# Solving set-valued optimization problems using a multiobjective approach

Gabriele Eichfelder*, Stefan Rocktäschel†

July 7, 2021

**Abstract**

Set-valued optimization using the set approach is a research topic of high interest due to its practical relevance and numerous interdependencies to other fields of optimization. However, it is a very difficult task to solve these optimzation problems even for specific cases. In this paper we study set-valued optimization problems and develop a multiobjective optimization problem that is strongly related to it. We prove that the set of weakly minimal solutions of this subproblem is closely related to the set of weakly minimal elements of the set-valued optimization problem and that these sets can get arbitrarily close in a certain sense. Subsequently, we introduce a concept of approximations of the solution set of the set-valued optimization problem. We define a quality measure in the image space that can be used to compare finite approximations of this kind and outline a procedure to enhance a given approximation. We conclude the paper with some numerical examples.

**Key Words: set-valued optimization, minimal value function, multiobjective optimization, approximation algorithms**

**Mathematics subject classifications (MSC 2010): 90C48, 90C29, 90C59**

*Institute of Mathematics, Technische Universität Ilmenau, Po 10 05 65, D-98684 Ilmenau, Germany, `gabriele.eichfelder@tu-ilmenau.de`

†Institute of Mathematics, Technische Universität Ilmenau, Po 10 05 65, D-98684 Ilmenau, Germany, `stefan.rocktaeschel@tu-ilmenau.de`

## 1 Introduction

Set-valued optimization is a very powerful tool. It can be considered as a generalization of scalar and vector optimization and has various applications, see [15]. Furthermore, since set optimization is such a general concept, there are many interdependencies to other important fields of research in the optimization community. Here we would like to mention interdependencies to robust optimization [6], bilevel optimization [18], as well as parametric and semi-infinite optimization [21]. Due to its high relevance, set optimization has recieved increasing interest and was studied intensively [15]. We use the set approach, as this is considered to be of specific practical relevance. Thereby, one compares the image sets of arguments as a whole using order relations on sets. There exist various order relations one could choose for this purpose with each of them having different scopes of application. We refer the reader to [13] for a detailed survey on different possible order relations on sets. In this paper we deal with the order relations examined by Kuroiwa [17]. These are namely the lower-type less, the upper-type less and the set less order relation. When comparing two image sets with each other the lower-type less relation focuses on the minimal elements of these sets for the comparison wheras the upper-type less relation focuses on the maximal elements of the image sets, and the set less relation combines both of these approaches.

Regarding algorithms for solving set-valued optimization problems, an algorithm for unconstrained set optimization problems with respect to the lower-type less order relation is introduced in [20]. This approach is for linear problems only and imposes further strong assumptions on the set-valued objective function. It focuses on computing a single solution. In [11], a derivative-free descent method is introduced. Thereby, it is assumed that the image sets are convex. The algorithm does not guarantee to find an optimal solution but rather tries to improve the function value in each descent step. The descending procedure relies on the comparison of image sets with respect to the respective set order relation. A vectorization result given in [12] and a discretization is the foundation for this. In [16], another descent method was introduced. This one relies on a nonlinear scalarization functional for the comparison of image sets. That allows to drop convexity assumptions on the image sets. However, also here the focus lies on finding a single solution only. In [8], an algorithm for finite families of sets is introduced. The algorithm finds all solutions of the set-valued optimization problem. However, often when investigating such problems one has to deal with infinitely many pre-image points and their corresponding image sets.

In this paper, we introduce an approach to solve set-valued optimization problems based on multiobjective optimization. The idea is inspired by Jahn's vector optimization problems introduced in [12]. However, we only consider finite dimensional subproblems. In that way, we can use tools from multiobjective optimization. The step from infinitely dimensional subproblems to finitely dimensional subproblems, of course, comes at a price. The solution sets of our subproblem and the original set-valued optimization problem do not coincide. However, we prove a result that allows us to get these two solution sets arbitrarily 'close' to each other. In this way, we reduce the set-valued optimization problem to a multiobjective optimization problem at the cost of an arbitrarily small error. For multiobjective optimization problems there are many solution approaches, which can then be applied.

The paper is organized as follows. In Section 2 we recall basic definitions, concepts and properties from vector and set optimization. We also introduce our basic notation and prove some short statments that we need in the following sections. In Section 3 we introduce the multiobjective subproblem that we are going to study and prove relations between the solution sets of that problem and the original set-valued optimization problem. Additionally, we introduce a quality measure in the image space for approximations of the set of weakly minimal elements of a set-valued optimization problem. In Section 4, we test our multiobjective approach on some numerical examples. Thereby, we obtain different approximations of the solution set of the initial set-valued optimization problems and compare these approximations based on the quality measure we introduced.

## 2 Preliminaries

We start this section by introducing the basic assumptions under which we will examine the set-valued optimization problems in the later sections.

**Assumption 2.1** *Let $X$ be a real normed space, let $Y$ be a real normed space partially ordered by a nontrivial closed pointed convex cone $C$, which is solid, i.e.,* $\operatorname{int}(C) \neq \emptyset$. *Let $S$ be a nonempty subset of $X$, and let $F\colon X \rightrightarrows Y$ be a set-valued map. It is assumed that $F(x)$ is nonempty, compact and convex for every $x \in S$.*

Under these assumptions we will study the set-valued optimization problem

$$\min_{x \in S} F(x) \text{ w.r.t. } C. \tag{SOP$^*$}$$

Note that some of our theoretical results would also hold under weaker assumptions, but for the sake of simplicity and readability, we usually refer to the assumptions given above. Since our aim is to examine multiobjective subproblems in order to find solutions of set-valued optimization problems, we start with basic notations and results from multiobjective optimization.

**Definition 2.2** *Let $Y$ be a real topological linear space partially ordered by a solid pointed convex cone $C \subseteq Y$. For $y^1, y^2 \in Y$ we define*

*(i)* $y^1 \leq_C y^2 :\Longleftrightarrow y^2 - y^1 \in C$,

*(ii)* $y^1 \lneq_C y^2 :\Longleftrightarrow y^2 - y^1 \in C \setminus \{0\}$,

*(iii)* $y^1 <_C y^2 :\Longleftrightarrow y^2 - y^1 \in \operatorname{int}(C)$.

*In case $Y = \mathbb{R}^m$ and $C = \mathbb{R}^m_+$ we may omit the subscript $C$.*

Note that the binary relation $\leq_C$ is a partial order, i.e., reflexive, transitive and anti-symmetric.
Furthermore, we use the standard concepts from vector optimization for the sets of minimal, maximal, weakly minimal and weakly maximal elements for subsets $A \subseteq Y$.

**Definition 2.3** *Let Assumption 2.1 be fulfilled and consider a set $A \subseteq Y$. We define*

$$\operatorname{Min}(A, C) := \{y \in A \mid (\{y\} - C) \cap A = \{y\}\} \text{ and}$$

$$\operatorname{Max}(A, C) := \{y \in A \mid (\{y\} + C) \cap A = \{y\}\},$$

*as well as*

$$\mathrm{wMin}(A, C) := \{y \in A \mid (\{y\} - \mathrm{int}(C)) \cap A = \emptyset\} \text{ and}$$
$$\mathrm{wMax}(A, C) := \{y \in A \mid (\{y\} + \mathrm{int}(C)) \cap A = \emptyset\}.$$

*In case* $Y = \mathbb{R}^m$ *and* $C = \mathbb{R}^m_+$ *we may omit the subscript* $C$.

The following proposition gives a well known property of the sets defined above that will be of use for us in some proofs.

**Proposition 2.4** *[10, Theorem 6.5] Let Assumption 2.1 be fulfilled. Then it holds* $\mathrm{Min}(F(x), C) \neq \emptyset$ *and* $\mathrm{Max}(F(x), C) \neq \emptyset$ *for all* $x \in S$.

In vector optimization, there are different solution concepts. With the following definition we introduce three of them. For all three we will show relations to different solution concepts of set-valued optimization in the later sections.

**Definition 2.5** *Let Assumption 2.1 be fulfilled. Consider* $h\colon X \to Y$, $\epsilon \in C$ *and the optimization problem*

$$\min_{x \in S} h(x) \text{ w.r.t. } C. \tag{VOP}$$

*For* $\overline{x} \in S$ *we say*

*(i) the element* $\overline{x}$ *is a minimal solution of* (VOP) *(w.r.t.* $\leq_C$*) if there is no* $x \in S$ *with* $h(x) \lneq_C h(\overline{x})$*, i.e., if* $h(\overline{x}) \in \mathrm{Min}(h(S), C)$*. The set of all minimal solution of* (VOP) *is denoted as* $\mathrm{argMin}(h, S, C)$.

*(ii) the element* $\overline{x}$ *is a weakly minimal solution of* (VOP) *(w.r.t.* $\leq_C$*) if there is no* $x \in S$ *with* $h(x) <_C h(\overline{x})$*, i.e., if* $h(\overline{x}) \in \mathrm{wMin}(h(S), C)$*. The set of all weakly minimal solutions of* (VOP) *is denoted as* $\mathrm{argwMin}(h, S, C)$.

*(iii) the element* $\overline{x}$ *is a weakly* $\epsilon$*-minimal solution of* (VOP) *(w.r.t.* $\leq_C$*) if there is no* $x \in S$ *with* $h(x) <_C h(\overline{x}) - \epsilon$*. The set of all weakly* $\epsilon$*-minimal solutions of* (VOP) *is denoted as* $\epsilon\mathrm{argwMin}(h, S, C)$.

*In case* $Y = \mathbb{R}^m$ *and* $C = \mathbb{R}^m_+$ *we may omit the subscript* $C$.

The definition of weakly $\epsilon$-minimal solutions of (VOP) is taken from [7]. Note that there are different concepts and definitions for weakly $\epsilon$-minimal solutions in the literature. We use this one, because it fits to the theoretical results we prove later on. Thereby, we often consider $\epsilon = \varepsilon \cdot e^p$, for an $\varepsilon \geq 0$ and the all-ones vector $e^p$ in $\mathbb{R}^p$,

$$e^p := (1, \dots, 1)^\top \in \mathbb{R}^p.$$

Next, we give notations and results for set-valued optimization problems. We start by recalling some binary relations that are used to compare sets with each other. Namely, we introduce the lower-type less, upper-type less and set less order relations. These relations represent different approaches one could make when comparing sets. While the lower-type less relation focuses on the minimal elements of the sets that have to be compared, the upper-type less relation focuses on their maximal elements, and the set less relation combines both of these approaches. For instance, in robust multiobjective optimization one studies the worst case and thus, one is interested in maximal elements. Hence, one is using the upper less relation there [5].
For more information on these and other order relations on sets see [13].

**Definition 2.6** *Let Assumption 2.1 be fulfilled. For nonempty subsets $A, B \subseteq Y$ we define the following order relations.*

$$\begin{aligned} A \preccurlyeq^l_C B &:\Longleftrightarrow B \subseteq A + C, \\ A \preccurlyeq^u_C B &:\Longleftrightarrow A \subseteq B - C, \\ A \preccurlyeq^s_C B &:\Longleftrightarrow A \preccurlyeq^l_C B \wedge A \preccurlyeq^u_C B. \end{aligned}$$

*In case $Y = \mathbb{R}^m$ and $C = \mathbb{R}^m_+$ we may omit the subscript $C$.*

Note that $\preccurlyeq^l_C$, $\preccurlyeq^u_C$ and $\preccurlyeq^s_C$ are reflexive and transitive and hence preorders. However, they are not antisymmetric.
Similar to the binary relations $\leq_C$ and $<_C$ in the vector optimization setting there is also a 'strict' correspondence for $\preccurlyeq^*_C$, $* \in \{l, u, s\}$. We define these binary relations in the following.

**Definition 2.7** *Let Assumption 2.1 be fulfilled. For nonempty subsets $A, B \subseteq Y$ we define*

$$\begin{aligned} A \prec^l_C B &:\Longleftrightarrow B \subseteq A + \operatorname{int}(C), \\ A \prec^u_C B &:\Longleftrightarrow A \subseteq B - \operatorname{int}(C), \\ A \prec^s_C B &:\Longleftrightarrow A \prec^l_C B \wedge A \prec^u_C B. \end{aligned}$$

Note that these relations are transitive, but in general neither reflexive nor antisymmetric. Furthermore, it is obvious that $A \prec^*_C B \Longrightarrow A \preccurlyeq^*_C B$ holds for all $A, B \subseteq Y$ and $* \in \{l, u, s\}$. Hence, $\prec^*_C$ is indeed more strict than $\preccurlyeq^*_C$.
Based on these binary relations for comparing sets, we now recall the set-valued optimization problem which we want to study. We introduce different solution concepts for it as well as we already did for the vector optimization problem (VOP). Under Assumption 2.1 and for $* \in \{l, u, s\}$ we consider the set-valued optimization problem

$$\min_{x \in S} F(x) \text{ w.r.t. } C. \tag{SOP$^*$}$$

Thereby, we use the solution concepts given by the following definition.

**Definition 2.8** *Let Assumption 2.1 be fulfilled, let $* \in \{l, u, s\}$. For $\overline{x} \in S$ we say*

*(i) the element $\overline{x}$ is a minimal solution of* (SOP$^*$) *if*

$$\forall x \in S \colon F(x) \preccurlyeq^*_C F(\overline{x}) \Longrightarrow F(\overline{x}) \preccurlyeq^*_C F(x). \tag{2.1}$$

*The set of all minimal solutions of* (SOP$^*$) *is denoted as* $\operatorname{argMin}^*(F, S, C)$.

*(ii) the element $\overline{x}$ is a weakly minimal solution of* (SOP$^*$) *if*

$$\forall x \in S \colon F(x) \prec^*_C F(\overline{x}) \Longrightarrow F(\overline{x}) \prec^*_C F(x).$$

*The set of all weakly minimal solutions of* (SOP$^*$) *is denoted as* $\operatorname{argwMin}^*(F, S, C)$.

*(iii) For $k^0 \in \operatorname{int}(C)$ and $\varepsilon \geq 0$ we say $\overline{x}$ is an $(\varepsilon, k^0)$-minimal solution of* (SOP$^*$) *if*

$$\nexists x \in S \colon F(x) \prec^*_C F(\overline{x}) - \varepsilon k^0.$$

*The set of all $(\varepsilon, k^0)$-minimal solutions of* (SOP$^*$) *is denoted as* $(\varepsilon, k^0)\operatorname{argwMin}^*(F, S, C)$.

*In case $Y = \mathbb{R}^m$ and $C = \mathbb{R}^m_+$ we may omit the subscript $C$.*

The first two solution concepts are well known and often used in set-valued optimization, see, for instance, [12] and [23]. The third one is a generalization of $\epsilon$-minimality from vector optimization and was studied in [4].
We conclude this section with a characterization of weakly minimal solutions of $(\text{SOP}^*)$, $* \in \{l, u, s\}$, which will be much easier to verify and be of use for proofs in later sections. Therefore, we need the following lemma.

**Lemma 2.9** *Let Assumption 2.1 be satisfied and let $A, B$ be nonempty subsets of $Y$.*

*(i) If $\text{wMin}(B, C) \neq \emptyset$, it holds $A \not\prec^l_C B \vee B \not\prec^l_C A$.*

*(ii) If $\text{wMax}(B, C) \neq \emptyset$ it holds $A \not\prec^u_C B \vee B \not\prec^u_C A$.*

*Proof.* For the proof of $(i)$ assume that there are nonempty sets $A, B \subseteq Y$ with $\text{wMin}(B, C) \neq \emptyset$ and $A \prec^l_C B$ and $B \prec^l_C A$. Then it holds

$$B \subseteq A + \text{int}(C) \subseteq B + \text{int}(C) + \text{int}(C) \subseteq B + \text{int}(C).$$

Since $\text{wMin}(B, C) \neq \emptyset$, there exists $\overline{y} \in \text{wMin}(B, C) \subseteq B$ and hence $\overline{y} = y + k$ for some $y \in B$ and $k \in \text{int}(C)$, which is a contradiction.
Statement $(ii)$ can be shown analogously. □

If we choose $A = B$ in Lemma 2.9 for any nonempty set $A \subseteq Y$ with $\text{wMin}(A, C) \neq \emptyset$ and $\text{wMax}(A, C) \neq \emptyset$, we obviously obtain $A \not\prec^*_C A$ for all $* \in \{l, u, s\}$.
The next lemma gives a characterization of weakly minimal solutions of $(\text{SOP}^*)$ for $* \in \{l, u, s\}$.

**Lemma 2.10** *Let Assumption 2.1 be fulfilled. For $\overline{x} \in S$ and $* \in \{l, u, s\}$ it holds that $\overline{x} \in S$ is a weakly minimal solution of $(\text{SOP}^*)$ if and only if there is no $x \in S$ with $F(x) \prec^*_C F(\overline{x})$.*

*Proof.* The reverse implication is obvious by definition.
For proving the implication assume by contradiction that $\overline{x} \in S$ is a weakly minimal solution of $(\text{SOP}^*)$ and there exists $x \in S$ with $F(x) \prec^*_C F(\overline{x})$. This implies $F(\overline{x}) \prec^*_C F(x)$ by definition. Using Proposition 2.4, we have $\emptyset \neq \text{Min}(F(x), C) \subseteq \text{wMin}(F(x), C)$ and $\emptyset \neq \text{Max}(F(x), C) \subseteq \text{wMax}(F(x), C)$. Using Lemma 2.9 we obtain a contradiction. □

# 3 Finite Dimensional Vectorization for set optimization

In this section, we introduce the multiobjective optimization problem on which we will focus for the rest of this paper. We show relations between that optimization problem and the set optimization problem $(\text{SOP}^*)$, $* \in \{l, u, s\}$. Since multiobjective optimization problems are well studied, these relations open up the possibility to use this knowledge in order to solve set optimization problems which are in general a lot harder to tackle.
To motivate and illustrate how we derive the multiobjective subproblem, we start with a theorem given by Jahn in [12]. This theorem connects set optimization and vector

optimization. Thereby, we use functionals from the dual space of $Y$. For a given cone $C$ the dual cone of $C$ is denoted as $C^*$, i.e.,

$$C^* := \{\ell \in Y^* \mid \ell(y) \geq 0 \text{ for all } y \in C\}.$$

Under Assumption 2.1 $Y$ is normed and hence, $Y^*$ is also normed using the dual norm. We define next, for $* \in \{l, u\}$, the function $v^*\colon X \to \mathbb{R}^{C^*}$ as

$$(v^*(x))(\ell) := \begin{cases} \inf_{y \in F(x)} \ell(y) & \text{if } * = l, \\ \sup_{y \in F(x)} \ell(y) & \text{if } * = u \end{cases}$$

for all $\ell \in C^*$, as well as $v^s\colon X \to \mathbb{R}^{C^*} \times \mathbb{R}^{C^*}$ by

$$v^s(x) := \begin{pmatrix} v^l(x) \\ v^u(x) \end{pmatrix},$$

where $\mathbb{R}^{C^*}$ denotes the space of all functions mapping from $C^*$ to $\mathbb{R}$. Since $F(x)$ is compact and $y \mapsto \ell(y)$ is continuous for all $x \in S$ and $\ell \in Y^*$, the infima and suprema are attained and finite.
The following theorem by Jahn states that rather than solving the optimization problem (SOP$^*$) for $* \in \{l, u, s\}$, we can solve the equivalent vector optimization problem (VOP$^*$) defined by

$$\min_{x \in S} v^*(x) \text{ w.r.t. } P^*, \tag{VOP$^*$}$$

where $P^l = P^u$ denotes the pointwise ordering cone in $\mathbb{R}^{C^*}$. That is

$$P^l = P^u := \{f \in \mathbb{R}^{C^*} \mid f(\ell) \geq 0 \text{ for all } \ell \in C^*\},$$

and $P^s := P^l \times P^u$.

**Theorem 3.1** *[12] Let Assumption 2.1 be satisfied and consider an element $\overline{x} \in S$. Then, the following statements hold:*

*(i) The element $\overline{x}$ is a minimal solution of (SOP$^l$) if and only if there is no $x \in S$ with*

$$\forall \ell \in C^* \setminus \{0_{Y^*}\}\colon \inf_{y \in F(x)} \ell(y) \leq \inf_{\overline{y} \in F(\overline{x})} \ell(\overline{y})$$
$$\wedge \exists \hat{\ell} \in C^* \setminus \{0_{Y^*}\}\colon \inf_{y \in F(x)} \hat{\ell}(y) < \inf_{\overline{y} \in F(\overline{x})} \hat{\ell}(\overline{y}),$$

*i.e., $\overline{x}$ is a minimal solution of (VOP$^l$).*

*(ii) The element $\overline{x}$ is a minimal solution of (SOP$^u$) if and only if $\overline{x}$ is a minimal solution of (VOP$^u$).*

*(iii) The element $\overline{x}$ is a minimal solution of (SOP$^s$) if and only if $\overline{x}$ is a minimal solution of (VOP$^s$).*

It directly follows that the sets of minimal solutions of the optimization problems (SOP$^*$) and (VOP$^*$) coincide for all $* \in \{l, u, s\}$.
In a similar way to the proof of the above theorem, we prove that also the set of weakly minimal solutions of these respective optimization problems coincide. Therefore, we need the following lemma which is an adaption of [12, Lemma 2.1] for the $\prec_C^*$ relation, $* \in \{l, u, s\}$.

**Lemma 3.2** *Let Assumption 2.1 be satisfied and consider two nonempty, compact, and convex sets $A, B \subseteq Y$. Then, the following statements hold:*

*(i)* $A \prec_C^l B$ *if and only if for all* $\ell \in C^* \setminus \{0_{Y^*}\}$ *it holds* $\inf_{y \in A} \ell(y) < \inf_{y \in B} \ell(y)$.

*(ii)* $A \prec_C^u B$ *if and only if for all* $\ell \in C^* \setminus \{0_{Y^*}\}$ *it holds* $\sup_{y \in A} \ell(y) < \sup_{y \in B} \ell(y)$.

*(iii)* $A \prec_C^s B$ *if and only if for all* $\ell \in C^* \setminus \{0_{Y^*}\}$ *it holds* $\inf_{y \in A} \ell(y) < \inf_{y \in B} \ell(y)$ *and* $\sup_{y \in A} \ell(y) < \sup_{y \in B} \ell(y)$.

*Proof.* For the proof of the implication of $(i)$ let $B \subseteq A + \text{int}(C)$ and $\ell \in C^* \setminus \{0_{Y^*}\}$ be chosen arbitrarily. Since $B$ is compact, there exist $y^B \in B$ with $\inf_{y \in B} \ell(y) = \ell(y^B)$. Since $B \subseteq A + \text{int}(C)$, there exist $y^A \in A$ and $k \in \text{int}(C)$ with $y^B = y^A + k$. Using that by [10, Lemma 3.21] it holds $\text{int}(C) = \{y \in Y \mid \ell(y) > 0 \text{ for all } \ell \in C^* \setminus \{0_{Y^*}\}\}$, we have

$$\inf_{y \in B} \ell(y) = \ell(y^B) = \ell(y^A + k) = \ell(y^A) + \ell(k) > \inf_{y \in A} \ell(y).$$

For the proof of the reverse implication of $(i)$ assume that there exists an element $y^B \in B$ with $y^B \notin A + \text{int}(C)$. By a separation theorem [10, Theorem 3.16] there exist $\hat{\ell} \in Y^* \setminus \{0_{Y^*}\}$ and $\alpha \in \mathbb{R}$ with

$$\hat{\ell}(y^B) \leq \alpha \leq \hat{\ell}(y) \text{ for all } y \in A + C.$$

This implies

$$\inf_{y \in B} \hat{\ell}(y) \leq \hat{\ell}(y^B) \leq \alpha \leq \inf_{y \in A + C} \hat{\ell}(y) = \inf_{y \in A} \hat{\ell}(y),$$

which is a contradiction.
Statement $(ii)$ can be shown analogously and statement $(iii)$ follows from $(i)$ and $(ii)$. □

The following theorem is a modification of Theorem 3.1 for weakly minimal solutions of (SOP$^*$) and follows from Lemma 2.10 and Lemma 3.2.

**Theorem 3.3** *Let Assumption 2.1 be satisfied and consider an element $\overline{x} \in S$. Then, the following statements hold:*

*(i) The element $\overline{x}$ is a weakly minimal solution of* (SOP$^l$) *if and only if there is no $x \in S$ with*

$$\forall \ell \in C^* \setminus \{0_{Y^*}\} \colon \inf_{y \in F(x)} \ell(y) < \inf_{\overline{y} \in F(\overline{x})} \ell(\overline{y}),$$

*i.e., $\overline{x}$ is a weakly minimal solution of* (VOP$^l$).

*(ii) The element $\overline{x}$ is a weakly minimal solution of* (SOP$^u$) *if and only if $\overline{x}$ is a weakly minimal solution of* (VOP$^u$).

*(iii) The element $\overline{x}$ is a weakly minimal solution of* (SOP$^s$) *if and only if $\overline{x}$ is a weakly minimal solution of* (VOP$^s$).

Note that in Theorem 3.1 and Theorem 3.3 we can restrict the functionals $\ell$ to

$$\ell \in K^* := \{\ell \in C^* \mid ||\ell|| = 1\} \subseteq C^* \setminus \{0_{Y^*}\}$$

and the results remain true.

### 3.1 A multiobjective subproblem

The obvious downside of the approach using (VOP$^*$), $* \in \{l, u, s\}$ is that this vector optimization problem has an infinite dimensional objective space and hence, is impossible to solve numerically. Jahn approaches this issue by a discretization and uses for instance 100 functionals to discretize the set $K^*$ for his numerical examples in [11] where $C = \mathbb{R}^2_+$. There are no error estimates and no guarantee of exactness given for this discretization. Our multiobjective approach tackles this problem. We consider finite subsets of elements $\ell \in K^*$ and obtain a finite dimensional objective space which allows us to apply methods from multiobjective optimization in order to solve the arising subproblems. Furthermore, we examine the error that arises by choosing this discretization and prove that a certain quality measure in the image space can still be achieved.

Under Assumption 2.1, and for finite sets $\mathcal{L},\mathcal{U} \subseteq K^*$ with $\mathcal{L} = \{\ell^1, ..., \ell^p\}$, $\mathcal{U} = \{\ell^{p+1}, ..., \ell^{p+q}\}$ and $p + q \geq 1$, we consider the multiobjective optimization problem

$$\min_{x\in S} f_{\mathcal{L},\mathcal{U}}(x) \text{ with } f_{\mathcal{L},\mathcal{U}}(x) := \begin{pmatrix} \inf_{y\in F(x)} \ell^1(y) \\ \vdots \\ \inf_{y\in F(x)} \ell^p(y) \\ \sup_{y\in F(x)} \ell^{p+1}(y) \\ \vdots \\ \sup_{y\in F(x)} \ell^{p+q}(y) \end{pmatrix} \text{ w.r.t. } \mathbb{R}^{p+q}_+. \qquad (\text{MOP}(\mathcal{L},\mathcal{U}))$$

Note that (MOP($\mathcal{L},\mathcal{U}$)) is a multiobjective optimization problem, where the objective function $f_{\mathcal{L},\mathcal{U}}$ has an image space of dimension $p+q$. Also note that under Assumption 2.1 the infima and suprema in the definition above are attained. In general, this optimization problem might not be solveable. However, if one assumes compactness of $S$ and continuity of $f_{\mathcal{L},\mathcal{U}}$, this is sufficient for the existence of (weakly) minimal solutions, see [10, Theorem 6.3]. Furthermore, it suffices to consider $\mathcal{L},\mathcal{U} \subseteq K^*$ since $C^*$ is a cone.

In this paper, our main focus is on examining the relationship between (MOP($\mathcal{L},\mathcal{U}$)) and (SOP$^*$). The following proposition shows that function values $f_{\mathcal{L},\mathcal{U}}(x)$ of points $x \in S$ carry information about the position of $F(x)$ in $Y$. This result serves as a motivation for our further investigations on this relationship. A more general result was formulated in [1, Section 2].

**Proposition 3.4** *Let Assumption 2.1 be fulfilled and let $\mathcal{L},\mathcal{U} \subseteq K^*$ be finite sets. Furthermore, we define $\mathcal{F}_{\mathcal{L},\mathcal{U}} \colon S \rightrightarrows Y$ by*

$$\mathcal{F}_{\mathcal{L},\mathcal{U}}(x) := \bigcap_{\ell\in\mathcal{L}} \{y \in Y \mid \ell(y) \geq f_{\{\ell\},\emptyset}(x)\} \cap \bigcap_{\ell\in\mathcal{U}} \{y \in Y \mid \ell(y) \leq f_{\emptyset,\{\ell\}}(x)\}.$$

*Then, it holds $F(x) \subseteq \mathcal{F}_{\mathcal{L},\mathcal{U}}(x)$ for all $x \in S$.*

*Proof.* Let $x \in S$ be arbitrarily given. Then, for every $\ell \in \mathcal{L}$ and for all $\hat{y} \in F(x)$ it holds $\ell(\hat{y}) \geq \inf_{y\in F(x)} \ell(y) = f_{\{\ell\},\emptyset}(x)$. Analogously, for every $\ell \in \mathcal{U}$ and for all $\hat{y} \in F(x)$ it holds $\ell(\hat{y}) \leq \sup_{y\in F(x)} \ell(y) = f_{\emptyset,\{\ell\}}(x)$. □

We illutrate this result with the following example.

**Example 3.5** *Let* $S = [0, \frac{\pi}{2}]$*,* $Y = \mathbb{R}^2$ *and* $C = \mathbb{R}^2_+$*. Furthermore, let*

$$\mathcal{L} = \{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), \tfrac{1}{\sqrt{2}}\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\}, \mathcal{U} = \emptyset.$$

*Consider the set-valued map* $F\colon S \to \mathbb{R}^2$ *given by*

$$F(x) := \{\left(\begin{smallmatrix}-cos(x)\\-sin(x)\end{smallmatrix}\right)\} + \tfrac{1}{4}\mathbb{B}_2,$$

*where* $\mathbb{B}_2 := \{y \in \mathbb{R}^2 \mid ||y||_2 \leq 1\}$*. Assume we are interested in the image set of* $\bar{x} = \frac{\pi}{6}$*. We obtain*

$$f_{\mathcal{L},\emptyset}(\bar{x}) = \left(-\frac{2\sqrt{3}+1}{4}, -\frac{1+\sqrt{2}+\sqrt{6}}{4}, -\frac{3}{4}\right)^\top =: (a_1, a_2, a_3)^\top.$$

*Proposition 3.4 states that*

$$\begin{aligned} F(\bar{x}) \subseteq \mathcal{F}_{\mathcal{L},\mathcal{U}}(\bar{x}) = &\{y \in Y \mid y_1 \geq a_1\} \\ &\cap \{y \in Y \mid \tfrac{y_1+y_2}{\sqrt{2}} \geq a_2\} \\ &\cap \{y \in Y \mid y_2 \geq a_3\}. \end{aligned}$$

*We illustrate* $F(\bar{x})$ *and* $\mathcal{F}_{\mathcal{L},\mathcal{U}}(\bar{x})$ *as well as some other image sets in Figure 1.*

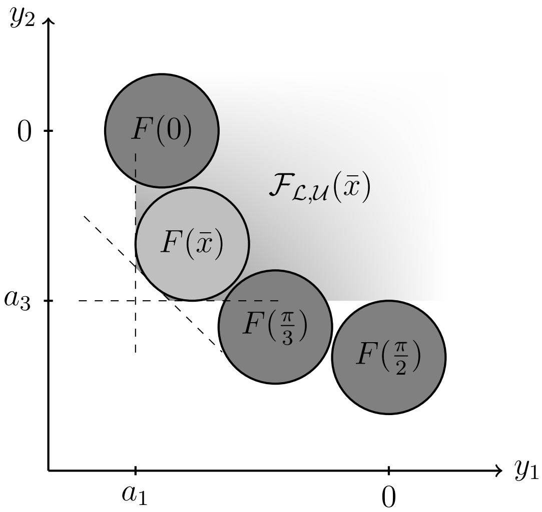


Figure 1: Image sets of $F$ from Example 3.5

Using Theorem 3.3, we obtain the following relation between the sets of weakly minimal solutions of (SOP$^*$), $* \in \{l, u, s\}$ and of (MOP($\mathcal{L}$,$\mathcal{U}$)).

**Theorem 3.6** *Let Assumption 2.1 be fulfilled. Consider finite sets* $\mathcal{L}, \mathcal{U} \subseteq K^*$ *with* $\mathcal{L} \cup \mathcal{U} \neq \emptyset$*. Then, the following statements hold.*

*(i) If* $\mathcal{L} \neq \emptyset$ *and* $\overline{x} \in S$ *is a weakly minimal solution of* (MOP($\mathcal{L}, \emptyset$))*, then* $\overline{x}$ *is a weakly minimal solution of* (SOP$^l$)*, i.e.,*

$$\operatorname{argwMin}(f_{\mathcal{L},\emptyset}, S, \mathbb{R}^p_+) \subseteq \operatorname{argwMin}^l(F, S, C).$$

*(ii)* *If $\mathcal{U} \neq \emptyset$ and $\overline{x} \in S$ is a weakly minimal solution of* (MOP($\emptyset$,$\mathcal{U}$))*, then $\overline{x}$ is a weakly minimal solution of* (SOP$^u$)*, i.e.,*

$$\operatorname{argwMin}(f_{\emptyset,\mathcal{U}}, S, \mathbb{R}^q_+) \subseteq \operatorname{argwMin}^u(F, S, C).$$

*(iii)* *If $\overline{x} \in S$ is a weakly minimal solution of* (MOP($\mathcal{L}$,$\mathcal{U}$))*, then $\overline{x}$ is a weakly minimal solution of* (SOP$^s$)*, i.e.,*

$$\operatorname{argwMin}(f_{\mathcal{L},\mathcal{U}}, S, \mathbb{R}^{p+q}_+) \subseteq \operatorname{argwMin}^s(F, S, C).$$

*Proof.* For the proof of $(i)$ assume that there exists $x \in S$ with $F(x) \prec^l_C F(\overline{x})$. Using Lemma 3.2, this implies $\inf_{y \in F(x)} \ell(y) < \inf_{y \in F(\overline{x})} \ell(y)$ for all $\ell \in C^* \setminus \{0_{Y^*}\}$. In particular it holds $\inf_{y \in F(x)} \ell^i(y) < \inf_{y \in F(\overline{x})} \ell^i(y)$ for all $i \in \{1, ..., p\}$. This implies $f_{\mathcal{L},\emptyset}(x) <_{\mathbb{R}^p_+} f_{\mathcal{L},\emptyset}(\overline{x})$, which implies that $\overline{x} \in S$ is not a weakly minimal solution of (MOP($\mathcal{L}$, $\emptyset$)).
Statements $(ii)$ and $(iii)$ can be shown analogously. □

One might think that a similar result would hold for the respective minimal solutions of (MOP($\mathcal{L}$,$\mathcal{U}$)) and (SOP$^*$). However, this is not the case as the following example shows. The image sets from this example are illustrated in Figure 2.

**Example 3.7** *Let $S = \{x^1, x^2\}$, $Y = \mathbb{R}^2$ and $C = \mathbb{R}^2_+$. Let $\mathcal{L} = \{\ell^1\} = \{\frac{1}{\sqrt{2}}\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)\}$ which is even a subset of* $\operatorname{int}(C^*)$*. Now, consider $F$ with $F(x^1) = \{\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)\}$ and $F(x^2) =$* $\operatorname{conv}(\{\left(\begin{smallmatrix}-1\\1\end{smallmatrix}\right), \left(\begin{smallmatrix}1\\-1\end{smallmatrix}\right)\})$.
*Then it is easy to verify that $f_{\mathcal{L},\emptyset}(x^1) = 0 = f_{\mathcal{L},\emptyset}(x^2)$ and $F(x^1) \subseteq F(x^2) + C$, but $F(x^2) \nsubseteq F(x^1) + C$. Hence,*

$$\operatorname{argMin}^l(f_{\mathcal{L},\emptyset}, S, \mathbb{R}^2) = \{x^1, x^2\} \nsubseteq \{x^2\} = \operatorname{argMin}^l(F, S, C).$$

Furthermore, it is necessary in $(i)$ that we set $\mathcal{U} = \emptyset$, and in $(ii)$ that we set $\mathcal{L} = \emptyset$ respectively. We show this for the first case in the following example. The image sets from this example are illustrated in Figure 3.

**Example 3.8** *Let $S = \{x^1, x^2\}$, $Y = \mathbb{R}^2$ and $C = \mathbb{R}^2_+$. Let $\mathcal{L} = \mathcal{U} = \{\ell^1\} = \{\frac{1}{\sqrt{2}}\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)\}$. Now, consider $F(x^1) = \{\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)\}$ and $F(x^2) = \operatorname{conv}(\{\begin{pmatrix}-1\\-1\end{pmatrix}, \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)\})$.*
*Then it is easy to verify that $f_{\mathcal{L},\mathcal{U}}(x^1) = \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$ and $f_{\mathcal{L},\mathcal{U}}(x^2) = \begin{pmatrix}-\sqrt{2}\\ \sqrt{2}\end{pmatrix}$. Furthermore, it holds $F(x^1) \subseteq F(x^2) + \operatorname{int}(C)$ and $F(x^2) \nsubseteq F(x^1) + \operatorname{int}(C)$. Hence,*

$$\operatorname{argwMin}^l(f_{\mathcal{L},\mathcal{U}}, S, \mathbb{R}^2) = \{x^1, x^2\} \nsubseteq \{x^2\} = \operatorname{argwMin}^l(F, S, C).$$

Another interesting question is under which assumptions $\operatorname{argwMin}(f_{\mathcal{L},\mathcal{U}}, S, \mathbb{R}^{|\mathcal{L}|+|\mathcal{U}|}_+) \neq \emptyset$ holds. If this condition would not be satisfied, the results from Theorem 3.6 would not be of any interest. A sufficient condition is that the set $f_{\mathcal{L},\mathcal{U}}(S)$ is compact, see [10, Theorem 6.3]. This is the case if $S$ is compact and the function $f_{\mathcal{L},\mathcal{U}}$ is continuous. The latter is the case if $F$ is continuous, see [2, Theorem 4.2.2] for a proof of this statement. The underlying concepts for continuity of set-valued mappings can be found in [2, Chapter 2.2]. Summarizing, continuity of $F$ and compactness of $S$ imply $\operatorname{argwMin}(f_{\mathcal{L},\mathcal{U}}, S, \mathbb{R}^{|\mathcal{L}|+|\mathcal{U}|}_+) \neq \emptyset$.
Compared to the approach using (VOP$^*$), $* \in \{l, u, s\}$ and Theorem 3.6 we also have

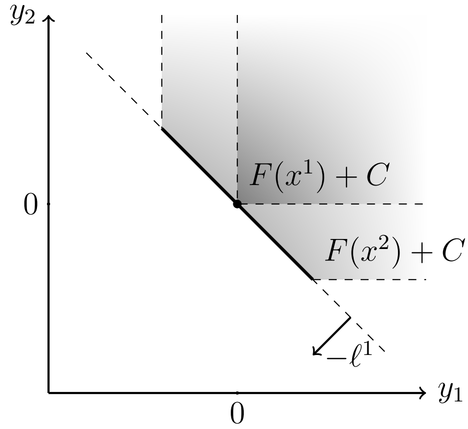


Figure 2: Image sets from Example 3.7

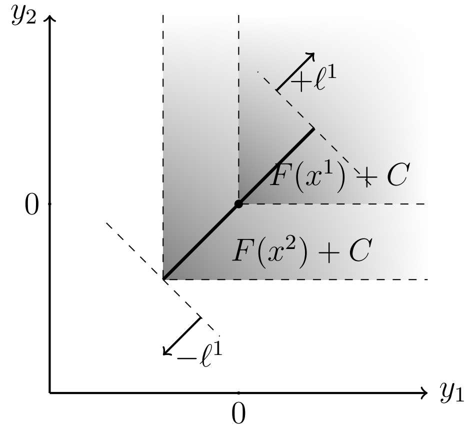


Figure 3: Image sets from Example 3.8

the property that every weakly minimal solution of our subproblem (MOP($\mathcal{L}$,$\mathcal{U}$)) is a weakly minimal solution of the set-valued optimization problem (SOP*). So this inclusion is preserved within our approach. Unfortunately, the reverse inclusion does not hold in our case as the following example shows. In particular, we show that for any given $\mathcal{L} \subseteq K^*$ of cardinality $p$ there exists a set optimization problem such that $\operatorname{argwMin}(f_{\mathcal{L},\emptyset}, S, \mathbb{R}^p_+) \not\supseteq \operatorname{argwMin}^l(F, S, C)$ holds.

**Example 3.9** *Let $S = \{x^1, x^2\}$, $(Y, ||\cdot||_Y) = (\mathbb{R}^2, ||\cdot||_2)$ and $C = \mathbb{R}^2_+$. Then we also have $||\cdot||_{Y^*} = ||\cdot||_2$. Let a finite set $\mathcal{L} \subseteq K^*$ with $p = |\mathcal{L}| \geq 2$ be arbitrarily chosen. Now, consider $F$ with $F(x^1) = \{y \in \mathbb{R}^2 \mid ||y||_2 \leq 1\}$ and $F(x^2) = \operatorname{conv}(\{-(1+\varepsilon)\ell \mid \ell \in \mathcal{L}\})$, where $\varepsilon := \frac{1}{2}(1 - \sqrt{1 - \frac{1}{4}\min_{i \neq j} ||\ell^i - \ell^j||_2^2})$.*
*One can show that in this case it holds $\operatorname{argwMin}^l(F, S, C) = \{x^1, x^2\}$, since $F(x^1) \not\preccurlyeq^l F(x^2)$ and $F(x^2) \not\preccurlyeq^l F(x^1)$. However, for every $\ell \in \mathcal{L}$ it holds $\inf_{y \in F(x^1)} \ell(y) = -1$ and $\inf_{y \in F(x^2)} \ell(y) = -1 - \varepsilon$. Hence, we have $f_{\mathcal{L},\emptyset}(x^1) = (-1)e^p$ and $f_{\mathcal{L},\emptyset}(x^2) = (-1-\varepsilon)e^p$. This implies $f_{\mathcal{L},\emptyset}(x^2) < f_{\mathcal{L},\emptyset}(x^1)$ and hence $\operatorname{argwMin}(f_{\mathcal{L},\emptyset}, S, \mathbb{R}^p_+) = \{x^2\}$. So not every weakly minimal solution of (SOP$^l$) is also weakly minimal for (MOP($\mathcal{L}$,$\mathcal{U}$)). But on the other hand, since $f_{\mathcal{L},\emptyset}(x^2) = f_{\mathcal{L},\emptyset}(x^1) - \varepsilon e^p$, we have $x^1 \in \epsilon\operatorname{argwMin}(f_{\mathcal{L},\emptyset}, S, \mathbb{R}^p_+)$ for $\epsilon = \varepsilon e^p$. So while $x^1$ might not be a weakly minimal element of (MOP($\mathcal{L}$, $\emptyset$)) it is an $\epsilon$-minimal element of this optimization problem.*
*To illustrate this example, we investigate the special case where*

$$\mathcal{L} = \{\ell^1, \ell^2, \ell^3\} = \{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), \tfrac{1}{\sqrt{2}}\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\}.$$

*In this case the image set $F(x^1)$ is the unit ball in $\mathbb{R}^2$ and the image set $F(x^2)$ is a triangle. The sets are illustrated in Figure 4.*

Please note that in this particular example the point $x^1 \in \operatorname{argwMin}^l(F, S, C)$ might not be an element of $\operatorname{argwMin}(f_{\mathcal{L},\emptyset}, S, \mathbb{R}^p_+)$, but indeed is an element of the set $(\varepsilon e^p)\operatorname{argwMin}(f_{\mathcal{L},\emptyset}, S, \mathbb{R}^p)$. So $x^1$ is not a weakly minimal solution of (MOP($\mathcal{L}$,$\mathcal{U}$)) but at least it is a weakly $\epsilon$-minimal solution of (MOP($\mathcal{L}$,$\mathcal{U}$)) for $\epsilon := \varepsilon e^p$. This is not a coincidence. We will show that for every $\varepsilon > 0$ there exist finite sets $\mathcal{L}, \mathcal{U} \subseteq K^*$ such that $\operatorname{argwMin}^*(F, S, C) \subseteq \epsilon\operatorname{argwMin}(f_{\mathcal{L},\mathcal{U}}, S, \mathbb{R}^{p+q}_+)$ holds for $\epsilon := \varepsilon e^{|\mathcal{L}|+|\mathcal{U}|}$. Together with the results from Theorem 3.6 we then have

$$\operatorname{argwMin}(f_{\mathcal{L},\mathcal{U}}, S, \mathbb{R}^{p+q}_+) \subseteq \operatorname{argwMin}^*(F, S, C) \subseteq \epsilon\operatorname{argwMin}(f_{\mathcal{L},\mathcal{U}}, S, \mathbb{R}^{p+q}_+), \qquad (3.1)$$

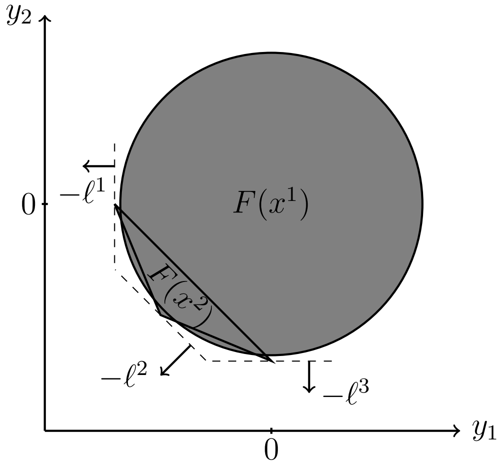


Figure 4: Image sets from Example 3.9

where $\mathcal{U} = \emptyset$ if $* = l$ and $\mathcal{L} = \emptyset$ if $* = u$. For the proof of this main result we need the following proposition which gives a characerization of weakly minimal solutions of a multiobjective optimization problem. It follows directly from the definition.

**Proposition 3.10** *Let Assumption 2.1 hold and consider a function* $h\colon X \to \mathbb{R}^p$, $\varepsilon > 0$ *and* $\epsilon := \varepsilon \cdot e^p$. *Then* $\bar{x}$ *is a weakly* $\epsilon$*-minimal solution of* $\min_{x\in S} h(x)$ *w.r.t.* $\mathbb{R}^p_+$, *i.e.,* $\bar{x} \in \epsilon\mathrm{argwMin}(h, S, \mathbb{R}^p_+)$, *if and only if*

$$\forall x \in S\colon \max_{i=1,\dots,p} \{h_i(x) - h_i(\bar{x})\} \geq -\varepsilon.$$

The following proposition shows that $f_{\{\ell\},\emptyset}(x)$ and $f_{\emptyset,\{\ell\}}(x)$ are Lipschitz continuous with respect to the parameter $\ell \in K^*$ for all $x \in S$ under some weak boundedness conditions.

**Proposition 3.11** *Let Assumption 2.1 be fulfilled. Then, the following statements hold.*

*(i) Suppose that the set* $\bigcup_{x\in S} \mathrm{wMin}(F(x), C)$ *is bounded. Define*

$$L^l := \sup\{\|y\| \mid y \in \bigcup_{x\in S} \mathrm{wMin}(F(x), C)\}.$$

*Then, for any* $\ell^1, \ell^2 \in C^*$ *and* $x \in S$, *we have* $|f_{\{\ell^1\},\emptyset}(x) - f_{\{\ell^2\},\emptyset}(x)| \leq L^l\|\ell^1 - \ell^2\|$, *i.e.,* $f_{\{\ell\},\emptyset}(x)$ *is Lipschitz-continuous in* $\ell$ *for all* $x \in S$.

*(ii) Suppose that the set* $\bigcup_{x\in S} \mathrm{wMax}(F(x), C)$ *is bounded. Define*

$$L^u := \sup\{\|y\| \mid y \in \bigcup_{x\in S} \mathrm{wMax}(F(x), C)\}.$$

*Then, for any* $\ell^1, \ell^2 \in C^*$ *and* $x \in S$, *we have* $|f_{\emptyset,\{\ell^1\}}(x) - f_{\emptyset,\{\ell^2\}}(x)| \leq L^u\|\ell^1 - \ell^2\|$, *i.e.,* $f_{\emptyset,\{\ell\}}(x)$ *is Lipschitz-continuous in* $\ell$ *for all* $x \in S$.

*Proof.* First, we observe that under our assumptions $L^l$ and $L^u$ are finite. Since $F(x)$ is compact, there exist $\overline{y}, \hat{y} \in \mathrm{wMin}(F(x), C)$ such that $f_{\{\ell^1\},\emptyset}(x) = \ell^1(\overline{y})$ and $f_{\{\ell^2\},\emptyset}(x) = \ell^2(\hat{y})$. It follows that

$$f_{\{\ell^1\},\emptyset}(x) \leq \ell^1(\hat{y}) = (\ell^1 - \ell^2)(\hat{y}) + \ell^2(\hat{y}) \leq \|\hat{y}\|\|\ell^1 - \ell^2\| + \ell^2(\hat{y}) \leq f_{\{\ell^2\},\emptyset}(x) + L^l\|\ell^1 - \ell^2\|$$

and analogously

$$f_{\{\ell^2\},\emptyset}(x) \leq \ell^2(\bar{y}) = (\ell^2-\ell^1)(\bar{y})+\ell^1(\bar{y}) \leq \|\bar{y}\|\|\ell^1-\ell^2\|+\ell^1(\bar{y}) \leq f_{\{\ell^1\},\emptyset}(x)+L^l\|\ell^1-\ell^2\|.$$

Hence, we have

$$|f_{\{\ell^1\},\emptyset}(x) - f_{\{\ell^2\},\emptyset}(x)| \leq L^l\|\ell^1-\ell^2\|.$$

Statement $(ii)$ can be shown analogously. □

We are now able to prove the following main theorem.

**Theorem 3.12** *Let Assumption 2.1 be fulfilled and $K^*$ be compact. Then, the following statements hold:*

*(i) If the set $\bigcup_{x\in S} \mathrm{wMin}(F(x), C)$ is bounded, then for every $\varepsilon > 0$ there exists a finite set $\mathcal{L} = \mathcal{L}(\varepsilon)$ such that*

$$\mathrm{argwMin}^l(F, S, C) \subseteq \epsilon\mathrm{argwMin}(f_{\mathcal{L},\emptyset}, S, \mathbb{R}^{|\mathcal{L}|}_+),$$

*where $\epsilon = \varepsilon e^{|\mathcal{L}|}$.*

*(ii) If the set $\bigcup_{x\in S} \mathrm{wMax}(F(x), C)$ is bounded, then for every $\varepsilon > 0$ there exists a finite set $\mathcal{U} = \mathcal{U}(\varepsilon)$ such that*

$$\mathrm{argwMin}^u(F, S, C) \subseteq \epsilon\mathrm{argwMin}(f_{\emptyset,\mathcal{U}}, S, \mathbb{R}^{|\mathcal{U}|}_+),$$

*where $\epsilon = \varepsilon e^{|\mathcal{U}|}$.*

*(iii) If the sets $\bigcup_{x\in S} \mathrm{wMin}(F(x), C)$ and $\bigcup_{x\in S} \mathrm{wMax}(F(x), C)$ are bounded, then for every $\varepsilon > 0$ there exist finite sets $\mathcal{L} = \mathcal{L}(\varepsilon)$ and $\mathcal{U} = \mathcal{U}(\varepsilon)$ such that*

$$\mathrm{argwMin}^s(F, S, C) \subseteq \epsilon\mathrm{argwMin}(f_{\mathcal{L},\mathcal{U}}, S, \mathbb{R}^{|\mathcal{L}|+|\mathcal{U}|}_+),$$

*where $\epsilon = \varepsilon e^{|\mathcal{L}|+|\mathcal{U}|}$.*

*Proof.* For the proof of $(i)$ let $\varepsilon > 0$. By assumption,

$$L^l = \sup\{\|y\| \mid y \in \bigcup_{x\in S} \mathrm{wMin}(F(x), C)\} < +\infty.$$

Without loss of generality we assume $L^l > 0$. Since $K^*$ is compact, there exists $\mathcal{L} = \{\ell^1, \ldots, \ell^p\} \subseteq K^*$ with $K^* \subseteq \mathcal{L} + \frac{\varepsilon}{4L^l}\mathbb{B}$, where $\mathbb{B}$ is the unit ball in $Y^*$. Now consider a fixed $\bar{x} \in \mathrm{argwMin}^l(F, S, C)$ and an element $x \in S$ arbitrarily chosen.
Because of Theorem 3.3, we know that $\sup_{\ell\in K^*}\{f_{\{\ell\},\emptyset}(x) - f_{\{\ell\},\emptyset}(\bar{x})\} \geq 0$. We can then pick $\hat{\ell} \in K^*$ with

$$f_{\{\hat{\ell}\},\emptyset}(x) - f_{\{\hat{\ell}\},\emptyset}(\bar{x}) \geq \sup_{\ell\in K^*}\{f_{\{\ell\},\emptyset}(x) - f_{\{\ell\},\emptyset}(\bar{x})\} - \frac{\varepsilon}{2} \geq -\frac{\varepsilon}{2}. \tag{3.2}$$

Furthermore, by construction, we can choose $\bar{\ell} \in \mathcal{L}$ with

$$\|\hat{\ell} - \bar{\ell}\| \leq \frac{\varepsilon}{4L^l}. \tag{3.3}$$

Taking into account Proposition 3.11 and inequalities (3.2) and (3.3), it follows that

$$\begin{aligned}\max_{\ell\in\mathcal{L}}\{f_{\{\ell\},\emptyset}(x)-f_{\{\ell\},\emptyset}(\bar{x})\} &\geq f_{\{\bar{\ell}\},\emptyset}(x)-f_{\{\bar{\ell}\},\emptyset}(\bar{x})\\ &\geq f_{\{\hat{\ell}\},\emptyset}(x)-L^l\|\hat{\ell}-\bar{\ell}\|-f_{\{\hat{\ell}\},\emptyset}(\bar{x})-L^l\|\hat{\ell}-\bar{\ell}\|\\ &\geq -\frac{\varepsilon}{2}-2L^l\|\hat{\ell}-\bar{\ell}\|\\ &\geq -\frac{\varepsilon}{2}-2L^l\frac{\varepsilon}{4L^l}\\ &= -\varepsilon.\end{aligned}$$

According to Proposition 3.10 this is equivalent to $\bar{x}\in\epsilon\text{argwMin}(f_{\mathcal{L},\emptyset},S,\mathbb{R}_+^{|\mathcal{L}|})$. Statement $(ii)$ can be shown analogously and statement $(iii)$ follows from $(i)$ and $(ii)$ by considering the cases where $\sup_{\ell\in K^*}\{f_{\{\ell\},\emptyset}(x)-f_{\{\ell\},\emptyset}(\bar{x})\}\geq 0$ or $\sup_{\ell\in K^*}\{f_{\emptyset,\{\ell\}}(x)-f_{\emptyset,\{\ell\}}(\bar{x})\}\geq 0$ and with

$$L^s:=\max\{L^l,L^u\}.$$

□

## 3.2 Approximating solution sets

In practice, one might not be able to solve the optimization problem (MOP($\mathcal{L}$,$\mathcal{U}$)) exactly but is rather only guaranteed to find approximate solutions. Hence, in the following theorem we examine relations between weakly $\epsilon$-minimal solutions of (MOP($\mathcal{L}$,$\mathcal{U}$)) and $(\varepsilon,k^0)$-minimal solutions of (SOP$^*$), $*\in\{l,u,s\}$.

**Theorem 3.13** *Let Assumption 2.1 be fulfilled. Let $\mathcal{L},\mathcal{U}\subseteq K^*$ be finite sets with $\mathcal{L}=\{\ell^1,...,\ell^p\}$, $\mathcal{U}=\{\ell^{p+1},...,\ell^{p+q}\}$ and $p+q\geq 1$. Consider $\varepsilon\geq 0$ and $k^0\in\text{int}(C)$. Furthermore, define*

$$\epsilon_{k^0}^{\mathcal{L}}:=\varepsilon\begin{pmatrix}\ell^1(k^0)\\ \vdots\\ \ell^p(k^0)\end{pmatrix},\ \epsilon_{k^0}^{\mathcal{U}}:=\varepsilon\begin{pmatrix}\ell^{p+1}(k^0)\\ \vdots\\ \ell^{p+q}(k^0)\end{pmatrix},\ \epsilon_{k^0}^{\mathcal{L},\mathcal{U}}:=\begin{pmatrix}\epsilon_{k^0}^{\mathcal{L}}\\ \epsilon_{k^0}^{\mathcal{U}}\end{pmatrix}.$$

*Then, the following statements hold:*

*(i) If $q=0$ and $\overline{x}\in S$ is a weakly $\epsilon_{k^0}^{\mathcal{L}}$-minimal solution of* (MOP($\mathcal{L}$,$\mathcal{U}$)), *then $\overline{x}$ is a $(\varepsilon,k^0)$-minimal solution of* (SOP$^l$), *i.e.,*

$$\epsilon_{k^0}^{\mathcal{L}}\text{argwMin}(f_{\mathcal{L},\emptyset},S,\mathbb{R}_+^p)\subseteq(\varepsilon,k^0)\text{argwMin}^l(F,S,C).$$

*(ii) If $p=0$ and $\overline{x}\in S$ is a weakly $\epsilon_{k^0}^{\mathcal{U}}$-minimal solution of* (MOP($\mathcal{L}$,$\mathcal{U}$)), *then $\overline{x}$ is a $(\varepsilon,k^0)$-minimal solution of* (SOP$^u$), *i.e.,*

$$\epsilon_{k^0}^{\mathcal{U}}\text{argwMin}(f_{\emptyset,\mathcal{U}},S,\mathbb{R}_+^q)\subseteq(\varepsilon,k^0)\text{argwMin}^u(F,S,C).$$

*(iii) If $\overline{x}\in S$ is a weakly $\epsilon_{k^0}^{\mathcal{L},\mathcal{U}}$-minimal solution of* (MOP($\mathcal{L}$,$\mathcal{U}$)), *then $\overline{x}$ is a $(\varepsilon,k^0)$-minimal solution of* (SOP$^s$), *i.e.,*

$$\epsilon_{k^0}^{\mathcal{L},\mathcal{U}}\text{argwMin}(f_{\mathcal{L},\mathcal{U}},S,\mathbb{R}_+^{p+q})\subseteq(\varepsilon,k^0)\text{argwMin}^s(F,S,C).$$

*Proof.* For the proof of $(i)$ assume that there exists $x \in S$ with $F(x) \prec^l_C F(\overline{x}) - \{\varepsilon k^0\}$. Using Lemma 3.2, this implies

$$\inf_{y \in F(x)} \ell(y) < \inf_{y \in F(\overline{x}) - \{\varepsilon k^0\}} \ell(y) = \inf_{y \in F(\overline{x})} \ell(y - \varepsilon k^0) = \inf_{y \in F(\overline{x})} \ell(y) - \varepsilon \ell(k^0)$$

for all $\ell \in C^* \setminus \{0_{Y^*}\}$. So, in particular it holds $\inf_{y \in F(x)} \ell^i(y) < \inf_{y \in F(\overline{x})} \ell^i(y) - \varepsilon \ell^i(k^0)$ for all $i \in \{1, ..., p\}$. This implies $f_{\mathcal{L},\emptyset}(x) <_{\mathbb{R}^p_+} f_{\mathcal{L},\emptyset}(\overline{x}) - \epsilon^{\mathcal{L}}_{k^0}$, which implies that $\overline{x} \in S$ is not a weakly $\epsilon^{\mathcal{L}}_{k^0}$-minimal solution of (MOP$(\mathcal{L}, \emptyset)$).
Statements $(ii)$ and $(iii)$ can be shown analogously. □

Theorem 3.13 states for instance that if one wants to obtain $(\varepsilon, k^0)$-minimal solutions of (SOP$^l$), $\epsilon^{\mathcal{L}}_{k^0}$-minimality of the solutions of (MOP$(\mathcal{L}, \emptyset)$) has to be guaranteed. If, on the other hand, one can only solve (MOP$(\mathcal{L}, \emptyset)$) to $\epsilon$-minimality for $\epsilon \in \mathbb{R}^p_+$, one is only guaranteed to find $(\varepsilon, k^0)$-minimal solutions of (SOP$^l$) if $\epsilon \leq \epsilon^{\mathcal{L}}_{k^0}$ holds.
Another issue that arises in practice is that multiobjective optimization problems can usually not be solved to full extend, i.e., one does not obtain the whole solution set $\text{argwMin}(f_{\mathcal{L},\mathcal{U}}, S, \mathbb{R}^{p+q}_+)$. Rather, a subset $\mathcal{A}$ of this set which, for instance, approximates the image of all minimal solutions well with respect to a certain quality measure is obtained. The arising question is whether such an approximation of the solution set of the multiobjective optimization problem yields a 'good' approximation of the solution set of the set-valued optimization problem. Moreover, it is of interest how to define a 'good' approximation in the first place. We propose a concept for this in the following definition. Thereby, concept $(i)$ is inspired by [19, Definition 1].

**Definition 3.14** *Let Assumption 2.1 be fulfilled and let $\varepsilon \geq 0$, $k^0 \in \text{int}(C)$ and finite sets $\mathcal{L}, \mathcal{U} \subseteq K^*$ of cardinalities $p$ and $q$ be given. The set $\mathcal{A} \subseteq S$ is called*

*(i) an $\varepsilon$-approximation of* (MOP$(\mathcal{L}, \mathcal{U})$) *if $\mathcal{A} \subseteq \text{argwMin}(f_{\mathcal{L},\mathcal{U}}, S, \mathbb{R}^{p+q}_+)$ and if for every $x \in S$ there exists $\bar{x} \in \mathcal{A}$ such that*

$$f_{\mathcal{L},\mathcal{U}}(\bar{x}) - \varepsilon e^{p+q} \leq_{\mathbb{R}^{p+q}_+} f_{\mathcal{L},\mathcal{U}}(x).$$

*(ii) an $(\varepsilon, k^0)$-approximation of* (SOP$^*$), *$* \in \{l, u, s\}$ if $\mathcal{A} \subseteq \text{argwMin}^*(F, S, C)$ and if for every $x \in S$ there exists $\bar{x} \in \mathcal{A}$ such that*

$$F(\bar{x}) - \{\varepsilon k^0\} \preccurlyeq^*_C F(x).$$

*For subsets $\mathcal{A}^1, \mathcal{A}^2$ of $S$ and $* \in \{l, u, s\}$ we define the approximation error of $\mathcal{A}^1$ given $\mathcal{A}^2$ as*

$$\text{Err}^*_{k^0}(\mathcal{A}^1 \mid \mathcal{A}^2) := \inf\{\varepsilon \geq 0 \mid \forall x \in \mathcal{A}^2 \exists \bar{x} \in \mathcal{A}^1 \colon F(\bar{x}) - \{\varepsilon k^0\} \preccurlyeq^*_C F(x)\}.$$

*The value $\text{Err}^*_{k^0}(\mathcal{A}^1 \mid S)$ is called approximation error of $\mathcal{A}^1$.*

The idea of these concepts is to pick only some weakly minimal points of the respective optimization problems for the approximation. Yet, these points should be good in the sense that if one just decreases their function values a little bit, the resulting points dominate the whole image set of the respective optimization problem.
A similar concept to $(i)$ has already been studied and algorithms have been developed to obtain such approximations. See, for instance, [19] and [22]. Here, we generalized

this idea to set-valued optimization problems and connect these concepts in this section.
The approximation error can be considered as a quality measure of a given approximation. Suppose that one compares two sets $\mathcal{A}^1, \mathcal{A}^2 \subseteq \operatorname{argwMin}^*(F,S,C)$. One would likely prefer $\mathcal{A}^1$ over $\mathcal{A}^2$ if and only if it has a smaller approximation error, i.e., $\operatorname{Err}^*_{k^0}(\mathcal{A}^1 \mid S) \leq \operatorname{Err}^*_{k^0}(\mathcal{A}^2 \mid S)$. Obviously, an $(\varepsilon, k^0)$-approximation $\mathcal{A}$ of (SOP$^*$) fulfills $\operatorname{Err}^*_{k^0}(\mathcal{A} \mid S) \leq \varepsilon$. We also have

$$\operatorname{Err}^*_{k^0}(\mathcal{A}^1 \mid \mathcal{A}^2) \leq \operatorname{Err}^*_{k^0}(\mathcal{A}^3 \mid \mathcal{A}^4) \tag{3.4}$$

for all sets $\mathcal{A}^1, \mathcal{A}^2, \mathcal{A}^3, \mathcal{A}^4 \subseteq S$ with $\mathcal{A}^1 \supseteq \mathcal{A}^3$ and $\mathcal{A}^2 \subseteq \mathcal{A}^4$. Also note that under Assumption 2.1 it holds $\operatorname{Err}^*_{k^0}(\operatorname{argwMin}^*(F,S,C) \mid S) = 0$ if $S$ is compact and $F$ is continuous ([9, Corollary 5.6]). In this case $\operatorname{argwMin}^*(F,S,C)$ is an $(0,k^0)$-approximation of (SOP$^*$) for all $* \in \{l,u,s\}$.
The following theorem states that, in fact, an $\varepsilon$-approximation of (MOP($\mathcal{L}$,$\mathcal{U}$)) is also an $(\tilde{\varepsilon}, k^0)$-approximation of (SOP$^*$), where $\tilde{\varepsilon}$ depends on $\varepsilon$.

**Theorem 3.15** *Let Assumption 2.1 be fulfilled, $k^0 \in \operatorname{int}(C)$ and $K^*$ be compact. Then, the following statements hold.*

*(i) Let $\mathcal{L} \subseteq K^*$ be a finite set and let $\varepsilon_1 > 0$ satisfy $K^* \subseteq \mathcal{L} + \varepsilon_1 \mathbb{B}$. Furthermore, let $\mathcal{A} \subseteq S$ be an $\varepsilon_2$-approximation of* (MOP($\mathcal{L}, \emptyset$))*. If $L^l < +\infty$, then $\mathcal{A}$ is an $(\tilde{\varepsilon}, k^0)$-approximation of* (SOP$^l$) *for*

$$\tilde{\varepsilon} = \bar{\varepsilon}^l(\varepsilon_1, \varepsilon_2, k^0) := \frac{\varepsilon_2 + 2L^l \varepsilon_1}{\inf_{\ell \in K^*} \ell(k^0)}.$$

*(ii) Let $\mathcal{U} \subseteq K^*$ be a finite set and let $\varepsilon_1 > 0$ satisfy $K^* \subseteq \mathcal{U} + \varepsilon_1 \mathbb{B}$. Furthermore, let $\mathcal{A} \subseteq S$ be an $\varepsilon_2$-approximation of* (MOP($\emptyset, \mathcal{U}$))*. If $L^u < +\infty$, then $\mathcal{A}$ is an $(\tilde{\varepsilon}, k^0)$-approximation of* (SOP$^u$) *for*

$$\tilde{\varepsilon} = \bar{\varepsilon}^u(\varepsilon_1, \varepsilon_2, k^0) := \frac{\varepsilon_2 + 2L^u \varepsilon_1}{\inf_{\ell \in K^*} \ell(k^0)}.$$

*(iii) Let $\mathcal{L},\mathcal{U} \subseteq K^*$ be finite sets and let $\varepsilon_1 > 0$ satisfy $K^* \subseteq \mathcal{L} + \varepsilon_1 \mathbb{B}$ and $K^* \subseteq \mathcal{U} + \varepsilon_1 \mathbb{B}$. Furthermore, let $\mathcal{A} \subseteq S$ be an $\varepsilon_2$-approximation of* (MOP($\mathcal{L}$,$\mathcal{U}$))*. If $L^s = \max\{L^l, L^u\} < +\infty$, then $\mathcal{A}$ is an $(\tilde{\varepsilon}, k^0)$-approximation of* (SOP$^s$) *for*

$$\tilde{\varepsilon} = \bar{\varepsilon}^s(\varepsilon_1, \varepsilon_2, k^0) := \frac{\varepsilon_2 + 2L^s \varepsilon_1}{\inf_{\ell \in K^*} \ell(k^0)} = \max\{\bar{\varepsilon}^l(\varepsilon_1, \varepsilon_2, k^0), \bar{\varepsilon}^u(\varepsilon_1, \varepsilon_2, k^0)\}.$$

*Proof.* First, we show that $\bar{\varepsilon}^*(\varepsilon_1, \varepsilon_2, k^0)$ is well defined for all $* \in \{l,u,s\}$. Since $K^*$ is compact, we have that $\inf_{\ell \in K^*} \ell(k^0) = \hat{\ell}(k^0)$ for some $\hat{\ell} \in K^*$. Furthermore, we have $k^0 \in \operatorname{int}(C)$ and hence, $\hat{\ell}(k^0) > 0$. This shows the well definedness of $\bar{\varepsilon}^*(\varepsilon_1, \varepsilon_2, k^0)$.
For the proof of $(i)$ first we observe that

$$\mathcal{A} \subseteq \operatorname{argwMin}(f_{\mathcal{L},\emptyset}, S, \mathbb{R}^p_+) \subseteq \operatorname{argwMin}^l(F,S,C)$$

holds by Theorem 3.6. Now, let $x \in S$ be arbitrarily chosen. Then, there exists $\bar{x} \in \mathcal{A}$ such that $f_{\{\ell\},\emptyset}(\bar{x}) \leq f_{\{\ell\},\emptyset}(x) + \varepsilon_2$ for all $\ell \in \mathcal{L}$. For every $\ell \in K^*$ there exists $\bar{\ell} \in \mathcal{L}$ with $||\ell - \bar{\ell}|| \leq \varepsilon_1$ by assumption. Hence, by Proposition 3.11 it holds

$$f_{\{\ell\},\emptyset}(x) - f_{\{\ell\},\emptyset}(\bar{x}) \geq f_{\{\bar{\ell}\},\emptyset}(x) - f_{\{\bar{\ell}\},\emptyset}(\bar{x}) - 2L^l ||\ell - \bar{\ell}|| \geq -\varepsilon_2 - 2L^l \varepsilon_1$$

for all $\ell \in K^*$. Then, for all $\ell \in K^*$ and $\tilde{\varepsilon} = \bar{\varepsilon}^l(\varepsilon_1, \varepsilon_2, k^0)$, it holds

$$\begin{aligned}
\inf_{y \in F(\bar{x}) - \{\tilde{\varepsilon} k^0\}} \ell(y) &= \inf_{y \in F(\bar{x})} \ell(y) - \tilde{\varepsilon}\ell(k^0) \\
&= f_{\{\ell\},\emptyset}(\bar{x}) - \frac{\varepsilon_2 + 2L^l\varepsilon_1}{\inf_{\tilde{\ell} \in K^*} \tilde{\ell}(k^0)} \ell(k^0) \\
&\leq f_{\{\ell\},\emptyset}(\bar{x}) - (\varepsilon_2 + 2L^l\varepsilon_1) \\
&\leq f_{\{\ell\},\emptyset}(x) \\
&= \inf_{y \in F(x)} \ell(y).
\end{aligned}$$

Using [12, Lemma 2.1] this implies $F(\bar{x}) - \{\tilde{\varepsilon} k^0\} \preccurlyeq_C^l F(x)$. Statements $(ii)$ and $(iii)$ can be shown analogously. □

Note that in Theorem 3.15 we have $\bar{\varepsilon}^*(\varepsilon_1, \varepsilon_2, k^0) \to 0$, $* \in \{l, u, s\}$ if $\varepsilon_1, \varepsilon_2 \to 0$. This goes in line with the intuition that choosing $\mathcal{L}$ and $\mathcal{U}$ such that (MOP($\mathcal{L}$,$\mathcal{U}$)) approximates (SOP$^*$) well (i.e., small $\varepsilon_1$) and then approximating (MOP($\mathcal{L}$,$\mathcal{U}$)) well (i.e., small $\varepsilon_2$) gives a good approximation of (SOP$^*$).
In general, it is hard to compute the approximation error of an arbitrary set $\mathcal{A} \subseteq S$. However, from Theorem 3.15 we obtain an upper bound $\bar{\varepsilon}^*(\varepsilon_1, \varepsilon_2, k^0)$ for the approximation error $\mathrm{Err}^*_{k^0}(\mathcal{A} \mid S)$ of $\mathcal{A}$ if one obtains this set from approximating the multi-objective optimization problem (MOP($\mathcal{L}$,$\mathcal{U}$)) as outlined in the theorem.
In a special case one might be able to calculate the values $\mathrm{Err}^*_{k^0}(\mathcal{A}^1 \mid \mathcal{A}^2)$ and $\mathrm{Err}^*_{k^0}(\mathcal{A}^2 \mid \mathcal{A}^1)$ when comparing two finite sets $\mathcal{A}^1$ and $\mathcal{A}^2$. We would then prefer $\mathcal{A}^1$ over $\mathcal{A}^2$ if and only if $\mathrm{Err}^*_{k^0}(\mathcal{A}^1 \mid \mathcal{A}^2) \leq \mathrm{Err}^*_{k^0}(\mathcal{A}^2 \mid \mathcal{A}^1)$. We apply these ideas in the following example which continues the special case from Example 3.9.

**Example 3.16** *We consider the sets $\mathcal{A}^1 = \{x^1\}$, $\mathcal{A}^2 = \{x^2\}$ and $k^0 = \frac{1}{\sqrt{2}}\binom{1}{1}$. Both sets are subsets of $\mathrm{argwMin}^l(F, S, C)$. From the construction it follows that for the given $\varepsilon = \frac{1}{2}(1 - \sqrt{1 - \frac{1}{4}\min_{i \neq j} \|\ell^i - \ell^j\|_2^2}) = \frac{2 - \sqrt{2+\sqrt{2}}}{4} \approx 0.0381$ we have $F(x^1) - \{\hat{\varepsilon} k^0\} \preccurlyeq^l_{\mathbb{R}^2_+} F(x^2)$ for $\hat{\varepsilon} = \sqrt{2}\varepsilon \approx 0.0538$ and that this relation will not hold for any $\tilde{\varepsilon} < \hat{\varepsilon}$. Hence,*

$$\mathrm{Err}^l_{k^0}(\mathcal{A}^1 \mid \mathcal{A}^2) = \hat{\varepsilon} \approx 0.0538.$$

*Furthermore, by doing some calculus one can check that*

$$\mathrm{Err}^l_{k^0}(\mathcal{A}^2 \mid \mathcal{A}^1) = \frac{\sqrt{82 - 31\sqrt{2}} - 6}{4} \approx 0.0443.$$

*So we would prefer aprroximation $\mathcal{A}^2$ over $\mathcal{A}^1$, since $\mathrm{Err}^l_{k^0}(\mathcal{A}^2 \mid \mathcal{A}^1) \leq \mathrm{Err}^l_{k^0}(\mathcal{A}^1 \mid \mathcal{A}^2)$. Since $S = \mathcal{A}^1 \cup \mathcal{A}^2$, we also have $\mathrm{Err}^l_{k^0}(\mathcal{A}^1 \mid S) = \mathrm{Err}^l_{k^0}(\mathcal{A}^1 \mid \mathcal{A}^2)$ and $\mathrm{Err}^l_{k^0}(\mathcal{A}^2 \mid S) = \mathrm{Err}^l_{k^0}(\mathcal{A}^2 \mid \mathcal{A}^1)$.*
*Now, we compare these parameters with the bounds we get from Theorem 3.15. One can calculate that $K^* \subseteq \mathcal{L} + \varepsilon_1\mathbb{B}$ for $\varepsilon_1 = \sqrt{2 - \sqrt{2 + \sqrt{2}}} \approx 0.3902$. Furthermore, we have $L^l = 1 + \varepsilon \approx 1.0381$ and $\inf_{\ell \in K^*} \ell(k^0) = \frac{1}{\sqrt{2}}$. Moreover, $\mathcal{A}^1$ is an $\varepsilon_2$-approximation of (MOP($\mathcal{L}$,$\emptyset$)) for $\varepsilon_2 = \varepsilon$. Theorem 3.15 states that $\mathcal{A}^1$ is an $(\tilde{\varepsilon}, k^0)$-approximation of (SOP$^l$) for*

$$\tilde{\varepsilon} = \bar{\varepsilon}^l(\varepsilon_1, \varepsilon_2, k^0) = \frac{\varepsilon_2 + 2L^l\varepsilon_1}{\inf_{\ell \in K^*} \ell(k^0)} \approx 1.1994.$$

*So we obtain the upper bound* $\mathrm{Err}^l_{k^0}(\mathcal{A}^1 \mid S) \leq 1.1994$ *from the theorem, which is not very sharp. On the other hand, remember that this holds in a fairly general setting. We do not need continuity of* $F$ *for example. Also, as already mentioned we can bring this bound arbitrarily close to zero.*

In the following, we describe how $\mathrm{Err}^*_{k^0}(\mathcal{A}^1 \mid \mathcal{A}^2)$ can be obtained for finite $\mathcal{A}^1, \mathcal{A}^2 \subseteq S$. We use the Tammer functional [15, Section 5.2] $\varphi_A \colon Y \to \bar{\mathbb{R}}$ defined by

$$\varphi_A(y) := \inf\{\gamma \in \mathbb{R} \mid y \in \gamma k^0 + A\}.$$

Next, we define $\phi^l_{k^0} \colon S \times Y \to \bar{\mathbb{R}}_+$ and $\phi^u_{k^0} \colon S \times Y \to \bar{\mathbb{R}}_+$ by

$$\begin{aligned}\phi^l_{k^0}(\bar{x}, y) &:= \inf\{\gamma \geq 0 \mid \bar{y} - \gamma k^0 \leq_C y, \bar{y} \in F(\bar{x})\} = \max\{0, \varphi_{-F(\bar{x})-C}(-y)\},\\ \phi^u_{k^0}(x, \bar{y}) &:= \inf\{\gamma \geq 0 \mid \bar{y} - \gamma k^0 \leq_C y, y \in F(x)\} = \max\{0, \varphi_{F(x)-C}(\bar{y})\}.\end{aligned}$$

Furthermore, we define $\psi^*_{k^0} \colon S \times S \to \mathbb{R}_+$ for all $* \in \{l, u, s\}$ by

$$\begin{aligned}\psi^l_{k^0}(x, \bar{x}) &:= \max_{y \in F(x)} \phi^l_{k^0}(\bar{x}, y),\\ \psi^u_{k^0}(x, \bar{x}) &:= \max_{\bar{y} \in F(\bar{x})} \phi^u_{k^0}(x, \bar{y}),\\ \psi^s_{k^0}(x, \bar{x}) &:= \max\{\psi^l_{k^0}(x, \bar{x}), \psi^u_{k^0}(x, \bar{x})\}.\end{aligned}$$

Under Assumption 2.1 the optimization problem for the calculation of $\phi^*_{k^0}(x, y)$ for all $* \in \{l, u\}$ and all $(x, y) \in S \times Y$ is a convex optimization problem. Moreover, the feasible sets of these respective optimization problems can be assumed to be compact without loss of generality, since all $F(x)$, $x \in S$ are compact and $C$ is closed. For the calculation of $\psi^*_{k^0}(x, \bar{x})$ for a given $* \in \{l, u, s\}$ and $(x, \bar{x}) \in S \times S$ one has to solve a maximizing problem over a convex set. The following lemma shows that the objective function $\phi^*_{k^0}(x, \cdot)$ is continuous and convex for all $x \in S$. Together with the compactness of all image sets $F(x)$ this guarantees well-definedness of $\phi^*_{k^0}(x, \bar{x})$ and more structure for its calculation. One could apply methods from concave minimization and convex maximization, respectively to obtain $\psi^*_{k^0}(x, \bar{x})$. See [3] for more information on these subjects.

**Lemma 3.17** *Let Assumption 2.1 be fulfilled,* $* \in \{l, u\}$ *and* $k^0 \in \mathrm{int}(C)$*. Then,* $\phi^*_{k^0}(x, \cdot)$ *is continuous, finite-valued and convex for every* $x \in S$*.*

*Proof.* Using [15, Theorem 5.2.3] we know that under our assumptions $\varphi_{-F(\bar{x})-C}(-y)$ and $\varphi_{F(x)-C}(y)$ are convex and finite valued in $y$. Using [15, Theorem 5.2.6] we obtain continuity in $y$ as well. The constant zero function is convex, finite valued and continuous as well and the maximum of two convex, finite valued and continuous functions is also convex, finite valued and continuous. □

The following theorem states how $\mathrm{Err}^*_{k^0}(\mathcal{A}^1 \mid \mathcal{A}^2)$ can be obtained for finite $\mathcal{A}^1, \mathcal{A}^2 \subseteq S$.

**Theorem 3.18** *Let Assumption 2.1 be fulfilled,* $* \in \{l, u, s\}$ *and* $k^0 \in \mathrm{int}(C)$*. Furthermore, let* $\mathcal{A}^1, \mathcal{A}^2 \subseteq S$ *be finite sets. Then, it holds*

$$\mathrm{Err}^*_{k^0}(\mathcal{A}^1 \mid \mathcal{A}^2) = \max_{x \in \mathcal{A}^2} \min_{\bar{x} \in \mathcal{A}^1} \psi^*_{k^0}(x, \bar{x}).$$

*Proof.* We only prove this theorem for $* = l$. The other cases can be shown analogously. Since $\mathcal{A}^1, \mathcal{A}^2$ are finite, $\max_{x\in\mathcal{A}^2} \min_{\bar{x}\in\mathcal{A}^1} \psi^l_{k^0}(x,\bar{x})$ is attained and equals some $\bar{\gamma} \in \mathbb{R}_+$. Then, for every $x \in \mathcal{A}^2$ it holds $\min_{\bar{x}\in\mathcal{A}^1} \psi^l_{k^0}(x,\bar{x}) \leq \bar{\gamma}$. This implies that for every $x \in \mathcal{A}^2$ there exists $\bar{x} \in \mathcal{A}^1$ with $\psi^l_{k^0}(x,\bar{x}) = \max_{y\in F(x)} \phi^l_{k^0}(\bar{x},y) \leq \bar{\gamma}$. It follows that for every $x \in \mathcal{A}^2$ there exists $\bar{x} \in \mathcal{A}^1$ such that for all $y \in F(x)$ it is

$$\min\{\gamma \geq 0 \mid \bar{y} - \gamma k^0 \leq_C y, \bar{y} \in F(\bar{x})\} \leq \bar{\gamma}.$$

This implies that for every $x \in \mathcal{A}^2$ there exists $\bar{x} \in \mathcal{A}^1$ such that for all $y \in F(x)$ there exists $\bar{y} \in F(\bar{x})$ such that $\bar{y} - \bar{\gamma}k^0 \leq_C y$. This is equivalent to

$$\forall x \in \mathcal{A}^2 \exists \bar{x} \in \mathcal{A}^1 \colon F(\bar{x}) - \{\bar{\gamma}k^0\} \preccurlyeq^l_C F(x).$$

Hence, we have $\mathrm{Err}^*_{k^0}(\mathcal{A}^1 \mid \mathcal{A}^2) \leq \max_{x\in\mathcal{A}^2} \min_{\bar{x}\in\mathcal{A}^1} \psi^*_{k^0}(x,\bar{x})$ and even equality, since the maximum is attained for some $x \in \mathcal{A}^2$. □

So, in general one would have to solve $|\mathcal{A}^1||\mathcal{A}^2|$ many maximizing problems of a continuous and convex objective function over a convex and compact set in order to compute $\mathrm{Err}^*_{k^0}(\mathcal{A}^1 \mid \mathcal{A}^2)$ for $* \in \{l,u\}$ and $2|\mathcal{A}^1||\mathcal{A}^2|$ many to compute $\mathrm{Err}^s_{k^0}(\mathcal{A}^1 \mid \mathcal{A}^2)$. For comparing two approximations, one would also have to compute $\mathrm{Err}^*_{k^0}(\mathcal{A}^2 \mid \mathcal{A}^1)$.
Finally, we outline a possibility to improve a given $(\varepsilon,k^0)$-approximation $\mathcal{A}$ of (SOP$^*$), $* \in \{l,u,s\}$. As we already pointed out in (3.4), if we add additional points $x \in \mathrm{argwMin}^*(F,S,C)$ to $\mathcal{A}$, the approximation error decreases. The following theorem further examines this idea. Therefore, we introduce the following optimization problem for $\mathcal{L},\mathcal{U} \subseteq K^*$ and $\hat{z} \in \mathbb{R}^{p+q}$.

$$\begin{aligned} \min\, & f_{\mathcal{L},\mathcal{U}}(x) \text{ w.r.t. } \mathbb{R}^{p+q} \\ \text{s.t. } & f_{\mathcal{L},\mathcal{U}}(x) \leq_{\mathbb{R}^{p+q}_+} \hat{z} \\ & x \in S. \end{aligned} \qquad (\mathrm{MOP}_{\hat{z}}(\mathcal{L},\mathcal{U}))$$

We derive this optimization problem by adding an additional constraint to (MOP($\mathcal{L},\mathcal{U}$)).

**Theorem 3.19** *Let Assumption 2.1 be fulfilled. Let $\mathcal{L} = \{\ell^1,...,\ell^p\} \subseteq K^*$ and $\mathcal{U} = \{\ell^{p+1},...,\ell^{p+q}\} \subseteq K^*$ with $p+q \geq 1$. Furthermore, let $* \in \{l,u,s\}$ and $\mathcal{A} \subseteq S$ be an $(\varepsilon,k^0)$-approximation of* (SOP$^*$)*. Let $\hat{z} \in \mathbb{R}^{p+q}$ and $\bar{x}$ be a weakly minimal solution of* (MOP$_{\hat{z}}$($\mathcal{L},\mathcal{U}$))*. Then, the following statements hold.*

*(i) If $* = l$ and $q = 0$, then $\mathcal{A}' := \mathcal{A} \cup \{\bar{x}\}$ is an $(\varepsilon,k^0)$-approximation of* (SOP$^l$)*.*

*(ii) If $* = u$ and $p = 0$, then $\mathcal{A}' := \mathcal{A} \cup \{\bar{x}\}$ is an $(\varepsilon,k^0)$-approximation of* (SOP$^u$)*.*

*(iii) If $* = s$, then $\mathcal{A}' := \mathcal{A} \cup \{\bar{x}\}$ is an $(\varepsilon,k^0)$-approximation of* (SOP$^s$)*.*

*Proof.* For the proof of $(i)$ first we show that $\bar{x}$ is a weakly minimal solution of (MOP($\mathcal{L},\emptyset$)). Otherwise, there would exist an element $x \in S$ with

$$f_{\mathcal{L},\emptyset}(x) <_{\mathbb{R}^p} f_{\mathcal{L},\emptyset}(\bar{x}) \leq_{\mathbb{R}^p} \hat{z}.$$

So $x$ would also be feasible for (MOP$_{\hat{z}}$($\mathcal{L},\emptyset$)), which is a contradiction.
Now, we apply Theorem 3.6 and conclude that $\bar{x}$ is a weakly minimal solution of (SOP$^l$). Therefore, the set $\mathcal{A}'$ is a subset of $\mathrm{argwMin}^l(F,S,C)$. Together with (3.4) we obtain that $\mathcal{A}'$ is an $(\varepsilon,k^0)$-approximation of (SOP$^l$).
Statements $(ii)$ and $(iii)$ can be shown analogously. □

In a special case, one can easily obtain points to add to an $(\varepsilon, k^0)$-approximation $\mathcal{A}$ of (SOP$^*$). If $S$ is compact and convex and $f_{\mathcal{L},\mathcal{U}}$ is continuous and convex, for every $r \in \mathbb{N}$ and $\lambda_i \geq 0$ for all $i \in \{1, ..., r\}$ with $\sum_{i=1}^r \lambda_i = 1$ and $x^1, ..., x^r \in \mathcal{A}$ it holds $\hat{x} := \sum_{i=1}^r \lambda_i x^i \in S$. With $\hat{z} := f_{\mathcal{L},\mathcal{U}}(\hat{x})$ the optimization problem (MOP$_{\hat{z}}(\mathcal{L},\mathcal{U})$) has a weakly minimal solution. One can simply apply a scalarization technique like weighted sum to this optimization problem to find a weakly minimal solution $\bar{x}$. One can then conclude that

$$f_{\mathcal{L},\mathcal{U}}(\bar{x}) \leq_{\mathbb{R}^{p+q}_+} \hat{z} = f_{\mathcal{L},\mathcal{U}}(\hat{x}) \leq_{\mathbb{R}^{p+q}_+} \sum_{i=1}^r \lambda_i f_{\mathcal{L},\mathcal{U}}(x^i).$$

Now, we apply these results to Example 3.5. Thereby, we show how Theorem 3.19 can be used in order to improve a given $(\varepsilon, k^0)$-approximation of (SOP$^*$).

**Example 3.20** *One can check that* $\operatorname{argwMin}^l(F, S, C) = S$. *We consider the set* $\mathcal{A} = \{0, \frac{\pi}{3}, \frac{\pi}{2}\} \subseteq \operatorname{argwMin}^l(F, S, C)$. *Doing some calculus, one can verify that $\mathcal{A}$ is an $(\varepsilon, k^0)$-approximation of* (SOP$^l$) *for* $\varepsilon = \operatorname{Err}^l_{k^0}(\mathcal{A} \mid S) = \frac{1}{\sqrt{2}} \approx 0.7071$.
*Now, we take* $x^1 = 0$, $x^2 = \frac{\pi}{3}$ *and* $\lambda_1 = \lambda_2 = \frac{1}{2}$ *and define* $\hat{x} := \lambda_1 x^1 + \lambda_2 x^2 = \frac{\pi}{6}$ *and* $\hat{z} := f_{\mathcal{L},\emptyset}(\hat{x})$. *We observe that* (MOP$_{\hat{z}}(\mathcal{L}, \emptyset)$) *has only one feasible point* $\bar{x} = \hat{x} = \frac{\pi}{6}$. *As a consequence $\bar{x}$ is a weakly minimal solution of* (MOP$_{\hat{z}}(\mathcal{L}, \emptyset)$). *By applying Theorem 3.19 we know that* $\mathcal{A}' := \mathcal{A} \cup \{\bar{x}\}$ *is also an $(\varepsilon, k^0)$-approximation of* (SOP$^l$). *We even have* $\operatorname{Err}^l_{k^0}(\mathcal{A}' \mid S) = \frac{\sqrt{3}-1}{2} \approx 0.3660 < 0.7071 \approx \operatorname{Err}^l_{k^0}(\mathcal{A} \mid S)$.

# 4 Numerical examples

In this section, we construct some set-valued optimization problems of which we know the sets of minimal elements. We formulate related multiobjective optimization problems (MOP$(\mathcal{L},\mathcal{U})$) and solve these in order to obtain approximations of the set-valued optimization problems. We compare different approximations by using the concept of the relative approximation error and visualize our results. Since we know the set of minimal solutions of the set-valued optimization problems, we can also give estimates for the approximation error of the approximations.
Similarly to the approach in [14, Chapter 10], we construct set-valued optimization problems from a multiobjective optimization problem. We do this in a way such that the set of minimal solutions of these optimization problems coincide. Therefore, let $X = Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, $k^0 = \frac{1}{\sqrt{2}}\binom{1}{1}$ and $S = \{x \in \mathbb{R}^2_+ \mid x_1^2 - 4x_1 + x_2 + 1.5 \leq 0\}$. Moreover, define $f \colon S \to \mathbb{R}^2$ by

$$f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} := \begin{pmatrix} \sqrt{1 + x_1^2} \\ x_1^2 - 4x_1 + x_2 \end{pmatrix}.$$

It is known [14, Example 10.4.2] that the set of minimal elements of $f$ over $S$ is given by

$$M := \{x \in \mathbb{R}^2_+ \mid x_1 \in [2 - \tfrac{\sqrt{10}}{2}, 2], x_2 = 0\}.$$

Furthermore, we define

$$\rho := \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}, H^l := \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, H^u := \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix}, H^s := \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Following the approach in [14, Chapter 10], we define the set-valued maps $F^*\colon S \to \mathbb{R}^2$ for all $* \in \{l, u, s\}$ by

$$F^*(x) := \{H^* f(x)\} + (\rho^\top f(x))\mathbb{B}_2.$$

Then, $F^*(x)$ is a ball for all $x \in S$ with center at $c^*(x) := H^* f(x)$ and radius $r(x) := \rho^\top f(x)$. Analogously to [14, Lemma 10.4.1] it holds

$$\operatorname{argMin}^*(F^*, S, \mathbb{R}^2_+) = M$$

for all $* \in \{l, u, s\}$. Now, for each $* \in \{l, u, s\}$ we constructed a set-valued optimization problem of which we know the set of minimal solutions.

Next, we aim to find weakly minimal solutions of these set-valued optimization problems using the multiobjective approach via (MOP($\mathcal{L}$,$\mathcal{U}$)). Afterwards, we compare the approximations that we obtained to the solution set $M$. We will also illustrate the effect of the choice of $\mathcal{L}$ and $\mathcal{U}$. Hence, we use two different sets for each optimization problem. This results in two different aprroximations per instance, which we can then compare by calculating their relative approximation errors. We also give estimates for the approximation error of these approximations by discretizing $M$ with 10000 equidistant points as

$$\hat{M} := \left\{ \begin{pmatrix} 2 - \frac{\sqrt{10}}{2} \\ 0 \end{pmatrix} + t \begin{pmatrix} \frac{\sqrt{10}}{2*9999} \\ 0 \end{pmatrix} \mid t \in \{0, 1, ..., 9999\} \right\}.$$

For a given approximation $\mathcal{A}$ we then estimate

$$\operatorname{Err}^*_{k^0}(\mathcal{A} \mid S) \approx \operatorname{Err}^*_{k^0}(\mathcal{A} \mid \hat{M}).$$

First, we investigate the case $* = l$.

We consider the two sets

$$\mathcal{L}^l_1 := \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\},$$

$$\mathcal{L}^l_2 := \left\{ \begin{pmatrix} \sin(\frac{\pi}{8}) \\ \cos(\frac{\pi}{8}) \end{pmatrix}, \begin{pmatrix} \sin(\frac{3\pi}{8}) \\ \cos(\frac{3\pi}{8}) \end{pmatrix} \right\}.$$

It holds $K^* \subseteq \mathcal{L}^l_1 + \varepsilon^l_1 \mathbb{B}_2$ for $\varepsilon^l_1 = \sqrt{2 - \sqrt{2}} \approx 0.7654$ and $K^* \subseteq \mathcal{L}^l_2 + \varepsilon^l_2 \mathbb{B}_2$ for $\varepsilon^l_2 = \sqrt{2 - \sqrt{2 + \sqrt{2}}} \approx 0.3902$.

We determine weakly minimal solutions of the multiobjective optimization problems (MOP($\mathcal{L}^l_1$, $\emptyset$)) and (MOP($\mathcal{L}^l_2$, $\emptyset$)) by epsilon-constraint method with 10 equidistant epsilons. Thereby, we obtained the subsets $\mathcal{A}^l_1 \subseteq S$ and $\mathcal{A}^l_2 \subseteq S$, respectively, of the set of weakly minimal solutions. By Theorem 3.6, these sets are also subsets of $\operatorname{argwMin}^l(F^l, S, \mathbb{R}^2_+)$. In this specific case it is easy to calculate $\psi^l_{k^0}(x, \bar{x})$. It holds

$$\psi^l_{k^0}(x, \bar{x}) = \sqrt{2} \max\{0, \max_{i=1,2}\{c_i(\bar{x}) - c_i(x)\} + r(x) - r(\bar{x})\}.$$

This allows to calculate the relative approximation errors as

$$\operatorname{Err}^l_{k^0}(\mathcal{A}^l_1 \mid \mathcal{A}^l_2) \approx 0.1384, \qquad \operatorname{Err}^l_{k^0}(\mathcal{A}^l_1 \mid \hat{M}) \approx 0.1604,$$

$$\operatorname{Err}^l_{k^0}(\mathcal{A}^l_2 \mid \mathcal{A}^l_1) \approx 0.1769, \qquad \operatorname{Err}^l_{k^0}(\mathcal{A}^l_2 \mid \hat{M}) \approx 0.2366.$$

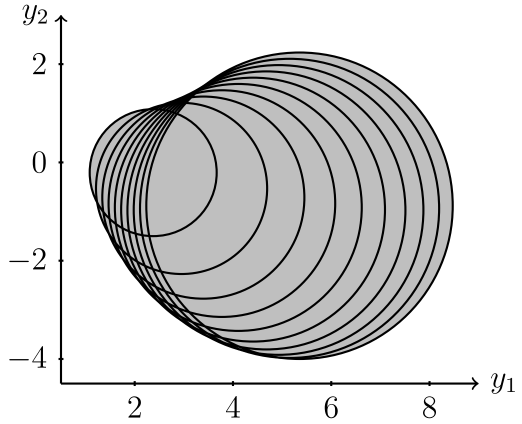


Figure 5: Image sets $F^l(x)$ for $x \in \mathcal{A}_1^l$

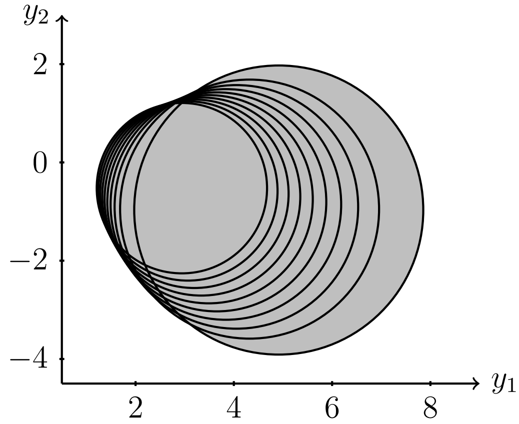


Figure 6: Image sets $F^l(x)$ for $x \in \mathcal{A}_2^l$

Hence, we would prefer $\mathcal{A}_1^l$ over $\mathcal{A}_2^l$. The image sets $F^l(x)$ for $x \in \mathcal{A}_1^l$ and $x \in \mathcal{A}_2^l$ are illustrated in Figure 5 and Figure 6, respectively.
For the case $* = u$, we also define $\mathcal{U}_1^u := \mathcal{L}_1^l$ and $\mathcal{U}_2^u := \mathcal{L}_2^l$. Consequently, we have $\varepsilon_1^u = \varepsilon_1^l$ and $\varepsilon_2^u = \varepsilon_2^l$.
We determine weakly minimal solutions of the multiobjective optimization problems (MOP($\emptyset$,$\mathcal{U}_1^u$)) and (MOP($\emptyset$,$\mathcal{U}_2^u$)) by epsilon-constraint method with 10 equidistant epsilons. Thereby, we obtained the subsets $\mathcal{A}_1^u \subseteq S$ and $\mathcal{A}_2^u \subseteq S$, respectively, of the set of weakly minimal solutions. By Theorem 3.6, these sets are also subsets of $\operatorname{argwMin}^u(F^u, S, \mathbb{R}^2_+)$. Here, we have

$$\psi^u_{k^0}(x, \bar{x}) = \sqrt{2} \max\{0, \max_{i=1,2}\{c_i(\bar{x}) - c_i(x)\} + r(\bar{x}) - r(x)\}.$$

We calculate the relative approximation errors as

$$\operatorname{Err}^u_{k^0}(\mathcal{A}_1^u \mid \mathcal{A}_2^u) \approx 0.1035, \qquad \operatorname{Err}^u_{k^0}(\mathcal{A}_1^u \mid \hat{M}) \approx 0.1604,$$

$$\operatorname{Err}^u_{k^0}(\mathcal{A}_2^u \mid \mathcal{A}_1^u) \approx 0.9150, \qquad \operatorname{Err}^u_{k^0}(\mathcal{A}_2^u \mid \hat{M}) \approx 0.9150.$$

Hence, we would prefer $\mathcal{A}_1^u$ over $\mathcal{A}_2^u$. The image sets $F^u(x)$ for $x \in \mathcal{A}_1^u$ and $x \in \mathcal{A}_2^u$ are illustrated in Figure 7 and Figure 8, respectively.
For the case $* = s$, we define

$$(\mathcal{L}_1^s, \mathcal{U}_1^s) := \left(\left\{\begin{pmatrix}0\\1\end{pmatrix}\right\}, \left\{\begin{pmatrix}1\\0\end{pmatrix}\right\}\right),$$

$$(\mathcal{L}_2^s, \mathcal{U}_2^s) := \left(\left\{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\right\}, \left\{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\right\}\right).$$

Consequently, we have $\varepsilon_1^s = \sqrt{2} \approx 1.4142$ and $\varepsilon_2^s = \sqrt{2 - \sqrt{2}} \approx 0.7654$.
We determine weakly minimal solutions of the multiobjective optimization problems (MOP($\mathcal{L}_1^s$,$\mathcal{U}_1^s$)) and (MOP($\mathcal{L}_2^s$,$\mathcal{U}_2^s$)) by epsilon-constraint method with 10 equidistant epsilons. Thereby, we obtained the subsets $\mathcal{A}_1^s \subseteq S$ and $\mathcal{A}_2^s \subseteq S$, respectively, of the set of weakly minimal solutions. By Theorem 3.6, these sets are also subsets of $\operatorname{argwMin}^s(F^s, S, \mathbb{R}^2_+)$. It holds

$$\psi^s_{k^0}(x, \bar{x}) = \max\{\psi^l_{k^0}(x, \bar{x}), \psi^u_{k^0}(x, \bar{x})\}.$$

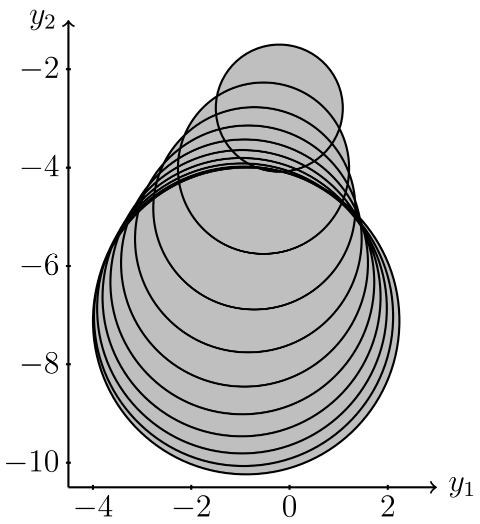


Figure 7: Image sets $F^u(x)$ for $x \in \mathcal{A}_1^u$

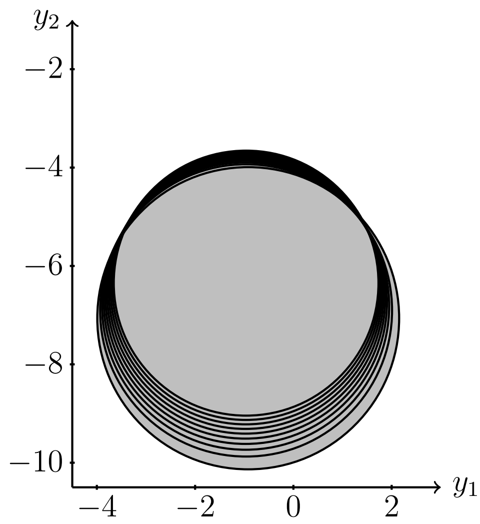


Figure 8: Image sets $F^u(x)$ for $x \in \mathcal{A}_2^u$

We calculate the relative approximation errors as

$$\mathrm{Err}^s_{k^0}(\mathcal{A}_1^s \mid \mathcal{A}_2^s) \approx 0.1157, \qquad \mathrm{Err}^s_{k^0}(\mathcal{A}_1^s \mid \hat{M}) \approx 0.1604,$$

$$\mathrm{Err}^s_{k^0}(\mathcal{A}_2^s \mid \mathcal{A}_1^s) \approx 0.2584, \qquad \mathrm{Err}^s_{k^0}(\mathcal{A}_2^s \mid \hat{M}) \approx 0.2937.$$

Hence, we would prefer $\mathcal{A}_1^s$ over $\mathcal{A}_2^s$. The image sets $F^s(x)$ for $x \in \mathcal{A}_1^s$ and $x \in \mathcal{A}_2^s$ are illustrated in Figure 9 and Figure 10, respectively.

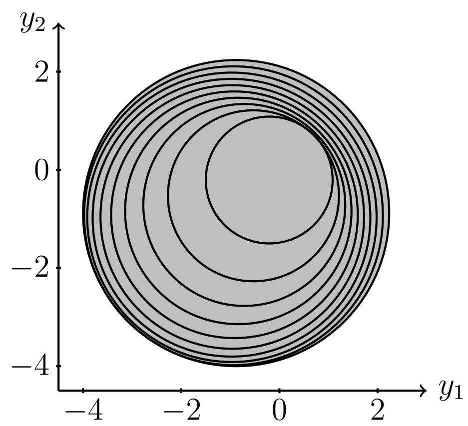


Figure 9: Image sets $F^s(x)$ for $x \in \mathcal{A}_1^s$

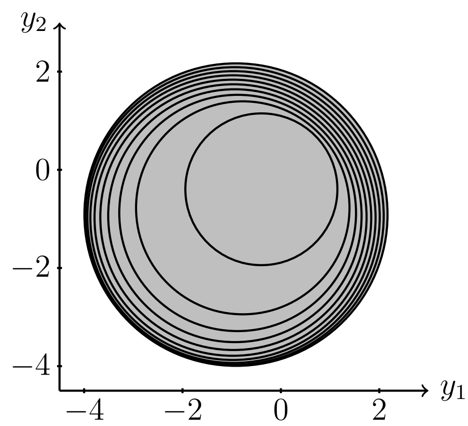


Figure 10: Image sets $F^s(x)$ for $x \in \mathcal{A}_2^s$

Note that we have $\mathrm{Err}^*_{k^0}(\mathcal{A}_1^* \mid \mathcal{A}_2^*) < \mathrm{Err}^*_{k^0}(\mathcal{A}_2^* \mid \mathcal{A}_1^*)$ despite $\varepsilon_1^* > \varepsilon_2^*$ for all $* \in \{l, u, s\}$. This shows again that the respective $\varepsilon_i^*$ are only used to obtain upper bounds for $\mathrm{Err}^*_{k^0}$ in Theorem 3.15. The actual values might obviously behave differently in comparison to their respective upper bounds.

# 5 Conclusion and Outlook

In this paper, we proposed a multiobjective optimization problem that can be used in order to find weakly minimal solutions of the set-valued optimization problem (SOP$^*$),

$* \in \{l, u, s\}$. In this way, we opened up a new approach to tackle set optimization with tools from multiobjective optimization. We have introduced a concept for approximating the solution set of a set valued optimization problem that uses an image space quality measure. We have shown that approximating the solution set of the multiobjective subproblem also yields an approximation of the solution set of the initial set-valued optimization problem. Thereby, the approximation error can be made arbitrarily small.
For future work it would be of interest to further examine implications of this relation between multiobjective and set-valued optimization.

## 6 Acknowledgements

We would like to thank Ernest Quintana from TU Ilmenau who contributed with fruitful discussions to the proof of Theorem 3.12. The authors would also like to thank the two anonymous referees whose suggestions and remarks helped to improve the quality of the paper. The work of the second author was funded by DFG, grant no. 392195690.

## References

[1] R. Baier, G. Eichfelder, and T. Gerlach. Optimality conditions for set optimization using a directional derivative based on generalized steiner sets. *Optimization*, 2020.

[2] B. Bank, J. Guddat, D. Klatte, B. Kummer, and K. Tammer. *Non-linear parametric optimization*. Springer, 1982.

[3] H. P. Benson. Concave minimization: theory, applications and algorithms. In *Handbook of global optimization*, pages 43–148. Springer, 1995.

[4] M. Dhingra and C. Lalitha. Approximate solutions and scalarization in set-valued optimization. *Optimization*, 66(11):1793–1805, 2017.

[5] G. Eichfelder, C. Krüger, and A. Schöbel. Decision uncertainty in multiobjective optimization. *Journal of Global Optimization*, 69(2):485–510, 2017.

[6] G. Eichfelder, J. Niebling, and S. Rocktäschel. An algorithmic approach to multiobjective optimization with decision uncertainty. *Journal of Global Optimization*, 77(1):3–25, 2020.

[7] A. Engau and M. M. Wiecek. Generating $\varepsilon$-efficient solutions in multiobjective programming. *European Journal of Operational Research*, 177(3):1566–1579, 2007.

[8] C. Günther, E. Köbis, and N. Popovici. Computing minimal elements of finite families of sets wrt preorder relations in set optimization. *Journal of Applied and Numerical Optimization*, 1(2):131–144, 2019.

[9] E. Hernández and L. Rodríguez-Marín. Existence theorems for set optimization problems. *Nonlinear Analysis: Theory, Methods & Applications*, 67(6):1726–1736, 2007.

[10] J. Jahn. Vector optimization: theory, applications, and extensions. 2011.

[11] J. Jahn. A derivative-free descent method in set optimization. *Computational Optimization and Applications*, 60(2):393–411, 2015.

[12] J. Jahn. Vectorization in set optimization. *Journal of Optimization Theory and Applications*, 167(3):783–795, 2015.

[13] J. Jahn and T. X. D. Ha. New order relations in set optimization. *Journal of Optimization Theory and Applications*, 148(2):209–236, 2011.

[14] A. A. Khan, E. Köbis, and C. Tammer. *Variational Analysis and Set Optimization: Developments and Applications in Decision Making.* CRC Press, 2019.

[15] A. A. Khan, C. Tammer, and C. Zălinescu. *Set-valued optimization.* Springer, 2015.

[16] E. Köbis and M. A. Köbis. Treatment of set order relations by means of a nonlinear scalarization functional: a full characterization. *Optimization*, 65(10):1805–1827, 2016.

[17] D. Kuroiwa. The natural criteria in set-valued optimization (nonlinear analysis and convex analysis). 1998.

[18] M. Pilecka. *Set-valued Optimization and its Application to Bilevel Optimization.* PhD thesis, Technische Universität Bergakademie Freiberg, 2016.

[19] H. Reuter. An approximation method for the efficiency set of multiobjective programming problems. *Optimization*, 21(6):905–911, 1990.

[20] C. Schrage and A. Löhne. An algorithm to solve polyhedral convex set optimization problems. *Optimization*, 62(1):131–141, 2013.

[21] O. Stein. *Bi-level strategies in semi-infinite programming*, volume 71. Springer Science & Business Media, 2013.

[22] K. Teichert. *A hyperboxing Pareto approximation method applied to radiofrequency ablation treatment planning.* PhD thesis, 2014.

[23] W. Zhang, S. Li, and K. L. Teo. Well-posedness for set optimization problems. *Nonlinear Analysis: Theory, Methods & Applications*, 71(9):3769–3778, 2009.