Robust Price Optimization of Multiple Products under Interval Uncertainties

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Abstract

In this paper, we solve the multiple product price optimization problem under interval uncertainties of the price sensitivity parameters in the demand function. The objective of the price optimization problem is to maximize the overall revenue of the firm where the decision variables are the prices of the products supplied by the firm. We propose an approach that yields optimal solutions under different variations of the estimated price sensitivity parameters. We adopt a robust optimization approach by building a data-driven uncertainty set for the parameters, and then construct a deterministic counterpart for the robust optimization model. The numerical results show that two objectives are fulfilled: the method reflects the uncertainty embedded in parameter estimations, and also an interval is obtained for optimal prices. We also conducted a simulation study to which we compared the results of our approach. The comparisons show that although robust optimization is deemed to be conservative, the results of the proposed approach show little loss compared to those from the simulation.

1   Introduction

In today's marketplace, price is considered as a key driver for optimizing revenue as it significantly affects the demand on products. Thus, the price optimization problem has long been studied as a significant decision-making problem for companies. There are numerous books, e.g., Phillips [23], Talluri and van Ryzin [27] and Özer and Phillips [22], that provide more comprehensive reviews on various price optimization models.

In this paper, we present a data-driven *robust optimization* (RO) based approach for the multi-product price optimization problem (MPPO) for a food supplier. The objective of the problem is to maximize the revenue of selling the products supplied by the firm while the decision variables are the prices of these products. We assume that for each

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product, a response function describes the demand for the product that depends on the price of the product itself as well as on the corresponding prices of its complementary and substitute products. The parameters of the demand function, which reflect the consumers' demand sensitivity to prices are typically estimated through various statistical approaches with associated confidence intervals. In practice, the price optimization problem is solved by assuming that the mid-points of the confidence intervals as the point estimators of the parameters to compute an approximation of the demands and the associated revenues at various prices. However, the solution obtained by this approximate problem may not reflect well the uncertainty involved in the parameter estimates.

In most of the existing studies, the demand functions with price variables are built with parameters whose values are obtained either from a priori knowledge or from exact estimates using existing data. Nevertheless, in practice the estimates of these parameters may take significantly different values in different circumstances, i.e., different customers may have different sensitivities to price, or customers may have different price reactions at different periods of the year. By neglecting such uncertainties, the demand functions may result in potentially inaccurate or even incorrect demand estimates at various prices. As suggested by a few researchers, e.g., Nahmias [21], Simchi-Levi et. al. [25] and Sheffi [24], many point estimates used in modern operations management decision models are wrong or meaningless. Instead, they recommend that the point estimates be replaced by their corresponding ranges to handle uncertainties.

Although in the field of revenue management, most conventional approaches aim to maximize the expected value of the revenue (e.g. Agrawal and Seshadri [1]), the risk-neutrality of the expected value may not well capture risk preferences of decision makers in many applications. Therefore, Levin et. al. [17], among the others, proposed the use of value-at-risk constraints to handle uncertainty in a more risk-averse fashion. However, one needs to estimate a probability distribution for the uncertain value and this often becomes a challenging task in practice. Furthermore, these parameters with probability information impose additional complexities to the underlying optimization models.

Our goal is to propose an approach to deal with the uncertainty involved in the estimated parameters of the demand functions. Our RO approach replaces point estimates by interval forecasts and requires no specific probability distribution. Instead, we choose a hyperplane uncertainty set based on constraining the deviation of uncertainty parameters from the nominal values. As an additional benefit, our RO approach addresses the risk-averse tendencies of decision makers.

To the best of our knowledge, the idea of RO was first introduced by Soyster [26]. Since then, especially with the advent of efficient algorithms to solve optimization problems, the field has attracted much more attention, and expanded significantly over the last few decades. RO addresses optimization problems with uncertainty in which the uncertainty model is not stochastic, but rather deterministic and set-based. For example, in 1995, Mulvey et. al. [20] developed a robust optimization approach to solve large-scale systems for a set of decision-making problems. The paper introduces a general framework for achieving a solution that is robust in terms of both feasibility and optimality with respect to *all* realizations of uncertain data. It also presents several applications solved using this approach.

The literature on computationally tractable RO models is rich. For instance, linear programming with ellipsoidal uncertainty sets has been addressed in [3], [4], [5], [14] and

[15]. In such a case, the optimization problem will correspond to a set of conic quadratic programming optimization problems. Alternatively, RO models with polyhedral uncertainty sets which can be formulated with linear/integer variables are also well-studied (see [4, 10, 11] for examples). For a thorough review of RO, we refer the interested readers to [7].

One of the most crucial steps in applying RO approaches is to construct an uncertainty set such that it contains all, or almost surely all, possible variations of uncertain data. Bound constraints and ellipsoidal uncertainty sets have been commonly used in many treatments [9]. However, these approaches are often ad-hoc, with emphases on sets that preserve computational tractability. Bertsimas and Brown [6] and Bertsimas et. al. [8] have proposed methods to construct polyhedral uncertainty sets using statistical hypothesis tests.

Previously, RO has been applied to several variations of the price optimization problem. Thiele [28] applied an RO approach to the pricing problem of a single product over a finite time horizon with a capacity limit. The uncertainty in the paper is the demand of the product which is assumed to be a function of price. The problem considered is different from the multiple products price optimization problem in our paper. In another paper, Thiele [29] proposed an RO approach for a problem with multiple products where a capacitated resource constraint is enforced, and the uncertainty is on the demand of the products. In contrast to the uncertainty set in both papers, our paper directly addresses the uncertainty on the sensitivity to the underlying prices in the demand functions which augments the uncertainty in the demand of the products.

In addition, Tien and Jaillet [18] studied an interesting pricing problem with general extreme value (GEV) choice models for customers. They assumed that the parameters of the choice model lie in an uncertainty set. They first consider an unconstrained pricing problem, then present an alternative version, and analyze the pricing problem with over-expected-revenue-penalties. For the Multinomial Logit (MNL) demand function and a rectangular uncertainty set, the RO problem can be converted to a deterministic one that can be solved efficiently.

The rest of our paper is organized as follows. In Section 2, we explain the multi-product price optimization model. The underlying uncertainty as well as the approach we use to handle the uncertainty are explained in Section 3. We then present our numerical experiments in Section 4. Finally, we conclude the paper with several findings and point out some potential future research directions in Section 5.

2   The Price Optimization Model

In this section, we first formulate the deterministic case of the multi-product price optimization problem which is modeled as a nonconvex quadratic problem with bilinear terms. We then present a relaxation of the problem into a linear program.

2.1   The Deterministic MPPO Model

For our MPPO, we assume a *linear price-response function* as suggested by Phillips [23]:

$$d(p) = \alpha + \beta p \tag{2.1}$$

where $\alpha > 0$ and $\beta < 0$ are the respective intercept and slope of the linear price-response function $d(p)$ which represents the sales volume as a function of price. The *satiating price* P, defined as the maximum allowed price at which demand drops to zero, is given by $P = -\alpha/\beta$. Phillips [23] points out that the *elasticity* of the linear price-response function is $-\beta p/(\alpha + \beta p)$ which ranges from 0 at $p = 0$ and approaches infinity as p approaches P.

In reality, it is well known that the demand of a product is often influenced by not only the price of itself, but also the prices of its complementary and substitute products. Therefore we extend the linear function (2.1) to incorporate the prices of complementary and substitute products. Let I represent the set of product indexes. The extended demand function for product i can be expressed by introducing additional price variables along with complementary/substitute products:

$$d_i(p) = \alpha_i + \beta_i p_i + \sum_{j \in C_i} \gamma_{ij} p_j \tag{2.2}$$

where p_j is the price of the complementary/substitute product j, γ_{ij} represents the corresponding cross-effect of product j on own product i, and C_i is the set of associated complementary/substitute products for product i.

The MPPO is to maximize the total revenue subject to a set of constraints:

$$\begin{aligned} \max_p \quad & \sum_{i \in I} p_i \cdot (\alpha_i + \beta_i p_i + \sum_{j \in C_i} \gamma_{ij} p_j) \\ \text{s.t.} \quad & Ap \leq b, \\ & l \leq p \leq u, \end{aligned} \tag{2.3}$$

where $p = (p_1, p_2, \ldots, p_n) \in \mathbb{R}^n$ (hence $n = |I|$), $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $l, u \in \mathbb{R}^n_+$ are finite. The constraints in (2.3) defined by matrix A and vector b represent business rules in practice.

We assume that the above problem is well-defined and *feasible*. We also assume that for $i \in I$, $\beta_i < 0$, so the demand of a product decreases when its price increases, and $\gamma_{ij} < 0$ for any complementary product $j \in C_i$ and $\gamma_{ij} > 0$ for any substitute product $j \in C_i$.

Thus, the objective function

$$f(p; \beta) = \sum_{i \in I} \left(\alpha_i p_i + \beta_i p_i^2 + \sum_{j \in C_i} \gamma_{ij} p_i p_j \right),$$

is, in general, a nonconvex quadratic function with bilinear terms.

2.2 A Relaxation of the Deterministic MPPO Model

For the quadratic and bilinear terms, p_i^2 and $p_i p_j$, in (2.3), we introduce two new sets of variables $x_i = -p_i^2$ and $y_{ij} = p_i \cdot p_j$, respectively. With this substitution, (2.3) is transformed into:

$$\max_{p,x,y} \quad \sum_{i \in I} \left(\alpha_i p_i - \beta_i x_i + \sum_{j \in C_i} \gamma_{ij} y_{ij} \right) \tag{2.4a}$$

$$\text{s.t.} \quad Ap \leq b, \tag{2.4b}$$

$$x_i \leq -p_i^2, \qquad\qquad\qquad i \in I, \tag{2.4c}$$

$$y_{ij} = p_i p_j, \qquad\qquad i \in I, j \in C_i, \tag{2.4d}$$

$$l \leq p \leq u. \tag{2.4e}$$

where the objective function (2.4a) is now linear, but the constraints (2.4c) and (2.4d) contain quadratic and bilinear terms, respectively.

Notice that for a given $i \in I$, since $\beta_i < 0$ and $p_i > 0$, $x_i < 0$ and $-\beta_i x_i < 0$. Because we are dealing with a maximization problem, it follows that x_i will attain its least negative value at $x_i = -p_i^2$ at optimality. Therefore, Problem (2.4) and Problem (2.3) are equivalent and consequently, we can deal with Problem (2.4) from this point on.

We now apply relaxations to Problem (2.4) to address the nonlinear constraints (2.4c) and (2.4d) and transform the problem into a more tractable linear program. First, we replace the quadratic term $-p_i^2$ by its piecewise linear approximation by discretizing its domain p_i into r segments:

$$l_i = \bar{p}_{i0} \leq \bar{p}_{i1} \leq \ldots \leq \bar{p}_{ir} = u_i.$$

Then at the kth knot:

$$\begin{aligned} p_i^2 &\approx -\bar{p}_{ik}^2 + (-2\bar{p}_{ik})(p_i - \bar{p}_{ik}) \\ &= \bar{p}_{ik}^2 - 2\bar{p}_{ik}p_i. \end{aligned}$$

Therefore, we can relax constraint (2.4c) with the following set of r supporting hyperplanes of p_i^2:

$$x_i \leq \bar{p}_{ik}^2 - 2\bar{p}_{ik}p_i \quad i \in C, k \in \{1, \ldots, r\}.$$

Figure 1 illustrates how this linearization technique to approximate the quadratic function $x_i = -p_i^2$.

Next, for constraint (2.4d), we relax the bilinear terms by introducing McCormick envelopes ([19, 2]) as follows:

$$\begin{aligned} y_{ij} &\geq l_j p_i + l_i p_j - l_i l_j, \\ y_{ij} &\leq u_j p_i + l_i p_j - l_i u_j, \\ y_{ij} &\leq l_j p_i + u_i p_i - l_j u_i, \\ y_{ij} &\geq u_j p_i + u_i p_j - u_i u_j. \end{aligned}$$
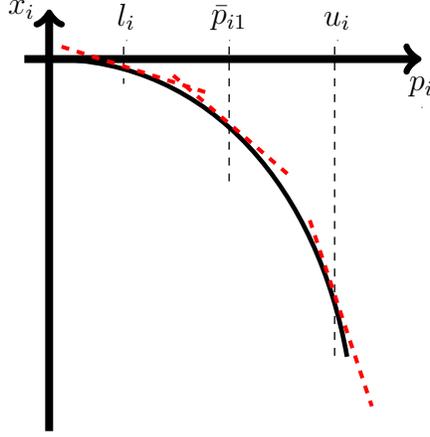
Figure 1: Linearizations of $x_i = -p_i^2$ at three points l_i, $\bar{p}_{i1}$ and u_i: dashed red lines are the linearizations

With the above relaxations, Problem (2.4) becomes:

$$\max_{p,x,y} \quad \sum_{i \in I} \left(\alpha_i p_i - \beta_i x_i + \sum_{j \in C_i} \gamma_{ij} y_{ij} \right) \tag{2.5a}$$

$$\text{s.t.} \quad Ap \leq b, \tag{2.5b}$$

$$x_i \leq \bar{p}_{ik}^2 - 2\bar{p}_{ik}p_i \qquad\qquad i \in I, k \in \{1, \ldots, r\}, \tag{2.5c}$$

$$y_{ij} \geq l_j p_i + l_i p_j - l_i l_j, \qquad\qquad i \in I, j \in C_i \tag{2.5d}$$

$$y_{ij} \leq u_j p_i + l_i p_j - l_i u_j, \qquad\qquad i \in I, j \in C_i \tag{2.5e}$$

$$y_{ij} \leq l_j p_i + u_i p_j - l_j u_i, \qquad\qquad i \in I, j \in C_i \tag{2.5f}$$

$$y_{ij} \geq u_j p_i + u_i p_j - u_i u_j, \qquad\qquad i \in I, j \in C_i \tag{2.5g}$$

$$l \leq p \leq u. \tag{2.5h}$$

3 Uncertainty

In our MMPO model, we assume that the uncertainty is on β_i and γ_{ij} which vary between their corresponding lower and upper bound values. Our goal is to find optimal price decisions so that the decisions remain optimal with respect to all possible realizations of β and γ. In this section, we first propose an RO approach to achieve the above goal. We then show how we construct an uncertainty set in building an RO model. Finally, we describe our method to solve the RO model.

3.1 The Robust Optimization Model for MPPO

The RO model is constructed by considering the worst-case scenario under all possible realizations of the uncertain parameter. Therefore, the RO counterpart of the objective function of Problem (2.5) is written as a *Max-Min* problem as follows:

$$\max_{p,x,y} \quad \min_{(\beta,\gamma)\in\mathcal{U}} \sum_{i\in I} \left(\alpha_i p_i - \beta_i x_i + \sum_{j\in C_i} \gamma_{ij} y_{ij} \right)$$

$$= \qquad \max_{p,x,y} \quad \left(\sum_{i\in I} \alpha_i p_i + \min_{(\beta,\gamma)\in\mathcal{U}} \sum_{i\in I} \left(-\beta_i x_i + \sum_{j\in C_i} \gamma_{ij} y_{ij} \right) \right)$$

Hence, Problem (2.5) under uncertainty becomes:

$$\max_{p,x,y} \quad \left(\sum_{i\in I} \alpha_i p_i + \min_{(\beta,\gamma)\in\mathcal{U}} \sum_{i\in I} \left(-\beta_i x_i + \sum_{j\in C_i} \gamma_{ij} y_{ij} \right) \right) \tag{3.1}$$

$$\text{s.t.} \quad (2.5b) - (2.5h),$$

where $\mathcal{U}$ is the uncertainty set of (β, γ). We will describe a method for constructing the uncertainty set later.

By introducing an ancillary variable η, we reformulate the model (3.1) as

$$\max_{p,x,y} \quad \eta + \sum_{i\in I} \alpha_i p_i$$

$$\text{s.t.} \quad \eta \leq \min_{(\beta,\gamma)\in\mathcal{U}} \left(\sum_{i\in I} \left(-\beta_i x_i + \sum_{j\in C_i} \gamma_{ij} y_{ij} \right) \right) \tag{3.2a}$$

$$(2.5b) - (2.5h).$$

3.2   Uncertainty Set Construction

We denote S_i^{RO} and T_{ij}^{RO} as two pre-determined numbers of uniformly distributed realizations $\hat{\beta}_{is}$ and $\hat{\gamma}_{ijs}$ for uncertain parameters β_i and γ_{ij}, respectively, from their corresponding interval estimates. Given the set of realizations, we define the following attributes for the uncertainty set for each product i:

- Mean of scenarios:

$$\bar{\beta}_i = \frac{\sum\limits_{s=1}^{S_i^{RO}} \hat{\beta}_{is}}{S_i^{RO}}.$$

$\bar{\beta}_i$ can be interpreted as the nominal value for the sensitivity parameter β_i. Likewise, for γ_{ij}, we have:

$$\bar{\gamma}_{ij} = \frac{\sum\limits_{s=1}^{T_{ij}^{RO}} \hat{\gamma}_{is}}{T_{ij}^{RO}}.$$

- Standard deviation of scenarios: The standard deviations of the samples of realizations for β and γ are calculated as follows.

$$\tilde{\beta}_i = \sqrt{\frac{\sum\limits_{s=1}^{S_i^{RO}} (\hat{\beta}_{is} - \bar{\beta}_i)^2}{S_i^{RO} - 1}},$$

and

$$\tilde{\gamma}_{ij} = \sqrt{\frac{\sum\limits_{s=1}^{T_{ij}^{RO}} (\hat{\gamma}_{ijs} - \bar{\gamma}_{ij})^2}{T_{ij}^{RO} - 1}}.$$

Now, for $k = \sum_{i \in I} |C_i|$ and writing $\boldsymbol{\beta}$ for the vectors of all β and $\boldsymbol{\gamma} \in \mathbb{R}^k$ for the vector of all γ_{ij}, we construct the uncertainty set $\mathcal{U}_\Delta$ as follows:

$$\mathcal{U}_\Delta = \left\{ \begin{bmatrix} \boldsymbol{\beta} \\ \boldsymbol{\gamma} \end{bmatrix} \in \mathbb{R}^{n+k} : \right.$$

$$\max_{i \in I} \frac{|\beta_i - \bar{\beta}_i|}{\tilde{\beta}_i} \leq \Delta, \tag{3.3a}$$

$$\hat{\beta}_i^{min} \leq \beta_i \leq \hat{\beta}_i^{max}, \ \forall i \in I, \tag{3.3b}$$

$$\max_{i \in I, j \in C_i} \frac{|\gamma_{ij} - \bar{\gamma}_{ij}|}{\tilde{\gamma}_{ij}} \leq \Delta, \tag{3.3c}$$

$$\left. \hat{\gamma}_{ij}^{min} \leq \gamma_{ij} \leq \hat{\gamma}_{ij}^{max}, \ \forall i \in I, j \in C_i \right\}, \tag{3.3d}$$

where $\hat{\beta}_i^{min} = \min_{s \in \{1,\ldots,S_i^{RO}\}} \{\hat{\beta}_{is}\}$, $\hat{\beta}_i^{max} = \max_{s \in \{1,\ldots,S_i^{RO}\}} \{\hat{\beta}_{is}\}$, $\hat{\gamma}_{ij}^{min} = \min_{s \in \{1,\ldots,T_{ij}^{RO}\}} \{\hat{\gamma}_{ijs}\}$ and $\hat{\gamma}_{ij}^{max} = \max_{s \in \{1,\ldots,T_{ij}^{RO}\}} \{\hat{\gamma}_{ijs}\}$.

Note that the constraints (3.3a) and (3.3c) in the above construction ensure that the maximum normalized deviation of the uncertain parameter β_i and γ_{ij}, respectively, from the nominal values among all the products remains under a certain positive scalar Δ. The parameter Δ is called *budget of uncertainty* [12]. Constraintd (3.3b) and (3.3d) also define bounds on individual β_i and γ_{ij}, respectively.

The budget of uncertainty constraint could also be specified by imposing a limit on the *total* amount of deviation across all products [13] as follows:

$$\sum_{i \in I} \frac{|\beta_i - \bar{\beta}_i|}{\tilde{\beta}_i} + \sum_{i \in I} \sum_{j \in C_i} \frac{|\gamma_{ij} - \bar{\gamma}_{ij}|}{\tilde{\gamma}_{ij}} \leq \Delta.$$

We therefore reformulate the uncertainty set as a polytope:

$$\mathcal{RU}_\Delta = \left\{ \begin{bmatrix} \boldsymbol{\beta} \\ \boldsymbol{\gamma} \\ \boldsymbol{\sigma}^\beta \\ \boldsymbol{\sigma}^\gamma \end{bmatrix} \in \mathbb{R}^{2n+2k} : \right.$$

$$\begin{aligned}
& \sigma_i^\beta \leq \Delta, && \forall i \in I, \\
& \sigma_{ij}^\gamma \leq \Delta, && \forall i \in I, j \in C_i, \\
& -\tilde{\beta}_i \cdot \sigma_i^\beta \leq \beta_i - \bar{\beta}_i \leq \tilde{\beta}_i \cdot \sigma_i^\beta, && \forall i \in I, \\
& \hat{\beta}_i^{min} \leq \beta_i \leq \hat{\beta}_i^{max}, && \forall i \in I, \\
& -\tilde{\gamma}_{ij} \cdot \sigma_{ij}^\gamma \leq \gamma_{ij} - \bar{\gamma}_{ij} \leq \tilde{\gamma}_{ij} \cdot \sigma_{ij}^\gamma, && \forall i \in I, j \in C_i, \\
& \hat{\gamma}_i^{min} \leq \gamma_{ij} \leq \hat{\gamma}_{ij}^{max}, && \left. \forall i \in I, j \in C_i \right\}
\end{aligned}$$

$$(3.4)$$

3.3 Solution Approach for the RO Model

Note that the right-hand side of the constraint (3.2a) constitutes an optimization problem. We substitute the uncertainty set $\mathcal{U}$ in (3.2a) with a hyperplane derived from (3.4), and yield the following optimization problem:

$$\min_{\beta,\gamma,\sigma^\beta,\sigma^\gamma} \quad \sum_{i \in I} \left(-\beta_i x_i + \sum_{j \in C_i} \gamma_{ij} y_{ij} \right)$$

$$\begin{aligned}
\text{s.t.} \quad & \sigma_i^\beta \leq \Delta, && \forall i \in I, && (3.5a) \\
& \sigma_{ij}^\gamma \leq \Delta, && \forall i \in I, j \in C_i, && (3.5b) \\
& \beta_i - \bar{\beta}_i \leq \tilde{\beta}_i \cdot \sigma_i^\beta, && \forall i \in I, && (3.5c) \\
& -\tilde{\beta}_i \cdot \sigma_i^\beta \leq \beta_i - \bar{\beta}_i, && \forall i \in I, && (3.5d) \\
& \hat{\beta}_i^{min} \leq \beta_i && \forall i \in I, && (3.5e) \\
& \beta_i \leq \hat{\beta}_i^{max}, && \forall i \in I, && (3.5f) \\
& \gamma_{ij} - \bar{\gamma}_{ij} \leq \tilde{\gamma}_{ij} \cdot \sigma_{ij}^\gamma, && \forall i \in I, j \in C_i, && (3.5g) \\
& -\tilde{\gamma}_{ij} \cdot \sigma_{ij}^\gamma \leq \gamma_{ij} - \bar{\gamma}_{ij}, && \forall i \in I, j \in C_i, && (3.5h) \\
& \hat{\gamma}_i^{min} \leq \gamma_{ij} && \forall i \in I, j \in C_i && (3.5i) \\
& \gamma_{ij} \leq \hat{\gamma}_{ij}^{max}, && \forall i \in I, j \in C_i && (3.5j) \\
& \sigma_i^\beta, \sigma_{ij}^\gamma \geq 0, && i \in I, j \in C_i.
\end{aligned}$$

By introducing dual multipliers μ_i^β, μ_{ij}^γ, π_{1i}^β, π_{2i}^β, λ_{1i}^β, λ_{2i}^β, π_{1ij}^γ, π_{2ij}^γ, λ_{1ij}^γ and λ_{2ij}^γ associated

with constraints (3.5a)-(3.5j), respectively, the dual problem becomes:

$$
\begin{aligned}
\max \quad & \sum_{i \in I} \left(-\Delta \mu_i^\beta - \bar{\beta}_i \pi_{1i}^\beta + \bar{\beta}_i \pi_{2i}^\beta + \hat{\beta}_i^{min} \lambda_{1i}^\beta - \hat{\beta}_i^{max} \lambda_{2i}^\beta \right) \\
& + \sum_{i \in I} \sum_{j \in C_i} \left(-\Delta \mu_{ij}^\gamma - \bar{\gamma}_i \pi_{1ij}^\gamma + \bar{\gamma}_i \pi_{2ij}^\gamma + \hat{\gamma}_{ij}^{min} \lambda_{1ij}^\gamma \right. \\
& \qquad\qquad \left. - \hat{\gamma}_{ij}^{max} \lambda_{2ij}^\gamma \right)
\end{aligned}
$$

$$
\begin{aligned}
\text{s.t.} \quad & -\pi_{1i}^\beta + \pi_{2i}^\beta + \lambda_{1i}^\beta - \lambda_{2i}^\beta = -x_i & i \in I, \\
& -\mu_i^\beta + \tilde{\beta}_i \pi_{1i}^\beta + \tilde{\beta}_i \pi_{2i}^\beta \le 0, & i \in I, \\
& -\pi_{1ij}^\gamma + \pi_{2ij}^\gamma + \lambda_{1ij}^\gamma - \lambda_{2ij}^\gamma = y_{ij} & i \in I, j \in C_i, \\
& -\mu_{ij}^\gamma + \tilde{\gamma}_{ij} \pi_{1ij}^\gamma + \tilde{\gamma}_{ij} \pi_{2ij}^\gamma \le 0, & i \in I, j \in C_i, \\
& \mu_i^\beta, \pi_{1i}^\beta, \pi_{2i}^\beta, \lambda_{1i}^\beta, \lambda_{2i}^\beta \ge 0. & i \in I, \\
& \mu_{ij}^\gamma, \pi_{1ij}^\gamma, \pi_{2ij}^\gamma, \lambda_{1ij}^\gamma, \lambda_{2ij}^\gamma \ge 0. & i \in I, j \in C_i.
\end{aligned}
\tag{3.6}
$$

In accordance with weak duality, any feasible solution to (3.6) provides a lower bound to (3.4). Therefore, we can enforce (3.2a) by adding the following constraint

$$
\begin{aligned}
\eta \le & \sum_{i \in I} \left(-\Delta \mu_i^\beta - \bar{\beta}_i \pi_{1i}^\beta + \bar{\beta}_i \pi_{2i}^\beta + \hat{\beta}_i^{min} \lambda_{1i}^\beta - \hat{\beta}_i^{max} \lambda_{2i}^\beta \right) + \\
& \sum_{i \in I} \sum_{j \in C_i} \left(-\Delta \mu_{ij}^\gamma - \bar{\gamma}_i \pi_{1ij}^\gamma + \bar{\gamma}_i \pi_{2ij}^\gamma + \hat{\beta}_i^{min} \lambda_{1ij}^\gamma - \hat{\beta}_i^{max} \lambda_{2ij}^\gamma \right)
\end{aligned}
$$

where $(\mu^\beta, \pi_{1.}^\beta, \pi_{2.}^\beta, \lambda_{1.}^\beta, \lambda_{2.}^\beta, \mu^\gamma, \pi_1^\gamma, \pi_2^\gamma, \lambda_1^\gamma, \lambda_2^\gamma)$ is feasible solution for (3.6). Therefore, the *Max-Min* problem can be reformulated as:

$$
Z_\Delta^* = \max \eta + \sum_{i \in I} \alpha_i p_i \tag{3.7a}
$$

$$
\begin{aligned}
\text{s.t.} \; \eta \le & \sum_{i \in I} \left(-\Delta \mu_i^\beta - \bar{\beta}_i \pi_{1i}^\beta + \bar{\beta}_i \pi_{2i}^\beta + \hat{\beta}_i^{min} \lambda_{1i}^\beta \right. \\
& \left. - \hat{\beta}_i^{max} \lambda_{2i}^\beta \right) + \sum_{i \in I} \sum_{j \in C_i} \left(-\Delta \mu_{ij}^\gamma - \bar{\gamma}_i \pi_{1ij}^\gamma + \bar{\gamma}_i \pi_{2ij}^\gamma \right.
\end{aligned}
$$

$$
\left. + \hat{\gamma}_{ij}^{min} \lambda_{1ij}^\gamma - \hat{\gamma}_{ij}^{max} \lambda_{2ij}^\gamma \right) \tag{3.7b}
$$

$$
\begin{aligned}
& -\pi_{1i}^\beta + \pi_{2i}^\beta + \lambda_{1i}^\beta - \lambda_{2i}^\beta = -x_i & i \in I, & \tag{3.7c} \\
& -\mu_i^\beta + \tilde{\beta}_i \pi_{1i}^\beta + \tilde{\beta}_i \pi_{2i}^\beta \le 0, & i \in I, & \tag{3.7d} \\
& -\pi_{1ij}^\gamma + \pi_{2ij}^\gamma + \lambda_{1ij}^\gamma - \lambda_{2ij}^\gamma = y_{ij} & i \in I, j \in C_i, & \tag{3.7e} \\
& -\mu_{ij}^\gamma + \tilde{\gamma}_{ij} \pi_{1ij}^\gamma + \tilde{\gamma}_{ij} \pi_{2ij}^\gamma \le 0, & i \in I, j \in C_i, & \tag{3.7f} \\
& (2.5b) \text{ - } (2.5h), & & \tag{3.7g} \\
& \mu_i^\beta, \pi_{1i}^\beta, \pi_{2i}^\beta, \lambda_{1i}^\beta, \lambda_{2i}^\beta \ge 0. & i \in I, & \tag{3.7h} \\
& \mu_{ij}^\gamma, \pi_{1ij}^\gamma, \pi_{2ij}^\gamma, \lambda_{1ij}^\gamma, \lambda_{2ij}^\gamma \ge 0. & i \in I, j \in C_i. & \tag{3.7i}
\end{aligned}
$$

The above formulation constitutes the RO model associated with a budget of uncertainty parameter Δ. This RO model is a conventional linear programming model that can be readily

solved using standard linear programming solvers. We report the results of our numerical experiments based on this model in the subsequent section.

4   Numerical Experiments

We implemented the RO model presented above using Python 3.5.2. We employed Gurobi 7.0 [16] to solve the linear programming model in (3.7). All tests reported were conducted on Amazon's Elastic Compute Cloud running Amazon Linux AMI 2018.03 on a machine with dual Intel(R) Xeon(R) 16-core E5-2686 v4 CPU @ 2.3GHz and 500GB memory.

4.1   Test Instance Description

We tested the proposed approach on 40 different instances from the data of a major food distributor in the United States. The company sells products to its customers via its geographically scattered divisions. Table 1 lists the name of the test instances (such that each instance represents a specific division), the number of products and number of cross terms (nonzero γ_{ij} terms) within a given time period in each instance.

Sample	# Products	# Cross Terms	Sample	# Products	# Cross Terms
S1	14,072	153	S2	11,563	247
S3	15,665	151	S4	1,148	19
S5	310	5	S6	1,975	23
S7	887	16	S8	987	15
S9	3,934	23	S10	2,078	35
S11	1,635	14	S12	2,008	16
S13	1,281	18	S14	629	10
S15	11,442	149	S16	2,940	79
S17	1,807	70	S18	924	53
S19	4,222	126	S20	1,675	26
S21	1,768	11	S22	333	22
S23	3,353	29	S24	1,232	34
S25	389,954	0	S26	288,719	0
S27	69,107	1,313	S28	141,475	3,409
S29	35,679	0	S30	298,750	0
S31	23,269	439	S32	9,518	201
S33	23,092	413	S34	23,249	481
S35	20,593	366	S36	18,862	232
S37	375,069	0	S38	271,054	0
S39	83,211	2,139	S40	112,555	2,686

Table 1: Statistics about instances: Sample name and number of products and number of cross terms

4.2 Parameter Values of the RO method

The value of Δ plays a crucial role in our tests. We select and test the values as follows. First, let us revisit the constraint involving Δ:

$$\max_{i \in I} \left\{ \frac{|\beta_i - \bar{\beta}_i|}{\tilde{\beta}_i} \right\} \leq \Delta, \quad \forall i \in I,$$

$$\max_{i \in I, j \in C_i} \frac{|\gamma_{ij} - \bar{\gamma}_{ij}|}{\tilde{\gamma}_{ij}} \leq \Delta, \quad \forall i \in I, j \in C_i.$$

Obviously, the minimum value for Δ is zero, in which case all the β_i values will be equal to their nominal value $\bar{\beta}_i$. We can also find the maximum value for Δ by:

$$\Delta_{max}^{\beta} = \max_{i \in I} \left\{ \max \left\{ \frac{|\hat{\beta}_i^{min} - \bar{\beta}_i|}{\tilde{\beta}_i}, \frac{|\hat{\beta}_i^{max} - \bar{\beta}_i|}{\tilde{\beta}_i} \right\} \right\},$$

and

$$\Delta_{max}^{\gamma} = \max_{i \in I, j \in C_i} \left\{ \max \left\{ \frac{|\hat{\gamma}_i^{min} - \bar{\gamma}_i|}{\tilde{\gamma}_i}, \frac{|\hat{\gamma}_i^{max} - \bar{\gamma}_i|}{\tilde{\gamma}_i} \right\} \right\}.$$

Therefore, we choose $\Delta_{max} = \max\{\Delta_{max}^{\beta}, \Delta_{max}^{\gamma}\}$. We then divide the interval $[0, \Delta_{max}]$ into 10 equally distanced segments, namely $[\Delta_1, \Delta_2], [\Delta_2, \Delta_3], \ldots, [\Delta_{10}, \Delta_{11}]$ where $\Delta_1 = 0$ and $\Delta_{11} = \Delta_{max}$, and test each of the Δ values in our subsequent experiments.

Other parameters of the model are S_i^{RO} and T_{ij}^{RO}, the numbers of samples used for calculating the means and variances for β_i and γ_{ij}. We set these two parameters to 50 and 5, respectively, reflecting the fact that the cross price elasticities are usually associated with smaller intervals compared with the own price elasticities, as found in our real data sets.

4.3 Test Results

To present our results, we show how changing the value of Δ affects the optimal value of the robust counterpart problem (3.7). To do this, we calculate the percent difference between the maximum and the minimum value of Z^* associated with values of Δ in problem (3.7). The maximum and minimum values of Z^* are attained when $\Delta = 0$ and $\Delta = \Delta_{max}$, i.e., the least conservative case and the most conservative case, respectively. We summarize the results in Table 2 by only displaying percent of decline from the least conservative to the most conservative case. We observe that the differences are all within in the range $(0.05\%, 6.96\%)$ with an average of 1.14%.

The distribution of the percent differences is depicted in the box plot of Figure 2. The detailed results are shown in the Appendix (Table 5).

Sample	Decline (%)	Sample	Decline (%)
S1	4.0869	S2	0.7784
S3	6.1058	S4	5.2174
S5	0.0964	S6	0.2366
S7	0.0844	S8	0.1391
S9	3.5418	S10	0.1005
S11	0.3744	S12	0.1979
S13	0.1200	S14	0.1398
S15	0.3392	S16	0.0951
S17	0.1019	S18	0.0716
S19	0.1092	S20	0.1906
S21	0.0898	S22	0.0757
S23	0.2508	S24	0.0681
S25	0.8945	S26	1.8451
S27	0.0501	S28	0.0691
S29	2.4146	S30	0.5939
S31	6.9645	S32	2.6059
S33	1.2579	S34	0.2168
S35	0.1478	S36	0.8719
S37	3.0746	S38	1.1271
S39	0.4145	S40	0.3026

Table 2: Percent decline from the least conservative to the most conservative solutions with RO approach.

4.4 Simulation Validation

We further validate our RO approach by comparing results from a set of Monte Carlo simulation tests with our RO solutions. We run a Monte Carlo simulations by taking S^{sim} uniformly distributed samples β_{is} from the interval $[\hat{\beta}_i^{min}, \hat{\beta}_i^{max}]$ and γ_{ijs} from $[\hat{\gamma}_{ij}^{min}, \hat{\gamma}_{ij}^{max}]$, and then solving each of the price optimization problems with given sampling values for β and γ:

$$f(\boldsymbol{\beta}_s, \boldsymbol{\gamma}_s) = \max_{p,x,y} \quad \sum_{i \in I} (\alpha_i p_i - \beta_{is} x_i + \sum_{j \in C_i} \gamma_{ij} y_{ij})$$
$$\text{s.t.} \quad (2.5b) - (2.5h). \tag{4.1}$$

The value of the simulation will then be the mean of optimal values from each replication, i.e. $Z_{sim}^* = \frac{1}{S^{sim}} \sum_s f(\boldsymbol{\beta}_s, \boldsymbol{\gamma}_s)$.

In these experiments, with $S^{sim} = 50$, we compare the values of the simulation Z_{sim}^* with the values of RO formulation (3.7). This comparison discloses the degree of conservativeness of the RO solutions. To perform such a comparison, we compare Z_{sim}^* with the least conservative case (achieved at Δ_1), the most conservative case Δ_{max} and the average across all Δ values, and calculate the corresponding difference ratios. For instance the difference ratio in the column under Δ_1 is attained from

$$\frac{Z_{sim}^* - Z_{\Delta_1}^*}{Z_{\Delta_1}^*}.$$

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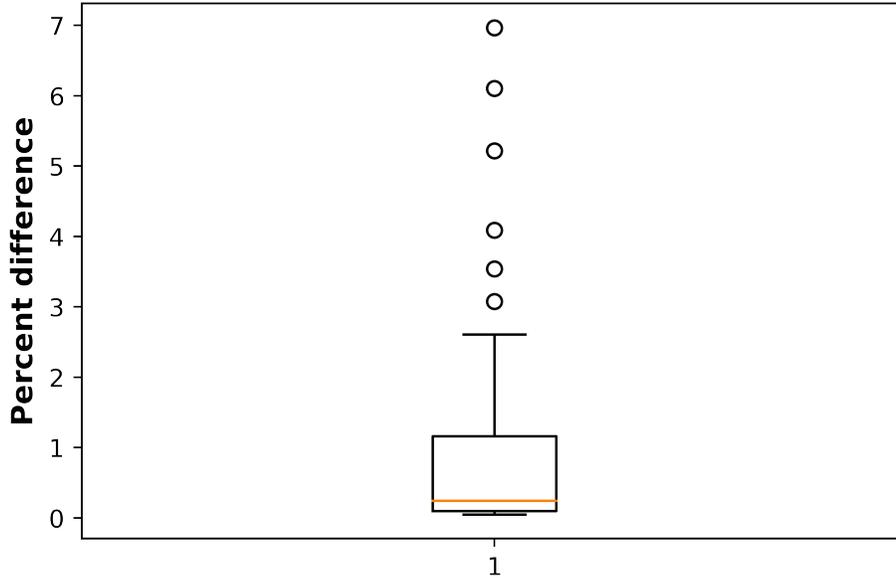
Figure 2: Boxplot of percent difference between least and most conservative solutions of RO problem for instances.

Table 3 summarizes these comparisons. while a negative value of difference ratio in the table implies that the RO value is better than the simulation, a positive value shows conversely. Note that most of the instances (35 out of 40) have positive values in Δ_{max} column, indicating RO results are worse than the simulation values. However, in most of such cases (30 out of 35) the difference ratios are less than 0.5%, and there are only three cases where the difference ratios are over 1%. This implies that for the most conservative scenarios, RO solution yields results with little losses as compared to the solutions from the simulation. On the other hand, in the least conservative case (under column Δ_1), most (31 of 40) differences are negative, demonstrating RO results are better than those from the simulations, while there are only nine instances where it is not and only in one of which the difference ratio is above 0.32%.

The other major benefit that we observe in our experiments is the gain from the computational time required for our RO approach. Table 4 lists the solution time of both RO approach and simulation. The solution times are significantly higher in simulation comparing to those using the RO approach. This is aligned with the expectations as the simulation needs to solve S^{sim} number of linear programs whereas the RO approach needs to only solve one linear program. The comparison of the solution time shows that the relative difference calculated by $\frac{t_{\text{Simulation}} - t_{\text{RO}}}{t_{\text{RO}}}$ has a geometric mean of 21.2 which means the simulation on average takes 21 times longer to solve. The advantage in computation times of our RO approach represents a significant benefit that could be achieved in many practical applications.

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Sample	Δ_1 (%)	Average (%)	Δ_{max} (%)	Sample	Δ_1 (%)	Average (%)	Δ_{max} (%)
S1	-1.8890	0.2668	0.9437	S2	-0.0190	0.2335	0.2986
S3	-2.8905	0.4543	1.9151	S4	0.2826	3.6168	5.0061
S5	-0.1005	-0.1198	-0.0184	S6	-0.1459	0.0224	0.0721
S7	0.0542	0.0965	0.1128	S8	0.1286	0.1395	0.1465
S9	-5.0707	-2.6889	-1.8811	S10	0.0430	0.1518	0.1716
S11	-0.2461	-0.0556	0.0774	S12	-0.0706	0.0239	0.1493
S13	-0.0518	0.1993	0.2539	S14	-0.0023	0.0169	0.0832
S15	-0.1280	0.0184	0.2112	S16	-0.0800	0.0541	0.1453
S17	-0.1146	-0.0004	0.0926	S18	0.0192	0.0281	0.0357
S19	-0.0865	-0.0006	0.0103	S20	-0.1491	-0.0081	0.0242
S21	-0.0953	0.0578	0.0941	S22	0.0062	0.0327	0.0401
S23	-0.2302	-0.0727	0.0346	S24	0.0034	0.0293	0.0658
S25	-0.6957	-0.1953	0.1140	S26	-1.5647	-0.1403	0.3698
S27	-0.0236	0.0398	0.0529	S28	-0.1683	-0.0924	-0.0981
S29	-1.8245	-03922	0.3490	S30	-0.5110	0.0049	0.1277
S31	-10.1855	-5.0828	-2.4091	S32	1.8656	2.6629	3.4037
S33	-0.2890	0.2004	0.3969	S34	-0.0104	0.0221	0.0562
S35	-0.0402	0.0493	0.0873	S36	-1.3137	-0.1453	0.296
S37	-4.7585	-3.0283	-2.0466	S38	-0.8910	-0.236	0.2993
S39	-0.0165	0.0100	0.1477	S40	0.3194	0.5114	0.6030

Table 3: Decline of the simulation optimal value from the RO models in the least conservative (Δ_1) and the most conservative (Δ_{max}). Also decline of the simulation optimal value from the average of the optimal values of the RO models across all Δ values.

Sample	RO (s)	Simulation (s)	Sample	RO (s)	Simulation (s)
S1	18.1	298.7	S2	17.5	323.5
S3	9.7	203.3	S4	0.2	6.3
S5	0.2	8.6	S6	0.4	14.6
S7	0.3	8.4	S8	0.3	10.4
S9	1.0	31.5	S10	0.7	22.8
S11	0.3	10.1	S12	0.3	11.6
S13	0.5	15.3	S14	0.1	3.5
S15	5.5	118.5	S16	1.4	39.7
S17	0.4	11.5	S18	0.2	5.5
S19	1.2	40.2	S20	0.4	11.3
S21	0.7	22.6	S22	0.1	1.9
S23	1.5	37.2	S24	0.2	7.7
S25	323.6	2746.9	S26	206.0	1759.1
S27	332.7	5622.9	S28	888.8	35712.6
S29	8.1	182.3	S30	187.0	1842.6
S31	106.0	1410.4	S32	23.2	395.4
S33	90.6	1152.9	S34	79.8	1259.7
S35	71.2	1044.9	S36	29.2	492.9
S37	313.9	2630.8	S38	177.8	1649.9
S39	436.6	12469.6	S40	436.6	12469.6

Table 4: Solution time for RO approach and simulation in seconds

5 Conclusions and Future Directions

In this paper, we investigate a multiple product price optimization problem with a linear demand function associated with prices on reference products as well as associated complementary/substitute products. The problem is associated with uncertainty where the sensitivity parameters on the product prices are expressed using uncertainty intervals. In contrast to the common practice of solving the problem with a point estimator of the interval, we devise a new robust optimization based approach to handle such uncertainty intervals.

In our proposed approach, we construct an uncertainty set based on all realizations of the parameters. The set depends on the budget of uncertainty parameter that controls the level of conservatism. With this set, a robust optimization formulation counterpart is developed that takes the uncertainty set into account, and it can be solved as a linear programming model. We further test our proposed model and solution approach based on several real data sets. As is expected, by increasing the budget of uncertainty value of the uncertainty set parameter, the optimal revenue decreases. An additional set of simulation experiments shows that although robust optimization is typically a more conservative approach, the losses of optimal values with the approach, in most instances, are quite insignificant. From the solution time perspective, while the simulation requires much longer time to solve, the proposed RO method, designed to address the uncertainty issue, can yield robust solutions in much shorter time at the price of being a little conservative.

We plan to further our research in several areas in the future. First, we currently use a uniform sampling approach from the intervals of the sensitivity parameters. However, in reality, some more refined information including posterior distribution may be collected and estimated for such parameters. A more appropriate sampling approach can be employed to better suit the posterior distribution of the parameters with uncertainty.

Also, in our robust optimization approach, we formulate the uncertainty set as explained in $\mathcal{U}_\Delta$, which was linearized in (3.4). However, there are some other approaches in the literature that might result in different approximation of the uncertainty associated with the parameters (for example [8].) Such new approximation methods are worth exploring in the future.

Finally, we find a simple linear demand response function appropriate for our application. More complicated demand response models, such as MNL demand functions, may be more appropriate and accurate in other price optimization applications. Hence, studying RO approach with such models could be an interesting future direction.

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Appendix

Table 5 presents each Δ value with its associated optimal objective function value (revenue) of each instance. For ease of exposition, the optimal revenue value (denoted as Z^*) is normalized so the base scenario with $\Delta = 0$ is set to zero. As we also expected, the larger the Δ value, the smaller Z_Δ^*, as the problem becomes more conservative.

Sample		\multicolumn{11}{c}{Optimal Values}										Decline (%)
		1	2	3	4	5	6	7	8	9	10	11	
S1	Δ	0.0	0.65	1.3	1.9	2.6	3.2	3.9	4.5	5.2	5.8	6.5	4.0869
	Z^*	1.000000	0.981036	0.964493	0.959536	0.959145	0.959132	0.959131	0.959131	0.959131	0.959131	0.959131	
S2	Δ	0.0	0.55	1.1	1.7	2.2	2.8	3.3	3.9	4.4	5.0	5.5	0.7784
	Z^*	1.000000	0.997144	0.994409	0.992639	0.992311	0.992245	0.992220	0.992216	0.992216	0.992216	0.992216	
S3	Δ	0.0	0.46	0.92	1.4	1.8	2.3	2.8	3.2	3.7	4.1	4.6	6.1058
	Z^*	1.000000	0.980973	0.962698	0.948051	0.940493	0.939001	0.938953	0.938943	0.938942	0.938942	0.938942	
S4	Δ	0.0	0.39	0.78	1.2	1.6	1.9	2.3	2.7	3.1	3.5	3.9	5.2174
	Z^*	1.000000	0.986918	0.973921	0.960932	0.948030	0.947895	0.947826	0.947826	0.947826	0.947826	0.947826	
S5	Δ	0.0	0.35	0.71	1.1	1.4	1.8	2.1	2.5	2.8	3.2	3.5	0.0964
	Z^*	1.000000	0.999791	0.999583	0.999374	0.999165	0.999040	0.999036	0.999036	0.999036	0.999036	0.999036	
S6	Δ	0.0	0.4	0.8	1.2	1.6	2.0	2.4	2.8	3.2	3.6	4.0	0.2366
	Z^*	1.000000	0.999404	0.998809	0.998223	0.997753	0.997637	0.997634	0.997634	0.997634	0.997634	0.997634	
S7	Δ	0.0	0.4	0.8	1.2	1.6	2.0	2.4	2.8	3.2	3.6	4.0	0.0844
	Z^*	1.000000	0.999798	0.999596	0.999395	0.999204	0.999156	0.999156	0.999156	0.999156	0.999156	0.999156	
S8	Δ	0.0	0.39	0.77	1.2	1.5	1.9	2.3	2.7	3.1	3.5	3.9	0.1391
	Z^*	1.000000	0.999667	0.999334	0.999002	0.998704	0.998611	0.998609	0.998609	0.998609	0.998609	0.998609	
S9	Δ	0.0	0.4	0.8	1.2	1.6	2.0	2.4	2.8	3.2	3.6	4.0	3.5418
	Z^*	1.000000	0.988791	0.977581	0.966854	0.964664	0.964583	0.964582	0.964582	0.964582	0.964582	0.964582	
S10	Δ	0.0	0.42	0.83	1.2	1.7	2.1	2.5	2.9	3.3	3.7	4.2	0.1005
	Z^*	1.000000	0.999753	0.999507	0.999260	0.999055	0.998999	0.998995	0.998995	0.998995	0.998995	0.998995	
S11	Δ	0.0	0.39	0.77	1.2	1.5	1.9	2.3	2.7	3.1	3.5	3.9	0.3744
	Z^*	1.000000	0.999164	0.998361	0.997558	0.996762	0.996259	0.996256	0.996256	0.996256	0.996256	0.996256	
S12	Δ	0.0	0.39	0.77	1.2	1.5	1.9	2.3	2.7	3.1	3.5	3.9	0.1979
	Z^*	1.000000	0.999501	0.999001	0.998502	0.998129	0.998021	0.998021	0.998021	0.998021	0.998021	0.998021	
S13	Δ	0.0	0.41	0.83	1.2	1.7	2.1	2.5	2.9	3.3	3.7	4.1	0.1200
	Z^*	1.000000	0.999697	0.999394	0.999091	0.998829	0.998801	0.998800	0.998800	0.998800	0.998800	0.998800	
S14	Δ	0.0	0.38	0.77	1.1	1.5	1.9	2.3	2.7	3.1	3.4	3.8	0.1398
	Z^*	1.000000	0.999686	0.999372	0.999058	0.998750	0.998604	0.998602	0.998602	0.998602	0.998602	0.998602	
S15	Δ	0.0	0.48	0.95	1.4	1.9	2.4	2.9	3.3	3.8	4.3	4.8	0.3392
	Z^*	1.000000	0.998885	0.997798	0.996985	0.996681	0.996632	0.996617	0.996609	0.996609	0.996608	0.996608	
S16	Δ	0.0	0.41	0.83	1.2	1.7	2.1	2.5	2.9	3.3	3.7	4.1	0.0951
	Z^*	1.000000	0.999764	0.999529	0.999293	0.999094	0.999049	0.999049	0.999049	0.999049	0.999049	0.999049	
S17	Δ	0.0	0.41	0.83	1.2	1.7	2.1	2.5	2.9	3.3	3.7	4.1	0.1019
	Z^*	1.000000	0.999742	0.999484	0.999226	0.999010	0.998982	0.998981	0.998981	0.998981	0.998981	0.998981	
S18	Δ	0.0	0.41	0.82	1.2	1.6	2.1	2.5	2.9	3.3	3.7	4.1	0.0716
	Z^*	1.000000	0.999821	0.999642	0.999464	0.999313	0.999285	0.999284	0.999284	0.999284	0.999284	0.999284	
S19	Δ	0.0	0.44	0.88	1.3	1.8	2.2	2.6	3.1	3.5	4.0	4.4	0.1092
	Z^*	1.000000	0.999714	0.999429	0.999143	0.998933	0.998908	0.998908	0.998908	0.998908	0.998908	0.998908	
S20	Δ	0.0	0.38	0.77	1.2	1.5	1.9	2.3	2.7	3.1	3.5	3.8	0.1906
	Z^*	1.000000	0.999560	0.999120	0.998679	0.998257	0.998097	0.998094	0.998094	0.998094	0.998094	0.998094	
S21	Δ	0.0	0.39	0.77	1.2	1.5	1.9	2.3	2.7	3.1	3.5	3.9	0.0898
	Z^*	1.000000	0.999793	0.999585	0.999378	0.999176	0.999104	0.999102	0.999102	0.999102	0.999102	0.999102	
S22	Δ	0.0	0.41	0.81	1.2	1.6	2.0	2.4	2.8	3.3	3.7	4.1	0.0757
	Z^*	1.000000	0.999818	0.999635	0.999453	0.999292	0.999246	0.999243	0.999243	0.999243	0.999243	0.999243	
S23	Δ	0.0	0.39	0.79	1.2	1.6	2.0	2.4	2.8	3.2	3.6	3.9	0.2508
	Z^*	1.000000	0.999300	0.998601	0.997901	0.997588	0.997496	0.997492	0.997492	0.997492	0.997492	0.997492	
S24	Δ	0.0	0.41	0.82	1.2	1.6	2.0	2.5	2.9	3.3	3.7	4.1	0.0681
	Z^*	1.000000	0.999831	0.999663	0.999494	0.999352	0.999320	0.999319	0.999319	0.999319	0.999319	0.999319	
S25	Δ	0.0	0.62	1.2	1.9	2.5	3.1	3.7	4.3	4.9	5.6	6.2	0.8945
	Z^*	1.000000	0.996451	0.992945	0.991132	0.991061	0.991056	0.991055	0.991055	0.991055	0.991055	0.991055	
S26	Δ	0.0	0.65	1.3	1.9	2.6	3.2	3.9	4.5	5.2	5.8	6.5	1.8451
	Z^*	1.000000	0.992429	0.985244	0.982216	0.981786	0.981554	0.981550	0.981550	0.981549	0.981549	0.981549	
S27	Δ	0.0	0.57	1.1	1.7	2.3	2.9	3.4	4.0	4.6	5.2	5.7	0.0501
	Z^*	1.000000	0.999812	0.999630	0.999518	0.999504	0.999501	0.999500	0.999500	0.999499	0.999499	0.999499	
S28	Δ	0.0	0.62	1.2	1.9	2.5	3.1	3.7	4.4	5.0	5.6	6.2	0.0691
	Z^*	1.000000	0.999720	0.999459	0.999331	0.999316	0.999311	0.999309	0.999309	0.999309	0.999309	0.999309	
S29	Δ	0.0	0.62	1.2	1.8	2.5	3.1	3.7	4.3	4.9	5.5	6.2	2.4146
	Z^*	1.000000	0.990004	0.981477	0.977098	0.976042	0.975912	0.975857	0.975854	0.975854	0.975854	0.975854	
S30	Δ	0.0	0.6	1.2	1.8	2.4	3.0	3.6	4.2	4.8	5.4	6.0	0.5939
	Z^*	1.000000	0.997535	0.995168	0.994144	0.994083	0.994070	0.994063	0.994061	0.994061	0.994061	0.994061	
S31	Δ	0.0	0.7	1.4	2.1	2.8	3.5	4.2	4.9	5.6	6.3	7.0	6.9645
	Z^*	1.000000	0.968040	0.940502	0.930946	0.930493	0.930380	0.930359	0.930356	0.930355	0.930355	0.930355	
S32	Δ	0.0	0.59	1.2	1.8	2.4	3.0	3.6	4.2	4.8	5.4	5.9	2.6059
	Z^*	1.000000	0.989968	0.981014	0.975376	0.974279	0.974051	0.973967	0.973953	0.973945	0.973941	0.973941	
S33	Δ	0.0	0.58	1.2	1.7	2.3	2.9	3.5	4.1	4.7	5.2	5.8	1.2579
	Z^*	1.000000	0.995042	0.990318	0.987787	0.987447	0.987430	0.987423	0.987421	0.987421	0.987421	0.987421	
S34	Δ	0.0	0.57	1.1	1.7	2.3	2.8	3.4	4.0	4.5	5.1	5.7	0.2168
	Z^*	1.000000	0.999182	0.998445	0.997991	0.997875	0.997843	0.997833	0.997832	0.997832	0.997832	0.997832	
S35	Δ	0.0	0.65	1.3	1.9	2.6	3.2	3.9	4.5	5.2	5.8	6.5	0.1478
	Z^*	1.000000	0.999383	0.998828	0.998545	0.998530	0.998523	0.998522	0.998522	0.998522	0.998522	0.998522	
S36	Δ	0.0	0.56	1.1	1.7	2.2	2.8	3.4	3.9	4.5	5.1	5.6	0.8719
	Z^*	1.000000	0.996699	0.993594	0.991802	0.991374	0.991324	0.991304	0.991291	0.991284	0.991281	0.991281	
S37	Δ	0.0	0.65	1.3	1.9	2.6	3.2	3.9	4.5	5.2	5.8	6.5	3.0746
	Z^*	1.000000	0.985972	0.972225	0.969320	0.969271	0.969257	0.969254	0.969254	0.969254	0.969254	0.969254	
S38	Δ	0.0	0.65	1.3	2.0	2.6	3.3	3.9	4.6	5.2	5.9	6.5	1.1271
	Z^*	1.000000	0.995359	0.990956	0.988909	0.988752	0.988731	0.988729	0.988729	0.988729	0.988729	0.988729	
S39	Δ	0.0	0.61	1.2	1.8	2.4	3.0	3.6	4.2	4.9	5.5	6.1	0.4145
	Z^*	1.000000	0.998126	0.996349	0.995896	0.995861	0.995856	0.995856	0.995855	0.995855	0.995855	0.995855	
S40	Δ	0.0	0.65	1.3	2.0	2.6	3.3	3.9	4.6	5.2	5.9	6.5	0.3026
	Z^*	1.000000	0.998797	0.997742	0.997134	0.996982	0.996976	0.996975	0.996974	0.996974	0.996974	0.996974	

Table 5: Δ and optimal values of the price optimization problem with robust approach.