Limit sets in global multiobjective optimization

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Abstract
Inspired by the recently introduced branch-and-bound method for continuous multiobjective optimization problems from G. Eichfelder, P. Kirst, L. Meng, O. Stein, A general branch-and-bound framework for continuous global multiobjective optimization, Journal of Global Optimization, 80 (2021) 195-227, we study for a general class of branch-and-bound methods in which sense the generated terminal enclosure and the terminal provisional nondominated set approximate the nondominated set when the termination accuracy is driven to zero. Our convergence analysis of the enclosures relies on constructions from the above paper, but is self-contained and also covers the mixed-integer case. The analysis for the provisional nondominated set is based on general convergence properties of the epsilon-nondominated set, and hence it is also applicable to other algorithms which generate such points. Furthermore, we discuss post processing steps for the terminal enclosure and provide numerical illustrations for the cases of two and three objective functions.

KEYWORDS
Enclosure, nondominated set, approximation, mixed-integer optimization, branch-and-bound, truncation

1. Introduction

In this paper we study the approximation properties of a class of solution algorithms for multiobjective optimization problems of the form

$$\min f(x) \quad \text{s.t.} \quad g(x) \leq 0, \; x \in X \qquad (MOP)$$

with a continuous vector-valued objective function $f : \mathbb{R}^n \to \mathbb{R}^m$, a continuous vector-valued inequality constraint function $g : \mathbb{R}^n \to \mathbb{R}^k$, and an n-dimensional box $X = [\underline{x}, \overline{x}]$ with $\underline{x}, \overline{x} \in \mathbb{R}^n$, $\underline{x} \leq \overline{x}$. The inequality $\leq$ as well as $<$ between vectors is always understood componentwise. For the most part of this paper we will consider the continuous problem MOP, but shall subsequently extend our results to the mixed-integer setting. We do not impose any convexity assumptions on the component functions of f or g so that, in particular, the set of feasible points $M = M(X) = \{x \in X \mid g(x) \leq 0\}$ of MOP is not necessarily convex. However, the compactness of M and the continuity of f yield a compact image set $Y := f(M)$.

The recently introduced branch-and-bound framework for the continuous problem MOP from [4] either identifies the case of an empty feasible set or, otherwise, aims to approximate the nondominated set

$$Y_N = \{y_N \in Y \mid \nexists\, y \in Y: \; y \le y_N, \; y \ne y_N\}$$

of Y by two simultaneous constructions. The first is a sequence of finite subsets of Y, namely the provisional nondominated sets $\mathcal{F}$. The second construction yields box enclosures

$$E(LB, UB) = (LB + \mathbb{R}^m_+) \cap (UB - \mathbb{R}^m_+) = \bigcup_{\substack{(\ell b, ub) \in LB \times UB \\ \ell b \le ub}} [\ell b, ub] \tag{1}$$

of $Y_N \cup \mathcal{F}$ with sequences of finite sets LB and UB. We recall the main properties of these lower and upper bounding sets in Section 2. The idea of using lower and upper bounding sets for multiobjective branch-and-bound methods goes back to [3,16] where such ideas have been developed for multi-objective integer linear optimization.

In [4, Alg. 1] the sequence of lower and upper bounding sets is constructed in such a way that some width $w(LB, UB)$ of the enclosures $E(LB, UB)$ (cf. Section 2) tends to zero. The algorithm terminates for $w(LB, UB) < \varepsilon$ with a prescribed tolerance $\varepsilon > 0$. This yields not only a terminal enclosure $E(LB, UB)$ of Y_N with width below ε, but the terminal provisional nondominated set $\mathcal{F}$ also forms a subset of the ε-nondominated set

$$Y^\varepsilon_N = \{y^\varepsilon_N \in Y \mid \nexists\, y \in Y: \; y \le y^\varepsilon_N - \varepsilon e, \; y \ne y^\varepsilon_N - \varepsilon e\}$$

of Y [4]. The latter result is in line with the approaches from [15,18] and, partly, [5,6]. It also indicates that the width $w(LB, UB)$ of the box enclosure $E(LB, UB)$ is a natural generalization of the gap in single-objective branch-and-bound.

The aim of this article is to study in which sense for ε descreasing to zero the terminal enclosure $E(LB, UB)$ (Section 3.1) and the terminal provisional nondominated set $\mathcal{F}$ (Section 3.2) approximate the nondominated set Y_N. We will show that the enclosure converges to the boundary of the upper image set $Y + \mathbb{R}^m_+$, restricted to some basic box. Also the elements of $\mathcal{F}$ converge to this boundary, where we can show linear speed of this convergence, while they may only converge arbitrarily slowly to the set of so-called weakly nondominated points of Y.

While our convergence analysis is motivated by the constructions from [4], we will use a set of general assumptions which also cover other branch-and-bound approaches for MOP. The analysis for $\mathcal{F}$ is based on general convergence properties of the ε-nondominated set Y^ε_N. Thus it is also applicable to other algorithms that generate points in Y^ε_N. In Section 4 we discuss how certain 'superfluous parts' of the terminal enclosure may be removed.

As the presented approximation analysis does not rely on the continuity of the decision variables x, it also applies to mixed-integer multi-objective optimization problems. Section 5 provides details and states the extended results explicitly, before Section 6 concludes the article with some final remarks.

2. Preliminaries

To ensure the enclosing property of a set $E(LB, UB)$ for the nondominated set Y_N of the (nonempty compact) set $Y = f(M)$ one needs to construct finite sets LB and UB such that the inclusion $Y_N \subseteq E(LB, UB) = (LB + \mathbb{R}^m_+) \cap (UB - \mathbb{R}^m_+)$ holds.

For the construction of an appropriate set UB, in each iteration of [4, Alg. 1] a provisional nondominated set $\mathcal{F}$ is available, that is, a finite and stable subset of Y. Stability means that $\mathcal{F}$ does not contain any two points y^1, y^2 with $y^1 \leq y^2$. For the following construction let $Z = [\underline{z}, \overline{z}]$ denote a sufficiently large box with $f(X) \subseteq \mathrm{int}(Z)$. Such a box exists by the compactness of X and the continuity of f. It allows to define the search region

$$S(\mathcal{F}) = \{z \in \mathrm{int}(Z) \mid \forall q \in \mathcal{F} : q \not\leq z\} = (\mathcal{F} + \mathbb{R}^m_+)^c \cap \mathrm{int}(Z)$$

of all points in $\mathrm{int}(Z) \setminus \mathcal{F}$ which are not dominated by any point from $\mathcal{F}$. With the finite set of local upper bounds $\mathrm{lub}(\mathcal{F}) \subseteq Z$ [12] the search region may be written as

$$S(\mathcal{F}) = \bigcup_{p \in \mathrm{lub}(\mathcal{F})} \{z \in \mathrm{int}(Z) \mid z < p\} = (\mathrm{lub}(\mathcal{F}) - \mathrm{int}(\mathbb{R}^m_+)) \cap \mathrm{int}(Z),$$

so that the choice $UB := \mathrm{lub}(\mathcal{F})$ guarantees $Y_N \subseteq S(\mathcal{F}) \cup \mathcal{F} \subseteq (UB - \mathbb{R}^m_+) \cap Z$. Note that the set $\mathrm{lub}(\mathcal{F})$ depends on the choice of the box Z, and that $UB \subseteq Z$ implies

$$UB \subseteq \overline{z} - \mathbb{R}^m_+. \tag{2}$$

For completeness, we provide a formal definition of local upper bounds, cf. [12].

Definition 2.1. *Let $\mathcal{F}$ be a finite and stable subset of $f(M)$. A set $\mathrm{lub}(\mathcal{F}) \subseteq Z$ is called local upper bound set (with respect to $\mathcal{F}$) if*

(i) $\forall z \in S(\mathcal{F}) : \exists p \in \mathrm{lub}(\mathcal{F}) : z < p$
(ii) $\forall z \in (\mathrm{int}(Z)) \setminus S(\mathcal{F}) : \forall p \in \mathrm{lub}(\mathcal{F}) : z \not< p$
(iii) $\forall p_1, p_2 \in \mathrm{lub}(\mathcal{F}) : p_1 \not\leq p_2$ *or* $p_1 = p_2$

We remark that, for the definition of the search region, $\mathrm{int}(Z)$ actually only needs to contain the set $f(M(X))$, while in the next step we shall also use the stronger inclusion $f(X) \subseteq \mathrm{int}(Z)$.

In [4] it is shown how lower bounding sets LB can be constructed in a branch-and-bound framework, such that (together with appropriate updates of UB) the width $w(LB, UB)$ of $E(LB, UB)$ (cf. the forthcoming equation (5)) tends to zero. By the construction from [4] each $\ell b \in LB$ is generated as an ideal point underestimator of $\widehat{Y}' + \mathbb{R}^m_+$ for a relaxation $\widehat{Y}' + \mathbb{R}^m_+$ of some partial upper image set $f(M(X')) + \mathbb{R}^m_+$ with a subbox $X' \subseteq X$ (recall that for a set $\widehat{Y}' + \mathbb{R}^m_+ \subseteq \mathbb{R}^m$ the point $\alpha \in \mathbb{R}^m$ with entries $\alpha_j := \min\{y_j \mid y \in \widehat{Y}' + \mathbb{R}^m_+\}$ is called ideal point of $\widehat{Y}' + \mathbb{R}^m_+$).

In [4] $(\widehat{Y}' + \mathbb{R}^m_+) \cap f(X') \neq \emptyset$ holds independently of the consistency of $M(X')$. This yields $\ell b \leq f(x)$ for some $x \in X' \subseteq X$ and hence

$$LB \subseteq \omega - \mathbb{R}^m_+, \tag{3}$$

where we call the point $\omega \in \mathbb{R}^m$ with entries

$$\omega_j := \max_{y \in f(X)} y_j$$

anti-ideal point of $f(X)$. In view of $f(X) \subseteq \text{int}(Z)$ it satisfies $\omega < \overline{z}$. See Figure 1 for an illustration of such an anti-ideal point.

We emphasize that the latter construction would not be possible with the set $f(M(X))$ in place of $f(X)$, because in the branch-and-bound framework from [4] lower bounds $\ell b \in LB$ can appear which are related to empty sets $M(X')$ whose relaxations are yet too coarse to verify their emptiness. Such lower bounds may not lie in the set $\omega - \mathbb{R}^m_+$ if ω denoted the anti-ideal point of $f(M(X))$, rather than that of $f(X)$.

While (3) implies $LB \subseteq \overline{z} - \mathbb{R}^m_+$, the inclusion $LB \subseteq \underline{z} + \mathbb{R}^m_+$ does not necessarily hold in the construction from [4], that is, $LB \subseteq Z$ may fail. However, in case of $\ell b \notin Z$, replacing ℓb by the componentwise maximum $\max(\ell b, \underline{z}) \geq \underline{z}$ yields improved lower bounds for the partial upper image sets $f(M(X'))+\mathbb{R}^m_+$ and may be used instead. Moreover, as the lower bounds from [4] converge to $f(X)$ and thus lie in Z anyway after sufficiently many branch-and-bound steps, in the sequel we will assume $LB \subseteq \underline{z} + \mathbb{R}^m_+$ without loss of generality. In view of (2) this entails particularly

$$E(LB, UB) = (LB + \mathbb{R}^m_+) \cap (UB - \mathbb{R}^m_+) \subseteq (\underline{z} + \mathbb{R}^m_+) \cap (\overline{z} - \mathbb{R}^m_+) = Z$$

for any of our choices LB and UB. Together with (3) it also yields $LB \subseteq [\underline{z}, \omega] \subseteq Z$. Figure 1 illustrates the construction.

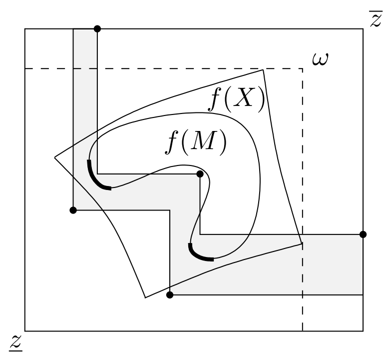


Figure 1.: Enclosure. The bold lines mark Y_N, the dots mark the points from $LB \cup UB$

From the lower bounding property $Y_N \subseteq LB + \mathbb{R}^m_+$ we obtain $Y_N + \mathbb{R}^m_+ \subseteq LB + \mathbb{R}^m_+$. As external stability is satisfied by [19, Th. 3.2.9], the inclusions $Y \subseteq Y_N + \mathbb{R}^m_+$ and hence

$$Y + \mathbb{R}^m_+ \subseteq Y_N + \mathbb{R}^m_+ \subseteq LB + \mathbb{R}^m_+ \subseteq \underline{z} + \mathbb{R}^m_+ \tag{4}$$

hold.

Finally, let us specify how in [4] the width of an enclosure $E(LB, UB)$ is measured. From a geometrical point of view it is defined as

$$w(LB, UB) = \max\{\|(y + te) - y\|_2/\sqrt{m} \mid t \geq 0,\ y, y + te \in E(LB, UB)\}, \tag{5}$$

but an algorithmically more tractable expression is

$$w(LB, UB) = \max\{s(\ell b, ub) \mid (\ell b, ub) \in LB \times UB,\ \ell b \le ub\},$$

where

$$s(\ell b, ub) := \min_{j=1,\dots,m} (ub_j - \ell b_j)$$

denotes the length of a shortest edge of the box $[\ell b, ub]$.

At the end of this section we collect the properties of the construction from [4] which are needed in the subsequent convergence analysis. The results of the present article thus hold for the class of all branch-and-bound algorithms satisfying Assumptions 1–4.

Assumption 1. *For the nondominated set Y_N of $Y = f(M)$ and for a finite stable set $\mathcal{F} \subseteq Y$ let the finite sets $LB, UB \subseteq \mathbb{R}^m$ possess the enclosing property*

$$Y_N \cup \mathcal{F} \subseteq E(LB, UB) = (LB + \mathbb{R}^m_+) \cap (UB - \mathbb{R}^m_+).$$

For $\varepsilon > 0$ and a width $w(LB, UB) < \varepsilon$ of $E(LB, UB)$ we have as a consequence $\mathcal{F} \subseteq Y_N^\varepsilon$.

Assumption 2. *For a box $Z = [\underline{z}, \overline{z}]$ with $f(X) \subseteq \text{int}(Z)$ and for a finite stable set $\mathcal{F} \subseteq Y$ the upper bounding set UB is defined as the set $\text{lub}(\mathcal{F})$ of local upper bounds induced by $\mathcal{F}$ on Z.*

Assumption 2 implies (2).

Assumption 3. *For the box $Z = [\underline{z}, \overline{z}]$ from Assumption 2 and for the anti-ideal point $\omega < \overline{z}$ of $f(X)$, the lower bounding set satisfies $LB \subseteq [\underline{z}, \omega]$.*

Assumption 3 entails (3) and, due to external stability, Assumptions 1 and 3 imply (4).

Assumption 4. *For each $\varepsilon > 0$ there are sets LB and UB satisfying Assumptions 1–3 and $w(LB, UB) < \varepsilon$.*

3. Approximation properties

This section studies if and how under Assumptions 1–4 enclosures $E(UB, LB)$ and provisional nondominated sets $\mathcal{F}$ approximate the nondominated set Y_N of Y for ε descreasing to zero. Observe that Y_N is known to be a subset of the boundary of the image set $Y = f(M)$ [2] so that, if Y is a topological manifold with boundary, the dimension of Y_N can be at most $m - 1$. At the same time, for $w(LB, UB) \to 0$ all boxes $[\ell b, ub]$ in (1) 'become flat at least in one direction' and thus at most $(m-1)$-dimensional. This fits well to the expected dimension of Y_N, so that one may expect that the enclosures $E(LB, UB)$ converge to Y_N in, for example, the Hausdorff metric.

However, Examples like [4, Test problems 9.2, 9.3] show that in general the enclosures $E(LB, UB)$ do not converge to Y_N for $w(LB, UB) \to 0$, but rather to some proper superset of Y_N. In Section 3.1 we will identify this superset and prove the asserted convergence.

Regarding provisional nondominated sets $\mathcal{F}$ recall that under Assumption 1 for $\varepsilon > 0$ and $w(LB, UB) < \varepsilon$ they form subsets of the sets of ε-nondominated points Y_N^ε. Approximation properties of points from Y_N^ε for the set of nondominated points Y_N have, to the best of our knowledge, not been discussed in the literature so far for the present concept of ε-nondominated points. The discussion in Section 3.2 contributes to closing this gap.

3.1. The limit set of the enclosures

This section shows that under Assumptions 1–4 the enclosures $E(LB, UB)$ actually approximate the set $B \cap Z$, where

$$B := \mathrm{bd}(Y + \mathbb{R}^m_+)$$

denotes the boundary of the upper image set $Y + \mathbb{R}^m_+$ with $Y = f(M)$, cf. Figure 2. Recall that, under our assumption of a compact set M and a continuous function f, the image set Y is compact and the upper image set is closed.

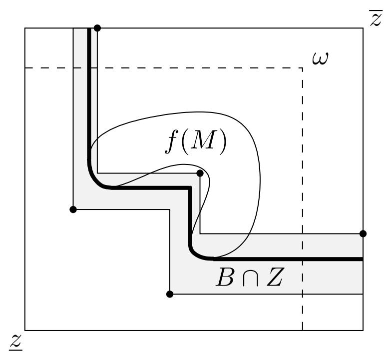


Figure 2.: Enclosure and the set $B \cap Z$

For the following it will be crucial that the set B coincides with the set of weakly nondominated points of the upper image set $Y + \mathbb{R}^m_+$,

$$\widehat{Y}_{wN} \;:=\; \{\widehat{y} \in Y + \mathbb{R}^m_+ \mid \nexists\, y \in Y + \mathbb{R}^m_+ :\; y < \widehat{y}\}.$$

This is related to more general results in [10, L. 4.13] and [14, Cor. 1.48], but for completeness we provide an elementary proof. Since the image set property of $Y = f(M)$ is not relevant for this proof, we consider a general set $Y \subseteq \mathbb{R}^m$.

Lemma 3.1. *For any nonempty compact set $Y \subseteq \mathbb{R}^m$ the sets $B = \mathrm{bd}(Y + \mathbb{R}^m_+)$ and $\widehat{Y}_{wN}$ coincide.*

Proof. To see the inclusion $\subseteq$ first note that the compactness of Y implies $B \subseteq Y + \mathbb{R}^m_+$. Furthermore, if for any $\bar{y}$ in B there is some $y \in Y + \mathbb{R}^m_+$ with $y < \bar{y}$, then some $z \in Y$ satisfies $z \leq y < \bar{y}$. This implies that all $\widetilde{y}$ from some neighborhood U of $\bar{y}$ fulfill $z \leq \widetilde{y}$, so that U is a subset of $Y + \mathbb{R}^m_+$. The latter contradicts the assumption that $\bar{y}$ is a boundary point of $Y + \mathbb{R}^m_+$.

For the proof of $\supseteq$ assume that some $\widehat{y} \in \widehat{Y}_{wN}$ is an interior point of $Y + \mathbb{R}^m_+$. Then, sufficiently close to $\widehat{y}$, there exists some $y \in Y + \mathbb{R}^m_+$ with $y < \widehat{y}$, in contradiction to the definition of $\widehat{Y}_{wN}$. Due to $\widehat{y} \in Y + \mathbb{R}^m_+$ the point $\widehat{y}$ must thus lie in B. $\square$

Since for $Y = f(M)$ all attainable points in $\widehat{Y}_{wN}$ satisfy

$$\begin{aligned} \widehat{Y}_{wN} \cap f(M) &= \{\widehat{y} \in f(M) \mid \nexists\, y \in f(M) + \mathbb{R}^m_+ :\; y < \widehat{y}\} \\ &= \{\widehat{y} \in f(M) \mid \nexists\, y \in f(M) :\; y < \widehat{y}\} \;=\; Y_{wN} \end{aligned}$$

and are, thus, weakly nondominated points of Y, Lemma 3.1 yields $Y_N \subseteq Y_{wN} \subseteq \widehat{Y}_{wN} = B$. Under Assumptions 1–4 we need to consider these inclusions relative to the superset Z of $f(X)$, that is,

$$Y_N \subseteq Y_{wN} \subseteq B \cap Z.$$

Figure 1 and Figure 2 illustrate that the sets Y_{wN} and $B \cap Z$ do not necessarily coincide (see also [4, Test problems 9.2, 9.3]). As announced we shall see that the enclosures $E(LB, UB)$ converge to $B \cap Z$, rather than to its possibly proper subsets Y_N or Y_{wN}.

In the following lemma the set Y can be chosen to be $Y = f(M)$ from Assumption 1 and Z to be the box with $f(X) \subseteq \mathrm{int}(Z)$ from Assumption 2. The assumptions of the lemma can thus be satisfied for MOP with $M(X) \neq \emptyset$ under Assumptions 1–3.

Lemma 3.2. *Let $Y \subseteq \mathbb{R}^m$ be nonempty and compact, let $Z = [\underline{z}, \overline{z}]$ be a box with $Y \subseteq \mathrm{int}(Z)$, let $LB \subseteq Z$ be a lower bounding set for Y_N, let $\mathcal{F}$ be a finite stable subset of Y, and define $UB = \mathrm{lub}(\mathcal{F})$ with the corresponding set of local upper bounds. Then the set $B = \mathrm{bd}(Y + \mathbb{R}^m_+)$ satisfies*

$$B \cap Z \subseteq E(LB, UB).$$

Proof. The inclusion $B \subseteq Y + \mathbb{R}^m_+$ and (4) entail

$$B \subseteq LB + \mathbb{R}^m_+. \tag{6}$$

Moreover, in view of $\mathcal{F} \subseteq Y$ we have $\mathcal{F} + \mathbb{R}^m_+ \subseteq Y + \mathbb{R}^m_+$ and therefore

$$(Y + \mathbb{R}^m_+)^c \cap \mathrm{int}(Z) \subseteq (\mathcal{F} + \mathbb{R}^m_+)^c \cap \mathrm{int}(Z) = S(\mathcal{F}) \subseteq (UB - \mathbb{R}^m_+) \cap Z.$$

The closedness of $(UB - \mathbb{R}^m_+) \cap Z$ thus implies

$$\mathrm{cl}\left((Y + \mathbb{R}^m_+)^c \cap \mathrm{int}(Z)\right) \subseteq (UB - \mathbb{R}^m_+) \cap Z. \tag{7}$$

In the remainder of the proof we shall show the inclusion

$$\mathrm{cl}\left((Y + \mathbb{R}^m_+)^c\right) \cap Z \subseteq \mathrm{cl}\left((Y + \mathbb{R}^m_+)^c \cap \mathrm{int}(Z)\right), \tag{8}$$

because from (7), (8) and from the identity $B = \mathrm{bd}\left((Y + \mathbb{R}^m_+)^c\right)$ we then obtain

$$B \cap Z \subseteq (UB - \mathbb{R}^m_+) \cap Z.$$

Together with (6) this yields $B \cap Z \subseteq (LB + \mathbb{R}^m_+) \cap (UB - \mathbb{R}^m_+) \cap Z = E(LB, UB)$, so that the assertion is shown.

For the proof of (8) choose any $y \in \mathrm{cl}\left((Y + \mathbb{R}^m_+)^c\right) \cap Z$. Then there exists a sequence $(y^k) \subseteq (Y + \mathbb{R}^m_+)^c$ with $\lim_k y^k = y \in Z$. As (y^k) does not necessarily lie in Z, in a first step we modify (y^k) to a sequence

$$(p^k) \subseteq (Y + \mathbb{R}^m_+)^c \cap Z \tag{9}$$

with

$$\lim_k p^k = y. \tag{10}$$

In a second step we will further change (p^k) to a sequence $(z^k) \subseteq (Y + \mathbb{R}^m_+)^c \cap \mathrm{int}(Z)$ with $\lim_k z^k = y$, so that (8) is shown.

Step 1: The construction of (p^k).

For each $k \in \mathbb{N}$ let p^k be the orthogonal projection of y^k to $\overline{z} - \mathbb{R}^m_+$. We need to show (9) and (10).

In view of $Z \subseteq \overline{z} - \mathbb{R}^m_+$, any $y^k \in Z$ satisfies $p^k = y^k \in (Y + \mathbb{R}^m_+)^c \cap Z$, so that for the proof of (9) it remains to verify $p^k \in (Y + \mathbb{R}^m_+)^c \cap Z$ for any $y^k \in Z^c$.

Thus let $y^k \in Z^c$. For the purpose of contradiction let us assume $p^k \in (Y + \mathbb{R}^m_+) \cup Z^c$. By construction we have $p^k \in \overline{z} - \mathbb{R}^m_+$, and (4) implies $p^k \in \underline{z} + \mathbb{R}^m_+$, so that we arrive at $p^k \in Z$ and, hence, $p^k \in Y + \mathbb{R}^m_+$. Since the orthogonal projection of y^k to $\overline{z} - \mathbb{R}^m_+$ can easily be calculated to coincide with the componentwise minimum $\min(y^k, \overline{z})$, this yields $y^k \geq \min(y^k, \overline{z}) = p^k \in Y + \mathbb{R}^m_+$ and, thus, $y^k \in Y + \mathbb{R}^m_+$. The latter contradicts the choice of y^k so that (9) is shown.

For the proof of (10) observe that p^k minimizes the function $\|p - y^k\|_2$ over $p \in \overline{z} - \mathbb{R}^m_+$. Due to $y \in Z$ this entails $\|p^k - y^k\|_2 \leq \|y - y^k\|_2$ and

$$\|p^k - y\|_2 \leq \|p^k - y^k\|_2 + \|y^k - y\|_2 \leq 2\|y^k - y\|_2.$$

The convergence of (y^k) to y thus implies (10).

Step 2: The construction of (z^k).

As seen in Step 1, for each $k \in \mathbb{N}$ the point p^k satisfies (9). In view of the compactness of Y the set $(Y + \mathbb{R}^m_+)^c$ is open, so that for each $k \in \mathbb{N}$ there exists some $\varepsilon^k \in (0, 1/k]$ such that the ball $B(p^k, \varepsilon^k)$ is contained in $(Y + \mathbb{R}^m_+)^c$. Since Z can be written as $\mathrm{cl}(\mathrm{int}\, Z)$, each such ball must also contain a point $z^k \in (Y + \mathbb{R}^m_+)^c \cap \mathrm{int}(Z)$. By $\|z^k - p^k\|_2 \leq \varepsilon^k \leq 1/k$ and

$$\|z^k - y\|_2 \leq \|z^k - p^k\|_2 + \|p^k - y\|_2 \leq 1/k + \|p^k - y\|_2$$

the convergence of (p^k) to y implies the convergence of (z^k) to y. This completes the proof. □

Due to Lemma 3.2, the Hausdorff distance of $B \cap Z$ and $E(LB, UB)$ collapses to the excess

$$\mathrm{ex}(E(LB, UB), B \cap Z) = \sup_{y \in E(LB,UB)} \mathrm{dist}(y, B \cap Z),$$

of $E(LB, UB)$ over $B \cap Z$, where $\operatorname{dist}(y, B \cap Z) = \inf_{b \in B \cap Z} \|b - y\|_2$ denotes the distance of y from $B \cap Z$. The subsequent Theorem 3.4 hence addresses a property of the Hausdorff distance.

Its proof relies on properties of the infimum $\varphi_A(a)$ of

$$\min_{t \in \mathbb{R}} t \quad \text{s.t.} \quad a + te \in A + \mathbb{R}^m_+ \tag{$D(a,A)$}$$

for a nonempty compact set $A \subseteq \mathbb{R}^m$ and a point $a \in \mathbb{R}^m$, where e denotes the all ones vector. The optimal value function φ_A is known as Tammer-Weidner functional [7]. Its following properties are consequences of more general results in [8, Prop. 2.3.4, Th. 2.3.1].

Lemma 3.3. *Let $A \subseteq \mathbb{R}^m$ be a nonempty compact set. Then the following assertions hold:*

a) For each $a \in \mathbb{R}^m$ the problem $D(a, A)$ is solvable with optimal point as well as optimal value $\varphi_A(a)$.
b) The function φ_A is continuous.
c) The identity $\{a \in \mathbb{R}^m \mid \varphi_A(a) \leq 0\} = A + \mathbb{R}^m_+$ holds.
d) The identity $\{a \in \mathbb{R}^m \mid \varphi_A(a) = \lambda\} = -\lambda e + \operatorname{bd}(A + \mathbb{R}^m_+)$ holds for any $\lambda \in \mathbb{R}$.

As a final preparation for Theorem 3.4, recall that the anti-ideal point ω of $f(X)$ from Assumption 3 satisfies $\omega < \overline{z}$. We call any $\widetilde{\omega} \in \mathbb{R}^m$ with $\omega \leq \widetilde{\omega}$ an anti-ideal point overestimator for $f(X)$. It may be chosen such that $\omega \leq \widetilde{\omega} < \overline{z}$ holds, so that the number

$$\delta := \min_{j=1,\ldots,m} (\overline{z}_j - \widetilde{\omega}_j) \tag{11}$$

is positive. Assumption 4 then allows to choose LB and UB with $w(LB, UB) \leq \delta/3$, as the subsequent theorem will presume. Note that (3) implies $LB \subseteq \omega - \mathbb{R}^m_+ \subseteq \widetilde{\omega} - \mathbb{R}^m_+$ for each lower bounding set LB satisfying Assumption 3.

Theorem 3.4. *Let the feasible set $M(X)$ of MOP be nonempty and let Assumptions 1–3 hold with sets Y, $\mathcal{F}$, $Z = [\underline{z}, \overline{z}]$, LB and UB. Furthermore, let Assumption 4 hold, and for some anti-ideal point overestimator $\widetilde{\omega}$ for $f(X)$ with $\widetilde{\omega} < \overline{z}$, let LB and UB be chosen such that $w(LB, UB) \leq \delta/3$ holds with δ from* (11). *Then the excess of $E(LB, UB)$ over $B \cap Z$ satisfies*

$$\operatorname{ex}(E(LB, UB), B \cap Z) \leq 3\sqrt{m}\, w(LB, UB),$$

i.e., the Hausdorff distance of $B \cap Z$ and $E(LB, UB)$ is bounded from above by $3\sqrt{m}\, w(LB, UB)$.

Proof. Choose any $y \in E(LB, UB)$. We will show the assertion by constructing some point $\widetilde{y} \in B \cap Z$ with

$$\|y - \widetilde{y}\|_2 \leq 3\sqrt{m}\, w(LB, UB), \tag{12}$$

since this implies

$$\operatorname{dist}(y, B \cap Z) \leq \|y - \widetilde{y}\|_2 \leq 3\sqrt{m}\, w(LB, UB)$$

and, thus, $\sup_{y\in E(LB,UB)} \operatorname{dist}(y, B\cap Z) \leq 3\sqrt{m}\, w(LB,UB)$.

For the construction of $\widetilde{y}$ we shall prove that (under some additional assumption, giving rise to the distinction of two cases) the set

$$T := \{t \in \mathbb{R} \mid y + te \in E(LB, UB)\}$$

contains some $\widetilde{t}$ with $\widetilde{y} := y + \widetilde{t}e \in B \cap Z$ and (12). We split the details of this proof into four steps.

Step 1: T is a closed interval.

The closedness of T is clear. Moreover, for any two scalars $t_0, t_1 \in T$ with $t_0 \leq t_1$ the points $y^0 := y + t_0 e$ and $y^1 := y + t_1 e$ are elements of $E(LB, UB)$, so that there are some $\ell b^0 \in LB$ and $ub^1 \in UB$ with $\ell b^0 \leq y^0$ and $y^1 \leq ub^1$. Hence, for any $\lambda \in (0,1)$ and the scalar $t_\lambda := (1-\lambda)t_0 + \lambda t_1$ the point $y^\lambda := y + t_\lambda e$ satisfies $\ell b^0 \leq y^0 \leq y^\lambda \leq y^1 \leq ub^1$. This means $y^\lambda \in [\ell b^0, ub^1]$ with $(\ell b^0, ub^1) \in LB \times UB$, $\ell b^0 \leq ub^1$ and, thus, $y^\lambda \in E(LB, UB)$. This yields $t_\lambda \in T$, so that T is convex and may be written as the closed interval $[\underline{t}, \overline{t}]$.

Note that $y \in E(LB, UB)$ yields $0 \in T$ and, thus, $\underline{t} \leq 0 \leq \overline{t}$, and that the closedness of $[\underline{t}, \overline{t}]$ implies that the points

$$\underline{y} := y + \underline{t}e \quad \text{and} \quad \overline{y} := y + \overline{t}e$$

are elements of $E(LB, UB)$. For later use also observe that for every $t \in T$ (5) entails

$$\sqrt{m}\,|t| = \|(y + te) - y\|_2 \leq \sqrt{m}\, w(LB, UB). \tag{13}$$

Step 2: $\underline{y} \in (Y + \mathbb{R}^m_+)^c \cup B$ holds.

Since $\underline{t}$ is a boundary point of T, also $\underline{y}$ is a boundary point of $E(LB, UB)$. More precisely, we have $\underline{y} \in E(LB, UB)$ and

$$y + te \notin E(LB, UB) = (LB + \mathbb{R}^m_+) \cap (UB - \mathbb{R}^m_+)$$

for all $t < \underline{t}$. Due to $y + te \in UB - \mathbb{R}^m_+$ for all $t < \overline{t}$ the point $\underline{y}$ actually lies in $\operatorname{bd}(LB + \mathbb{R}^m_+)$.

In view of (4) we have $\operatorname{int}(Y + \mathbb{R}^m_+) \subseteq \operatorname{int}(LB + \mathbb{R}^m_+)$. Consequently $\underline{y} \in \operatorname{bd}(LB + \mathbb{R}^m_+)$ implies

$$\underline{y} \in \big(\operatorname{int}(LB + \mathbb{R}^m_+)\big)^c \subseteq \big(\operatorname{int}(Y + \mathbb{R}^m_+)\big)^c = (Y + \mathbb{R}^m_+)^c \cup B.$$

Step 3: $\overline{y} < \overline{z}$ implies $\overline{y} \in Y + \mathbb{R}^m_+$.

Similarly as in Step 2, since $\overline{t}$ is a boundary point of T, the point $\overline{y}$ lies on the boundary of $E(LB, UB)$ and, in particular, on $\operatorname{bd}(UB - \mathbb{R}^m_+)$. We recall the definition and characterization of the search region

$$S(\mathcal{F}) = (\mathcal{F} + \mathbb{R}^m_+)^c \cap \operatorname{int}(Z) = (UB - \operatorname{int}(\mathbb{R}^m_+)) \cap \operatorname{int}(Z).$$

It is not hard see that this identity of $(\mathcal{F}+\mathbb{R}^m_+)^c$ and $UB-\text{int}(\mathbb{R}^m_+)$ relative to $\text{int}(Z)$ carries over to an identity relative to $\overline{z}-\text{int}(\mathbb{R}^m_+)$ (but not relative to $\overline{z}-\mathbb{R}^m_+$). Thus, also the sets $\text{bd}\left((\mathcal{F}+\mathbb{R}^m_+)^c\right)$ and $\text{bd}(UB-\text{int}(\mathbb{R}^m_+))$ coincide relative to $\overline{z}-\text{int}(\mathbb{R}^m_+)$, so that, under our assumption $\overline{y}<\overline{z}$,

$$\begin{aligned}\overline{y}\in \text{bd}(UB-\mathbb{R}^m_+)\cap(\overline{z}-\text{int}(\mathbb{R}^m_+)) &= \text{bd}(UB-\text{int}(\mathbb{R}^m_+))\cap(\overline{z}-\text{int}(\mathbb{R}^m_+))\\ = \text{bd}\left((\mathcal{F}+\mathbb{R}^m_+)^c\right)\cap(\overline{z}-\text{int}(\mathbb{R}^m_+)) &= \text{bd}(\mathcal{F}+\mathbb{R}^m_+)\cap(\overline{z}-\text{int}(\mathbb{R}^m_+))\end{aligned}$$

holds. Hence the closedness of $\mathcal{F}+\mathbb{R}^m_+$ and $\mathcal{F}\subseteq Y$ yield $\overline{y}\in\mathcal{F}+\mathbb{R}^m_+\subseteq Y+\mathbb{R}^m_+$.

Step 4: There exists some $\widetilde{y}\in B\cap Z$ with (12).

We consider two subcases.

Case 4.1: $y<\overline{z}-w(LB,UB)e$.

In view of (13) and $\overline{t}\geq 0$ we have $\overline{t}\leq w(LB,UB)$, so that $\overline{y}=y+\overline{t}e$ implies $\overline{y}\leq y+w(LB,UB)e<\overline{z}$, and Step 3 entails $\overline{y}\in Y+\mathbb{R}^m_+$. The latter observation, Step 2 and Lemma 3.3c),d) yield $\varphi_Y(\underline{y})\geq 0$ and $\varphi_Y(\overline{y})\leq 0$ for the Tammer-Weidner functional φ_Y. Since φ_Y is continuous by Lemma 3.3b), also the function

$$\sigma(\lambda):=\varphi_Y\left((1-\lambda)\underline{y}+\lambda\overline{y}\right)$$

is continuous. Due to $\sigma(0)=\varphi_Y(\underline{y})\geq 0$ and $\sigma(1)=\varphi_Y(\overline{y})\leq 0$ the intermediate value theorem thus guarantees the existence of some $\widetilde{\lambda}\in[0,1]$ with $\sigma(\widetilde{\lambda})=0$. The latter means that the point $\widetilde{y}:=(1-\widetilde{\lambda})\underline{y}+\widetilde{\lambda}\overline{y}$ fulfills $\varphi_Y(\widetilde{y})=0$. By Lemma 3.3d) this is equivalent to $\widetilde{y}\in\text{bd}(Y+\mathbb{R}^m_+)=B$. In addition, $\underline{y},\overline{y}\in Z$ and the convexity of Z imply $\widetilde{y}\in Z$.

It is not hard to see that $\widetilde{y}=y+\widetilde{t}e$ holds with $\widetilde{t}:=(1-\widetilde{\lambda})\underline{t}+\widetilde{\lambda}\overline{t}\in T$, so that (13) yields

$$\|y-\widetilde{y}\|_2=\|\widetilde{t}e\|_2=\sqrt{m}\,|\widetilde{t}|\leq\sqrt{m}\,w(LB,UB). \tag{14}$$

Hence the assertion of Step 4 is shown in this case. Figure 3 illustrates the construction.

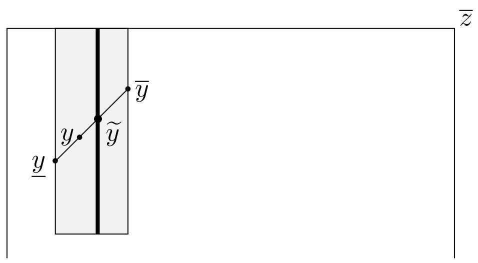


Figure 3.: Construction in Case 4.1. The bold line marks B

Case 4.2: $y\not<\overline{z}-w(LB,UB)e$.

Figure 4 shows that, unlike in Case 4.1, under the current assumption we may have $\overline{y}\in\text{bd}(\overline{z}-\mathbb{R}^m_+)$ and the line segment between $\underline{y}$ and $\overline{y}$ may not contain any $\widetilde{y}\in B\cap Z$.

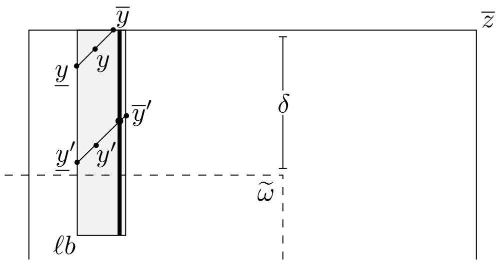


Figure 4.: Construction in Case 4.2

As a remedy, we shift the point y 'away from $\mathrm{bd}(\overline{z} - \mathbb{R}^m_+)$' to some $y' \in E(LB, UB)$, such that the existence of some corresponding $\widetilde{y} \in B \cap Z$ and (12) are guaranteed.

In fact, we define the index sets $J_< := \{j \mid y_j < \overline{z}_j - w(LB, UB)\}$ and $J_\geq := \{j \mid y_j \geq \overline{z}_j - w(LB, UB)\}$ and consider the point y' with $y'_j = y_j$, $j \in J_<$, and $y'_j = y_j - 2w(LB, UB)$, $j \in J_\geq$. Then it holds

$$y' < \overline{z} - w(LB, UB)e. \tag{15}$$

Moreover, y' lies in $E(LB, UB)$ since it actually lies in the same boxes as y. In fact, let $y \in [\ell b, ub]$ with $(\ell b, ub) \in LB \times UB$, $\ell b \leq ub$. This implies $y' \leq y \leq ub$. In addition to that, for each $j \in J_<$ it holds $y'_j = y_j \geq \ell b_j$, and due to $w(LB, UB) \leq \delta/3$ each $j \in J_\geq$ satisfies

$$\begin{aligned} y'_j = y_j - 2w(LB, UB) &\geq \overline{z}_j - 3w(LB, UB) \geq \overline{z}_j - \delta \\ = (\overline{z}_j - \widetilde{\omega}_j) - \min_{i=1,\ldots,m} (\overline{z}_i - \widetilde{\omega}_i) &+ \widetilde{\omega}_j \geq \widetilde{\omega}_j \geq \omega_j \geq \ell b_j, \end{aligned}$$

where the last inequality holds by (3).

We may thus proceed as in Case 4.1: we define the set $T' := \{t \in \mathbb{R} \mid y' + te \in E(LB, UB)\}$ and deduce from Step 1 that it is a closed interval $[\underline{t}', \overline{t}']$. The point $\underline{y}' := y' + \underline{t}'e$ lies in $(Y + \mathbb{R}^m_+)^c \cup B$ by Step 2. From (13) and (15) we obtain $\overline{y}' := y' + \overline{t}'e < \overline{z}$, so that Step 3 yields $\overline{y}' \in Y + \mathbb{R}^m_+$. By the intermediate value theorem for φ_Y this implies the existence of some $\widetilde{y} \in B \cap Z$ on the connecting line segment of $\underline{y}'$ and $\overline{y}'$, cf. Figure 4.

The distance of $\widetilde{y}$ from y satisfies

$$\|y - \widetilde{y}\|_2 \leq \|y - y'\|_2 + \|y' - \widetilde{y}\|_2$$

with

$$\|y - y'\|_2^2 = \sum_{j \in J_\geq} (y_j - y'_j)^2 = \sum_{j \in J_\geq} (2w(LB, UB))^2 \leq 4m\, w(LB, UB)^2$$

and, like in (14),

$$\|y' - \widetilde{y}\|_2 \leq \sqrt{m}\, w(LB, UB).$$

This results in (12) and completes the proof. □

The main consequence of Theorem 3.4 for branch-and-bound methods meeting Assumptions 1–4 is that for sequences (LB^k) and (UB^k) of lower and upper bounding sets with $\lim_k w(LB^k, UB^k) = 0$ the enclosing sets $E(LB^k, UB^k)$ of the nondominated set Y_N converge to $B \cap Z$ in the Hausdorff metric (at linear speed in $w(LB^k, UB^k)$).

At first glance it may seem that this is a purely qualitative result, since the anti-ideal point ω of $f(X)$, its overestimators $\widetilde{\omega}$ and, thus, δ from (11) are not explicitly known. For some prescribed termination tolerance ε with $w(LB^k, UB^k) < \varepsilon$ it is then unclear if $\varepsilon \leq \delta/3$ and its consequence $\text{ex}(E(LB^k, UB^k), B \cap Z) < 3\sqrt{m}\,\varepsilon$ from Theorem 3.4 hold.

This, however, can easily be fixed by choosing the box $Z = [\underline{z}, \overline{z}]$ appropriately, as described in the following corollary. It does without the explicit knowledge of the anti-ideal point ω of $f(X)$ or of a general overestimator $\widetilde{\omega}$ in the computation of δ from (11). Note that, on the other hand, the appearance of ω in Assumption 3 is not restrictive. Furthermore, in practical applications the vector f often possesses factorable entries, so that a box Z' with $f(X) \subseteq \text{int}(Z')$ can be constructed by interval arithmetic.

Corollary 3.5. *Let the feasible set $M(X)$ of MOP be nonempty, let $Z' = [\underline{z}', \overline{z}']$ be a box with $f(X) \subseteq \text{int}(Z')$, for some $\bar{\varepsilon} > 0$ define $Z := [\underline{z}', \overline{z}' + 3\bar{\varepsilon}e]$ and let Assumptions 1–3 hold with sets Y, $\mathcal{F}$, Z, LB and UB. Furthermore, let Assumption 4 be satisfied, and for any $\varepsilon \in (0, \bar{\varepsilon}]$ let LB and UB be chosen such that $w(LB, UB) \leq \varepsilon$ holds. Then the excess of $E(LB, UB)$ over $B \cap Z$ satisfies*

$$\text{ex}(E(LB, UB), B \cap Z) \leq 3\sqrt{m}\,\varepsilon.$$

Proof. We will show that $\widetilde{\omega} := \overline{z}'$ is an anti-ideal point overestimator for $f(X)$ satisfying the assumptions of Theorem 3.4. In fact, the box $Z = [\underline{z}, \overline{z}]$ with $\underline{z} := \underline{z}'$ and $\overline{z} := \overline{z}' + 3\bar{\varepsilon}e$ clearly satisfies $f(X) \subseteq \text{int}(Z)$, in view of $f(X) \subseteq \text{int}(Z')$ the point $\widetilde{\omega} = \overline{z}'$ is an anti-ideal point overestimator for $f(X)$, and we have $\overline{z} - \widetilde{\omega} = 3\bar{\varepsilon}e > 0$. Moreover, in view of

$$\delta = \min_{j=1,\ldots,m} (\overline{z}_j - \widetilde{\omega}_j) = 3\bar{\varepsilon}$$

all $\varepsilon \in (0, \bar{\varepsilon}]$ fulfill $\varepsilon \leq \bar{\varepsilon} \leq \delta/3$, so that $w(LB, UB) \leq \varepsilon$ implies $w(LB, UB) \leq \delta/3$. Theorem 3.4 thus entails $\text{ex}(E(LB, UB), B \cap Z) \leq 3\sqrt{m}\, w(LB, UB) \leq 3\sqrt{m}\,\varepsilon$. □

3.2. Convergence of ε-nondominated points

This section treats approximation properties of the ε-nondominated set. As under Assumption 1 for $w(E(LB, UB)) < \varepsilon$ the provisional nondominated set $\mathcal{F}$ is a subset of the ε-nondominated set, the results imply insights into its convergence behaviour, but our results are also applicable to study the results of other algorithms which generate ε-nondominated points.

As a first observation, Figure 5 illustrates that even for a convex set $Y = f(M)$ one must not expect points from Y_N^ε to converge to the nondominated set Y_N for ε decreasing to zero, but only to the weakly nondominated set Y_{wN}. In fact, we shall show that for any nonempty compact set $Y \subseteq \mathbb{R}^m$ and ε decreasing to zero the sets Y_N^ε converge to Y_{wN} in the Hausdorff metric. As for any $\varepsilon > 0$ the inclusion $Y_{wN} \subseteq Y_N^\varepsilon$ is

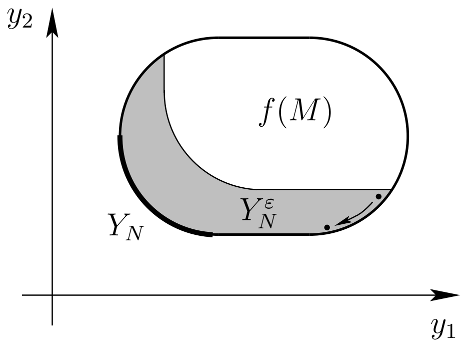


Figure 5.: ε-nondominated points

easy to see, the Hausdorff distance reduces to the excess

$$\mathrm{ex}(Y_N^\varepsilon, Y_{wN}) = \sup_{y_N^\varepsilon \in Y_N^\varepsilon} \mathrm{dist}(y_N^\varepsilon, Y_{wN})$$

of Y_N^ε over Y_{wN}.

We would like to note that it is possible to show convergence to the nondominated set Y_N, rather than to the weakly nondominated set Y_{wN} only, if one employs a definition for ε-nondominated points in which the fixed direction e is replaced by the union over arbitrary directions in $\mathbb{R}^m_+ \setminus \{0\}$, see [20]. In [9] convergence for various concepts of ε-nondominance is examined. Only for those concepts being defined by strictly monotone functionals, convergence to the nondominated set is obtained. For a notion of ε-nondominated points which is slightly stronger than the present one, results concerning their convergence to the weakly nondominated set are given in [9, Sec. 3.1], but for completeness we also provide a proof using our slightly weaker notion.

Proposition 3.6. *For any nonempty compact set $Y \subseteq \mathbb{R}^m$ we have*

$$\lim_{\varepsilon \searrow 0} \mathrm{ex}(Y_N^\varepsilon, Y_{wN}) = 0.$$

Proof. Since for any $0 < \varepsilon_1 \leq \varepsilon_2$ the inclusion $Y_N^{\varepsilon_1} \subseteq Y_N^{\varepsilon_2}$ holds, each monotonically decreasing zero sequence (ε_k) leads to a monotonically decreasing sequence $(\mathrm{ex}(Y_N^{\varepsilon_k}, Y_{wN}))$. As the latter sequence is also bounded from below by zero, it converges to some nonnegative limit.

Assume that we have $\lim_k \mathrm{ex}(Y_N^{\varepsilon_k}, Y_{wN}) = c > 0$ for some monotonically decreasing zero sequence (ε_k). Then, for each $k \in \mathbb{N}$ there exists some $y^k \in Y_N^{\varepsilon_k}$ with $\mathrm{dist}(y^k, Y_{wN}) \geq c/2$. As the sequence (y^k) lies in the compact set Y, without loss of generality we may assume that it converges to some $\bar{y} \in Y$. Due to, e.g., [17], the distance function $\mathrm{dist}(\cdot, Y_{wN})$ is continuous for the nonempty closed set Y_{wN}, so that we arrive at $\mathrm{dist}(\bar{y}, Y_{wN}) \geq c/2 > 0$ and, in particular, $\bar{y}$ does not lie in Y_{wN}.

In view of $\bar{y} \in Y$, there exists some $y \in Y$ with $y < \bar{y}$. As also the sequence $(y^k - \varepsilon_k e)$ converges to $\bar{y}$, for some sufficiently large k we find $y < y^k - \varepsilon_k e$, in contradiction to the choice $y^k \in Y_N^{\varepsilon_k}$. Therefore, for each monotonically decreasing zero sequence (ε_k) we obtain $\lim_k \mathrm{ex}(Y_N^{\varepsilon_k}, Y_{wN}) = 0$, and the assertion is shown. □

Since the terminal provisional nondominated set $\mathcal{F}$ for $w(E(LB, UB)) < \varepsilon$ forms a subset of Y_N^ε under Assumption 1, Proposition 3.6 ensures that the maximal distance of its elements from Y_{wN} tends to zero for ε descreasing to zero. On the other hand, while the set Y_{wN} is a subset of Y_N^ε, it is of course not a subset of $\mathcal{F}$, so that we do not obtain a result about the Hausdorff distance of $\mathcal{F}$ from Y_{wN}. In particular, it is possible that the elements of $\mathcal{F}$ only approximate a small part of Y_{wN}.

Despite the positive convergence result from Proposition 3.6, Figure 5 also indicates that the approximation of Y_{wN} by points in Y_N^ε may be arbitrarily slow in ε, even for convex sets $Y = f(M)$. On the other hand, next we show that there is a potentially fruitful relation between the sets Y_N^ε and $B = \mathrm{bd}(Y + \mathbb{R}^m_+)$, as for the approximation of B by points from Y_N^ε we can establish a linear rate of convergence.

Proposition 3.7. *For any nonempty compact set $Y \subseteq \mathbb{R}^m$ and for each $\varepsilon > 0$ we have*

$$\mathrm{ex}(Y_N^\varepsilon, B) \leq \sqrt{m}\,\varepsilon.$$

Proof. By Lemma 3.3a), for any $y_N^\varepsilon \in Y_N^\varepsilon$ the problem $D(y_N^\varepsilon, Y)$ possesses the unique optimal point $\widehat{t} := \varphi_Y(y_N^\varepsilon)$. Lemma 3.3c) and $y_N^\varepsilon \in Y$ yield $\widehat{t} \leq 0$. Furthermore, the definition of Y_N^ε rules out that any $t < -\varepsilon$ is a feasible point of $D(y_N^\varepsilon, Y)$ so that we obtain $\widehat{t} \in [-\varepsilon, 0]$. With Lemma 3.3d) for $\lambda = \widehat{t}$ we obtain $\widehat{y} := y_N^\varepsilon + \widehat{t}e \in B$. The identity $\|\widehat{y} - y_N^\varepsilon\| = \sqrt{m}\,|\widehat{t}| \leq \sqrt{m}\,\varepsilon$ then implies $\mathrm{dist}(y_N^\varepsilon, B) \leq \sqrt{m}\,\varepsilon$ and, thus, the assertion. □

Note that neither B nor $B \cap Z$ are subsets of Y_N^ε, so that Proposition 3.7 does not address the Hausdorff distance, and potentially not all elements of B are approximated. Still, the application of Proposition 3.7 to the output of any algorithm satisfying Assumptions 1–4 yields that for ε decreasing to zero the elements of the provisional nondominated set $\mathcal{F} \subseteq Y_N^\varepsilon$ converge to B at linear speed. If upon termination a point from $\mathcal{F}$ turns out to be close to some $\widehat{y} \in B \cap (f(M))^c$, one may try to design a post processing step to move it to some point in Y_N or in Y_{wN}.

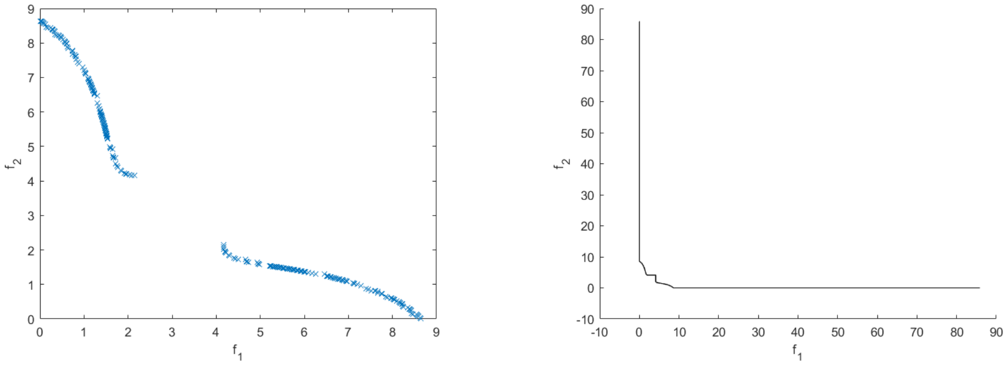
4. Postprocessing steps for the terminal enclosure

Algorithms meeting Assumptions 1–4 like Algorithm 1 from [4] provide the terminal provisional nondominated set $\mathcal{F}$ and the terminal enclosure $E(LB, UB)$ as two simultaneous approximations of the nondominated set Y_N. While Proposition 3.6 makes sure that the maximal distance of elements in $\mathcal{F}$ from Y_{wN} tends to zero, it is not clear how well the shape of Y_{wN} may be (approximately) recovered from the shape of $\mathcal{F}$. Since by Lemma 3.2 the terminal enclosure contains Y_{wN}, one may hope that the information from $E(LB, UB)$ regarding the shape of Y_{wN} complements the information from $\mathcal{F}$. On the other hand, the terminal enclosure even contains the superset $B \cap Z$ of Y_{wN} which may be significantly larger than Y_{wN}, so that information about the shape of Y_{wN} may be hard to retrieve from $E(LB, UB)$.

For example, for the bicriteria problem

$$DEB2DK: \quad \min \begin{pmatrix} r(x)\sin(x_1\pi/2) \\ r(x)\cos(x_1\pi/2) \end{pmatrix} \quad \text{s.t.} \quad 0 \leq x_1, x_2 \leq 1$$

with $r(x) = (5+10(x_1-0.5)^2+\cos(4\pi x_1))(1+9x_2)$ from [1] in its modified form from [4], Figure 6 shows the terminal provisional nondominated set and the terminal enclosure generated by [4, Alg. 1] for $\varepsilon = 0.1$. It illustrates that, as expected, the approximation



(a) Terminal provisional nondominated set $\mathcal{F}$ (b) Terminal enclosure $E(LB, UB)$

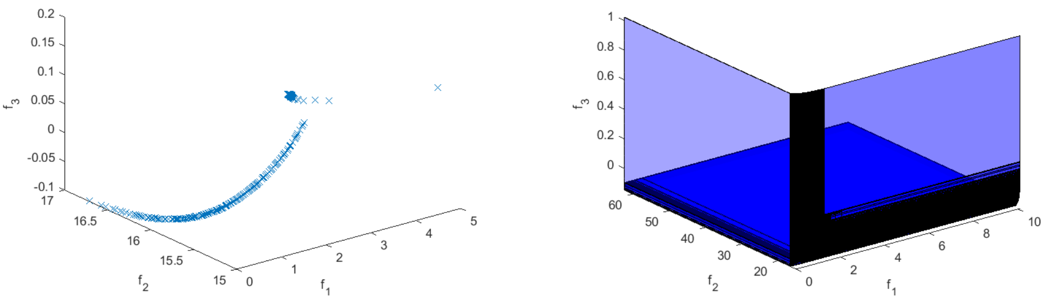
Figure 6.: Solution of DEB2DK with $\varepsilon = 0.1$

of $B \cap Z$ by the terminal enclosure can lead to a significantly larger set than the approximation of Y_{wN} by the terminal provisional nondominated set. In Figure 6b this appears to be mainly due to 'lengthy' boxes which extend along the coordinate axes from the nondominated set to the boundary of the computed box $Z = [0, 85]^2$, and some more lengthy boxes which 'fill the gap' between the two connected components of the nondominated set. As a consequence, in Figure 6b the details of the enclosure of the actual nondominated set are barely recognizable.

Figure 7 shows the corresponding output for the three criteria problem

$$VFM: \quad \min \begin{pmatrix} 0.5(x_1^2 + x_2^2)^2 + \sin(x_1^2 + x_2^2) \\ \frac{(3x_1 - 2x_2 + 4)^2}{8} + \frac{(x_1 - x_2 + 1)^2}{27} + 15 \\ \frac{1}{x_1^2 + x_2^2 + 1} - 1.1 \exp(-x_1^2 - x_2^2) \end{pmatrix} \quad \text{s.t.} \quad -3 \leq x_1, x_2 \leq 3$$

from [21].



(a) Terminal provisional nondominated set $\mathcal{F}$ (b) Terminal enclosure $E(LB, UB)$

Figure 7.: Solution of VFM with $\varepsilon = 0.05$

Also in Figure 7b the details of the enclosure of the actual nondominated set are barely visible, since lengthy boxes extend from the nondominated set to the boundary of the box $Z = [(-1, 15, -1.05)^\top, (10, 61.9, 1)^\top]$.

In both of the above examples, it seems to make sense to delete or at least to truncate lengthy boxes. In fact, in the examples boxes seem to be lengthy since their upper boundary vectors ub are located far away from Y_N, for example on the boundary of Z. Recall that ℓb is generated as an ideal point underestimator of $\widehat{Y}' + \mathbb{R}^m_+$ for a relaxation $\widehat{Y}' + \mathbb{R}^m_+$ of some partial upper image set $f(M(X')) + \mathbb{R}^m_+$ with a subbox $X' \subseteq X$. Therefore, if a lengthy box $[\ell b, ub]$ contains image points from Y at all, they may rather be expected to lie close to ℓb than to ub. Following this observation it makes sense to truncate $[\ell b, ub]$ by replacing ub with a vector $ub' \leq ub$, $ub' \neq ub$.

In Section 4.1, we suggest a truncation algorithm for the bicriteria case. Unfortunately its ideas cannot be extended straightforwardly to the case $m \geq 3$, which we explain in Section 4.2. We formulate a heuristic procedure instead. Section 4.3 complements these considerations by a rescaling technique for the objective values.

4.1. A box truncation algorithm for bicriteria problems

In this section we consider the bicriteria case $m = 2$. Given lower and upper bounding sets LB and UB with $w(LB, UB) < \varepsilon$ for some $\varepsilon > 0$, let $[\ell b, ub]$ with $\ell b \in LB$, $ub \in UB$ and $\ell b \leq ub$ be some box from the terminal enclosure $E(LB, UB)$. Then at least one of the two edge lengths of $[\ell b, ub]$ lies below ε. We consider $[\ell b, ub]$ 'lengthy' if one of the edge lengths exceeds $\bar{c}\varepsilon$ for some parameter $\bar{c} \gg 1$. We will comment on the choice of $\bar{c}$ below. Let $\ell, s \in \{1, 2\}$ denote the indices of the long and short edge, respectively, so that $ub_\ell - \ell b_\ell \geq \bar{c}\varepsilon$ and $ub_s - \ell b_s < \varepsilon$ hold. We suggest to truncate such lengthy boxes $[\ell b, ub]$ to $[\ell b, ub']$ with $ub'_s := ub_s$ and $ub'_\ell < ub_\ell$.

To avoid the deletion of points from $Y_N \cap [\ell b, ub]$ in this truncation step, one may try to compute the value $y_\ell^{\max}$ of the largest y_ℓ-component of points in $Y \cap [\ell b, ub]$ and put $ub'_\ell := y_\ell^{\max}$. This would, however, involve the global solution of a possibly nonconvex optimization problem for each lengthy box, which we wish to avoid in view of its large algorithmic effort.

Instead, in the following we suggest an algorithmically feasible procedure for a sufficiently deep truncation which, at the same time, guarantees that the deleted part $[\ell b, ub] \setminus [\ell b, ub']$ of a lengthy box at least does not contain any 'relevant' nondominated points in the following sense. Recall that a nondominated point $\bar{y}$ of $Y \subseteq \mathbb{R}^m$ is called Geoffrion properly nondominated for Y if there exists some constant $K > 0$ such that for all $j \in \{1, \ldots, m\}$ and all $y \in Y$ with $y_j < \bar{y}_j$ there exists some $i \in \{1, \ldots, m\}$ with $y_i > \bar{y}_i$ and the trade-off bound

$$\frac{\bar{y}_j - y_j}{y_i - \bar{y}_i} \leq K.$$

For $m = 2$ we will show that ub'_ℓ can be chosen such that $[\ell b, ub] \setminus [\ell b, ub']$ may only contain Geoffrion properly nondominated points $\bar{y}$ with a large trade-off bound K. In view of this large trade-off, these $\bar{y}$ may be irrelevant for practical applications.

Indeed, we put $ub'_\ell := \ell b_\ell + c'\varepsilon$ for some cut-off parameter $1 \leq c' < \bar{c}$ (Fig. 8). The lower bound for c' avoids a truncation which is so deep that the new edge length $ub'_\ell - \ell b_\ell$ falls below ε. The construction of c' involves the solution of an auxiliary optimization problem which aims at a point $y^\star \in Y \cap [\ell b, ub]$ with $y^\star_s = ub_s$ and

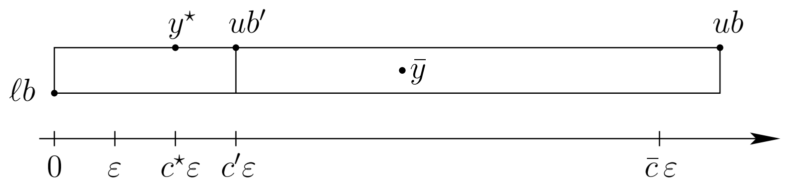


Figure 8.: Truncation for $m = 2$, $\ell = 1$, and $s = 2$

minimal y_ℓ-component:

$$\min_y \; y_\ell \quad \text{s.t.} \quad y \in Y \cap [\ell b, ub], \; y_s = ub_s, \; y_\ell \leq \ell b_\ell + (\bar{c}/2)\varepsilon. \tag{P_ℓ}$$

Note that also P_ℓ is in general nonconvex. However, here the global solution of P_ℓ is not necessary, but any of its feasible points $y^\star$ will subsequently lead to some statement about Geoffrion properly nondominated points. This statement, though, is strongest possible if $y^\star$ is a minimal point of P_ℓ. We choose the upper bound determined by $\bar{c}/2$ to guarantee a sufficient truncation in the subsequent construction from Proposition 4.1.

The problem P_ℓ is solvable if and only if its feasible set is nonempty. In case of an empty feasible set one may check if already $Y \cap [\ell b, ub]$ is empty and then delete the whole box $[\ell b, ub]$. If $Y \cap [\ell b, ub]$ is nonempty while P_ℓ does not possess feasible points, we do not truncate this box. An alternative option would be to increase the factor $\bar{c}/2$ in the last constraint of P_ℓ to an appropriate value close to $\bar{c}$ and to replace ub_s in the subsequent discussion by the maximal y_s-value of points in $Y \cap [\ell b, ub]$. The latter, however, would again require the global solution of a possibly nonconvex optimization problem which we wish to avoid in the present construction.

Proposition 4.1. *Let $y^\star$ be a minimal point of P_ℓ, define $c^\star := (y^\star_\ell - \ell b_\ell)/\varepsilon$, choose $c^K \in (\max\{0, 1-c^\star\}, \bar{c}-c^\star)$, and put $c' := c^\star + c^K$, $ub'_\ell := \ell b_\ell + c'\varepsilon$ as well as $ub'_s := ub_s$. Then $[\ell b, ub']$ is a proper subset of $[\ell b, ub]$ with $ub'_\ell - \ell b_\ell \geq \varepsilon$, and $[\ell b, ub] \setminus [\ell b, ub']$ does not contain any Geoffrion proper nondominated point of Y with trade-off bound $K \leq c^K$.*

Proof. By the definition of P_ℓ and of $c^\star$ we have $0 \leq c^\star \leq \bar{c}/2$ so that $\bar{c} - c^\star$ is positive. Due to $\bar{c} > 1$ we also have $\bar{c} - c^\star > 1 - c^\star$ which implies that the choice of c^K from the required interval is possible. The relation

$$ub'_\ell = \ell b_\ell + (c^\star + c^K)\varepsilon < \ell b_\ell + \bar{c}\varepsilon \leq ub_\ell$$

proves that $[\ell b, ub']$ is a proper subset of $[\ell b, ub]$. Furthermore, we have

$$ub'_\ell = \ell b_\ell + (c^\star + c^K)\varepsilon \geq \ell b_\ell + \max\{c^\star, 1\}\varepsilon$$

which implies $ub'_\ell - \ell b_\ell \geq \varepsilon$.

Let $\bar{y} \in [\ell b, ub] \setminus [\ell b, ub']$ be a Geoffrion proper nondominated point of Y with trade-off bound K. Next we show that $y^\star_\ell < \bar{y}_\ell$ and $y^\star_s > \bar{y}_s$ hold. In fact, $c^K > 0$ implies

$$y^\star_\ell = \ell b_\ell + c^\star\varepsilon < \ell b_\ell + (c^\star + c^K)\varepsilon = ub'_\ell < \bar{y}_\ell,$$

and equality in the relation $\bar{y}_s \leq ub_s = y^\star_s$ would lead to the contradiction that $y^\star$

dominates $\bar{y}$. With the choices $y := y^\star$, $j := \ell$ and $i := s$ in the definition of Geoffrion proper nondominance this yields

$$K \geq \frac{\bar{y}_\ell - y^\star_\ell}{y^\star_s - \bar{y}_s} > \frac{ub'_\ell - y^\star_\ell}{\varepsilon} = c^K$$

which shows the assertion. □

Proposition 4.1 allows two approaches for the choice of the parameter c^K. Firstly, one may be interested in truncating $[\ell b, ub]$ 'cautiously', such that for a given $K > 0$ the deleted part $[\ell b, ub] \setminus [\ell b, ub']$ does not contain any Geoffrion proper nondominated point with trade-off bound up to K. By Proposition 4.1 this is possible for any $K \in (\max\{0, 1 - c^\star\}, \bar{c} - c^\star)$ with the choice $c^K := K$. In particular, if one expects a small value of $c^\star$ (i.e., $y^\star_\ell$ is expected to lie close to ℓb_ℓ), $\bar{c}$ should be chosen slightly larger than K. In any case, this construction results in a truncation with $ub'_\ell = \ell b_\ell + (c^\star + K)\varepsilon$, so that for K close to $\bar{c} - c^\star$ the truncation is not deep but, on the contrary, the remaining part of the box is 'close to lengthy'.

Figure 9a shows the result of a cautious truncation for the problem DEB2DK with $\varepsilon = 0.1$ and $\bar{c} = 10$. From the 1,674 boxes in the terminal enclosure 22 are identified as lengthy. All corresponding problems P_ℓ are solvable, and for each box and its value $c^\star$ we chose a value c^K slightly below $10 - c^\star$. This resulted in a trade-off bound of at least $K = 9.66$ in each box, and a cut-off value of at least $c' = 9.91$. As expected the latter is close to $\bar{c} = 10$ so that the truncated boxes are 'close to lengthy'.

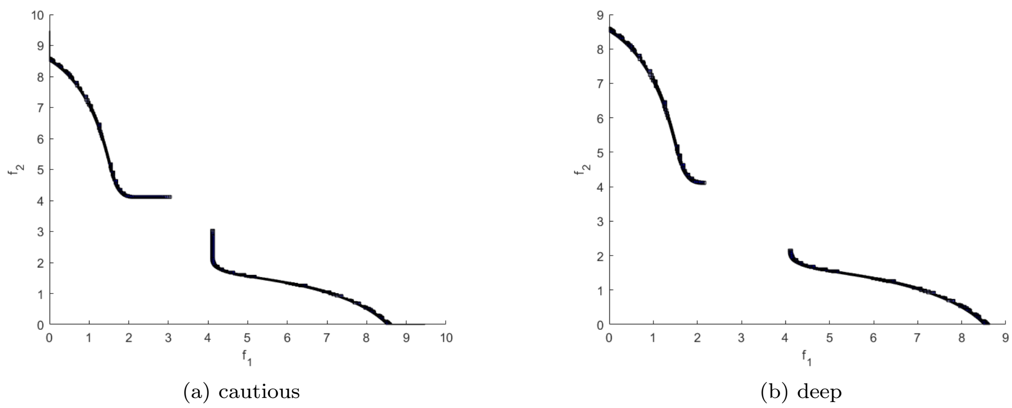


Figure 9.: Truncated enclosures of DEB2DK with $\varepsilon = 0.1$ and $\bar{c} = 10$

Secondly, one may be interested in a 'deep' truncation of the box $[\ell b, ub]$ with c' close to 1, say $c' = c^\star + c^K \leq 1 + \delta$ with some small $\delta \in (0, \bar{c} - 1)$. The choice $c^K > \max\{0, 1 - c^\star\}$ then requires $\delta > c^\star - 1$, and only trade-off bounds $K \leq 1 + \delta - c^\star$ can be ruled out for Geoffrion proper nondominated points in the deleted part of the box. Even in the most favorable case $y^\star_\ell = \ell b_\ell$ only values $K \leq 1 + \delta$ are possible.

For the above setting of DEB2DK with $\varepsilon = 0.1$ and $\bar{c} = 10$ we chose c^K close to $\max\{0, 1 - c^\star\}$ for each of the 22 lengthy boxes, resulting in cut-off values of at most $c' = 1.09$ and trade-off bounds as low as $K = 0.84$. Figure 9b illustrates the corresponding truncated enclosure.

While the approximations of Y_N by the provisional nondominated set from Figure 6a and by the deeply truncated enclosure from Figure 9b show a close resemblance, we emphasize that our previous discussion does not rule out that some elements of Y_N may lie, for example, in the additional enclosure parts from Figure 9a.

4.2. A box truncation heuristic for more than two objectives

As in the previous section let $w(LB, UB) < \varepsilon$ for some $\varepsilon > 0$ and let $[\ell b, ub]$ with $\ell b \in LB$, $ub \in UB$ and $\ell b \leq ub$ be some box from the terminal enclosure $E(LB, UB)$. While the shortest edge length of $[\ell b, ub]$ still lies below ε, Figure 7b illustrates that in the presence of more than two objective functions lengthy boxes may not only extend in a single dimension. Therefore we define the index set of all such directions, $J_\ell(\ell b, ub) := \{j \mid ub_j - \ell b_j \geq \bar{c}\varepsilon\}$ with some $\bar{c} \gg 1$.

In generalization of our approach for the case $m = 2$ we suggest to truncate all boxes $[\ell b, ub]$ with $J_\ell(\ell b, ub) \neq \emptyset$ along all directions with index in $j \in J_\ell(\ell b, ub)$ by replacing ub_j with $ub'_j := \ell b_j + c'\varepsilon$ for some $1 \leq c' < \bar{c}$. For $j \notin J_\ell(\ell b, ub)$ we put $ub'_j := ub_j$ and obtain the truncated box $[\ell b, ub']$.

Unfortunately our assessment of the 'quality' of nondominated points $\bar{y}$ which possibly lie in the deleted part $[\ell b, ub] \setminus [\ell b, ub']$ of a lengthy box does not carry over to this construction for the case $m > 2$. The main reason is that for points $\bar{y} \in [\ell b, ub] \setminus [\ell b, ub']$ not all, but only some indices $j \in J_\ell(\ell b, ub)$ may satisfy $\bar{y}_j > ub'_j$. Thus a straightforward generalization of our discussion from the case $m = 2$ would require the concept of a substantially nondominated point from [13] (which is identical to the concept of a strongly proper nondominated point introduced later in [11]): a nondominated point $\bar{y}$ of Y is called substantially nondominated for Y if there exists some constant $K > 0$ such that for all $i, j \in \{1, \ldots, m\}$ and all $y \in Y$ with $y_j < \bar{y}_j$ and $y_i > \bar{y}_i$ the trade-off satisfies

$$\frac{\bar{y}_j - y_j}{y_i - \bar{y}_i} \leq K.$$

Substantial nondominance strengthens the notion of Geoffrion proper nondominance, where not for all pairs of i and j the trade-off needs to be bounded, but only for each j there must exist some i with this property. For bicriteria problems both notions obviously coincide. In [13] it is shown, however, that for $m > 2$ substantially nondominated points rarely exist, so that this concept does not seem to be suitable for practical applications.

In addition, even if we used this concept and if we were able to determine a point $y^\star \in Y \cap [\ell b, ub]$ with $y^\star_s = ub_s$ for the index s of some edge with $ub_s - \ell b_s < \varepsilon$ (e.g., a shortest edge), a point $\bar{y} \in [\ell b, ub] \setminus [\ell b, ub']$ with $\bar{y}_s = ub_s$ would not necessarily be dominated by $y^\star$, so that not even the relation $\bar{y}_s < y^\star_s$ can be deduced along the lines of the case $m = 2$.

Therefore, in the case $m > 2$ we suggest to choose constants $1 \leq c' \leq \bar{c}$ with $c' \approx 1$ and $\bar{c} \gg 1$ heuristically. For the case $m = 3$ their choice may take the effect of their sizes on the graphical output into account. Figure 10 shows the postprocessed version of the box enclosure from Figure 7b with the parameter choices $c' = 2$ and $\bar{c} = 10$. This results in the identification and truncation of 6,569 out of the 9,813 boxes forming the terminal enclosure.

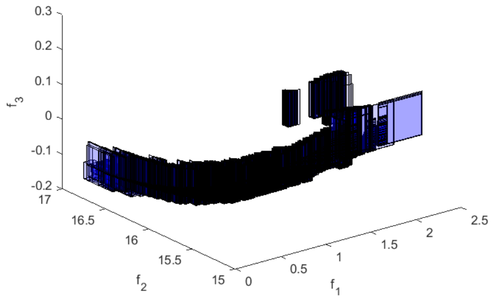


Figure 10.: Truncated enclosure of VFM with $c' = 2$ and $\bar{c} = 10$

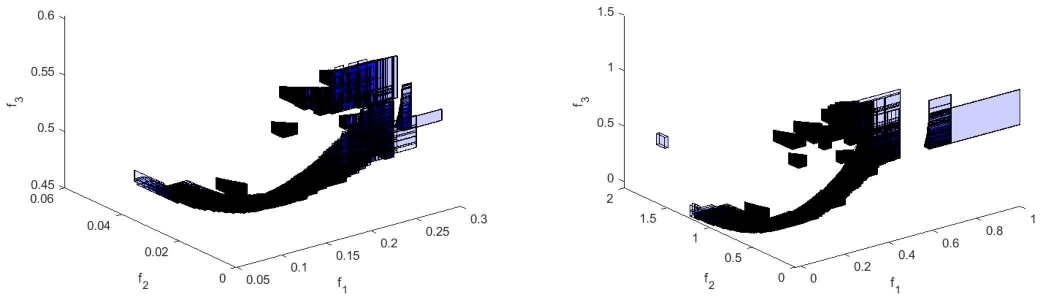
4.3. Rescaling the objective functions

In Figure 10 another undesirable effect appears, as the lengthy boxes in direction of the objective function f_3 are not satisfactorily truncated. The reason for this effect is the significantly different scaling of the f_3 axis compared to the f_1 and f_2 axes, resulting in truncated boxes which appear to be stretched in f_3-direction.

As usual in parts of the literature on multi-objective optimization, the algorithmic approach from [4] uses the all ones vector e as an interior direction of the ordering cone $\mathbb{R}^m_+$ for discarding tests, for the definition of ε-nondominated points as well as for the geometric definition (5) of the enclosure's width $w(LB, UB)$. This choice of the direction vector, as opposed to a vector with different positive entries, assumes that the objective functions generate values on similar scales. While the algorithmic approach also works if this implicit assumption is violated, difficulties may arise in the geometric interpretation of its output, as in Figure 10.

In such cases we suggest to rescale the objective functions. Appropriate scaling factors may either be obtained a-priori or a-posteriori. The a-priori option uses information from the box $Z = [\underline{z}, \overline{z}]$ with $f(X) \subseteq \text{int}(Z)$ which is computed at the beginning of the underlying branch-and-bound method. If the ratio $\max_j(\overline{z}_j - \underline{z}_j)/\min_j(\overline{z}_j - \underline{z}_j)$ is sufficiently large, it replaces each objective function f_j by $(f_j - \underline{z}_j)/(\overline{z}_j - \underline{z}_j)$, so that the rescaled objective function vector maps to the unit cube. In problem VFM with $\underline{z} = (-1.1, 14.5, -1)^\top$ and $\overline{z} = (10.1, 62.4, 1)^\top$ the above ratio is about 24 and indicates that a-priori rescaling may be useful. Figure 11a shows the result for $\varepsilon = 0.005$. All but 31 of the 21,783 boxes in the terminal enclosure are truncated.

If the approximation of $f(X)$ by Z is too coarse or if the nondominated set still appears to exhibit different scales along different image space coordinate directions, an a-posteriori and, thus, more elaborate option is to run the branch-and-bound method for a first time with the original function vector f, determine the smallest enclosing box $P = [\underline{p}, \overline{p}]$ of the generated provisional nondominated set and, if the ratio $\max_j(\overline{p}_j - \underline{p}_j)/\min_j(\overline{p}_j - \underline{p}_j)$ is sufficiently large, run the algorithm a second time for the scaled functions $(f_j - \underline{p}_j)/(\overline{p}_j - \underline{p}_j)$. The points $\underline{p}$ and $\overline{p}$ may be interpreted as a provisional ideal point and a provisional nadir point of MOP. The provisional nondominated set of the rescaled problem then lies approximately in the unit cube. In problem VFM the box P corresponding to Figure 7a possesses the vertices $\underline{p} = (0.016, 15, -0.1)^\top$ and $\overline{p} = (4.51, 16.7, 0.18)^\top$, resulting in a ratio between longest and shortest edge length of about 16. Thus also a-posteriori scaling may make sense. Figure 11b shows the result for $\varepsilon = 0.05$, where 28,105 out of 28,508 boxes have been truncated. While qualitatively both approximations in Figure 11 look similar, as expected the a-posteriori rescaling leads to less distortion in the f_3-direction.



(a) a-priori (b) a-posteriori

Figure 11.: Truncated enclosures for rescalings of VFM with $c' = 2$ and $\bar{c} = 10$

5. Extension to the mixed-integer setting

While Assumptions 1–4 and the resulting convergence analysis in Section 3 are motivated by [4, Alg. 1] for the continuous problem MOP, the continuity of the variables is not necessary for the validity of the main results of the present paper.

In fact, consider a multiobjective mixed-integer optimization problem of the form

$$\min f(x) \quad \text{s.t.} \quad g(x) \leq 0,\ x \in X,\ x_i \in \mathbb{Z},\ i \in I \qquad (MOMIP)$$

where, in addition to the assumptions on the problem MOP, we denote by $I \subseteq \{1, \ldots, n\}$ an index set of integer variables. The feasible set of *MOMIP* is

$$M = M(X) = \{x \in X \mid x_i \in \mathbb{Z},\ i \in I,\ g(x) \leq 0\},$$

and in the case $I = \emptyset$ the problem *MOMIP* collapses to the problem MOP.

The two main reasons for the extendability of the convergence results from Section 3 to the mixed-integer case are that we did not impose any convexity assumptions on the problem MOP, and that all required notions as well as the results are formulated exclusively in the image space.

To be more precise, the image set $Y = f(M(X))$ is also compact for *MOMIP*, its nondominated set Y_N can be approximated by box enclosures $E(LB, UB)$ with finite sets LB, UB, and by provisional nondominated sets $\mathcal{F}$. No changes are required in the definitions of the search region $S(\mathcal{F})$ and of the set of local upper bounds $\text{lub}(\mathcal{F})$. Hence, Assumptions 1–4 can be formulated for *MOMIP*. Note that (4) still follows from Assumptions 1 and 3 since external stability is guaranteed by the compactness of Y.

As a consequence, the class of branch-and-bound methods for *MOMIP* satisfying Assumptions 1–4 possesses the approximation properties from Section 3. For the purpose of easier reference we explicitly state the main results.

Theorem 5.1. *Let the feasible set $M(X)$ of MOMIP be nonempty and let Assumptions 1–3 hold with sets Y, $\mathcal{F}$, $Z = [\underline{z}, \overline{z}]$, LB and UB.Furthermore, let Assumption 4 hold, and for some anti-ideal point overestimator $\widetilde{\omega}$ for $f(X)$ with $\widetilde{\omega} < \overline{z}$, let LB and UB be chosen such that $w(LB, UB) \le \delta/3$ holds with δ from* (11)*. Then the excess of $E(LB, UB)$ over $B \cap Z$ satisfies*

$$\mathrm{ex}(E(LB, UB), B \cap Z) \le 3\sqrt{m}\, w(LB, UB).$$

Corollary 5.2. *Let the feasible set $M(X)$ of MOMIP be nonempty, let $Z' = [\underline{z}', \overline{z}']$ be a box with $f(X) \subseteq \mathrm{int}(Z')$, for some $\bar{\varepsilon} > 0$ define $Z := [\underline{z}', \overline{z}' + 3\bar{\varepsilon}e]$ and let Assumptions 1–3 hold with sets Y, $\mathcal{F}$, Z, LB and UB. Furthermore, let Assumption 4 be satisfied, and for any $\varepsilon \in (0, \bar{\varepsilon}]$ let LB and UB be chosen such that $w(LB, UB) \le \varepsilon$ holds. Then the excess of $E(LB, UB)$ over $B \cap Z$ satisfies*

$$\mathrm{ex}(E(LB, UB), B \cap Z) \le 3\sqrt{m}\, \varepsilon.$$

Furthermore, Proposition 3.6 ensures that the maximal distance of the terminal provisional nondominated set $\mathcal{F}$ from Y_{wN} tends to zero for ε descreasing to zero. While this convergence may be arbitrarily slow in ε, Proposition 3.7 implies that the maximal distance of the terminal provisional nondominated set $\mathcal{F}$ from B tends to zero for ε descreasing to zero at linear speed.

6. Final remarks

In this paper we have studied approximation properties of the terminal enclosure and of the terminal provisional nondominated set for branch-and-bound algorithms satisfying Assumptions 1–4 and termination accuracies tending to zero. Our analysis applies to continuous as well as to mixed-integer multi-objective optimization problems. For the boundary B of the upper image set of $Y = f(M)$ and a box container Z of Y, we have seen that the terminal enclosure converges in the Hausdorff metric to $B \cap Z$, and that the maximal distance of elements in the terminal provisional nondominated set to the weakly nondominated set of Y tends to zero. While the latter convergence may be arbitrarily slow, we established linear convergence of these distances to the set $B \cap Z$. This illustrates the central role of the upper image set's boundary for our convergence proofs, which may also turn out to be useful in other approximation considerations in multiobjective optimization.

All concepts and results from this paper make use of the componentwise structure of the natural ordering in the image space $\mathbb{R}^m$. It would be of interest to extend the concept of an enclosure and the ideas behind the examined class of algorithms to more general partial orderings defined by, for instance, polyhedral ordering cones. For doing so an extension of the definition of local upper bound sets is required, and, what is more, algorithms for calculating these sets.

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