Boole-Bonferroni Inequalities to Approximately Determine Optimal Arrangements

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Abstract

We consider the problem of laying out several objects in an equal number of predefined positions. Objects are allowed finitely many orientations, can overlap each other, and must be arranged contiguously. We are particularly interested in the case when the evaluation of the dimensions of the objects requires computational or physical effort. We develop a notion of an arrangement of objects that achieves these requirements. Then, we provide two optimization models that further determine particular notions of the arrangement. The first optimization model is an exact formulation, however it requires an exponentially many inputs. In the second model, we employ classical Boole-Bonferroni inequalities to approximate the lengths of objects. This model requires only quadratically many inputs. We explore the connection of our problem to other combinatoric problems, such as the traveling salesman problem and the bin-packing problem. Finally, we describe how our model generalizes a mathematical puzzle that was recently proposed in the literature.

Keywords: Boole-Bonferroni, packing problem, combinatorics, arrangements, binary optimization, approximations

1. Background

We study the problem of determining an "optimal" layout of physical objects in a plane. Objects can be oriented in different ways. This problem is related to fundamental combinatorial optimization problems involving arrangement of finitely many items. These include the assignment problem, cutting stock problem, and the bin-packing problem; see, e.g., [1]. The problem finds application in a wide breadth of research in specific settings as well, e.g., optimal layouts of furniture [2], development of patterning exercises for childhood education [3], layout-planning for construction site facilities [4], arrangement of boxes in a visual space [5], and several layout problems in the chemical industry [6].

Specifically of interest to us is the situation when the dimensions of the objects in a layout are not known apriori; thus, evaluations are required for determining these measurements. As we quantify later in this work, performing such measurements to determine layouts involves an astronomical number of evaluations even for a handful of objects. Our aim is to determine a *minimum-length arrangement* of items, a term we make precise below. We develop two optimization models in this regard.

Our first model provides an exact formulation that achieves this notion of a minimum-length arrangement. However, this model requires both an exponential number of decision variables as well as evaluations to determine the dimensions of the objects. Next, motivated by the use of classical Boole-Bonferroni styled inequalities, we approximately determine the dimensions of objects in a layout. Then, we develop a second optimization model that uses these approximate dimensions to further determine an approximation to the minimum-length arrangement. The latter model employs decision variables that are several orders of magnitudes fewer than the exact formulation and even fewer evaluations to determine the approximate dimensions.

Boole-Bonferroni styled inequalities are frequently studied to determine approximations to the union of events. This is especially the case when the union involves a considerable number of events and cannot be precisely determined. For an introduction, as well as an extensive survey of such inequalities, see, e.g., [7]. Our work is inspired from these ideas. Such Boole-Bonferroni inequalities have also been used in a similar vein in approximating joint chance-constrained stochastic optimization models; see, e.g., [8, 9]. In these works, first an approximation to the joint chance-constraint using these classical inequalities is constructed. This approximation is then used to bound the chance-constrained optimization model. Instead of a single binary decision variable that determines whether a joint chance-constraint is satisfied at all times, the approximating models employ separate binary decision variables for each of the times. The corresponding optimization models that approximate the true problem have more decision variables, however they may benefit from lower computational effort; see, e.g., [9, 10, 11]. Our work is in a similar spirit, except our approximating models also benefit from significantly fewer decision variables. We provide a detailed discussion of this issue in Section 3.

The structure of this article is as follows. In Section 2, we present some mathematical definitions that describe our problem. In Section 3, we present the two optimization models and provide a discussion on the computational effort. In Section 4 we provide an application of our models in a specific setting. In Section 5, we present some concluding remarks.

2. Definitions

We consider a set of objects, $i \in I$, that require fitting in a set of pre-defined positions, $k \in K$. Each object occupies a single position. The meaning of a position is subtle, and we explain this via an example. Consider three objects to be fit in a rectangle-shaped box of sufficient width. We say an object is in position k_1, k_2, or k_3 when it occupies the left, center, or right positions in the box, respectively. If $|I| < |K|$, empty spaces are allowed between the objects; while, if $|I| > |K|$, then some positions occupy multiple objects. Throughout this work, we consider the $|I| = |K|$ case. We also assume position is filled by only one object that ensures all objects touch each other. Further, we do not consider the heights of the objects; i.e., the fitting does not require a lid. Thus, we are reduced to a problem of positioning objects in a two-dimensional plane. Then, this problem is similar to the cutting and packing optimization problem with irregular shaped objects; the latter problem is known to be $\mathcal{NP}$-

complete [12, 13].

There are infinitely many possibilities of orienting a single object, by rotating it, on a plane. Previous works consider rotations by 90° [14, 15], 120° [16], continuous rotations [17], or no rotations [18]. To this end, we consider object i at location k can be oriented in finitely many ways given by the set $j \in J$. The following definition summarizes this discussion.

Definition 1. *We define an* `Arrangement` *as a linear ordering of objects, where object i is allowed $j = 1, \ldots |J|$ orientations, occupies length b_{ij}, and two adjoining objects $i, i'; i \neq i'$ in orientations j, j' occupy a total length of no more than $b_{ij} + b_{i,j,i',j'}$.*

Corollary 1. *There are a total of $|J|^{|I|-1} \times |I|!$* `Arrangements`.

Proof. With $|I| = |K|$ objects and positions, and $|J|$ orientations, there are $|J|^{|I|} \times |I|!$ possible `Arrangements`. However, $|J|$ of these are repeated as we can view the container from different sides. Hence, the result follows. $\square$

Intuitively, Definition 1 restricts an `Arrangement` to a positioning of objects with no spacing in between them. In other words, adjoining objects have (non-negative) amounts of overlap; e.g., two L-shaped objects in an `Arrangement` can overlap nearly completely or not at all in different orientations. This motivates the definition of the length of several adjoining objects.

Definition 2. *We define the length of r, $r \leq |I|$ objects, $i = 1, \ldots, i_r$, in an* `Arrangement` *where object i_p is in orientation j_{i_p} at position k_p, $p = 1, \ldots, r$, as $\mathbb{L}_r(i_1, j_{i_1}; i_2, j_{i_2}; \ldots; i_r, j_{i_r})$.*

For brevity, we let $l \in L$ denote an `Arrangement` of $|I|$ objects and len_l denote its corresponding length; i.e., $\mathbb{L}_{|I|}(i_1, j_{i_1}; i_2, j_{i_2}; \ldots; i_{|I|}, j_{i_{|I|}}) = \text{len}_l$. Further, we have $\mathbb{L}_1(i, j) = b_{ij}$ and $\mathbb{L}_2(i, j; i', j') = d_{i,j,i',j'}$. Irregularly shaped objects in an `Arrangement` require an exponential number of measurements to determine all the possible lengths. The following corollary to Definition 2 quantifies this exponential growth.

Corollary 2. *Determining all values of* $\mathbb{L}_r(i_1, j_{i_1}; i_2, j_{i_2}; \ldots; i_r, j_{i_r})$ *requires* ${}^{|I|}P_r |J|^{r-1}$ *evaluations; here* ${}^{n}P_k = \frac{n!}{(n-k)!}$.

Proof. The proof is similar to that of Corollary 1. The exponent of $r-1$ arises since $|J|$ arrangements are mirror images. □

Definition 3. *We define an* `Optimal Arrangement` *as an* `Arrangement` *that minimizes* $\mathbb{L}_{|I|}(i_1, j_{i_1}; i_2, j_{i_2}; \ldots; i_{|I|}, j_{i_{|I|}})$.

Corollary 2 leads to Corollary 1 when $|I| = r$. Then, it follows from Definition 3 that determining the minimum length of all the objects in an `Arrangement` requires $|J|^{|I|-1} \times |I|!$ measurements; and, we search for the minimum of these. In Section 3.2, we present an optimization model to this end.

The main goal of this work is to exploit Boole-Bonferroni inequalities to determine *approximate* lengths of arrangements. Instead of performing the astronomical number of evaluations required to determine the length of an arrangement, len_l, we seek to determine lengths of only one or two adjoining objects. In other words, we approximate len_l using $\mathbb{L}_1(i_1, j_{i_1})$ and $\mathbb{L}_2(i_1, j_{i_1}; i_2, j_{i_2})$ alone. Our proposal finds merit in at least two situations:

(i) First, if $|I|$ is large, determining len_l requires an exponential number of measurements; see, Corollary 1. This issue is similar to that we discuss in Section 1 on the calculation of a union of a large number of events; see, e.g., [8]. Further, each of these measurements requires significant repeated effort — first, laying objects in arrangements by a human or a machine, and then measuring them. Such repeated experiments also result in measurement errors; see, e.g., [19]. Determining $\mathbb{L}_1(i_1, j_{i_1})$ and $\mathbb{L}_2(i_1, j_{i_1}; i_2, j_{i_2})$ requires only a linear and quadratic number of evaluations with respect to the number of objects, respectively; see, Corollary 2.

(ii) Second, if the objects have large lengths, positioning them to determine len_l requires large amounts of space that might not be physically available. Also, the errors in such measurements of large dimensions can be substantial; see, e.g., [20].

To this end, we first use the Boole-Bonferroni inequalities with the inclusion-exclusion principle and relate the true length of an `Arrangement` with that of its elements; see, Theorem 1. Then, in Section 3.2 we present our second optimization model that uses this approximate length as an input.

Theorem 1. $\mathbb{L}_{|I|}(i_1, j_{i_1}; i_2, j_{i_2}; \ldots; i_{|I|}, j_{i_{|I|}}) \approx \sum_{p=1}^{|I|-1} \mathbb{L}_2(i_p, j_{i_p}; i_{p+1}, j_{i_{p+1}}) - \sum_{p=2}^{|I|-1} \mathbb{L}_1(i_p, j_{i_p})$ $= \sum_{p=1}^{|I|-1} d_{i_p, j_{i_p}, i_{p+1}, j_{i_{p+1}}} - \sum_{p=2}^{|I|-1} b_{i_p, j_{i_p}}$.

Proof. The proof follows directly from the inclusion-exclusion principle of the classical Boole-Bonferroni inequalities for the union of sets; see, e.g., [21]. For sets $A_t, t \in T$, we have $\mathbb{P}(\bigcup_{t \in T} A_t) = S_1 - S_2 + \cdots (-1)^{T-1} S_T$, where, $S_k = \sum_{1 \leq n_1 < \cdots < n_k \leq |T|} \mathbb{P}(A_{n_1} \cap \cdots \cap A_{n_k})$. □

3. Optimization Models

3.1. Notation

Indices / Sets

$i \in I$	set of objects
$j \in J$	set of orientations
$k \in K$	set of positions
$l \in L$	set of `Arrangements`

Data

$b_{i,j}$	length of object i in orientation j
$d_{i,j,i',j'}$	combined length of object i in orientation j next to object i' in orientation j', $i \neq i'$
len_l	total length of an `Arrangement`

Decision Variables

z_l	=1 if `Arrangement` l is an `Optimal Arrangement`; else 0
$x_{i,j,k}$	=1 if object i is in orientation j at position k; else 0

$y_{i,j,k,i',j',k+1}$ =1 if objects i and i' are in orientation j and j' at position k and $k+1$, respectively; else 0

3.2. Optimization Models

We define a decision variable, z_l, that takes value 0 or 1; a value of 1 indicates arrangement l is an `Optimal Arrangement`. Then, we have the following optimization model:

$$
\begin{aligned}
z^{\text{EXACT}} = \min_{z} \quad & \sum_{l \in L} \text{len}_l z_l && \text{(1a)} \\
& \sum_{l \in L} z_l = 1 && \text{(1b)} \\
& z_l \in \{0, 1\}, \forall l \in L. && \text{(1c)}
\end{aligned}
$$

We denote the optimization model (1) as `EXACT`. The objective function of `EXACT` determines an `Optimal Arrangement`, while equations (1b) and (1c) ensure a single arrangement is chosen. `EXACT` is a trivial optimization model — given the input data, len_l, it simply determines $l^* = \text{argmin}\{\text{len}_l\}$. However, as shown in Corollary 1, `EXACT` requires as input an exponential number of inputs for the $|L|$ values of len_l.

This phenomena of simple mathematical programs relying on an exponential number of data inputs is well-known in combinatorial optimization. For example, a traveling salesman problem determines the minimum tour length over all possible tours; however there are an exponential number of tours to begin with. Another example is the cutting stock problem where patterns are cut from a roll of paper to minimize wastage; again there are an exponential number of patterns to begin with, however given the patterns the solution is easy. In general, given all the exponential number of extreme points of an integer program the minimum is quickly determined in linear time; see, e.g., [1] for more examples.

Using the approximation in Theorem 1, we now present an optimization model that provides an approximation for `EXACT`. To this end, let $x_{i,j,k}$ be a binary variable that takes value 1 if object i is present in orientation j at position k. Then, we have the following optimization model:

$$
\begin{aligned}
z^{\text{APPROX}} = \quad & \min_{x,y} \sum_{\substack{i,i' \in I, j,j' \in J, k \in K; \\ i \neq i', k \neq |K|}} d_{i,j,i',j'} y_{i,j,k,i',j',k+1} - \sum_{\substack{i \in I, j \in J, k \in K; \\ 1 < k < |K|}} b_{i,j} x_{i,j,k} && \text{(2a)} \\
\text{s.t.} \quad & \sum_{i \in I, j \in J} x_{i,j,k} = 1, \quad \forall k \in K && \text{(2b)} \\
& \sum_{j \in J, k \in K} x_{i,j,k} = 1, \quad \forall i \in I && \text{(2c)} \\
& \forall i, i' : i \neq i' \in I, j, j' \in J, k : k \neq |K| \in K && \\
& \qquad y_{i,j,k,i',j',k+1} \leq x_{i,j,k} && \text{(2d)} \\
& \qquad y_{i,j,k,i',j',k+1} \leq x_{i',j',k+1} && \text{(2e)} \\
& \qquad y_{i,j,k,i',j',k+1} \geq x_{i,j,k} + x_{i',j',k+1} - 1 && \text{(2f)} \\
& x_{i,j,k} \in \{0,1\}, \qquad \forall i \in I, j \in J, k \in K && \text{(2g)} \\
& y_{i,j,k,i',j',k+1} \in \{0,1\}, \quad \forall i, i' \in I, j, j' \in J, k, k' \in K. && \text{(2h)}
\end{aligned}
$$

We denote the optimization model (2) as `APPROX`. The first term in the objective function (2a) denotes the quantity $d_{i,j,i',j'} x_{i,j,k} x_{i',j',k+1}$. Constraints (2d)-(2f) linearize the term $x_{i,j,k} x_{i',j',k+1}$, by introducing the binary variable $y_{i,j,k,i',j',k+1}$, with its Mccormick envelope [22]; i.e., y is 1 if an only if both the x are 1. Thus, $y_{i,j,k,i',j',k+1}$ takes value 1 if and only if objects i and i' are present in orientations j and j' at positions k and $k+1$, respectively. Then, the objective function follows from Theorem 1 and seeks to determine the x and y corresponding to the arrangement of minimum length. Constraints (2b) and (2c) ensure that at each position only one (object, orientation) pair is placed and each object is placed at only one (position, orientation) pair, respectively. Constraints (2g) and (2h) enforce the binary restrictions on x and y, respectively.

Next, we discuss the size of the optimization models and the effort required in determining the parameters. Recall that $|I| = |K|$. `EXACT` requires $|J|^{|I|-1} \cdot |I|!$ decision variables and the same number of measurements. `APPROX` requires $|I|^2 \cdot |J| + |I|^2 \cdot (|I|-1)^2 \cdot |J|^2$ decision variables, but only $|I| \cdot |J| + |I| \cdot (|I|-1) \cdot |J|^2$

measurements. As demonstrated in Table 1, the computational effort required to set up EXACT scales quickly to astronomical numbers even for a modest number of objects and orientations. The number of measurements required for APPROX are always fewer than those required for EXACT. The number of binary variables required by EXACT far outgrows those required by APPROX, except for very small numbers of objects and orientations. Specifically, the number of variables in APPROX is more than EXACT only if: (i) there is just one allowed orientation and at most six objects, or (ii) there are at most six orientations and at most four objects.

Objects	Orientations	Variables		Measurements	
$\|I\|$	$\|J\|$	EXACT	APPROX	EXACT	APPROX
4	2	1.92E+02	6.08E+02	1.92E+02	5.60E+01
5	2	1.92E+03	1.65E+03	1.92E+03	9.00E+01
10	2	1.86E+09	3.26E+04	1.86E+09	3.80E+02
20	2	1.28E+24	5.78E+05	1.28E+24	1.56E+03
50	2	1.71E+79	2.40E+07	1.71E+79	9.90E+03
4	5	3.00E+03	3.68E+03	3.00E+03	3.20E+02
5	5	7.50E+04	1.01E+04	7.50E+04	5.25E+02
10	5	7.09E+12	2.03E+05	7.09E+12	2.30E+03
20	5	4.64E+31	3.61E+06	4.64E+31	9.60E+03
50	5	5.40E+98	1.50E+08	5.40E+98	6.15E+04

Table 1: Comparison of computational requirements for EXACT and APPROX. For details, see Section 3.2.

4. A Motivating Example

The research in this work was motivated by a mathematical puzzle proposed by Ninjbat in a recent work [23]. In this work, Ninjbat presents a game called "The Four Strongest" that requires toys of four Mongolian legendary animals —

Dragon (D), Tiger (T), Lion (L), and Garuda (G) — fit in a wooden container. Each of the four toys are allowed two orientations; we denote these orientations as $\uparrow$ and $\downarrow$, respectively. After some simplifications, Ninjbat solves this game with an exhaustive enumeration of all the possibilities. To this end, Ninjbat manually arranges all the four toys in two orientations each and checks whether they fit in the container. Then, he concludes that only one of the several combinations provides an arrangement with a "comfortable" placing of the toys; for details, see [23]. We refer to this puzzle as Ninjbat's Puzzle.

Ninjbat's Puzzle is a special case of the problem we present in Section 3.2 with $|I| = 4, |J| = 2$. The first row of Table 1 presents the dimensions of Ninjbat's Puzzle. The original toys used in Ninjbat's Puzzle are housed at the National Museum of Mongolia, and we do not have the exact measurements. However, we gratefully received images of the projected bases of the toys via private communication [24]; these help us with ballpark measurements of the toys. In the Appendix, we provide details on how we measure the toys, as well as images of the projected bases of the toys. As we mention in Table 1, we conduct 56 evaluations to determine the inputs while Ninjbat conducts 192 [23].

We use the modeling language GAMS with CPLEX 34.3.0 to solve Ninjbat's Puzzle using model (2). We obtain an optimal solution of T, D, L, G in orientations $\uparrow$, $\uparrow$, $\uparrow$, $\downarrow$, respectively, that provides a total length of 29.08 cms. Ninjbat claims a different solution as optimal — D, L, T, G in orientations $\downarrow$, $\uparrow$, $\downarrow$, $\uparrow$, respectively; this solution provides a slightly worse length of 31.54 cms with our measurements.

5. Conclusion

We study the problem of determining an optimal layout of objects in a plane. This problem is related to several well-studied problems in combinatorics. Even when objects are allowed only one orientation, the general version of this problem is known to be hard. We propose two models to solve this problem. The first model relies on an exponential number of inputs, but, given the inputs, is trivial to solve. The second model relies on the well-known Boole-Bonferroni approximations in set theory, and employs inputs that are several orders of magnitude fewer than the first model. The bounds on the errors in such approximations of the union of events is an open question in probability theory. Future research could also explore connections of the approximations we present within existing methods to solve packing and arrangement problems. Finally, we demonstrate how our model generalizes a previously studied traditional puzzle. Several popular puzzles — such as Sokoban and Sudoku — can be formulated as mathematical problems; see, e.g., [25]. In this regard, we also seek to fulfill a philosophical goal: the development of a traditional game into a binary optimization program.

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