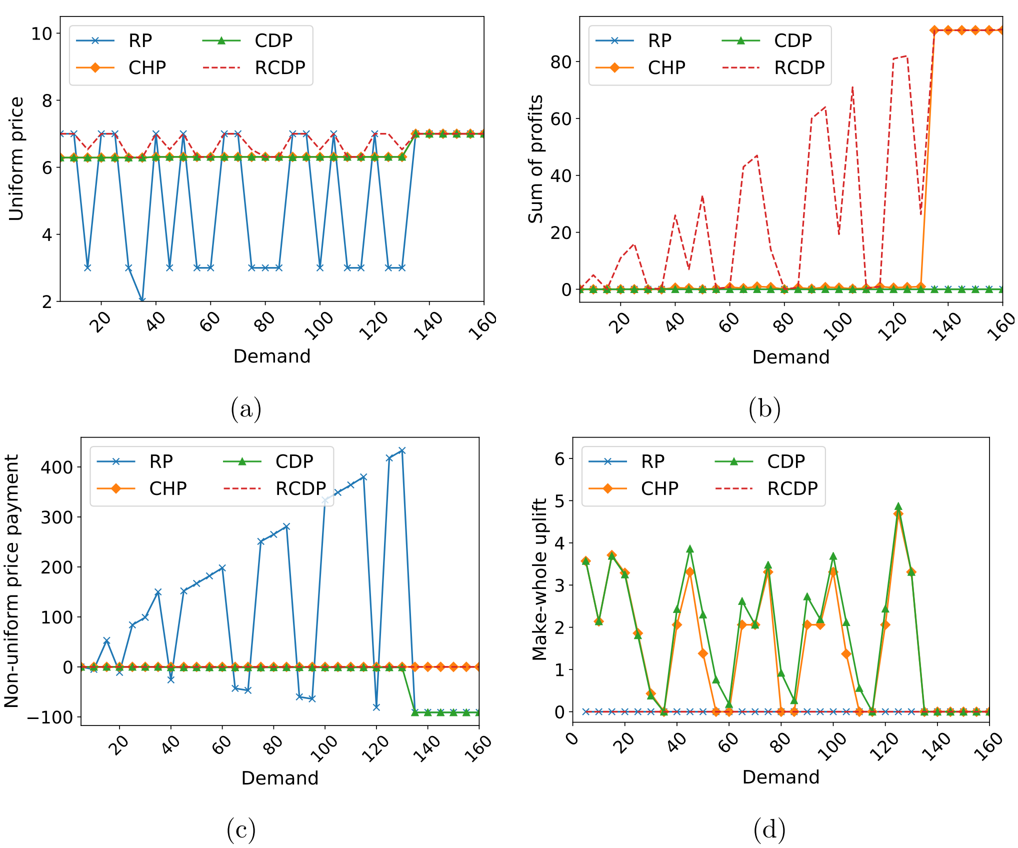


Figure 1 **Comparison of different pricing schemes for (a) uniform prices, (b) sum of profits, (c) payments from non-uniform prices, (d) make-whole uplift payments.**

Mosek solves all instances in under ten seconds. Because the cutting plane algorithm is significantly slower, we only use it when the demand level is less than 100. When the demand level is higher, we present results from the SDP approximation.

Figure 1a shows that a small change in demand level can result in significant volatility in RP. This observation is consistent with results in the literature. Interestingly, CHP and CDP are equivalent for all demand levels. RCDP is higher than CDP for lower demand, and equals COP when the demand is high.

In Figure 1b, we find that RP and CDP have zero profit for all instances. CHP generates near-zero profits at lower demand level and higher profits at higher demand levels. RCDP generates the highest profits among all pricing schemes, and match the profits of CHP at higher demand levels.

In Figure 1c, both CHP and RCDP have no non-uniform prices due to the fact that both only use uniform prices corresponding to the demand constraint. CDP produces near-zero negative non-uniform prices at low demand levels, and more negative prices at higher demand levels. RP produces volatile and large non-uniform prices in many instances. As explained by O'Neill et al. (2005), the negative non-uniform prices are used to discourage the entry of marginal plants when it is uneconomic to do so. In practice, utilities usually disregard such negative prices.

Figure 1d shows that RP requires zero make-whole payment, which is also consistent with the results in Azizan et al. (2020). RCDP ensures revenue adequacy and thus also needs no make-whole payment. CHP requires make-whole payments, as expected, because Lagrangian duals of MILPs do not in general have strong duality. Interestingly, CDP also requires make-whole payments in many instances, which could be because bounds imposed on entries of Ω are restrictive.

The advantages of RCDP are that it does not rely on non-uniform/make-whole uplift payments to ensure revenue adequacy, and its uniform prices are less volatile than RP.

6.2. Pricing in Electricity Markets

In this section we compare pricing schemes for unit commitment, as described in Section 4. Note that in the experiments of Sections 6.2 and 6.3 we use the formulation in (8) for the $\mathcal{UC}$ problem. Our test instances are based on the adapted California ISO dataset of Guo et al. (2022). The parameters for the generators are listed in Table 2, the first three of which are coal and the rest natural gas.

In Table 3 we list the instances used in Sections 6.2 and 6.3. "Coals", "NGs", and "Hours" are respectively the number of coal generators, natural gas generators and hours. To ensure the instances are feasible with fewer generators in the system, we scale down the load by dividing it with the value in "Scale". "Bound" is the bound on the absolute values of the entries in Ω. A larger bound improves accuracy while slowing down convergence.

Table 2 Generator Parameters

	Gen. 1	Gen. 2	Gen. 3	Gen. 4	Gen. 5	Gen. 6
c_g^p	25.0	25.5	26.0	44.7	44.7	44.7
c_g^u	140.9	140.9	140.9	86.3	86.3	86.3
$p_g^{\min}$	297	238	198	198	198	198
$p_g^{\max}$	620	496	413	620	620	620

Table 3 Instances for Electricity Markets

Case	Coals	NGs	Hours	Scale	Bound
1	2	0	4	26	5000
2	2	0	4	16	5000
3	1	1	4	26	5000
4	1	3	4	12	5000
5	3	1	4	12	5000
6	1	1	8	26	5000
7	1	3	8	12	3000
8	1	1	12	26	5000
9	1	3	12	12	3000
10	1	1	24	26	3000

We first consider two generators over four hours with Scale equals 26, which we refer to as Case 1. We set $d_t = [508, 644, 742, 776]$ and use Generators 1 and 2 in Table 2. We use the cutting plane algorithm of Section 5.1 to solve the CDP and RCDP COPs. Note that in this case, Mosek fails to solve the SDP approximation of the RCDP problem (returns "UNKNOWN_RESULT_STATUS"), while the cutting plane algorithm converges in 244.47 seconds. When using the cutting plane algorithm, we set the bound for each element of the copositive matrix to 5000, and add constraint $\text{Tr}(\mathbf{x}^*\mathbf{x}^{*\top}\Omega) \geq 0$.

We compare the following aspects of RP, CHP, CDP and RCDP and present the results in Table 4:

1. Generator profits.
2. The payments from non-uniform prices (absolute value).
3. The make-whole uplift payments.
4. The percentage increase in total payment compared with RP.

RP results in zero profit for both generators, but relies on the revenue from non-uniform prices for Generator 2. CHP uses a make-whole uplift payment to avoid a loss for Generator 2. Interestingly, CDP results in positive profit for Generator 1, and uses a make-whole uplift equal to Generator 1's profit to cover the loss of Generator 2. As expected, CDP is revenue neutral in aggregate, but not for individual generators. RCDP is the only pricing scheme that has no non-uniform or make-whole uplift payments. It also ensures the generators with lower costs receive higher profits, which is

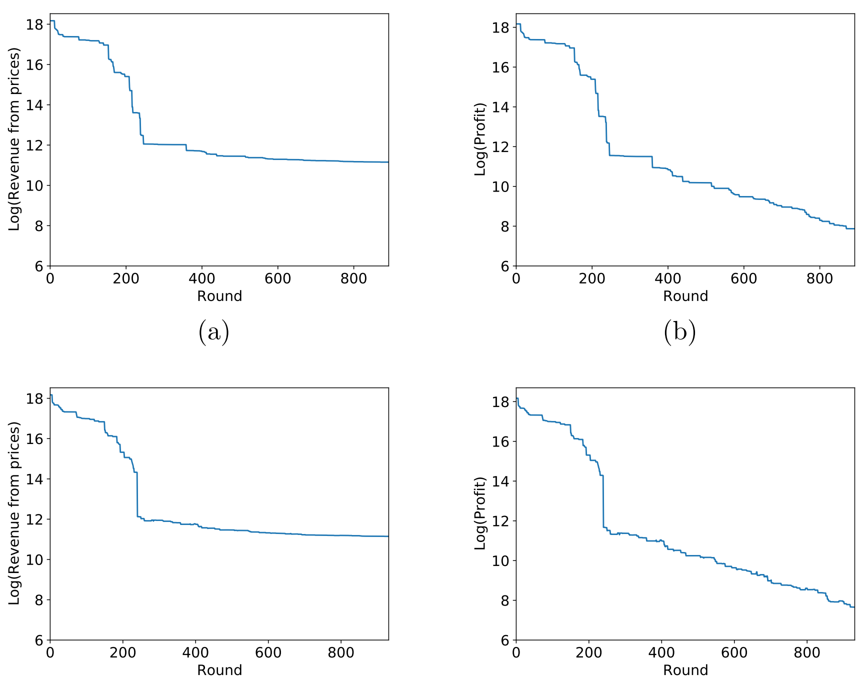
Table 4 Comparison of Pricing Schemes for Case 1

	RP	CHP	CDP	RCDP
Gen. 1 profit	0	839.4	259.6	1183.8
Gen. 2 profit	0	0	0	0
Total profit	0	839.4	259.6	1183.8
Non-uniform payment	497.9	0	0	0
Make-whole uplift	0	95.3	259.6	0
vs RP payment (%)	0	1.2	0.4	1.8

desirable and to be expected in a market with only uniform prices. The marginal generator has zero profit, which is similar to the outcome of shadow pricing-based schemes in convex markets.

Consumers pay less under RP, while schemes that mainly rely on uniform prices, such as CHP and RCDP, have higher payments. This is because RP has non-uniform prices and thus can lower the payment via price discrimination.

In Figure 2 we show the revenue and profit in each iteration of the cutting plane algorithm (without the constraint $\mathrm{Tr}(\mathbf{x}^*\mathbf{x}^{*\top}\Omega) \geq 0$) for UC Case 1. We observe that for both CDP and RCDP, the revenue and profit decrease as more cuts are added.



(a) (b) (c) (d)

Figure 2 Trends of the cutting plane algorithm for UC Case 1 in (a) revenue of CDP, (b) profit of CDP, (c) revenue of RCDP, (d) profit of RCDP.

For the second example we use Case 8, which has two generators over 12 hours. In Table 5, we again compare the profit, generator dependent payment, make-whole uplift and total payment for each pricing scheme. In this case, Mosek is unable to solve the SDP approximation for the RCDP COP, and the SOCP-based cutting plane algorithm solves the problems to 16.13% (CDP) and 0.91% (RCDP) optimality gaps after 2 hours (where optimality gap is calculated based on the optimal $\mathcal{UC}$ solution, as explicitly defined in the next section).

Table 5 Comparison of Pricing Schemes for Case 8

	RP	CHP	CDP	RCDP
Gen. 1 profit	0	43146.1	27453.0	47588.4
Gen. 2 profit	0	0	2825.7	0
Total profit	0	43146.1	30278.7	47588.4
Non-uniform payment	23977.8	0	42345.7	0
Make-whole uplift	0	56.9	0	0
vs RP payment (%)	0	23.0	16.1	25.3

When the CDP or RCDP problem is not solved to optimality, the SO is still budget balanced, and generators remain revenue adequate in aggregate. However, in this case the prices may not support individual rationality even if $\mathbf{\Lambda}^* = \mathbf{0}$ at optimality. Without individual rationality, and consequently, a competitive equilibrium, the generators may not have the incentive to follow the social-welfare maximizing decisions enforced by the SO.

Although the prices from CDP and RCDP are not optimal, we observe that similar to Case 1, the total revenue from uniform and availability prices, as well as the profit, have decreasing trends with more cuts added, as shown in Figure 3. This indicates that adding more cutting planes brings the prices closer to satisfying the theoretical properties of CDP and RCDP.

The results for Case 8 are similar to Case 1. RCDP is the only pricing scheme that does not result in non-uniform or uplift payments. The consumer payments are higher under CHP and RCDP. Note that in practice, the payment under RP is usually higher than our computed value because negative non-uniform prices are ignored (O'Neill et al. 2005).

Results from the other instances in our experiments are similar to the two examples above. Note that a large optimality gap upon termination (see Table 7 for optimality gaps) can lead to higher total payment, which highlights the importance of efficient COP algorithms.

6.3. Computational Performance

In this section we compare the computational performance of our cutting plane algorithms for the instances in Table 3. All experiments in this section are completed on Linux workstations with Intel Xeon CPUs, 28 cores and 125 GB memory.

Table 6 and Table 7 show the computational performance for instances in Table 3. The optimality gap ("Gap") is the normalized difference between the COP dual and the original MILP:

$$\frac{\mathrm{opt}(\mathcal{UC}^{\mathrm{COP}}) - \mathrm{opt}(\mathcal{UC})}{\mathrm{opt}(\mathcal{UC})}.$$

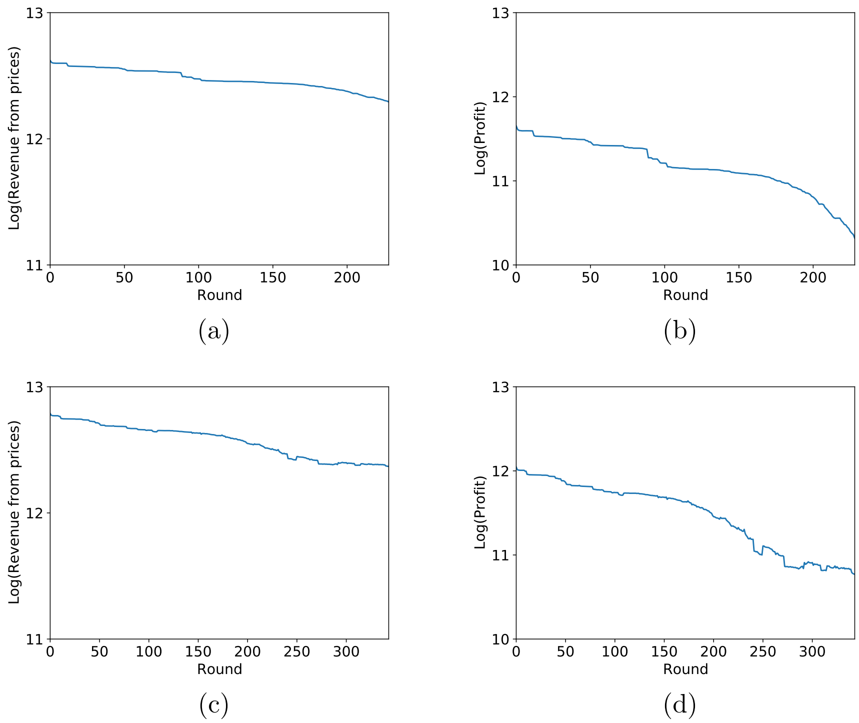


Figure 3 Trends of the cutting plane algorithm for UC Case 8 in (a) revenue of CDP, (b) profit of CDP, (c) revenue of RCDP, (d) profit of RCDP.

In the tables "LP" and "SOC" correspond to results from the cutting plane algorithm with LP and SOCP master problems, respectively. "#Iter" reports the total number of iterations. All instances are run with a time limit of 2 hours. Smaller instances such as Cases 1-3 converge within the time limit, so we report their computation time ("Time") in Table 6, while instances in Table 7 did not terminate automatically within the time limit. Note that we use the constraint $\mathrm{Tr}(\mathbf{x}^*\mathbf{x}^{*\top}\Omega) \geq 0$ for Cases 1-3, and leave it out in other instances so as to compare optimality gaps more accurately when the algorithm does not converge.

If the algorithm terminates automatically with no optimality gap, then strong duality holds and we have the optimal solution. If the algorithm terminates automatically with a negative optimality gap, then it could be because (i) the imposed bounds on entries of Ω are restrictive; (ii) SOC constraints (23b) are restrictive; or (iii) individual revenue adequacy constraints (15b) are restrictive. Due to possible negative optimality gaps, when the algorithm does not terminate automatically, it can be difficult to tell from the optimality gap how close the solution is to optimality. Nonetheless, our results show that when the optimality gap is low, CDP and RCDP lead to lower total payment. Regardless of the optimality gap, RCDP remains the only scheme that does not need non-uniform or uplift payments.

Table 6 Time (seconds), Optimality Gap (%) and Number of Iterations of Cases 1-3

	CDP LP			CDP SOC			RCDP LP			RCDP SOC		
Case	Time	Gap	#Iter	Time	Gap	#Iter	Time	Gap	#Iter	Time	Gap	#Iter
1	187.4	0	1051	80.2	-0.34	121	244.5	0	1303	50.4	0	55
2	195.3	0	1038	89.1	0	62	249.5	0	1339	76.2	0	55
3	536.7	-4.97	1635	68.5	-7.28	131	319.5	-4.18	1310	78.0	-7.34	146

Table 7 Optimality Gap (%) and Number of Iterations of Cases 4-10

	CDP LP		CDP SOC		RCDP LP		RCDP SOC	
Case	Gap	#Iter	Gap	#Iter	Gap	#Iter	Gap	#Iter
4	14.41	3279	3.32	212	15.43	3212	3.77	207
5	14.80	3750	5.85	230	15.03	2985	4.15	243
6	9.08	2979	-1.25	310	4.95	3247	-0.93	394
7	88.78	1171	18.71	99	110.13	909	17.63	109
8	49.04	3895	16.13	228	27.06	5496	0.91	344
9	106.28	1549	21.35	102	130.37	1734	21.52	132
10	80.94	1724	17.96	224	132.40	1826	18.12	250

The SOCP-based cutting plane algorithm outperforms its counterpart in all instances, leading to either less computation time or smaller optimality gaps (if the time limit is met). The SOCP-based algorithm also requires fewer iterations, which indicates the tightness of the SOCP master problem. In each case, the computational performance for CDP and RCDP are mostly similar when solved with the same algorithm. The only exception is Case 9, where RCDP is solved to much smaller optimality gaps than CDP using both algorithms. This could be because the extra individual revenue adequacy constraints under RCDP restrict the feasible region and tighten the master problems.

The computational performance of the algorithms is more likely to be affected by the number of generators than the length of the time horizon. For example, compared with Case 6, Case 4 has twice as many generators and half the number of hours, and Case 6 converges faster than Case 4. The algorithms behave similarly on instances with the same time horizon and number of generators.

In Table 8 we compare solution times for the master problem ("Master") and subproblem ("Sub") for Cases 4-10. In all instances more time is spent on the MILP subproblems than the LP/SOCP master problems. The SOCP master problem usually takes less time in total than its LP counterpart. It also takes much fewer iterations to converge. On the other hand, the subproblems in the SOCP-based cutting plane algorithm take much longer to solve compared with the original algorithm. This shows that with better master problem solutions, the subproblem becomes more difficult to solve.

Table 8 Master Problem and Subproblem Solving Time (hours) of Cases 4-10

	CDP LP		CDP SOC		RCDP LP		RCDP SOC	
Case	Master	Sub	Master	Sub	Master	Sub	Master	Sub
4	0.08	1.92	0.03	1.97	0.07	1.93	0.02	1.98
5	0.09	1.91	0.02	1.98	0.07	1.93	0.03	1.97
6	0.07	1.93	0.02	1.98	0.07	1.93	0.05	1.95
7	0.07	1.93	0.02	1.98	0.05	1.95	0.02	1.98
8	0.22	1.78	0.02	1.98	0.36	1.64	0.05	1.95
9	0.20	1.80	0.14	1.86	0.22	1.78	0.27	1.73
10	0.25	1.77	0.06	1.94	0.24	1.76	0.08	1.92

7. Conclusion

MBQPs can be equivalently written as CPPs, which are NP-hard but convex. Given an MBQP, we straightforwardly derive its dual COP. Due to convexity, if a constraint qualification is satisfied, the CPP and COP have strong duality. This provides a new and general notion of duality for discrete optimization problems.

We use this perspective to design a new pricing mechanism for nonconvex electricity markets, which has several useful theoretical properties. One direction of future study is the design of economic mechanisms for other nonconvex markets, e.g., surge pricing in transportation. To enable implementation, we design a new cutting plane algorithm for COPs, which we use to solve moderate-size nonconvex electricity markets instances as a proof of concept.

There are several promising avenues of future work. It would be interesting to explore the properties of CDP and RCDP after incorporating network constraints, as those constraints are straightforward to accommodate in the CPP model and do not affect strong duality, yet they will change the structure of the pricing mechanism, e.g., by making prices differ across locations. In addition, our COP-based pricing schemes make it possible to model market equilibrium in discrete markets as KKT conditions, which can then be used in applications such as modeling strategic behaviors in energy markets. Finally, it may be possible to improve the cutting plane algorithm by deriving conditions under which the algorithm is guaranteed to terminate. Our work shows that COP-based pricing schemes are of potentially significant value to electricity and other discrete markets, and will become more practical in the future as solution algorithms improve.

Acknowledgments

The authors thank the Department Editor, Associate Editor and three anonymous referees for valuable comments. They gratefully acknowledge Clemson University for the allotment of compute time on the Palmetto Cluster.

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Appendix A: SDP Approximation for COP

Let $\mathcal{S}_n^+$ be the n-dimensional positive semidefinite (PSD) cone and $\mathcal{N}_n$ be the cone of n-dimensional entrywise nonnegative matrices, then we have $\mathcal{C}_n^* \subseteq \mathcal{S}_n^+ \cap \mathcal{N}_n$ and $\mathcal{S}_n^+ + \mathcal{N}_n \subseteq \mathcal{C}_n$ (Dür 2010). Taking advantage of these relationships, we can relax and approximately solve CPP as an optimization problem over the intersection of $\mathcal{S}_n^+$ and $\mathcal{N}_n$ cones, and restrict the COP problem as an optimization over the Minkowski sum of $\mathcal{S}_n^+$ and $\mathcal{N}_n$ cones.

In our work, we reformulate MBQP models to CPPs using the method by Burer (2009), to utilize the dual COP problems. We summarize the relationship between all those mathematical programming problems in Figure 4.

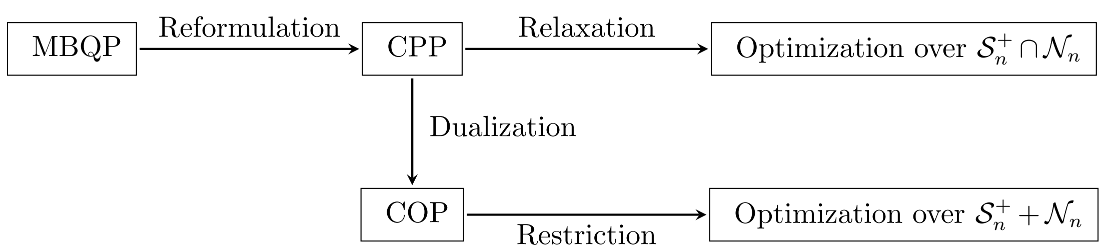

Figure 4 Relationships between MBQP, CPP, COP and their approximations

We need to solve COPs in this work. One method often used in the literature, as mentioned in Section 5.2, is to use the relationship $\mathcal{S}_n^+ + \mathcal{N}_n \subseteq \mathcal{C}_n$ for approximation. More specifically, we replace the conic constraint $\Omega \in \mathcal{C}_n$ with the following restriction:

$$V + N = \Omega$$
$$N \geq 0$$
$$V \in \mathcal{S}_n^+,$$

which can be solved with SDP solvers such as Mosek and SeDuMi.

Another method to obtain a solution of a COP is to solve its dual CPP problem using a commercial solver, then query the duals of CPP constraints via the solver. However, there is not any solver that directly solves CPPs, so we instead solve an SDP relaxation of the CPP problem, then query the duals of the SDP relaxation. More specifically, we relax the conic constraint $X \in \mathcal{C}_n^*$ to the following constraints:

$$X \in \mathcal{S}_n^+$$
$$X \geq 0,$$

which can then be solved with SDP solvers.

Appendix B: Proofs

Theorem 1. *CDP balances the aggregate revenue and the aggregate cost of the generators. It also balances the budget of the SO.*

Proof of Theorem 1. Fix all primal and dual variables at the optimal values. Multiplying constraints (11b) - (11e) by their corresponding dual variables yields

$$\sum_{g\in\mathcal{G}} \lambda_t^* p_{gt}^* = \lambda_t^* d_t \qquad \forall t\in\mathcal{T} \tag{26b}$$
$$\phi_{jgt}^* \mathbf{a}_{jgt}^\top \mathbf{x}^* = \phi_{jgt}^* b_{jgt} \qquad \forall j=1,...,m, g\in\mathcal{G}, t\in\mathcal{T} \tag{26c}$$
$$\sum_{g\in\mathcal{G}} \Lambda_t^* X_{gt,gt}^{pp*} + 2\sum_{g_1<g_2, g_1,g_2\in\mathcal{G}} \Lambda_t^* X_{g_1t,g_2t}^{pp*} = \Lambda_t^* d_t^2 \quad \forall t\in\mathcal{T} \tag{26d}$$
$$\Phi_{jgt}^* \mathrm{Tr}(\mathbf{a}_{jgt}\mathbf{a}_{jgt}^\top X^*) = \Phi_{jgt}^* b_{jgt}^2 \qquad \forall j=1,...,m, g\in\mathcal{G}, t\in\mathcal{T}. \tag{26e}$$

Summing all the constraints in (26), we obtain

$$\begin{aligned}&\sum_{t\in\mathcal{T}}\left(\sum_{g\in\mathcal{G}}\left(\lambda_t^* p_{gt}^* + \Lambda_t^* X_{gt,gt}^{pp*} + \sum_{j=1}^{m}\left(\phi_{jgt}^*\mathbf{a}_{jgt}\mathbf{x}^* + \Phi_{jgt}^*\mathrm{Tr}(\mathbf{a}_{jgt}\mathbf{a}_{jgt}^\top X^*)\right)\right)\right.\\&\left.+2\sum_{g_1<g_2,g_1,g_2\in\mathcal{G}}\Lambda_t^* X_{g_1t,g_2t}^{pp*}\right)\\&=\sum_{t\in\mathcal{T}}\left(d_t\lambda_t^* + d_t^2\Lambda_t^* + \sum_{j=1}^{m}\sum_{g\in\mathcal{G}}\left(b_{jgt}\phi_{jgt}^* + b_{jgt}^2\Phi_{jgt}^*\right)\right)\end{aligned} \tag{27}$$

where the left-hand side and the right-hand side are respectively equal to $\Pi^{\mathrm{G}} := \sum_{g\in\mathcal{G}}\sum_{t\in\mathcal{T}}\pi_{gt}^{\mathrm{G}}$ and $\Pi^{\mathrm{L}} := \sum_{t\in\mathcal{T}}\pi_t^{\mathrm{L}}$ (using the definitions of π_{gt}^{G} and π_t^{L} in Definition 1). This shows that the total amount collected, Π^{G}, is equal to the total amount paid, Π^{L}, thus CDP is budget balanced for the SO.

Since strong duality holds for $\mathcal{UC}^{\mathrm{CPP}}$, and considering the optimal objective of $\mathcal{UC}^{\mathrm{COP}}$ equals π^{L}, we have:

$$\pi^{\mathrm{G}} = \sum_{g\in\mathcal{G}}\sum_{t\in\mathcal{T}}\left(c_g^p p_{gt}^* + c_g^u u_{gt}^*\right).$$

Therefore, for the generators the total payment from the SO equals the total cost. □

Theorem 2. *If the price $\Lambda^* = 0$, then CDP is individually rational.*

Proof of Theorem 2. In $\mathcal{UC}^{\mathrm{CPP}}$ we dualize demand constraints (11b) and lifted demand constraints (11d) with their respective optimal dual prices λ_t^* and Λ_t^*. We obtain the following partial Lagrangian relaxation problem (Faye and Roupin 2007, Bomze 2015):

$$\begin{aligned}\min &\sum_{g\in\mathcal{G}}\sum_{t\in\mathcal{T}}\left(c_g^p p_{gt} + c_g^u u_{gt}\right) + \sum_{t\in\mathcal{T}}\lambda_t^*\left(d_t - \sum_{g\in\mathcal{G}} p_{gt}\right)\\&+\sum_{t\in\mathcal{T}}\Lambda_t^*\left(d_t^2 - \sum_{g\in\mathcal{G}}\left(X_{gt,gt}^{pp} + \sum_{g'\in -g} X_{gt,g't}^{pp}\right)\right)\end{aligned} \tag{28a}$$
$$\text{s.t. } (11c), (11e) - (11g). \tag{28b}$$

Since we have strong duality (Cifuentes et al. 2024), and because the Lagrangian multipliers are fixed at their optimal values, an optimal solution $(\mathbf{x}^*, X^*)$ for CPP (11) is also optimal for its Lagrangian relaxation (28).

Because $\Lambda^*=0$, the objective function is equivalent to

$$\sum_{g\in\mathcal{G}}\sum_{t\in\mathcal{T}}\left(c_g^p p_{gt}+c_g^u u_{gt}\right)+\sum_{t\in\mathcal{T}}\lambda_t^*\left(d_t-\sum_{g\in\mathcal{G}}p_{gt}\right), \tag{29}$$

which transforms the CPP (28) to the form of $\mathcal{P}^{\text{CPP}}$, with (29), (11c), (11e), (11f) and (11g) respectively corresponding to (4a), (4b), (4c), (4d) and (4e). Due to the equivalence between $\mathcal{P}^{\text{MBQP}}$ and $\mathcal{P}^{\text{CPP}}$ (The key assumption (1) of Burer (2009) is satisfied by (11c)), (28) is equivalent to the following MILP:

$$\begin{aligned}
\min\quad & \sum_{g\in\mathcal{G}}\sum_{t\in\mathcal{T}}\left(c_g^p p_{gt}+c_g^u u_{gt}\right)+\sum_{t\in\mathcal{T}}\lambda_t^*\left(d_t-\sum_{g\in\mathcal{G}}p_{gt}\right) && \text{(30a)}\\
\text{s.t.}\quad & \text{(11c)} && \text{(30b)}\\
& \mathbf{x}\in\mathbb{R}_+^n && \text{(30c)}\\
& z_{gt}\in\{0,1\} \qquad \forall g\in\mathcal{G}, t\in\mathcal{T}. && \text{(30d)}
\end{aligned}$$

Ignoring the constant term $\lambda_t^* d_t$ in the objective, problem (30) can be separated into individual optimization problems, one per generator. Solving problem (30) is equivalent to solving the following maximization problem for all $g\in\mathcal{G}$:

$$\begin{aligned}
\max\quad & \sum_{t\in\mathcal{T}}\left(\lambda_t^* p_{gt}-c_g^p p_{gt}-c_g^u u_{gt}\right) && \text{(31a)}\\
\text{s.t.}\quad & \mathbf{a}_{jgt}^\top\mathbf{x}=b_{jgt} \qquad \forall t\in\mathcal{T}, j=1,\cdots,m && \text{(31b)}\\
& \mathbf{x}\in\mathbb{R}_+^n && \text{(31c)}\\
& z_{gt}\in\{0,1\} \qquad \forall t\in\mathcal{T}. && \text{(31d)}
\end{aligned}$$

When $\Lambda^*=0$, (31) is the same as the profit-maximization problem (13) (except for the constant terms $\phi_{jgt}^* b_{jgt}$ and $\Phi_{jgt}^* b_{jgt}^2$). Therefore, if a solution $(\mathbf{x}^*, X^*)$ is optimal for CPP (11), then $\mathbf{x}^*$ also solves (13) optimally (If CPP (11) has multiple solutions, we consider a solution with integral $\mathbf{x}^*$). In other words, under CDP, the generators' profit-maximizing actions align with the centrally optimal dispatch. □

We now define some useful notation. Let $\mathbf{x}_g$ be a subvector of $\mathbf{x}$ corresponding to generator g, X_g be a submatrix of X corresponding to generator g. We define a feasible region $\mathcal{X}_g$ for $(\mathbf{x}_g, X_g)$, which includes constraints (11c), (11e), (11f) for generator g, and the conic constraint $\begin{bmatrix}1 & \mathbf{x}_g^\top\\ \mathbf{x}_g & X_g\end{bmatrix}\in\mathcal{C}_{n_g+1}^*$. In addition, let $\hat{\mathcal{X}}_g=\{(\mathbf{x}_g, X_g)|\text{(13b)}-\text{(13d)}; X_g=\mathbf{x}_g\mathbf{x}_g^\top\}$.

Lemma 2. *Consider the following CPP obtained by removing (11d) from $\mathcal{UC}^{CPP}$:*

$$\begin{aligned}
\min\quad & \sum_{g\in\mathcal{G}}\sum_{t\in\mathcal{T}}\left(c_g^p p_{gt}+c_g^u u_{gt}\right) && \text{(32a)}\\
\text{s.t.}\quad & \text{(11b)} && \text{(32b)}\\
& \text{(11c)}, \text{(11e)}-\text{(11g)}. && \text{(32c)}
\end{aligned}$$

The optimal objective value of (32) is equivalent to opt($\mathcal{UC}^{CPP'}$).

Proof. Since $\mathcal{UC}^{\text{CPP}'}$ replaces $Y \in \mathcal{C}^*_{n+1}$ by imposing completely positive constraints on principal submatrices of Y, it is a relaxation of (32). Thus, opt($\mathcal{UC}^{\text{CPP}'}$) is no more than the optimal objective value of (32). We now prove the reverse direction.

We relax constraint (32b) in (32) with its optimal shadow price $\boldsymbol{\lambda}^{**}$ and obtain the following partial Lagrangian relaxation problem:

$$\min \sum_{g\in\mathcal{G}}\sum_{t\in\mathcal{T}} \left(c^p_g p_{gt} + c^u_g u_{gt}\right) + \sum_{t\in\mathcal{T}} \lambda^{**}_t \left(d_t - \sum_{g\in\mathcal{G}} p_{gt}\right) \tag{33a}$$

$$\text{s.t. } (11c), (11e) - (11g). \tag{33b}$$

Because (32) has a linear objective function, strong duality holds (Cifuentes et al. 2024), and thus (32) has the same optimal objective as (33). Due to the equivalence between $\mathcal{P}^{\text{MBQP}}$ and $\mathcal{P}^{\text{CPP}}$, (33) is equivalent to the following MILP:

$$\min \quad \sum_{g\in\mathcal{G}}\sum_{t\in\mathcal{T}} \left(c^p_g p_{gt} + c^u_g u_{gt}\right) + \sum_{t\in\mathcal{T}} \lambda^{**}_t \left(d_t - \sum_{g\in\mathcal{G}} p_{gt}\right) \tag{34a}$$

$$\text{s.t.} \quad (11c) \tag{34b}$$

$$\mathbf{x} \in \mathbb{R}^n_+ \tag{34c}$$

$$z_{gt} \in \{0,1\} \qquad \forall g \in \mathcal{G}, t \in \mathcal{T}. \tag{34d}$$

Ignoring the constant terms $\lambda^{**}_t d_t, t \in \mathcal{T}$, (34) can be decomposed by generators. For each generator's problem, we can reformulate it to a CPP due to the equivalence between $\mathcal{P}^{\text{MBQP}}$ and $\mathcal{P}^{\text{CPP}}$. We then aggregate those CPPs and add back the constant terms $\lambda^{**}_t d_t, t \in \mathcal{T}$ to obtain the following CPP:

$$\min_{(\mathbf{x}_g, X_g)\in\mathcal{X}_g, g\in\mathcal{G}} \quad \sum_{g\in\mathcal{G}}\sum_{t\in\mathcal{T}} \left(c^p_g p_{gt} + c^u_g u_{gt}\right) + \sum_{t\in\mathcal{T}} \lambda^{**}_t \left(d_t - \sum_{g\in\mathcal{G}} p_{gt}\right). \tag{35}$$

Problem (35) is a relaxation of $\mathcal{UC}^{\text{CPP}'}$, and it has the same optimal objective value as (32). Thus, the optimal objective value of (32) is no more than opt($\mathcal{UC}^{\text{CPP}'}$). $\square$

We now introduce Lemma 3, which is used in the proof of Proposition 1.

Lemma 3. $\text{conv}(\hat{\mathcal{X}}_g) = \mathcal{X}_g$.

Proof. As a reminder, $\mathcal{X}_g = \left\{(\mathbf{x}_g, X_g) \middle| (11c), (11e), (11f) \text{ for generator } g; \begin{bmatrix} 1 & \mathbf{x}_g^\top \\ \mathbf{x}_g & X_g \end{bmatrix} \in \mathcal{C}^*_{n_g+1}\right\}$ and $\hat{\mathcal{X}}_g = \{(\mathbf{x}_g, X_g) | (13b) - (13d); X_g = \mathbf{x}_g \mathbf{x}_g^\top\}$.

Burer (2009)'s Corollary 2.5 indicates that if the set $\mathcal{P} = \{\mathbf{x}_g | (13b) - (13d)\}$ is bounded, then

$$\text{conv}\left\{\begin{bmatrix} 1 & \mathbf{x}_g^\top \\ \mathbf{x}_g & \mathbf{x}_g\mathbf{x}_g^\top \end{bmatrix} \middle| \mathbf{x}_g \in \mathcal{P}\right\} = \left\{\begin{bmatrix} 1 & \mathbf{x}_g^\top \\ \mathbf{x}_g & X_g \end{bmatrix} \middle| (\mathbf{x}_g, X_g) \in \mathcal{X}_g\right\},$$

which is equivalent to $\text{conv}(\hat{\mathcal{X}}_g) = \mathcal{X}_g$.

To ensure the boundedness of $\mathcal{P}$, note that the feasible region of $\mathcal{UC}$ is usually bounded. If not, it can be modified to become bounded. For example, in the UC model (8), both p_{gt} and z_{gt} are bounded, while the boundedness of u_{gt} is implied by (8c) and the objective. Thus, we can impose $u_{gt} \in [0,1]$ to make $\mathcal{P}$ bounded, without changing the solution(s) of the problem. $\square$

For the convenience of the readers, we include the Shapley-Folkman Lemma as Lemma 4. For its proof, see Proposition 5.7.1 of the textbook by Bertsekas (2009).

LEMMA 4 **(Shapley-Folkman Lemma)**. *Let $\mathcal{S}_g, g \in \mathcal{G}$, be nonempty subsets of $\mathbb{R}^{|\mathcal{T}|+1}$, with $|\mathcal{G}| > |\mathcal{T}|+1$, and let $\mathcal{S} = \mathcal{S}_1 + \cdots + \mathcal{S}_{|\mathcal{G}|}$. Then every vector $\boldsymbol{\sigma} \in \text{conv}(\mathcal{S})$ can be represented as $\boldsymbol{\sigma} = \sum_{g\in\mathcal{G}} \boldsymbol{\sigma}_g$, where $\boldsymbol{\sigma}_g \in \text{conv}(\mathcal{S}_g)$ for all $g \in \mathcal{G}$, and $\boldsymbol{\sigma}_g \notin \mathcal{S}_g$ for at most $|\mathcal{T}|+1$ indices g.*

PROPOSITION 1. *Assume $|\mathcal{G}| > |\mathcal{T}|+1$. There exists an optimal solution of $\mathcal{UC}^{CPP'}$, which we denote as $(\bar{\mathbf{x}}, \bar{X})$, where at most $|\mathcal{T}|+1$ generators have non-integral solutions for $\bar{z}_{gt}, t \in \mathcal{T}$.*

Proof. We use the Shapley-Folkman Lemma to show that there exists an optimal solution of $\mathcal{UC}^{\text{CPP}'}$, such that the number of generators with non-integral solutions is bounded by 1 plus the number of constraints (11b).

Similar to the method used in the proof of Theorem 1 in (Vujanic et al. 2014), we first construct a set $\mathcal{S}$ of constraint-cost tuples:

$$\mathcal{S} = \left\{ (s_1, ..., s_{|\mathcal{T}|+1})^\top \middle| s_j = \sum_{g\in\mathcal{G}} p_{gj}, j = 1, ..., |\mathcal{T}|; s_{|\mathcal{T}|+1} = \sum_{g\in\mathcal{G}} \sum_{t\in\mathcal{T}} \left(c_g^p p_{gt} + c_g^u u_{gt}\right), \forall (\mathbf{x}_g, X_g) \in \hat{\mathcal{X}}_g, g \in \mathcal{G} \right\}, \tag{36}$$

where $s_j, \forall j = 1, ..., |\mathcal{T}|$ are the constraint functions of constraints (11b), which are the only constraints in $\mathcal{UC}^{\text{CPP}'}$ that are not separable by generators. The term $s_{|\mathcal{T}|+1}$ is the cost function of $\mathcal{UC}^{\text{CPP}'}$. Note that when defining $\mathcal{S}$ we use the feasible region $\hat{\mathcal{X}}_g$ instead of its convex hull $\mathcal{X}_g$.

The set $\mathcal{S}$ equals the Minkowski sum $\mathcal{S}_1 + \cdots + \mathcal{S}_{|\mathcal{G}|}$, where $\forall g \in \mathcal{G}$:

$$\mathcal{S}_g = \left\{ (s_1^g, ..., s_{|\mathcal{T}|+1}^g)^\top \middle| s_j^g = p_{gj}, j = 1, ..., |\mathcal{T}|; s_{|\mathcal{T}|+1}^g = \sum_{t\in\mathcal{T}} \left(c_g^p p_{gt} + c_g^u u_{gt}\right), \forall (\mathbf{x}_g, X_g) \in \hat{\mathcal{X}}_g \right\}. \tag{37}$$

We can obtain the set $\text{conv}(\mathcal{S}_g)$ by replacing $\hat{\mathcal{X}}_g$ in (37) with $\mathcal{X}_g$. To prove this, we first define a linear transformation $h : \hat{\mathcal{X}}_g \to \mathbb{R}^{|\mathcal{T}|+1}$, which maps an element of $\hat{\mathcal{X}}_g$ to $\boldsymbol{\sigma}_g \in \mathcal{S}_g$, i.e., $\boldsymbol{\sigma}_g = h(\mathbf{x}_g, X_g)$ (we omit an explicit definition of $h(\cdot,\cdot)$ as it is straightforward by the definition of $\mathcal{S}_g$). In addition, the notation $h(\mathcal{A})$ means applying the linear transformation h to all elements of the set $\mathcal{A}$. Since convex combinations commute with linear transformations, we have $\text{conv}(\mathcal{S}_g) = \text{conv}(h(\hat{\mathcal{X}}_g)) = h(\text{conv}(\hat{\mathcal{X}}_g)) = h(\mathcal{X}_g)$ (the last equality is due to Lemma 3). Since $\text{conv}(\mathcal{S}) = \text{conv}(\mathcal{S}_1) + ... + \text{conv}(\mathcal{S}_{|\mathcal{G}|})$, this result also indicates that $\text{conv}(\mathcal{S})$ can be obtained by replacing $\hat{\mathcal{X}}_g$ in (36) with $\mathcal{X}_g$. That is to say,

$$\text{conv}(\mathcal{S}) = \left\{ (s_1, ..., s_{|\mathcal{T}|+1})^\top \middle| s_j = \sum_{g\in\mathcal{G}} p_{gj}, j = 1, ..., |\mathcal{T}|; s_{|\mathcal{T}|+1} = \sum_{g\in\mathcal{G}} \sum_{t\in\mathcal{T}} \left(c_g^p p_{gt} + c_g^u u_{gt}\right), \forall (\mathbf{x}_g, X_g) \in \mathcal{X}_g, g \in \mathcal{G} \right\}. \tag{38}$$

By (38), we can find $\bar{\boldsymbol{\sigma}} = (\bar{s}_1, ..., \bar{s}_{|\mathcal{T}|+1})^\top \in \text{conv}(\mathcal{S})$ that corresponds to an optimal solution $(\bar{\mathbf{x}}, \bar{X})$ of $\mathcal{UC}^{\text{CPP}'}$. Since the solution is feasible to $\mathcal{UC}^{\text{CPP}'}$, we have $\bar{s}_j = \sum_{g\in\mathcal{G}} \bar{p}_{gj} = d_j, \forall j = 1, ..., |\mathcal{T}|$; Since it is optimal, we also have $\bar{s}_{|\mathcal{T}|+1} = \sum_{g\in\mathcal{G}} \sum_{t\in\mathcal{T}} \left(c_g^p \bar{p}_{gt} + c_g^u \bar{u}_{gt}\right) = \text{opt}(\mathcal{UC}^{\text{CPP}'})$.

Due to Shapley-Folkman Lemma, $\bar{\boldsymbol{\sigma}}$ can be written as $\bar{\boldsymbol{\sigma}} = \sum_{g\in\mathcal{G}} \bar{\boldsymbol{\sigma}}_g$, where $\bar{\boldsymbol{\sigma}}_g \in \text{conv}(\mathcal{S}_g) \setminus \mathcal{S}_g$ for at most $|\mathcal{T}|+1$ indices $g \in \mathcal{G}$, while the rest of $\bar{\boldsymbol{\sigma}}_g$'s are in $\mathcal{S}_g$. This result indicates that there exists an optimal solution $(\bar{\mathbf{x}}, \bar{X})$, which ensures that $d_j = \sum_{g\in\tilde{\mathcal{G}}} \bar{s}^g_j + \sum_{g\in\tilde{\mathcal{G}}^c} \bar{s}^g_j, \forall j = 1, ..., |\mathcal{T}|$ and $\text{opt}(\mathcal{UC}^{\text{CPP}'}) = \sum_{g\in\tilde{\mathcal{G}}} \bar{s}^g_{|\mathcal{T}|+1} + \sum_{g\in\tilde{\mathcal{G}}^c} \bar{s}^g_{|\mathcal{T}|+1}$, where $\tilde{\mathcal{G}}$ is the index set for $\bar{\boldsymbol{\sigma}}_g \in \text{conv}(\mathcal{S}_g) \setminus \mathcal{S}_g$ and $\tilde{\mathcal{G}}^c = \mathcal{G} \setminus \tilde{\mathcal{G}}$. Since $|\tilde{\mathcal{G}}| \leq |\mathcal{T}|+1$, we have $(\bar{\mathbf{x}}_g, \bar{X}_g) \in \mathcal{X}_g \setminus \hat{\mathcal{X}}_g$ for at most $|\mathcal{T}|+1$ generators. This indicates that at most $|\mathcal{T}|+1$ generators have non-integral solutions for $\bar{z}_{gt}, t \in \mathcal{T}$, as the difference between $\mathcal{X}_g$ and $\hat{\mathcal{X}}_g$ is that $\hat{\mathcal{X}}_g$ additionally impose integrality constraints on $\bar{z}_{gt}$. □

PROPOSITION 2. *Assume $|\mathcal{G}| > |\mathcal{T}|+1$. Then a CDP that pays generators using prices $(\bar{\boldsymbol{\lambda}}, \bar{\boldsymbol{\phi}}, \mathbf{0}, \bar{\boldsymbol{\Phi}})$ and the primal optimal solution $(\bar{\mathbf{x}}, \bar{X})$ ensures individual rationality of at least $|\mathcal{G}| - |\mathcal{T}| - 1$ generators.*

Proof. Proposition 1 establishes that there exists an optimal solution $(\bar{\mathbf{x}}, \bar{X})$ of $\mathcal{UC}^{\text{CPP}'}$, where at least $|\mathcal{G}| - |\mathcal{T}| - 1$ generators have integral solutions for $\bar{z}_{gt}, t \in \mathcal{T}$. We show that for any generator g in this subset (denoted as $\tilde{\mathcal{G}}^c$), the subvector $\bar{\mathbf{x}}_g$ is an optimal solution to the generator's profit maximization problem (13) under the prices $(\bar{\boldsymbol{\lambda}}, \bar{\boldsymbol{\phi}}, \mathbf{0}, \bar{\boldsymbol{\Phi}})$.

We use $\bar{\boldsymbol{\lambda}}$ to relax its corresponding constraint and obtain a partial Lagrangian relaxation of $\mathcal{UC}^{\text{CPP}'}$. Since $\mathcal{UC}^{\text{CPP}'}$ has a linear objective function, strong duality holds (Cifuentes et al. 2024), and thus $(\bar{\mathbf{x}}, \bar{X})$ is also optimal to this partial Lagrangian relaxation. After dropping the constant terms, the partial Lagrangian relaxation can be decomposed by g. For each g we have

$$\min_{(\mathbf{x}_g, X_g)\in\mathcal{X}_g} \quad \sum_{t\in\mathcal{T}} \left(c^p_g p_{gt} + c^u_g u_{gt}\right) - \sum_{t\in\mathcal{T}} \bar{\lambda}_t p_{gt}, \tag{39}$$

which is optimally solved by $(\bar{\mathbf{x}}_g, \bar{X}_g)$.

Note that the objective function of (39) can be equivalently converted to that of (13) by changing the sense and adding constant terms. Since $\bar{\mathbf{x}}_g$ leads to an optimal objective value for (39), it should also be optimal to (13) as long as it is feasible to it. This is true for $g \in \tilde{\mathcal{G}}^c$, as for those generators, $\bar{z}_{gt}, t \in \mathcal{T}$ are integral and thus $\bar{\mathbf{x}}_g$ satisfies all constraints of the profit-maximization problem (13). Therefore, for $g \in \tilde{\mathcal{G}}^c$, the prices $(\bar{\boldsymbol{\lambda}}, \bar{\boldsymbol{\phi}}, \mathbf{0}, \bar{\boldsymbol{\Phi}})$ and quantities $(\bar{\mathbf{x}}, \bar{X})$ ensure individual rationality. □

PROPOSITION 3. *Under a CDP that pays generators using prices $(\bar{\boldsymbol{\lambda}}, \bar{\boldsymbol{\phi}}, \mathbf{0}, \bar{\boldsymbol{\Phi}})$ and the primal optimal solution $(\bar{\mathbf{x}}, \bar{X})$, the total subsidy for covering the cost of generators that are not individually rational is $\mathcal{O}((|\mathcal{T}|+1)|\mathcal{T}|)$, and the over-production at each time period is $\mathcal{O}(|\mathcal{T}|+1)$*

Proof. Consider a generator $\hat{g}$ that is not individually rational. It has at least one time period with a non-integral solution. Assume $\hat{t}$ is such a time period. Since $(\bar{\boldsymbol{\lambda}}, \bar{\boldsymbol{\phi}}, \mathbf{0}, \bar{\boldsymbol{\Phi}})$ supports a non-integral solution for $\hat{g}$ which is infeasible, we obtain a feasible solution by rounding up the non-integral solution for $\bar{z}_{\hat{g}\hat{t}}$.

The subsidy for generator $\hat{g}$ at $\hat{t}$ equals the total cost increase due to the rounding of $\bar{z}_{\hat{g}\hat{t}}$. Since this rounding affects feasibility of $\bar{p}_{\hat{g}\hat{t}}$ only via constraint (8d), we can bound the increase in the production level by $p^{\min}_{\hat{g}}$, corresponding to an increase of $c^p_{\hat{g}} p^{\min}_{\hat{g}}$ in the production cost. In addition, the increase in $\bar{u}_{\hat{g}\hat{t}}$ is bounded by 1. Thus, the total increase in cost for $\hat{g}$ and $\hat{t}$ is bounded by $c^p_{\hat{g}} p^{\min}_{\hat{g}} + c^u_{\hat{g}}$. Since in the worst case the solution for $\hat{g}$ could be non-integral for $|\mathcal{T}|$ time periods, and there are at most $|\mathcal{T}|+1$ generators with non-integral solutions, the total subsidy needed for covering the cost is at most $\max_{g\in\mathcal{G}} \left(c^p_g p^{\min}_g + c^u_g\right) (|\mathcal{T}|+1)|\mathcal{T}| = \mathcal{O}((|\mathcal{T}|+1)|\mathcal{T}|)$.

To obtain the bound on over-production, note that the optimal solution of $\mathcal{UC}^{\text{CPP}'}$ clears the market. Thus, if rounding up non-integral solutions increases the production levels, then the increased production levels are over-production. We have shown that when rounding up the non-integral solution of a generator $\hat{g}$ at a time period, the increase in the production level is bounded by $p_{\hat{g}}^{\min}$. Thus, the total over-production at each time period after rounding up all fractional solutions is at most $(|\mathcal{T}|+1)\max_{g\in\mathcal{G}} p_g^{\min} = \mathcal{O}(|\mathcal{T}|+1)$. □

Proposition 4. *RCDP pricing balances the budget of the SO.*

Proof of Proposition 4. According to equation (27) in the proof of Theorem 1, we have

$$\sum_{t\in\mathcal{T}}\left(\sum_{g\in\mathcal{G}}\left(\lambda_t^* p_{gt}^* + \Lambda_t^* X_{gt,gt}^{pp*} + \sum_{j=1}^m \left(\phi_{jgt}^* \mathbf{a}_{jgt}\mathbf{x}^* + \Phi_{jgt}^* \mathrm{Tr}(\mathbf{a}_{jgt}\mathbf{a}_{jgt}^\top X^*)\right)\right) + 2\sum_{g_1<g_2, g_1,g_2\in\mathcal{G}} p_{g_1t}^* p_{g_2t}^* \Lambda_t^*\right)$$
$$= \sum_{t\in\mathcal{T}}\left(d_t\lambda_t^* + d_t^2\Lambda_t^* + \sum_{j=1}^m\sum_{g\in\mathcal{G}}\left(b_{jgt}\phi_{jgt}^* + b_{jgt}^2\Phi_{jgt}^*\right)\right). \quad (40)$$

The payment that the SO collects at hour t is $\lambda_t^* d_t + \Lambda_t^* d_t^2$, while the amount it pays generators at hour t is $\sum_{g\in\mathcal{G}}\left(\lambda_t^* p_{gt}^* + \Lambda_t^* X_{gt,gt}^{pp*} + \sum_{g'\in -g} p_{gt}^* p_{g't}^* \Lambda_t^*\right)$. We can prove that those two expressions are equivalent, by eliminating terms containing availability prices, i.e., $\sum_{j=1}^m \left(\phi_{jgt}^* \mathbf{a}_{jgt}\mathbf{x}^* + \Phi_{jgt}^* \mathrm{Tr}(\mathbf{a}_{jgt}\mathbf{a}_{jgt}^\top X^*)\right)$ and $\sum_{j=1}^m\sum_{g\in\mathcal{G}}\left(b_{jgt}\phi_{jgt}^* + b_{jgt}^2\Phi_{jgt}^*\right)$ from either side of (40), as they are equivalent. □

Proposition 5. *RCDP incentivizes the optimal solution of $\mathcal{UC}$ if the optimal value of (15) equals the optimal value of $\mathcal{UC}$.*

Proof of Proposition 5. Since an optimal solution of (15) is in $\mathcal{F}^{\mathrm{COP}}$, it is feasible for $\mathcal{UC}^{\mathrm{COP}}$. If the optimal solution corresponds to an objective value that equals the optimal value of $\mathcal{UC}^{\mathrm{COP}}$, then it is also optimal to $\mathcal{UC}^{\mathrm{COP}}$. An optimal solution of $\mathcal{UC}^{\mathrm{COP}}$ incentivizes the optimal solution of $\mathcal{UC}$. □

Proposition 7. *If the optimal value of $\mathcal{SP}(\bar{\Omega})$ is nonzero, then (21) cuts off $\bar{\Omega}$.*

Proof of Proposition 7. Because of (20c) and the fact that $q=1$, there is at least one element in u that is nonzero, i.e. $\beta = \{i|u_i = 1, i = 1,...,n_c\} \neq \emptyset$. Denote $\Omega_{\beta\beta}$ as the submatrix of Ω that consists of rows and columns with indices in β. Similarly we define the subvector $\mathbf{z}_\beta$. From the fact that the optimal objective $\bar{w}>0$, we know from constraint (20b) that $\bar{\Omega}_{\beta\beta}\bar{\mathbf{z}}_\beta < \mathbf{0}$ and thus $\bar{\mathbf{z}}_\beta \neq \mathbf{0}$. Therefore, $\bar{\mathbf{z}}_\beta^\top \bar{\Omega}_{\beta\beta}\bar{\mathbf{z}}_\beta < 0$. Also, let $\alpha = \{1,...,n_c\}\setminus\beta$, then $\bar{\mathbf{z}}_\alpha = \mathbf{0}$ due to (20e), which means $\bar{\mathbf{z}}^\top\bar{\Omega}\bar{\mathbf{z}} = \bar{\mathbf{z}}_\beta^\top\bar{\Omega}_{\beta\beta}\bar{\mathbf{z}}_\beta < 0$. Thus, $\bar{\Omega}$ violates the cut (21). □

Appendix C: Individual Revenue Adequacy with Both Uniform and Availability Prices

If we include both uniform and availability prices in the revenue adequacy constraints, we have the following pricing problem to solve:

$$\max \sum_{t\in\mathcal{T}}\left(d_t\lambda_t + d_t^2\Lambda_t + \sum_{j=1}^m\sum_{g\in\mathcal{G}}(b_{jgt}\phi_{jgt} + b_{jgt}^2\Phi_{jgt})\right) \quad (41a)$$

$$\text{s.t.} \sum_{t\in\mathcal{T}}\left(p_{gt}^*\lambda_t + p_{gt}^{*2}\Lambda_t + \sum_{g'\in\mathcal{G}\setminus\{g\}} p_{gt}^* p_{g't}^*\Lambda_t + \sum_{j=1}^m\left(\mathbf{a}_{jgt}^\top\mathbf{x}^*\phi_{jgt} + \mathrm{Tr}(\mathbf{a}_{jgt}\mathbf{a}_{jgt}^\top X^*)\Phi_{jgt}\right)\right)$$
$$\geq \sum_{t\in\mathcal{T}}\left(c_g^p p_{gt}^* + c_g^u u_{jgt}^*\right) \qquad \forall g\in\mathcal{G} \quad (41b)$$

$$(\boldsymbol{\lambda},\boldsymbol{\phi},\boldsymbol{\Lambda},\boldsymbol{\Phi},\boldsymbol{\delta},\Omega)\in\mathcal{F}^{\mathrm{COP}}. \quad (41c)$$

Again, we use $(p^*_{gt})^2$ in place of $X^{pp*}_{gt,gt}$ and $p^*_{gt}p^*_{g't}$ in place of $X^{pp*}_{gt,g't}$.

If (41) is feasible, then prices from (41) should satisfy revenue neutrality for generators. This is because if we sum up left-hand sides of (41b) over g, then the value equals the objective (41a) (proved by (27)), which represents the total revenue of generators. One the other hand, (41) can be viewed as imposing extra constraints on the original CDP problem (12), whose objective value, according to weak duality, is no more than the total costs $\sum_{g\in\mathcal{G}}\sum_{t\in\mathcal{T}}\left(c^p_g p^*_{gt} + c^u_g u^*_{gt}\right)$. But (41) also restricts the total revenue of generators to be no less than the total costs. Therefore, a feasible solution of (41) should ensure revenue neutrality for generators.

In addition, we actually have revenue neutrality for every generator, so each generator is paid exactly its cost. To understand why this result is true, assume towards contradiction if any generator has a strictly positive profit, then because of revenue neutrality of the whole system, some other generator must have a strictly negative profit, which violates (41b).

In comparison, we do not have revenue neutrality for individual generators in RCDP, i.e., in that case generators could have strictly positive profits.

Similar to the case of RCDP, with this pricing scheme, it can be proved that (41) is guaranteed to be feasible if its dual problem has an interior.

Appendix D: COP Algorithms Comparison for Maximum Clique Problem

We compare the performance of our cutting plane algorithm with other COP algorithms in the literature on the maximum clique problem, which is commonly used to benchmark COP algorithms. In the maximum clique problem, one tries to find the maximum clique number of a graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$, which is equivalent to finding the stability number of $\mathcal{G}$'s complementary graph, $\bar{\mathcal{G}} = (\mathcal{N}, \bar{\mathcal{E}})$. Let ω be the maximum clique number of $\mathcal{G}$. We can formulate the maximum clique problem as the following MILP, which finds the stability number of graph $\bar{\mathcal{G}}$:

$$\omega = \max \quad \sum_{i=1}^{n} x_i \tag{42a}$$

$$\text{s.t.} \quad x_i + x_j \le 1 \quad \forall (i,j) \in \bar{\mathcal{E}} \tag{42b}$$

$$x_i \in \{0,1\} \quad \forall i = 1, ..., n. \tag{42c}$$

Let A be the adjacency matrix of $\mathcal{G}$. Then we have $A = Q - \bar{A}$, where $Q = \mathbf{e}\mathbf{e}^\top - I$, and $\bar{A}$ is the adjacency matrix of $\bar{\mathcal{G}}$. Applying this relationship to the COP model in Corollary 2.4 of De Klerk and Pasechnik (2002), we obtain the following COP model for the maximum clique number of $\mathcal{G}$:

$$\omega = \min \quad \lambda \tag{43a}$$

$$\text{s.t.} \quad \lambda(\mathbf{e}\mathbf{e}^\top - A) - \mathbf{e}\mathbf{e}^\top = Y \tag{43b}$$

$$Y \in \mathcal{C}_n. \tag{43c}$$

In our experiment we use 10 max-clique problem instances from the second DIMACS challenge (DIMACS 1992). We compare the following ways of solving the COP (43):

(1) Approximately solve the COP with SDP, as shown in Section A. This is the method suggested by De Klerk and Pasechnik (2002). Note that the SDP approximation is a restriction to the dual COP model, and thus overestimates the maximum clique number of $\mathcal{G}$. We use Mosek 9.1.2 to solve the SDPs.

(2) Exactly solve the COP with the cutting plane algorithm of Section 5.

(3) Exactly solve the COP with the simplicial partition method of Bundfuss and Dür (2009).

We use Linux workstation with 3.6 Hz Intel Core i9-9900K CPUs and 128 GB memory for experiments using methods (1) and (2). We directly cite the results for Method (3) from Bundfuss and Dür (2009) for comparison, as it is not straightforward to implement and the source code is not available.

Note that we can strengthen the master problem in our cutting plane algorithm by providing bounds for Y in the initialization stage. Since the maximum clique number ω cannot exceed the number of total nodes, $|\mathcal{N}|$, and elements of $\mathbf{e}\mathbf{e}^\top$ are all 1's while elements of A are either 0 or 1, from constraints (43b) we have that the elements of Y should be in the range of $[-1, |\mathcal{N}|-1]$.

We present the results of the comparison between methods (1) and (2) in Table 9, where we list the number of nodes, $|\mathcal{N}|$, number of edges, $|\mathcal{E}|$, and the maximum clique number of the graph, ω for each instance. We measure the computational performance of the SDP approximation in terms of the objective ("Obj"), optimality gap ("Gap", compared with the true ω), and the computational time ("Time"). For our cutting plane algorithm we list the computational time and the number of iterations needed for convergence; we do not list the objectives because the cutting plane algorithm always converges to an exact solution.

Table 9 Algorithm Comparison for Maximum Clique Dual COP Model

					Mosek			Cutting plane	
Instance	$\|\mathcal{N}\|$	$\|\mathcal{E}\|$	$\frac{\|\mathcal{E}\|}{\|\mathcal{N}\|}$	ω	Obj	Gap(%)	Time(sec)	Time(sec)	#Iter
c-fat200-1	200	1534	7.67	12	12	0	566.81	13.87	2
c-fat200-2	200	3235	16.18	24	24	0	638.72	18.90	2
c-fat200-5	200	8473	42.37	58	60.35	3.89	606.33	12.19	2
hamming6-2[a]	64	1824	28.50	32	32	0	1.51	6.05	2
hamming6-4[a]	64	704	11.00	4	4	0	1.59	1.55	4
johnson8-2-4	28	210	7.50	4	4	0	0.20	9.53	2
johnson8-4-4	70	1855	26.50	14	14	0	2.47	11.82	2
johnson16-2-4[a,b]	120	5460	45.50	8	8	0	31.88	62.75	2
keller4	171	9435	55.18	11	13.47	18.34	426.16	-	-
MANN_a9	45	918	20.40	16	17.48	8.47	0.45	547.62	2

[a] Obtained by setting $q=\bar{\omega}$ in the separation problem, see text for explanation.
[b] Obtained by early termination of the separation problem.

When solving instances "hamming6-2", "hamming6-4" and "johnson16-2-4" with cutting planes, we encounter some very hard separation problems that take a long time to solve. To speed up the process, we use instead a strengthened version of the separation problem with $q=\bar{\omega}$ in constraint (20c) (Anstreicher 2021), where $\bar{\omega}$ is the current master problem solution for ω. Even with this enhancement, the instance "johnson16-2-4" still has a hard separation problem which achieves a nonzero lower bound early on (thus proving that the matrix is not copositive), but cannot converge after an extended period of time. In this case we set a time limit of 1 minute and use the non-converged solution. Finally, when solving the instance "keller4" we encounter a very hard separation problem after a few iterations. CPLEX fails to find a feasible solution for this separation problem, and we had to stop the algorithm due to memory usage. However, we still obtain useful information from the master problem objective, which upper bounds the COP objective. When we stopped the algorithm for "keller4" the master problem objective was 11, which is the correct value of ω, and is therefore superior to the bound provided by the SDP approximation (13.47).

We observe that in some instances, the SDP approximation fails to provide the correct maximum clique number. Also, in certain instances such as "c-fat200-1", "c-fat200-2" and "c-fat200-3", the cutting plane algorithm is faster than the SDP approximation. Both methods tend to struggle with instances that have higher $\frac{|\mathcal{E}|}{|\mathcal{N}|}$ ratios, though there are exceptions such as "c-fat200-1", where the $\frac{|\mathcal{E}|}{|\mathcal{N}|}$ ratio is relatively low, yet both methods need longer time to solve.

We also compare our algorithm with the simplicial partition method of Bundfuss and Dür (2009), which to the best of our knowledge is the only exact algorithm for general linear COPs in the literature. Because of the difference in computer setups, this comparison is not an exact one, and is meant to provide a general idea to the readers. Bundfuss and Dür (2009) also solve the maximum clique instances from the second DIMACS challenge. They report that their computation time for "johnson8-2-4" and "hamming6-4" are respectively 1 minutes 33 seconds and 57 minutes 52 seconds. In all other instances, their algorithm produces relatively loose bounds within two hours. Their algorithm was implemented in C++ on a Pentium IV, 2.8GHz Linux machine with 1GB RAM. Since Intel Core i9-9900K CPUs can be up to 7.4 times faster than Pentium IV, 2.8GHz CPUs (UserBenchmark 2023), we compare the two algorithms by multiplying the computation time of the cutting plane algorithm by 7.4. We find that our algorithm performs better in all instances.

Note that the cutting plane algorithm terminates in very few iterations in almost all test instances. This is not generally the case with the cutting plane algorithm when solving other COP problems. One reason for this is the use of a strong formulation of the maximum clique problem. For example, if we use the weaker COP formulation (44) below, then the cutting plane algorithm takes longer to terminate: the simplest instance (in terms of the number of nodes and edges) "johnson8-2-4" now takes 200.64 seconds and 690 iterations.

$$\omega = \min \quad \lambda \tag{44a}$$

$$\text{s.t.} \quad \lambda I + \sum_{(i,j)\in\bar{\mathcal{E}}} x_{ij} E_{ij} - \mathbf{e}\mathbf{e}^\top = Y \tag{44b}$$

$$Y \in \mathcal{C}_n. \tag{44c}$$

Here $E_{ij} \in \mathbb{R}^{n\times n}$ is a matrix with ones at ith row and jth column and jth row and ith column, and with zeros in all other positions.

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