Time-Varying Semidefinite Programming: Geometry of the Trajectory of Solutions

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Abstract

In many applications, solutions of convex optimization problems must be updated on-line, as functions of time. In this paper, we consider time-varying semidefinite programs (TV-SDP), which are linear optimization problems in the semidefinite cone whose coefficients (input data) depend on time. We are interested in the geometry of the solution (output data) trajectory, defined as the set of solutions depending on time. We propose an exhaustive description of the geometry of the solution trajectory. As our main result, we show that only 6 distinct behaviors can be observed at a neighborhood of a given point along the solution trajectory. Each possible behavior is then illustrated by an example.

1 Introduction

A *semidefinite program* (SDP) is a convex constrained optimization problem wherein one wants to optimize a linear objective function over the intersection of the cone of positive semidefinite matrices with an affine space. In this paper, a *time-varying semidefinite program* (TV-SDP) is an SDP

$$\begin{aligned} \min_{X \in \mathbb{S}^n} \quad & C(t) \bullet X \\ s.t. \quad & \mathcal{A}(X) = b(t) \\ & X \succeq 0 \end{aligned} \tag{P_t}$$

whose coefficients depend on a time parameter t belonging to a given real open interval $T = (t_i, t_f) \subseteq \mathbb{R}$.

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The goal of (P_t) is to optimize a time-varying linear objective over a non-linear time-varying feasible region, whose dependence is restricted to the right-hand side of the equality constraints. The objective is to minimize the scalar product $C(t) \bullet X$ between two matrices of $\mathbb{S}^n$, the vector space of symmetric matrices of size n with real entries. The time-varying feasible region is an intersection of the semidefinite cone $\mathbb{S}^n_+ = \{X \in \mathbb{S}^n \mid v^T X v \geq 0,\ \forall v \in \mathbb{R}^n\}$ with a time-varying affine subspace described by linear equations. The notation $X \succeq 0$ is a shortcut for $X \in \mathbb{S}^n_+$. The notation $\mathcal{A}(X) = b(t)$ models linear equations that X must also satisfy: $A_i \bullet X = b_i(t)$ for $i = 1, \ldots, m$, where $A_i \in \mathbb{S}^n$ are given matrices and $b_i(t)$ are given time-varying scalars. Thus, problem (P_t) is convex and its feasible region at any value of parameter $t \in T$ is an affine section of the semidefinite cone, often referred to as a *spectrahedron*. In Section 2, we present our notation in more detail.

TV-SDP appears in numerous applications. For example, in power systems, semidefinite programming relaxations of the so-called alternating-current optimal power flow (ACOPF) are very successful, cf. Lavaei and Low (2011). Tracking of a trajectory of solutions to ACOPF with supply and demand varying over time is crucial for a transmission system operator, who decides on the activation of ancillary services to balance the transmission system, cf. Liu et al. (2018). In general, our goal is to understand certain properties of such solution trajectories, which would make it possible to design algorithms for TV-SDP with guarantees on their performance.

Background and Contribution

SDP can be thought of as a generalization of linear programming (LP) with a number of applications in data science. Anjos and Lasserre (2011) offered a snapshot of the state of the art in the areas of SDP, conic optimization, and polynomial optimization. Polynomial optimization problems can be approximated via a hierarchy of SDP problems of increasing size, developed by Lasserre (2001), also known as the moments - sum of squares (SOS) hierarchy. Many problems in control theory can be reduced to solving polynomial equations, polynomial inequalities, or polynomial differential equations, and they can hence be often solved approximately by the moment-SOS hierarchy, see Henrion et al. (2020) for a recent overview. Applications in theoretical computer science include approximation algorithms for fundamental problems like the Max-Cut problem or coloring problems, quantum information theory, robust learning, and estimation.

The geometry of SDP, that is, the geometry of the feasible region of an SDP problem, is well understood. We refer to (Wolkowicz et al. 2012, Chapter 3) for an excellent overview. Likewise, solution regularity (duality, Slater's condition, uniqueness of the solution, strict complementarity, non-degeneracy) and its prerequisites are well understood; see for example Alizadeh et al. (1997) where the relation between uniqueness of the solution, non-degeneracy of the solutions and strict complementarity is discussed.

Here, our purpose is to study the behavior of the trajectory of the solutions to TV-SDP. Around points of the trajectory satisfying strict complementarity and uniqueness, by means of the implicit function theorem, one can show that the trajectory defines a branch of a smooth curve (Theorem 2.26). When this fails to happen, a number of irregular behaviors may arise. The main result of this paper (Theorem 3.11) consists of a complete classification of such points. So far, to the best of our knowledge, a complete classification of types of behavior of points making up the trajectory of solutions has not been proposed. Here, we suggest one based on a purely logical construction, whose definitions use set-valued analysis. In particular, we use the Painlevé-Kuratowski extension of the notion of continuity to the case of set-valued functions, so as to reason about continuity properties at values of the parameter, when there are multiple solutions. Informally, we now define the types of points that our classification comprises. This is based on the geometry of the trajectory of solutions parametrized over a time interval. Before a given time, we assume that the trajectory is a continuous curve. Then at the time of interest, we can distinguish between the following situations:

- **Regular point:** the trajectory is single-valued and Fréchet differentiable;
- **Non-differentiable point:** the trajectory is single-valued but not continuously differentiable;
- **Continuous bifurcation point:** the trajectory splits into several distinct branches. This results in a loss of uniqueness which still preserves continuity;
- **Discontinuous isolated multiple point:** a loss of continuity causes a loss of uniqueness of the solution, implying a multiple-valued solution. After the point, uniqueness is restored, and hence the loss of uniqueness is isolated;
- **Discontinuous non-isolated multiple point:** a loss of continuity causes a loss of uniqueness of the solution, implying a multiple-valued solution. After the point, uniqueness is not restored hence the loss of uniqueness is not isolated;
- **Irregular accumulation point:** accumulation point of a set made of either bifurcation points or discontinuous isolated multiple points.

The formal definitions of the points discussed above can be found in Def. 3.1, 3.3, 3.4, 3.5, 3.6, and 3.7

We believe that a first contribution of this paper is precisely the definition of these types of points. In this respect, our approach was deeply inspired by (Guddat et al. 1990, Chapter 2) where a classification of solutions to univariate parametric nonlinear constrained optimization problems (NLPs) is proposed. There, critical points satisfying first-order optimality (or KKT) conditions are considered. Under precise algebraic conditions, these points are "non-degenerate" (see Remark 3.10). The local behavior of such points is then shown to be regular. If a critical point is instead "degenerate" then, according to which algebraic condition is satisfied, the point is classified into four different types. Our approach is the same in spirit, in that we also start by considering algebraic conditions ensuring a regular behavior. As a main difference, we classify irregular points according to the behavior of the trajectory of solutions at the point considered rather than according to different sets of algebraic conditions (see Remark 3.10). The main result that we present in this paper is Theorem 3.11, which we informally state here.

Theorem (Informal statement of Theorem 3.11). Under assumptions of Linear Independence Constraint Qualification (LICQ, cf. Assum. 1), existence of Slater's point (cf. Assum. 2) and continuity of the data with respect to time (cf. Assum. 3):

(i) If the trajectory has at least a non-singular point (cf. Def. 2.24), then the trajectory is comprised of only regular points (cf. Def. 3.1), non-differentiable points (cf. Def. 3.3), or isolated multiple points (cf. Def. 3.5). In other words, bifurcation (cf. Def. 3.4) points, non-isolated discontinuous multiple points (cf. Def. 3.6), and irregular accumulation points (cf. Def. 3.7) cannot appear.

(ii) Otherwise, the trajectory can be comprised of points of all six types described above.

Previous Work

Following the pioneering contribution of Goldfarb and Scheinberg (1999), who first studied the properties of the optimum as a function of a varying parameter and extended the concept of the optimal partition from LP to SDP, a number of important papers appeared recently. In decreasing order of generality of the dependence of problem data (coefficients) on the parameter (time):

• Al-Salih and Bohner (2018) studied LP on time scales, which allows for the mixing of difference and differential operators in a broad class of extensions of LP models. While very elegant, mathematically, it seems non-trivial to extend this approach to TV-SDP;

• Wang et al. (2009) studied a broad family of parametric optimization problems, which are known as separated continuous conic programming (SCCP). They developed a strong duality theory for SCCP and proposed a polynomial-time approximation algorithm that solves an SCCP to any required accuracy. This algorithm does not, however, seem easy to extend to TV-SDP;

• Mohammad-Nezhad (2019), Hauenstein et al. (2019) and Mohammad-Nezhad and Terlaky (2020) are perhaps the closest to our work, in spirit. Their dependence of the problem on the data is assumed to be linear, which is a more restrictive assumption than the one we use. Moreover, they do not provide a complete characterization of the possible behaviors of the trajectory of the solutions. In part, we build upon their theoretical results, but instead of building upon the concepts of non-linearity intervals, invariancy intervals, and transition points, we use a purely set-valued analysis approach;

• El Khadir (2020) and Ahmadi and Khadir (2021) are perhaps the closest to our work, in name. They studied the setting where the data vary with known polynomials of the parameter and showed that under a strict feasibility assumption, restricting the solutions to be polynomial functions of the parameter does not change the optimal value of the TV-SDP. They also provided a sequence of SDP problems that give upper bounds on the optimal value of a TV-SDP converging to the optimal value. In contrast, we use a more general setting, where we only assume continuity of the map from the parameter to the problem data. Moreover, we provide a complete geometric characterization of the solutions trajectory.

2 Preliminaries

In this section, we expose the tools needed to state and prove our main result. In Subsection 2.1 we first review geometric properties of SDP. In Subsection 2.2 we survey continuity properties of the optimal and feasible sets of TV-SDP, considered as set-valued maps, in terms of inner and outer semicontinuity and Painlevé-Kuratowski continuity, to adopt the notions of Rockafellar and Wets (2009). Then, in Subsection 2.3 we show that the existence of a unique pair of strictly complementary primal and dual solutions at a value of the time parameter $\hat{t}$ implies that there is a neighbourhood of $\hat{t}$ where both the primal and dual optimal trajectory have a regular behavior. Finally, we observe that under fairly weak assumptions, among which the existence of a non-singular point in the parameterization interval, the number of points where strict complementarity or uniqueness is lost is finite.

2.1 SDP optimality conditions and properties

In this section, we assume for notational simplicity that the data b, C are not parameter-dependent. Let us review geometric properties of SDP in primal form

$$\begin{aligned} \min_{X \in \mathbb{S}^n} \quad & C \bullet X \\ s.t. \quad & \mathcal{A}(X) = b \\ & X \succeq 0 \end{aligned} \tag{P}$$

and dual form

$$\begin{aligned} \max_{y \in \mathbb{R}^m,\, Z \in \mathbb{S}^n} \quad & b^T y \\ s.t. \quad & \mathcal{A}^*(y) + Z = C \\ & Z \succeq 0. \end{aligned} \tag{D}$$

Remark 2.1. Without any loss of generality we assume that the linear operator $\mathcal{A}$ is surjective (see Assum. 1). Then, given a matrix $Z \in \mathbb{S}^n$ satisfying the dual constraint $\mathcal{A}^*(y) + Z = C$ for some $y \in \mathbb{R}^m$, y can be uniquely determined by solving the linear system $(\mathcal{A}\mathcal{A}^*)y = \mathcal{A}(C - Z)$. We exploit this fact and when discussing a dual point (y, Z) we omit y and refer to a dual point simply as a matrix $Z \in \mathbb{S}^n$.

We call a matrix X satisfying the constraints of (P) a *primal feasible point*, a matrix Z satisfying the constraints of (D) a *dual feasible point*, a pair of matrices (X, Z) satisfying the constraints of (P, D) a *primal-dual feasible point.* We call a solution X^* to (P) a *primal optimal point*, a solution Z^* to (D) a *dual optimal point*, a solution (X^*, Z^*) to (P, D) a *primal-dual optimal point.* First, we recall that for the primal-dual pair of TV-SDPs (P, D) a set of first order optimality sufficient conditions is available. Given two matrices X and Z of $\mathbb{S}^n$, their scalar product is denoted by $X \bullet Z = \mathrm{trace}(XZ) = \sum_{i,j=1}^{n} X_{i,j} Z_{i,j}$.

Definition 2.2 (KKT conditions)**.** A primal-dual feasible point $(X, Z) \in \mathbb{S}^n \times \mathbb{S}^n$ satisfies the *Karush-Kuhn-Tucker conditions* (KKT) for (P, D) if

$$\begin{aligned} &\mathcal{A}(X) = b \\ &\mathcal{A}^*(y) + Z = C \\ &X, Z \succeq 0 \\ &X \bullet Z = 0 \end{aligned} \tag{KKT}$$

for some $y \in \mathbb{R}^m$.

It is well-known that for a convex optimization problem, KKT conditions are sufficient for optimality.

Proposition 2.3. *Any primal-dual feasible point (X, Z) satisfying the KKT conditions is optimal for (P, D).*

Definition 2.4 (Strict feasibility)**.** We say that *strict feasibility* holds for (P) (or that (P) is *strictly feasible*) if there exists an interior point of the primal feasible region. That is, there exists a matrix $X \succ 0$ satisfying $\mathcal{A}(X) = b$. Similarly, strict feasibility holds for (D) (or (D) is *strictly feasible*) if there exists an interior point of the dual feasible region. That is, there exist $y \in \mathbb{R}^m$ and a matrix $Z \succ 0$ satisfying $\mathcal{A}^*(y) + Z = C$.

Under strict feasibility, the KKT conditions are also necessary.

Theorem 2.5 (See Theorem 2.3.1 in (Drusvyatskiy et al. 2017) or Theorem 2.2 in (De Klerk 2006))**.** *If strict feasibility holds for both (P) and (D), then for any primal-dual optimal point (X, Z), the duality gap is zero, i.e. $X \bullet Z = 0$ and (P) and (D) have the same optimal value. That is, strong duality is a necessary condition for optimality.*

From $X \bullet Z = 0$ and $X, Z \succeq 0$ it follows that $XZ = ZX = 0$. In particular, $X, Z \succeq 0$ commute and they are therefore simultaneously digonalizable, i.e., they share a basis of orthonormal eigenvectors: $X = Q\Lambda Q^T$, $\Lambda =$

$\operatorname{diag}(\lambda_1,\dots,\lambda_n)$, $Z = Q\Omega Q^T$, $\Omega = \operatorname{diag}(\omega_1,\dots,\omega_n)$, $QQ^T = I_n$, where $\lambda_1,\dots,\lambda_n$ are the eigenvalues of X, $\omega_1,\dots,\omega_n$ the eigenvalues of Z and the columns of Q are the common eigenvectors of X, Z. Since $XZ = 0$, it follows that $\lambda_i\omega_i = 0$ for all $i = 1,\dots,n$. This latter equation expresses complementary slackness in terms of the eigenvalues of X and Z. Let $r = \operatorname{rank}(X)$ and $s = \operatorname{rank}(Z)$, then complementarity implies $r + s \le n$. We can order the common basis of eigenvectors so that $X = Q\operatorname{diag}(\lambda_1,\dots,\lambda_r,0,\dots,0)Q^T = Q_P\Lambda_P Q_P^T$, $Z = Q\operatorname{diag}(0,\dots,0,\omega_{n-s+1},\dots,\omega_n)Q^T = Q_D\Omega_D Q_D^T$, where $\Lambda_P = \operatorname{diag}(\lambda_1,\dots,\lambda_r)$, $\Omega_D = \operatorname{diag}(\omega_{n-s+1},\dots,\omega_n)$, $\lambda_1,\dots,\lambda_r$ are the strictly positive eigenvalues of X and the columns of Q_P are the eigenvectors relative to $\lambda_1,\dots,\lambda_r$. Analogously, $\omega_{n-s+1},\dots,\omega_n$ are the strictly positive eigenvalues of Z and the columns of Q_D are the eigenvectors relative to $\omega_{n-s+1},\dots,\omega_n$. Defining Q_N as the matrix formed by the columns of Q that are not in Q_P nor in Q_D, we get a partition of the columns of Q so that

$$Q = [Q_P, Q_N, Q_D]. \tag{1}$$

Definition 2.6 (Strict complementarity)**.** A primal-dual optimal point (X, Z) is said to be *strictly complementary* if $\operatorname{rank}(X) + \operatorname{rank}(Z) = r + s = n$. In terms of the eigenvalues of X and Z, this is equivalent to the following condition:

$$\begin{aligned} \lambda_i\omega_i &= 0, \\ \lambda_i > 0 &\iff \omega_i = 0, \end{aligned} \qquad \text{for all } i = 1,\dots,n. \tag{2}$$

A primal-dual problem (P, D) satisfies *strict complementarity* if there exists a strictly complementary primal-dual optimal point (X, Z).

We now introduce the definitions of primal and dual non-degeneracy. These conditions guarantee both primal and dual uniqueness of the solution, see Alizadeh et al. (1997).

Definition 2.7 (Primal non-degeneracy)**.** We say that a primal feasible point X is *primal non-degenerate* if

$$\mathcal{N}(\mathcal{A}) + \mathcal{T}_X = \mathbb{S}^n,$$

where $\mathcal{N}(\mathcal{A}) = \{Y \in \mathbb{S}^n | A_i \bullet Y = 0 \text{ for all } i = 1,\dots,m\}$ and

$$\mathcal{T}_X = \left\{ Q \begin{pmatrix} U & V \\ V^T & 0 \end{pmatrix} Q^T \;\middle|\; U \in \mathbb{S}^r, V \in \mathbb{R}^{r\times(n-r)} \right\}$$

is the tangent space at X in $\mathbb{S}^n_+$ with $r = \operatorname{rank}(X)$.

Definition 2.8 (Dual non-degeneracy)**.** We say that a dual feasible point Z is *dual non-degenerate* if

$$\mathcal{R}(\mathcal{A}) + \mathcal{T}_Z = \mathbb{S}^n,$$

where $\mathcal{R}(\mathcal{A}) = \operatorname{span}(A_1,\dots,A_m)$ and

$$\mathcal{T}_Z = \left\{ Q \begin{pmatrix} 0 & V \\ V^{\mathrm{T}} & W \end{pmatrix} Q^{\mathrm{T}} \;\middle|\; W \in \mathbb{S}^s,\ V \in \mathbb{R}^{(n-s)\times s} \right\}$$

is the tangent space at Z in $\mathbb{S}^n_+$, $s = \operatorname{rank}(Z)$.

Proposition 2.9 (Theorem 6 in Alizadeh et al. (1997)). *Let X be a primal feasible point with* $\operatorname{rank}(X) = r$ *and let $Q_P \in \mathbb{R}^{n\times r}$, $Q_{ND} = [Q_N, Q_D] \in \mathbb{R}^{n\times(n-r)}$ as defined in* (1)*. Then X is primal non-degenerate if and only if the following matrices are linearly independent in $\mathbb{S}^n$:*

$$\begin{pmatrix} Q_P^{\mathrm{T}} A_k Q_P & Q_P^{\mathrm{T}} A_k Q_{ND} \\ Q_{ND}^{\mathrm{T}} A_k Q_P & 0 \end{pmatrix}, \quad \textit{for } k = 1, \ldots, m. \tag{3}$$

Proposition 2.10 (Theorem 9 in Alizadeh et al. (1997)). *Let Z be a dual feasible point with* $\operatorname{rank}(Z) = s$ *and let $Q_{PN} = [Q_P, Q_N] \in \mathbb{R}^{n\times(n-s)}$, $Q_D \in \mathbb{R}^{n\times s}$ as defined in* (1)*. Then Z is dual non-degenerate if and only if the following matrices span $\mathbb{S}^{n-s}$:*

$$Q_{PN}^{\mathrm{T}} A_k Q_{PN}, \quad \textit{for } k = 1, \ldots, m. \tag{4}$$

Our interest in non-degeneracy is motivated by the following results.

Theorem 2.11 (Theorems 7 and 10 in Alizadeh et al. (1997)). *If there exists a primal non-degenerate optimal point, then there exists a unique dual optimal point. Conversely, if there exists a dual non-degenerate optimal point, then there exists a unique primal optimal point.*

Definition 2.12 (Non-degeneracy). We say that a primal-dual feasible point (X, Z) is *non-degenerate* if X is primal non-degenerate and Z is dual non-degenerate.

Corollary 2.13. *If (X^*, Z^*) is a primal-dual non-degenerate optimal point then (X^*, Z^*) is the unique primal-dual optimal point for (P, D).*

Under strict complementarity, the converse of Theorem 2.11 holds true:

Theorem 2.14 (Theorem 11 in Alizadeh et al. (1997)). *Let (P, D) have a strictly complementary primal-dual optimal point (X^*, Z^*). Then X^* is a unique primal optimal solution if and only if Z^* is dual non-degenerate. Conversely, Z^* is a unique dual optimal solution if and only if X^* is primal non-degenerate.*

In other words, under strict complementarity the non-degeneracy of the dual (primal) problem is equivalent to the uniqueness of the primal (dual) problem.

2.2 Set-valued analysis for TV-SDP

We are interested in studying the trajectories of solutions to the primal TV-SDP

$$\begin{aligned} \min_{X\in\mathbb{S}^n} \quad & C(t) \bullet X \\ s.t. \quad & \mathcal{A}(X) = b(t) \\ & X \succeq 0 \end{aligned} \tag{P$_t$}$$

with time parameter $t \in T = (t_i, t_f) \subset \mathbb{R}$. For a given value of t the dual TV-SDP is

$$\begin{aligned} \max_{y\in\mathbb{R}^m,\, Z\in\mathbb{S}^n} \quad & b(t)^T y \\ s.t. \quad & \mathcal{A}^*(y) + Z = C(t) \\ & Z \succeq 0. \end{aligned} \tag{D$_t$}$$

Definition 2.15 (Set-valued maps)**.** A *set-valued map* F from a set T to another set X maps a point $t \in T$ to a non-empty subset of $F(t) \subseteq X$. In symbols:

$$\begin{aligned} F : T &\rightrightarrows X \\ t &\mapsto F(t) \subseteq X. \end{aligned}$$

We say that F is *single-valued* at $t \in T$ if $F(t)$ is a singleton. We say that F is *multi-valued* at $t \in T$ if $F(t)$ is neither empty nor a singleton.

Given a primal-dual pair of TV-SDPs (P_t, D_t), we can now define the primal and dual *feasible set maps*:

$$\begin{aligned} \mathcal{P}(t) &= \{X \in \mathbb{S}^n \mid \mathcal{A}(X) = b(t),\ X \succeq 0\}, \\ \mathcal{D}(t) &= \{Z \in \mathbb{S}^n \mid \mathcal{A}^*(y) + Z = C(t),\ y \in \mathbb{R}^m,\ Z \succeq 0\}. \end{aligned}$$

The primal and dual *optimal value functions* are defined as

$$\begin{aligned} p^*(t) &= \min_{X\in\mathbb{S}^n} \{C(t) \bullet X \mid \mathcal{A}(X) = b(t),\ X \succeq 0\}, \\ d^*(t) &= \max_{Z\in\mathbb{S}^n} \{b(t)^T y \mid \mathcal{A}^*(y) + Z = C(t),\ y \in \mathbb{R}^m,\ Z \succeq 0\}. \end{aligned}$$

Finally, the primal and dual *optimal set maps* are

$$\begin{aligned} \mathcal{P}^*(t) &= \{X \in \mathcal{P}(t) \mid C(t) \bullet X = p^*(t)\}, \\ \mathcal{D}^*(t) &= \{Z \in \mathcal{D}(t) \mid b(t)^T y = d^*(t),\ \mathcal{A}^*(y) + Z = C(t),\ y \in \mathbb{R}^m\}. \end{aligned}$$

Continuity properties of set-valued maps can be defined in terms of outer and inner limits, leading to the notion of Painlevé-Kuratowski continuity. First, we introduce the notion of inner and outer limits of a set-valued map:

Definition 2.16 (Inner and outer limits)**.** Given a set-valued mapping $F : T \rightrightarrows X$, its *inner limit* at $\hat{t} \in T$ is denoted by $\liminf_{t\to\hat{t}} F(t)$ and defined as

$$\left\{\hat{x} \mid \forall\{t_k\}_{k=1}^{\infty} \subseteq T \text{ such that } t_k \to \hat{t},\ \exists\{x_k\}_{k=1}^{\infty} \subseteq X,\ x_k \to \hat{x} \text{ and } x_k \in F(t_k)\right\},$$

while its *outer limit* at $\hat{t} \in T$ is denoted by $\limsup_{t\to\hat{t}} F(t)$ and is defined as

$$\left\{\hat{x} \mid \exists\{t_k\}_{k=1}^{\infty} \subseteq T \text{ such that } t_k \to \hat{t},\ \exists\{x_k\}_{k=1}^{\infty} \subseteq X,\ x_k \to \hat{x} \text{ and } x_k \in F(t_k)\right\}.$$

Definition 2.17 (Painlevé-Kuratowski continuity)**.** Let $F : T \rightrightarrows X$ be a set-valued map. We say that F is *outer semicontinuous* at $\hat{t} \in T$ if

$$\limsup_{t\to\hat{t}} F(t) \subseteq F(\hat{t}).$$

We say that F is *inner semicontinuous* at $\hat{t} \in T$ if

$$\liminf_{t \to \hat{t}} F(t) \supseteq F(\hat{t}).$$

Finally, we say that F is *Painlevé-Kuratowski continuous* at $\hat{t}$ if it is both outer and inner semicontinuous at $\hat{t}$.

Remark 2.18 (Continuity). Note that a single-valued map $F : T \to X$ is continuous in the usual sense at a point $x \in X$ if and only if it is Painlevé-Kuratowski continuous at $x \in X$ as a multi-valued map $F : T \rightrightarrows X$. Thus, without ambiguity, we will refer to Painlevé-Kuratowski continuity simply as continuity.

First, we list some continuity results on the feasible and optimal set maps. The proof of Theorem 2.21 in the primal version is an original contribution of this paper.

Theorem 2.19 (Example 5.8 in (Rockafellar and Wets 2009))**.** *If $b(t)$ and $C(t)$ are continuous functions of t, then the feasible set maps $\mathcal{P}(t)$ and $\mathcal{D}(t)$ are outer semicontinuous at any $t \in T$.*

Theorem 2.20 (Theorem 8 in (Hogan 1973))**.** *If $b(t)$ and $C(t)$ are continuous functions of t, then the optimal set maps $\mathcal{P}^*(t)$ and $\mathcal{D}^*(t)$ are outer semicontinuous at any $t \in T$.*

Naturally, we now investigate the inner continuity of the feasible and optimal set maps. This property turns out to always hold for the primal and dual feasible set maps $\mathcal{P}(t)$ and $\mathcal{D}(t)$ which are then continuous. We have:

Theorem 2.21. *Assume that strict feasibility holds at any $t \in T$, that the linear operator $\mathcal{A}$ is surjective, and that $b(t)$ and $C(t)$ are continuous functions of t. Then the set-valued maps $\mathcal{P}(t)$ and $\mathcal{D}(t)$ are continuous for every $t \in T$.*

Proof. For the dual case, we refer to Lemma 1 in (Hauenstein et al. 2019). We prove the primal case, in the more general case where the right hand side $b(t)$ is continuous. Fix $\hat{t} \in T$ and $\hat{X} \in \mathcal{P}(\hat{t})$. Given a sequence of times $\{t_k\}_{k=1}^{\infty}$ with $t_k \to \hat{t}$, we will construct a convergent sequence $X_k \to \hat{X}$ so that $X_k \in \mathcal{P}(t_k)$ for all sufficiently large values of k. If $\hat{X} \succ 0$ we define

$$X_k := \hat{X} + \mathcal{A}^*(\mathcal{A}\mathcal{A}^*)^{-1}\left(b(t_k) - b(\hat{t})\right).$$

The definition is well posed because under the assumptions of the theorem the operator $\mathcal{A}$ has full rank, thus $\mathcal{A}\mathcal{A}^*$ is invertible. Clearly, $\mathcal{A}(X_k) = b(t_k)$. Furthermore, $\|X_k - \hat{X}\|_F = \|\mathcal{A}^*(\mathcal{A}\mathcal{A}^*)^{-1}\left(b(t_k) - b(\hat{t})\right)\| \leq C_{\mathcal{A}}\|b(t_k) - b(\hat{t})\| \to 0$ for some constant $C_{\mathcal{A}}$ and by continuity of $b(t)$, so that $X_k \to \hat{X}$ and $X_k \succeq 0$ for sufficiently large k. If $\hat{X} \succeq 0$ and its smallest eigenvalue $\lambda_{\min}(\hat{X})$ is zero, we define

$$X_k := (1 - \alpha_k)\hat{X} + \alpha_k\bar{X} + \mathcal{A}^*(\mathcal{A}\mathcal{A}^*)^{-1}\left(b(t_k) - b(\hat{t})\right)$$

for a fixed $\bar{X} \in \mathcal{P}(\hat{t})$ such that $\bar{X} \succ 0$, which exists by the strict feasibility assumption, and for a sequence $\{\alpha_k\}_{k=1}^{\infty} \subseteq [0, 1]$ which we shall conveniently define in the following. Clearly, $\mathcal{A}(X_k) = b(t_k)$ and hence we only need to prove that $X_k \succeq 0$ or, equivalently, that

$$\lambda_{\min}\left((1-\alpha_k)\hat{X} + \alpha_k \bar{X} + \mathcal{A}^*(\mathcal{A}\mathcal{A}^*)^{-1}\left(b(t_k) - b(\hat{t})\right)\right) \geq 0,$$

which holds if

$$\alpha_k \lambda_{\min}(\bar{X}) + \lambda_{\min}\left(\mathcal{A}^*(\mathcal{A}\mathcal{A}^*)^{-1}\left(b(t_k) - b(\hat{t})\right)\right) \geq 0.$$

Rearranging:

$$\alpha_k \geq -\frac{\lambda_{\min}\left(\mathcal{A}^*(\mathcal{A}\mathcal{A}^*)^{-1}\left(b(t_k) - b(\hat{t})\right)\right)}{\lambda_{\min}(\bar{X})}.$$

We then define $\alpha_k := \max\{0, \beta_k\}$, where

$$\beta_k := -\frac{\lambda_{\min}\left(\mathcal{A}^*(\mathcal{A}\mathcal{A}^*)^{-1}\left(b(t_k) - b(\hat{t})\right)\right)}{\lambda_{\min}(\bar{X})}.$$

For sufficiently large k, $\beta_k \leq 1$, so that $\{\alpha_k\}_{k=1}^{\infty} \subseteq [0,1]$ and thus $X_k \in \mathcal{P}(t_k)$, since $\beta_k \to 0$, $\alpha_k \to 0$ and $X_k \to \hat{X}$. □

However, in general, it is not true that the optimal set maps $\mathcal{P}^*(t)$ and $\mathcal{D}^*(t)$ are inner semicontinuous. Still, the set of $t \in T$ such that $\mathcal{P}^*(t)$ or $\mathcal{D}^*(t)$ fails to be inner semicontinuous, is of *first category*, i.e., countable and nowhere dense.

Theorem 2.22 (Theorem 1 in (Hauenstein et al. 2019))**.** *The subset of points $t \in T$ at which $\mathcal{P}^*(t)$ or $\mathcal{D}^*(t)$ fails to be continuous is the union of countably many sets that are nowhere dense in T, in particular, it has empty interior.*

However, if the optimal set is single-valued, then it is continuous everywhere:

Proposition 2.23 (Corollary 8.1 in (Hogan 1973))**.** *Assume that strict feasibility holds at any $t \in T$ and that $b(t)$ and $C(t)$ are continuous functions of t. If $\mathcal{P}^*(t)$ is single-valued at some $\hat{t}$, then $\mathcal{P}^*(t)$ is continuous at $\hat{t}$. The same holds for $\mathcal{D}^*(t)$.*

2.3 Regularity properties of the TV-SDP optimal set map

Given a primal-dual pair of TV-SDPs (P_t, D_t), we denote a primal-dual point by (X, Z, t). If at a fixed value of the parameter $\hat{t} \in T$ there exists a primal-dual non-degenerate optimal point (X^*, Z^*), then, by Proposition 2.11 (X^*, Z^*) is a unique primal-dual optimal point, and by Proposition 2.23, around $\hat{t}$ the primal and dual optimal set maps are continuous single-valued functions. Under strict complementarity, these functions are analytic. In the following, we provide details of this fact.

The optimality conditions (KKT) for (X, Z, t) to be a solution of (P_t, D_t) at a fixed value of the parameter $t \in T$ can be equivalently written as

$$F(X, y, Z, t) := \begin{pmatrix} \tilde{\mathcal{A}} \operatorname{vec}(X) - b(t), \\ \tilde{\mathcal{A}}^T y + \operatorname{vec}(Z) - \operatorname{vec}(C(t)) \\ \frac{1}{2} \operatorname{vec}(XZ + ZX) \end{pmatrix} = 0, \tag{5}$$

$$X, Z \succeq 0 \tag{6}$$

for some $y \in \mathbb{R}^m$, where $\tilde{\mathcal{A}} := (\operatorname{vec}(A_1), \ldots \operatorname{vec}(A_m))^T$ and $\operatorname{vec}(X)$ denotes a linear map stacking the upper triangular part of X, where the off-diagonal entries are multiplied by $\sqrt{2}$:

$$\operatorname{vec}(X) := \left(X_{11}, \sqrt{2}X_{12}, \ldots, \sqrt{2}X_{1n}, X_{22}, \sqrt{2}X_{23}, \ldots, \sqrt{2}X_{2n}, \ldots, X_{nn}\right)^T$$

so that $X \bullet X = \operatorname{vec}(X)^T \operatorname{vec}(X)$.

Definition 2.24 (Singular points)**.** We say that a point (X, Z, t) is *singular* if the Jacobian w.r.t (X, y, Z) of F at (X, Z, t)

$$J_F(X, y, Z, t) = \begin{pmatrix} \tilde{\mathcal{A}} & 0 & 0 \\ 0 & \tilde{\mathcal{A}}^T & I_{\tau(n)} \\ Z \otimes_s I_n & 0 & I_n \otimes_s X \end{pmatrix} \tag{7}$$

is not invertible. Otherwise, we say that (X, Z, t) is *non-singular.*

The following lemma gives equivalent conditions for a primal-dual point (X, Z, t) to be non-singular.

Lemma 2.25 (Theorem 3.1. in (Alizadeh et al. 1998))**.** *A primal-dual optimal point (X, Z, t) is non-singular if and only if strict complementarity holds and (X, Z, t) is non-degenerate.*

Note that under strict complementarity Theorem 2.14 holds. Therefore, the Jacobian of F is non-singular at (X, Z, t) if and only if (X, Z) is a *unique* primal-dual optimal point satisfying strict complementarity. We use this result in the following theorem.

Theorem 2.26. *Let (P_t, D_t) be a primal-dual pair of TV-SDPs parametrized over a time interval T and $\hat{t} \in T$ a fixed value of the time parameter. Suppose that (X^*, Z^*) is a unique primal-dual optimal and strictly complementary point for $(P_{\hat{t}}, D_{\hat{t}})$. Then there exists $\varepsilon > 0$ and a unique continuously differentiable curve $(X^*(\cdot), Z^*(\cdot))$ defined on $(\hat{t} - \varepsilon, \hat{t} + \varepsilon)$ such that $(X^*(t), Z^*(t))$ is a unique and strictly complementary primal-dual optimal point to (P_t, D_t) for all $t \in (\hat{t} - \varepsilon, \hat{t} + \varepsilon)$.*

Proof. By Lemma 2.25, we are under the hypothesis of the Implicit Function Theorem, so that there exists $\varepsilon' > 0$ and a unique continuously differentiable curve $(X^*(\cdot), y^*(\cdot), Z^*(\cdot))$ on $(\hat{t} - \varepsilon', \hat{t} + \varepsilon')$ such that $F(X^*(t), y^*(t), Z^*(t), t) = 0$

for all $t \in (\hat{t} - \varepsilon', \hat{t} + \varepsilon')$. In order to show that $(X^*(t), Z^*(t))$ is a primal-dual optimal point for (P_t, D_t) we also need to prove that $X^*(t), Z^*(t) \succeq 0$ for all $t \in (\hat{t} - \varepsilon', \hat{t} + \varepsilon')$. If this was not true, then at least one between the primal and dual problem would be infeasible, as it would violate the KKT conditions, which under Assumption 2 are necessary conditions for optimality. This would contradict Assumption 2, which ensures that both P_t and D_t must be feasible. Thus, $(X^*(t), Z^*(t))$ is a primal-dual optimal point for (P_t, D_t) for all $t \in (\hat{t} - \varepsilon', \hat{t} + \varepsilon')$. Finally, by continuity, for small enough $\varepsilon \leq \varepsilon'$, $(X^*(t), Z^*(t))$ is a unique strictly complementary primal-dual solution for (P_t, D_t) for all $t \in (\hat{t} - \varepsilon, \hat{t} + \varepsilon)$. □

Under weak assumptions, one can improve upon Theorem 2.22 so that the number of singular points of (5) is finite.

Theorem 2.27. *For the primal-dual TV-SDPs (P_t, D_t), assume that there exists a value of the parameter $\hat{t} \in T$ at which strict complementarity and non-degeneracy of the primal-dual optimal point hold. Then the set of values of the time parameter t at which the primal-dual optimal point is either not unique or not strictly complementary is finite.*

Proof. Let us elaborate upon the proof techniques that (Hauenstein et al. 2019) use to prove their Theorem 2. We first define the *algebraic set*

$$C := \{(X, y, Z, t) \in \mathbb{C}^{\tau(n)} \times \mathbb{C}^m \times \mathbb{C}^{\tau(n)} \times \mathbb{C} \mid F(X, y, Z, t) = 0,\ \det(J_F(X, y, Z, t)) = 0\}.$$

An algebraic set is a set defined by a finite number of polynomial equations on an algebraically closed field. Note that the equations defining C are considered in $\mathbb{C}$, which is the algebraic closure of $\mathbb{R}$. In particular, C is a *constructible set.* A constructible set is a member of the smallest family of sets which contains the algebraic sets and is also closed under complementation, finite unions, and finite intersections. Furthermore, the projection of a constructible set is a constructible set itself (Theorem 1.32 in (Basu 2017)), so that the projection of C on the t coordinate

$$C_P = \Big\{t \in \mathbb{C} \mid \exists (X, y, Z, t) \in C\Big\}$$

is a constructible set in $\mathbb{C}$. At this point, we exploit the fact that any constructible set of $\mathbb{C}$ is either a finite set or the complement of a finite set (Exercise 1.3 in (Basu 2017)). By the hypothesis and Theorem 2.26 the complement of C_P contains an open neighborhood of $\hat{t}$ where $F(X, y, Z, t) = 0$ and $\det(J_F(X, y, Z, t)) \neq 0$. This neighborhood is contained in the complement of C_P and it is not finite (it is a open interval with positive measure). Hence C_P is a finite set. Since

$$\{t \in T \mid \exists (X, y, Z) \in \mathbb{S}^n \times \mathbb{S}^n \text{ s.t. } F(X, y, Z, t) = 0,\ \det(J_F(X, y, Z, t)) = 0\} \subseteq C_P,$$

the set of values of the parameter at which there exists a singular point for the Jacobian of F is also finite. Application of Lemma 2.25 yields the final result. □

Thus, under the assumption of Theorem 2.27, the values of t at which strict complementarity or uniqueness of the primal-dual solution is lost is finite. In particular, the values of t at which $\mathcal{P}^*(t)$ or $\mathcal{D}^*(t)$ fails to be inner semicontinuous (and hence fails to be continuous) are finite. It also implies that wherever $\mathcal{P}^*(t)$ defines a continuous curve of unique optima, the values of t at which $\mathcal{P}^*(t)$ fails to be differentiable are finite. The same holds for $\mathcal{D}^*(t)$.

3 A complete classification of optimal points

The focus of our study is first put on values t^* of the time parametrization interval T at which strict complementarity or uniqueness of the primal-dual optimal point is lost. In other words, these are singular points preceded by non-singular points. By Theorem 2.27 such points are finite. There, the trajectory described by the primal and dual optimal sets can exhibit a restricted number of irregular behaviors. If, instead, all primal-dual optimal points (X, Z, t) are singular for every $t \in T$, the number of possible types of irregular behaviors grows. In our main Theorem 3.11, we provide a complete classification of these behaviors under both cases. The object of our study is the trajectory of solutions to the primal TV-SDP (P_t), that is, the primal optimal set map. Every result that we propose can be clearly transposed to the dual case.

We first adopt the following standard assumptions:

Assumption 1 (LICQ). The m matrices $\{A_i\}_{i=1,\ldots,m}$ are linearly independent in $\mathbb{S}^n$, so that the linear operator $\mathcal{A}$ is surjective. This condition is known as the *linear independence constraint qualification* (LICQ).

This assumption can be made without any loss of generality at the cost of Gaussian elimination of the redundant constraints and it allows us to describe the dual solution just in terms of matrix Z (see Remark 2.1).

Assumption 2 (Slater's condition). For every $t \in T$, problem (P_t) and its dual (D_t) are strictly feasible.

It is otherwise possible to project the primal and dual problems and their feasible sets onto a smaller subspace, so that this property holds on T (Goldfarb and Scheinberg 1999). This assumption is standard in the SDP literature (Goldfarb and Scheinberg (1999), Ahmadi and Khadir (2021), Hauenstein et al. (2019)). Slater's condition guarantees that the primal and dual optimal sets $\mathcal{P}^*(t)$ and $\mathcal{D}^*(t)$ are non-empty and bounded for any $t \in T$.

Assumption 3 (Data continuity). Data $b(t)$ and $C(t)$ depend continuously on the time parameter t.

This assumption is quite general compared to those usually found in the TV-SDP literature, where the data are often assumed to vary linearly with respect to the time parameter. This linearity assumption is standard when one studies sensitivity properties, so that the perturbation can be assumed to be linear. Instead, our purpose is to give a geometric characterization of the points of the trajectory of solutions, in which case we can keep a high degree of generality by just assuming continuity of the data, without any further differentiability requirement.

Assumptions 1, 2, and 3 ensure that:

(a) There is no duality gap: $p^*(t) = d^*(t)$ for all $t \in T$.

(b) The primal and dual optimal faces $\mathcal{P}^*(t)$, $\mathcal{D}^*(t)$ are non-empty and uniformly bounded for all $t \in T$. In other words, (P_t) and (D_t) are both feasible and bounded.

(c) The optimal set maps are outer semicontinuous at any $t \in T$.

(d) The subset of T where the optimal set map fails to be inner semicontinuous has empty interior and it is the union of countably many sets that are nowhere dense in T.

Equipped with the results of the previous section, we introduce a classification into six different types of primal optimal points according to the behavior of the optimal set map at these points. Our purpose is to study irregularities arising after an interval where the optimal set map has regular behavior. We hence classify points for which the optimal set map on a left neighborhood is unique and thus continuous.

Let (P_t, D_t) be a primal-dual pair of TV-SDPs parametrized by $t \in T$. For a fixed $t^* \in T$, we consider a primal optimal point (X^*, t^*) for (P_{t^*}). Based on the behavior of the primal optimal set map $\mathcal{P}^*(t)$ at t^*, we can distinguish between six different cases. According to this cases we classify the primal point (X^*, t^*) into six different types. This can be done analogously for the dual case.

Definition 3.1 (Regular point)**.** At a *regular point* (X^*, t^*), $\mathcal{P}^*(t^*) = \{X^*\}$ and there exists $\varepsilon > 0$ such that

- $\mathcal{P}^*(t)$ is single-valued and continuous for every $t \in (t^* - \varepsilon, t^* + \varepsilon)$, for some $\varepsilon > 0$,
- $\mathcal{P}^*(t)$ is Fréchet differentiable at t^*.

Remark 3.2. Note that a primal optimal point (X^*, t^*) for (P_{t^*}) for which there exists a dual optimal point (Z^*, t^*) for (D_{t^*}) such that (X^*, Z^*, t^*) is a non-singular point for (P_{t^*}, D_{t^*}), is necessarily a regular point. This follows directly from Theorem 2.26 and Lemma 2.25. The converse does not hold in general.

Definition 3.3 (Non-differentiable point)**.** At a *non-differentiable point* (X^*, t^*), $\mathcal{P}^*(t^*) = \{X^*\}$ and there exists $\varepsilon > 0$ such that

- $\mathcal{P}^*(t)$ is single-valued and continuous for every $t \in (t^* - \varepsilon, t^* + \varepsilon)$,
- $\mathcal{P}^*(t)$ is *not* Fréchet differentiable at t^*.

Definition 3.4 (Continuous bifurcation point)**.** At a *continuous bifurcation point* (X^*, t^*), $\mathcal{P}^*(t^*) = \{X^*\}$ and there exists $\varepsilon > 0$ such that

- $\mathcal{P}^*(t)$ is continuous at any $t \in (t^* - \varepsilon, t^* + \varepsilon)$,
- $\mathcal{P}^*(t)$ is single-valued for every $t \in (t^* - \varepsilon, t^*]$,
- $\mathcal{P}^*(t)$ is multi-valued for every $t \in (t^*, t^* + \varepsilon)$.

In particular, there exist at least two distinct continuous curves

$$\begin{array}{rcl} X_1 : (t^*, t^* + \varepsilon) & \to & \mathbb{S}^n \\ t & \mapsto & X_1(t) \end{array} \qquad \begin{array}{rcl} X_2 : (t^*, t^* + \varepsilon) & \to & \mathbb{S}^n \\ t & \mapsto & X_2(t) \end{array}$$

such that $X_1(t)$ and $X_2(t)$ are two distinct points of $\mathcal{P}^*(t)$ for every $t \in (t^*, t^* + \varepsilon)$ and $\lim_{t \to t^{*+}} X_1(t) = \lim_{t \to t^{*+}} X_2(t) = X^*$. In this sense, a continuous bifurcation point can be thought as a continuous loss of uniqueness from a single branch into two or more branches.

Definition 3.5 (Discontinuous isolated multiple point)**.** At a *discontinuous isolated multiple point* (X^*, t^*), $X^* \in \mathcal{P}^*(t^*)$ and there exists $\varepsilon > 0$ such that

- $\mathcal{P}^*(t)$ is single-valued and continuous for every $t \in (t^* - \varepsilon, t^*) \cup (t^*, t^* + \varepsilon)$,
- $\mathcal{P}^*(t)$ is multi-valued at t^*.

In particular, a loss of inner semicontinuity and uniqueness of $\mathcal{P}^*(t)$ occurs at t^*, but it is isolated and, relatively to $(\hat{t} - \varepsilon, \hat{t} + \varepsilon)$, it is a measure-zero set.

Definition 3.6 (Discontinuous non-isolated multiple point)**.** At a *discontinuous non-isolated multiple point* (X^*, t^*), $X^* \in \mathcal{P}^*(t^*)$ and there exists $\varepsilon > 0$ such that

- $\mathcal{P}^*(t)$ is continuous at any $t \in (t^* - \varepsilon, t^*) \cup (t^*, t^* + \varepsilon)$,
- $\mathcal{P}^*(t)$ is single-valued for every $t \in (t^* - \varepsilon, t^*)$,
- $\mathcal{P}^*(t)$ is multi-valued for every $t \in [t^*, t^* + \varepsilon)$.

In particular, a loss of inner continuity of $\mathcal{P}^*(t)$ occurs at t^*. This is not an isolated point, but rather an element of a positive-dimensional set of points at which the solution is not unique.

Definition 3.7 (Irregular accumulation point)**.** At an *irregular accumulation point* (X^*, t^*), $X^* \in \mathcal{P}^*(t^*)$ and there exists $\varepsilon > 0$ such that

- $\mathcal{P}^*(t)$ is single-valued and continuous for every $t \in (t^* - \varepsilon, t^*)$

and for any $\delta > 0$ at least one of the following is true:

- there exists a sequence of times $\{t_k\}_{k=1}^{\infty} \subseteq (t^*, t^* + \delta)$ at which a loss of inner semicontinuity occurs and $\lim_{k \to \infty} t_k = t^*$. At these times, either a discontinuous isolated multiple point or a discontinuous non-isolated multiple point appears.

- there exists a sequence of times $\{t_k\}_{k=1}^{\infty} \subseteq (t^*, t^*+\delta)$ at which a continuous bifurcation occurs and $\lim_{k\to\infty} t_k = t^*$.

Remark 3.8. The above definitions consider points whose sufficiently small *left* time neighborhood consists of all regular points. By a change of sign of the parameter, the definition clearly extends to points whose sufficiently small *right* time neighborhood consists of all regular points.

Remark 3.9 (Existence of a continuous selection). The optimal set map is continuous in a neighborhood of a regular, non-differentiable, or a continuous bifurcation point. Instead, at a discontinuous isolated or non-isolated multiple point (Definitions 3.5 and 3.6), a loss of inner semicontinuity occurs. For such points (X^*, t^*) it holds $\liminf_{t\to t^*-} \mathcal{P}^*(t) \neq \mathcal{P}^*(t^*)$. However, in both cases, clearly only one of the following is true:

$$(A) \lim_{t\to t^*+} \mathcal{P}^*(t) = \mathcal{P}^*(t^*),$$
$$(B) \liminf_{t\to t^*+} \mathcal{P}^*(t) \neq \mathcal{P}^*(t^*).$$

In case (A), one can select a continuous curve $(t^*-\varepsilon, t^*+\varepsilon) \ni t \mapsto X(t) \in \mathbb{S}^n$ such that $X(t) \in \mathcal{P}^*(t)$ for every $t \in (t^*-\varepsilon, t^*+\varepsilon)$, while in case (B) such a curve does not exist. Furthermore, for a discontinuous isolated multiple point under case (A), such a curve is unique. Also note that in case (A) it might be impossible to select a curve that is differentiable at t^*.

Remark 3.10 (Comparison with Guddat et al. (1990)). The definition of the six different types of points was inspired by (Guddat et al. 1990, Chapter 2), where a classification of solutions to univariate parametric non-linear constrained optimization problems was proposed. There, critical primal-dual points satisfying first-order optimality (or KKT) conditions are considered. These points are defined as *non-degenerate* if strict complementarity holds as well as the invertibility of the Hessian of the Lagrangian restricted to the tangent space at the point. We remark that this notion of non-degeneracy does not coincide with that of primal and dual non-degeneracy defined in Definitions 2.7 and 2.8. However, one can still identify an algebraic resemblance between primal non-degeneracy as defined in 2.7 and the non-singularity of the Hessian of the Lagrangian.

In the terminology that we used, the notion of *non-degeneracy* adopted by Jongen in Guddat et al. (1990) is analogous to *non-singularity*, as defined in Definition 2.24, as they both guarantee the applicability of the implicit function theorem, hence ensuring a regular behavior (Theorem 2.4.2 in Guddat et al. (1990)). Around these points the optimal set can be parametrized by means of a single parameter and the parameterization is a diffeomorphism. If a critical point is instead *degenerate* then, according to which algebraic condition is not satisfied by such points, these are classified in four different types. Instead, we classified irregular points according to the behavior of the trajectory of solutions at the point considered, focusing at the possible local topological structure of points

Theorem 3.11 (Main result)**.** *For a primal-dual pair of TV-SDPs (P_t, D_t), let Assumptions 1, 2, and 3 hold. Then, there are two exhaustive scenarios:*

(*i*) *Suppose that at some time in T there exists a non-singular (cf. Def. 2.24) primal-dual optimal point. Then, along the parametrization interval T the number of points in times at which there is a non-differentiable point (cf. Def. 3.3) or a discontinuous isolated multiple point (cf. Def. 3.5) for $\mathcal{P}^*(t)$ or $\mathcal{D}^*(t)$ is finite. All the other points are regular points (cf. Def. 3.1) for both $\mathcal{P}^*(t)$ and $\mathcal{D}^*(t)$.*

(*ii*) *Suppose that for every time $t \in T$, every primal-dual optimal point (X, Z, t) for (P_t, D_t) is singular (cf. Def. 2.24). This happens only when at any $t \in T$ either the solution to (P_t, D_t) is not unique or the solution is unique but X and Z are not strictly complementary. Let $t^* \in T$. If $\mathcal{P}^*(t)$ is unique for every $t \in (t^* - \varepsilon', t^*)$ for some $\varepsilon' > 0$ and $X^* \in \mathcal{P}^*(t^*)$, then (X^*, t^*) must be a point of a type defined in Definitions 3.1, 3.3, 3.4, 3.5, 3.6, or 3.7. The same holds for $\mathcal{D}^*(t)$*

Proof.

case (*i*) By Theorem 2.27, the hypothesis implies that the number of values of $t \in T$ at which there exists an optimal primal-dual singular point for (5) is finite. Let S denote the set of such values. First, let $t^*_{ns} \in T \setminus S$. Then there exists an optimal primal-dual non-singular point $(X^*_{ns}, Z^*_{ns}, t^*_{ns})$. By Theorem 2.26, both (X^*_{ns}, t^*_{ns}) and (Z^*_{ns}, t^*_{ns}) are regular points (cf. Def. 3.1 and Rem. 3.2). Now consider $t^*_s \in S$. Then there exists an optimal primal-dual singular point (X^*_s, Z^*_s, t^*_s). If at t^*_s a loss of inner semicontinuity for $\mathcal{P}^*$ occurs then $\mathcal{P}^*(t^*_s)$ is multi-valued, hence (X^*_s, t^*_s) is a discontinuous isolated multiple point (cf. Def. 3.5). The same holds in the dual version for $\mathcal{D}^*$ and (Z^*_s, t^*_s). If instead at t^*_s continuity of $\mathcal{P}^*$ is preserved, then $\mathcal{P}^*(t^*_s)$ is a singleton. According to whether $\mathcal{P}^*$ is differentiable at t^*_s or not, (X^*_s, t^*_s) is a regular point or a non-differentiable point (cf. Def. 3.3). Since $\mathcal{P}^*(t^*_s)$ is a singleton, a loss of differentiability only happens when t^*_s is in S; that is, when either $\mathcal{D}^*(t^*_s)$ is multi-valued or strict complementarity between X^*_s and Z^*_s fails (this follows from Lemma 2.25). The same holds in the dual version for $\mathcal{D}^*$ and (Z^*_s, t^*_s).

case (*ii*) First, let $t^* \in T$ and $X^* \in \mathcal{P}^*(t^*)$. By hypothesis, there exists $\varepsilon' > 0$ such that $\mathcal{P}^*(t)$ is single-valued and continuous for every $t \in (t^* - \varepsilon', t^*)$. Let us perform a first binary case partition:

A $\mathcal{P}^*(t^*)$ is a single-valued (and thus equal to $\{X^*\}$).

B $\mathcal{P}^*(t^*)$ is multi-valued.

Then, we also define a three-way case partition, independent from the previous one:

1 there exists $\varepsilon'' > 0$ such that $\mathcal{P}^*(t)$ is single-valued for every $t \in (t^*, t^* + \varepsilon'')$.

2 there exists $\varepsilon'' > 0$ $\mathcal{P}^*(t)$ is multi-valued for every $t \in (t^*, t^* + \varepsilon'')$.

3 for every $\delta > 0$ there exists $t', t'' \in (t^*, t^* + \delta)$ such that $\mathcal{P}^*(t')$ is single-valued and $\mathcal{P}^*(t'')$ is multi-valued.

Combining the two partitions, we obtain one consisting of six cases:

A1 in this case $\mathcal{P}^*(t)$ is a single-valued function defined in $(t^* - \varepsilon, t^* + \varepsilon)$, where $\varepsilon := \min\{\varepsilon', \varepsilon''\}$, which is hence continuous by Proposition 2.23. According to whether $\mathcal{P}^*(t)$ is differentiable at t^* or not, (X^*, t^*) is a regular point or a non-differentiable point.

A2 if there exists $\varepsilon'' > 0$ such that $\mathcal{P}^*(t)$ is continuous at any $t \in (t^* - \varepsilon', t^* + \varepsilon'')$ then by definition (X^*, t^*) is a continuous bifurcation point (Definition 3.4). Otherwise, for every $k \in \mathbb{N}$ there must exist a point $t_k \in (t^*, t^* + \frac{1}{k})$ such that a loss of inner semicontinuity occurs a t_k. Hence, (X^*, t^*) is an irregular accumulation point (Definition 3.7).

A3 if there exists $\varepsilon'' > 0$ such that $\mathcal{P}^*(t)$ is continuous at any $t \in (t^* - \varepsilon', t^* + \varepsilon'')$ then, as for any $\delta > 0$ a continuous switch from unique to non-unique solutions must occur, we can construct a sequence of times $\{t_k\}_{k=1}^{\infty}$ at which a continuous bifurcation occurs converging to t^*. Otherwise, we can proceed as in case **A2** and construct a sequence of times at which a loss of inner semicontinuity occurs converging to t^*. Hence, (X^*, t^*) is an irregular accumulation point.

B1 in this case, simply by definition, (X^*, t^*) is a discontinuous isolated multiple point (Definition 3.5).

B2 if there exists $\varepsilon'' > 0$ such that $\mathcal{P}^*(t)$ is continuous at any $t \in (t^* + \varepsilon'')$, by definition (X^*, t^*) is a discontinuous non-isolated multiple point (type 3.5). Otherwise, for every $k \in \mathbb{N}$ there exists a point $t_k \in (t^*, t^* + \frac{1}{k})$ such that a loss of inner semicontinuity occurs a t_k. Hence, (X^*, t^*) is an irregular accumulation point.

B3 the same discussion as in **A3**, (X^*, t^*) is hence an irregular accumulation point.

□

To prove that any type of point that we defined can actually appear, in the following section we exhibit an example of each type.

4 Examples

Regular, non-differentiable and discontinuous isolated multiple points

For $t \in T = (-2, 3)$, consider the primal TV-SDP

$$\begin{aligned} \min \ & tx + ty + z \\ s.t. \ & \begin{pmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{pmatrix} \succeq 0. \end{aligned} \tag{P_t^1}$$

The feasible region is known as Cayley's spectrahedron. We have:

$$\mathcal{P}^*(t) = \begin{cases} \begin{pmatrix} 1 & -t/2 & -t/2 \\ -t/2 & 1 & \frac{t^2}{2} - 1 \\ -t/2 & \frac{t^2}{2} - 1 & 1 \end{pmatrix} & \text{for } t \in (-2, 2) \setminus \{0\}, \\ \left\{ \begin{pmatrix} 1 & a & b \\ a & 1 & -1 \\ b & -1 & 1 \end{pmatrix} \middle| \begin{array}{c} a + b = 0 \\ a, b \in [-1, 1] \end{array} \right\} & \text{at } t = 0, \\ \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix} & \text{for } t \in [2, 3). \end{cases}$$

In both intervals $(-2, 0)$ and $(0, 2)$, the solution to (P_t^1) is unique and the trajectory describes a parabolic differentiable curve. All points are hence regular (Def. 3.1). In $[2, 3)$, the trajectory is constant and hence all its points are also regular.

At $t = 0$ instead there is a loss of uniqueness, as $\mathcal{P}^*(0)$ is a one-dimensional face of Cayley's spectrahedron. Thus, $t = 0$ is a discontinuous isolated multiple point (Def. 3.5), as uniqueness is holding before for $t \in (-2, 0)$ and after for $t \in (0, 3)$.

Moreover, $t = 2$ is a non-differentiable point (Def. 3.3). Indeed:

$$\frac{d}{dt}\mathcal{P}^*(t)|_{t=2^-} = \begin{pmatrix} 0 & -0.5 & -0.5 \\ -0.5 & 0 & 2 \\ -0.5 & 2 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{d}{dt}\mathcal{P}^*(t)|_{t=2^+}.$$

Note that this example covers part (i) of Theorem 3.11, as for all points in $t \in (-2, 0)$ there exists a unique solution (X, y, Z) to (P_t, D_t) such that (X, y, Z, t) is non-singular (Def. 2.24).

Let us show this in more details. Consider the TV-SDP dual to (P_t^1)

$$\begin{aligned} &\max \ x+y+z \\ &s.t. \quad \begin{pmatrix} -x & t/2 & t/2 \\ t/2 & -y & 1/2 \\ t/2 & 1/2 & -z \end{pmatrix} \succeq 0. \end{aligned} \tag{D_t^1}$$

The optimal set map for (D_t^1) is

$$\mathcal{D}^*(t) = \begin{cases} \begin{pmatrix} t^2/2 & -t/2 & -t/2 \\ -t/2 & 0.5 & 0.5 \\ -t/2 & 0.5 & 0.5 \end{pmatrix} & \text{for } t \in (-2,2), \\ \\ \begin{pmatrix} t & t/2 & t/2 \\ t/2 & (t-1)/2 & 0.5 \\ t/2 & 0.5 & (t-1)/2 \end{pmatrix} & \text{for } t \in [2,3). \end{cases}$$

For $t \in (-2, 0)$ the primal-dual pair of solutions is strictly complementary, as the rank of the primal solution is 2 and the rank of the dual solution is 1. Being both unique solutions for every $t \in (-2, 0)$, we conclude by Lemma 2.25.

Notice that the singular points for the parameterization interval $T = (-2, 3)$ are 0 and 2. Indeed, at $t = 0$ there is a loss of uniqueness, while at $t = 2$ there is a loss of strict complementarity (the rank of both primal and dual solution is 1).

Continuous bifurcation point

For $t \in T = (-1, 1)$, consider the primal TV-SDP

$$\begin{aligned} \min \ & x_{11} \\ s.t. \ & x_{44} - x_{33} = 0 \\ & x_{22} = 1 \\ & 2x_{12} + x_{33} + x_{44} = t \\ & X \succeq 0 \end{aligned} \tag{P_t^2}$$

for which

$$\mathcal{P}^*(t) = \begin{cases} \begin{pmatrix} t^2/4 & t/2 & 0 & 0 \\ t/2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \text{for } t \in (-1, 0], \\ \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & a & b \\ 0 & a & t/2 & c \\ 0 & b & c & t/2 \end{pmatrix} \middle| \begin{array}{c} a^2+b^2+c^2 \le \frac{t^2}{4} + t \\ \frac{t}{2}(a^2+b^2) + c^2 - 2abc \le \frac{t^2}{4} \end{array} \right\} & \text{for } t \in (0, 1). \end{cases}$$

$\mathcal{P}^*(t)$ is continuous for every $t \in (-1, 1)$, it is single-valued for every $t \in (-1, 0]$, and it is multi-valued for every $t \in (0, 1)$, being there a 3-dimensional face. Hence $t = 0$ is a continuous bifurcation point for (P_t^2) according to Def. 3.4.

Recall that when there exists a continuous bifurcation point it is necessary that all the points of the parameterization interval are singular according to Def. 2.24. In other words, at any point $t \in (-1, 1)$ either the primal solution is not unique, or the dual solution is not unique, or both primal and dual solutions are unique but not strictly complementary. Indeed, the dual TV-SDP to (P_t^2) is

$$\begin{aligned} &\max \; y + tz \\ &s.t. \quad \begin{pmatrix} 1 & -z & 0 & 0 \\ -z & -y & 0 & 0 \\ 0 & 0 & -x-z & 0 \\ 0 & 0 & 0 & x-z \end{pmatrix} \succeq 0, \end{aligned} \qquad (D_t^2)$$

which is equivalent to $\max\{y + tz \mid y + z^2 \le 0, \; -z \le x \le z\}$ and for which

$$\mathcal{D}^*(t) = \begin{cases} \left\{ \begin{pmatrix} 1 & -t/2 & 0 & 0 \\ -t/2 & t^2/4 & 0 & 0 \\ 0 & 0 & -a & 0 \\ 0 & 0 & 0 & a-t \end{pmatrix} \middle| \; a \in [t, 0] \right\} & \text{for } t \in (-1, 0), \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \text{for } t \in [0, 1). \end{cases}$$

$\mathcal{D}^*(t)$ is continuous for every $t \in (-1, 1)$, multi-valued for every $t \in (-1, 0)$, and it is single-valued for every $t \in [0, 1)$, being there a 1-dimensional face. Thus,

$t = 0$ is a continuous bifurcation point for (D_t^2), according to Def. 3.4.

In particular, a pair of primal-dual solutions for (P_t^2, D_t^2) is not unique for every $t \in (-1, 1) \setminus \{0\}$; for $t = 0$, there is a unique pair of primal-dual solutions for which however strict complementarity does not hold.

Discontinuous non-isolated multiple points

For $t \in T = (-1, 1)$, consider the TV-SDP

$$\begin{aligned} & \min \ tx + ty + z \\ & s.t. \quad \begin{pmatrix} 1 & x & y & 0 \\ x & 1 & z & 0 \\ y & z & 1 & 0 \\ 0 & 0 & 0 & 1 + x + y + z \end{pmatrix} \succeq 0 \end{aligned} \qquad (P_t^3)$$

for which

$$\mathcal{P}^*(t) = \begin{cases} \begin{pmatrix} 1 & -t/2 & -t/2 & 0 \\ -t/2 & 1 & \frac{t^2}{2} - 1 & 0 \\ -t/2 & \frac{t^2}{2} - 1 & 1 & 0 \\ 0 & 0 & 0 & \frac{t^2}{2} - t \end{pmatrix} & \text{for } t \in (-1, 0), \\ \left\{ \begin{pmatrix} 1 & a & b & 0 \\ a & 1 & -1 & 0 \\ b & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \;\middle|\; \begin{array}{c} a + b = 0 \\ a, b \in [-1, 1] \end{array} \right\} & \text{for } t \in [0, 1). \end{cases}$$

$\mathcal{P}^*(t)$ is continuous for every $t \in (-1, 1) \setminus \{0\}$, it is single-valued for every $t \in (-1, 0]$, and it is multi-valued for every $t \in [0, 1)$, as for every $t \in [0, 1)$ the optimal face at t is 2-dimensional. A loss of inner semincontinuity occurs at $t = 0$. Hence, $t = 0$ is a discontinuous non-isolated multiple point, according to Def. 3.6.

Irregular accumulation points

For $t \in T = (-1, 1)$, consider the TV-SDP

$$\begin{aligned} & \min \ f(t)(x - y) + z \\ & s.t. \quad \begin{pmatrix} 1 & x & y & 0 & 0 \\ x & 1 & z & 0 & 0 \\ y & z & 1 & 0 & 0 \\ 0 & 0 & 0 & g(t) & x - y \\ 0 & 0 & 0 & x - y & g(t) \end{pmatrix} \succeq 0 \end{aligned} \qquad (P_t^4)$$

where

$$f(t) := \begin{cases} t \sin \frac{\pi}{t} & \text{if } t > 0, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and } g(t) := \begin{cases} 2t & \text{if } t > 0, \\ 0 & \text{otherwise.} \end{cases}$$

For $t \leq 0$ the feasible region is the intersection between Cayley's spectrahedron and the plane $x - y = 0$. For $t > 0$ the feasible region is the intersection between Cayley's spectrahedron and the region $x - y \in [-2t, 2t]$. Expressing the solutions of (P_t^4) in terms of the variables $x(t), y(t), z(t)$, we have:

$$(x(t), y(t), z(t)) = \begin{cases} (0, 0, -1) & \text{for } t \in (-1, 0], \\ (t, -t, -1) & \text{for } t \in \left(\frac{1}{2k-1}, \frac{1}{2k}\right), \ k = 1, 2, \ldots \\ \{(\alpha, -\alpha, -1) \mid \alpha \in [-t, t]\} & \text{for } t = \frac{1}{k}, \qquad k = 1, 2, \ldots \\ (-t, t, -1) & \text{for } t \in \left(\frac{1}{2k}, \frac{1}{2k+1}\right), \ k = 1, 2, \ldots \end{cases}$$

For every $t \in (-1, 0]$, $\mathcal{P}^*(t)$ is continuous and single-valued. The parameter sequence $\{t_k\}_{k=1}^{\infty} \subseteq (0, 1]$ defined by $t_k := \frac{1}{k}$ is such that $\lim_{k \to \infty} t_k = 0$ and at each t_k a loss of inner semicontinuity occurs. Hence, $t = 0$ is an irregular accumulation point, according to Def. 3.7

In the following, we also provide an example of an accumulation point for a sequence of continuous bifurcation points. For $t \in (-1, 1)$, consider the TV-SDP

$$\begin{aligned} & \min \ z \\ & s.t. \quad \begin{pmatrix} 1 & x & y & 0 & 0 \\ x & 1 & z & 0 & 0 \\ y & z & 1 & 0 & 0 \\ 0 & 0 & 0 & 2h(t) & x - y \\ 0 & 0 & 0 & x - y & 2h(t) \end{pmatrix} \succeq 0, \end{aligned} \tag{P_t^5}$$

where

$$h(t) := \begin{cases} t \sin^2 \frac{\pi}{t} & \text{if } t > 0, \\ 0 & \text{otherwise.} \end{cases}$$

For $t \leq 0$ and for $t = 1/k, \ k = 1, 2, \ldots$ the feasible region is the intersection between Cayley's spectrahedron and the plane $x - y = 0$, while for $t \in (1/k, 1/(k+1))\,, \ k = 1, 2, \ldots$ the feasible region is the intersection between Cayley's spectrahedron and the region $x - y \in [-2h(t), 2h(t)]$. Writing the solutions of (P_t^5) in terms of the variables $x(t), y(t), z(t)$, we have:

$$(x(t), y(t), z(t)) = \begin{cases} (0, 0, -1) & \text{for } t \in (-1, 0], \\ \{(\alpha, -\alpha, -1) \mid \alpha \in [-h(t), h(t)]\} & \text{for } t \in \left(\frac{1}{k}, \frac{1}{k+1}\right), \ k = 1, 2, \ldots \\ (0, 0, -1) & \text{for } t = \frac{1}{k}, \qquad k = 1, 2, \ldots \end{cases}$$

For every $t \in (-1, 1)$, $\mathcal{P}^*(t)$ is continuous. The parameter sequence $\{t_k\}_{k=1}^{\infty} \subseteq (0, 1]$ defined by $t_k := \frac{1}{k}$ is such that $\lim_{k \to \infty} t_k = 0$ and each t_k is a continuous bifurcation point. Hence, $t = 0$ is an irregular accumulation point, according to Def. 3.7.

5 Discussion

Our approach draws upon a long history of work in parametric optimization. In particular, the pioneering work of (Guddat et al. 1990, Chapter 2) outlined a classification of solutions to univariate parametric non-linear constrained optimization problems. There, precise algebraic conditions are shown for points satisfying first-order optimality conditions to be *non-degenerate* (see Remark 3.10). These points exhibit a regular behavior. For *degenerate* points, four different types are defined according to which subset of non-degeneracy conditions is violated. Analogously, our approach also starts by considering algebraic conditions that ensure a regular behavior, but our classification of irregular points was made according to the local topological behavior of the trajectory of solutions at the point considered, rather than according to different sets of algebraic conditions.

We notice that regular points and discontinuous isolated multiple points, defined as in Definitions 3.1 and 3.5 respectively, were first identified by Hauenstein et al. (2019) within the optimal partition approach to parametric analysis for linearly parametrized SDP. Furthermore, non-differentiable points (Definition 3.3) can be easily derived from their results. Our work can hence be seen as a completion of the effort of Hauenstein et al. (2019). Likewise, in our analysis, part (i) of Theorem 3.11 relies on Theorem 2.27 and Theorem 2.26. There, the proof of Theorem 2.27 uses the technique of Hauenstein et al. (2019), while Theorem 2.26 uses the implicit function theorem. Theorem 3.11 part (ii) essentially suggests that the implicit function theorem does not hold anywhere, which allows for a broader range of possible behaviors.

From the point of view of formulating a TV-SDP, the key insight of Hauenstein et al. (2019) and ours is that even seemingly strong and standard assumptions such as the continuity of the data and primal-dual Slater's condition are not sufficient to prevent pathological behavior. We presented a complete characterization of such behaviors. Thereby, we showed that guaranteeing the existence of a non-singular point along the trajectory suffices to prevent highly pathological behaviors. However, this does not prevent from a finite number of losses of differentiability or isolated losses of uniqueness to occur.

From the point of view of algorithm design, two insights can be given: if one can guarantee that the conditions of Theorem 4.4 part (i) are satisfied, algorithms for time-varying optimization need not consider the less obvious behaviors corresponding to Definitions 3.4, 3.6, and 3.7. If, however, we would like to develop a solver for the case where only LICQ (Assumption 1), Slater's condition (Assumption 2), and data continuity (Assumption 3) are satisfied, we need to consider some rather pathological behaviors, such as irregular accumulation points (Definition 3.7). We very much hope that his leads to the development of practical algorithms for time-varying semidefinite programming.

6 Conclusion

We used set-valued analysis to describe and study the trajectory of solutions to TV-SDP. The analysis we carried out brought us to define six different types of points, according to the local structure of the solutions trajectory. Our main result consists in proving that under standard assumptions, there are no other types of points.

One could extend our research by weakening our assumptions: continuity of the data dependence on the parameter, and Slater's primal and dual condition throughout the parameterization interval. These requirements avoid highly degenerate situations. In particular, without continuity of the data, one can expect the trajectory to potentially present a lot of irregularities, e.g., it may fail to be both inner and outer semicontinuous, while, as Theorem 2.20 shows, under the continuity of the data outer semicontinuity is ensured. When Slater's condition is lost, two additional forms of degenerate behavior might occur: the optimal value may not be attained at any feasible point, or there may be a strictly positive duality gap between the primal and dual optimal values. It is not clear whether there could be other types too, perhaps akin to irregular accumulation points. Finally, the properties of specific classes of trajectories of solutions in specific applications may be of considerable interest.

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