Integer Programming Methods for Solving Binary Interdiction Games

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Abstract

This paper studies a general class of interdiction problems in which the solution space of both the leader and follower are characterized by two discrete sets denoted the leader's *strategy set* and the follower's *structure set*. In this setting, the interaction between any strategy-structure pair is assumed to be binary, in the sense that the strategy selected by the leader either interacts or not with the follower's choice of structure and, if it does, then the structure becomes unavailable for the follower. There are many interdiction games defined by this type of setup, including problems where the leader wishes to attack some type of network structures, such as shortest paths, minimum spanning trees, and minimum dominating sets, among others. We study a general set-covering type of formulation that can be used for solving this type of problems and analyze several properties of the convex hull of its solutions. We develop a wide class of valid inequalities that generalizes several others that have appeared in the literature in recent years. We provide conditions for them to be facet-defining and conclude with a general discussion about their separation. Several examples of problems in the context of network interdiction are presented to help with the exposition.
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1 Introduction and Motivation

Interdiction problems are adversarial Stackelberg games played between two actors—a *leader* and a *follower*—who sequentially solve interdependent optimization problems with conflicting objectives. The leader, who plays first, conducts a set of interdiction actions aimed to optimally deteriorate the follower's objective. In turn, the follower reacts by optimizing their own problem parameterized by the actions taken by the leader.

In this paper, we are interested in interdiction problems in which the solution spaces for both the leader and follower are characterized by two discrete sets that we denote the leader's *strategy set* Π and the follower's *structure set* Ω. We refer to the follower's actions as structures because in most

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interdiction problems found in the literature, the follower's feasible solutions are often composed of sets of elements satisfying some structural properties. In this type of setting, we associate the leader's strategies with a *cost function* $c : \Pi \to \mathbb{R}^+$, representing the cost that the leader incurs by selecting a given strategy from Π. Similarly, we define a *weight function* $w : \Omega \to \mathbb{R}^+$ associated with the follower's structures and assume without loss of generality that the follower's optimization problem consists of finding a structure $T \in \Omega$ of minimum weight.

The pairwise interactions between the leader's strategies and follower's structures define the specific rules of the interdiction game. Depending on the context, the leader's strategies are often set to target the follower's objective function w, thereby increasing the follower's perceived weight of the structures; or, alternatively, such strategies are aimed at impairing some of the elements that compose the structures in Ω, thus making them unusable for the follower. We will focus on problems defined by the latter type of interaction.

We study a wide class of interdiction problems for which the interaction between any action pair $(U, T) \in \Pi \times \Omega$ can be described by an *interdiction function* $\mathcal{I}$ that indicates the way in which the strategies in Π affect the structures in Ω. In general, a strategy $U \in \Pi$ is said to interact with structure $T \in \Omega$ if U impairs (partially or fully) some elements comprising T. As part of this type of problems, we concentrate on the special case where a strategy $U \in \Pi$ interacting with a structure $T \in \Omega$ results directly in T becoming unavailable for the follower (i.e., strategy U *blocks* structure T). We denote this type of interdiction games as binary and formally define them as follows.

Definition 1 (Binary Interdiction Game (BIG))**.** Given a strategy set Π for the leader and a structure set Ω for the follower, an interdiction game is binary if the interaction between the two players can be described by the interdiction function $\mathcal{I} : \Pi \times \Omega \to \{0, 1\}$ such that, for every strategy-structure pair $(U, T) \in \Pi \times \Omega$,

$$\mathcal{I}(U,T) = \begin{cases} 1, & \text{if strategy } U \text{ blocks the follower from selecting structure } T \\ 0, & \text{otherwise.} \end{cases}$$

Any problem that fits into this definition is called an instance of BIG.

There are multiple interdiction problems defined by this type of natural binary interaction that have appeared in the literature in recent years. Of notable importance are several instances of *Network Interdiction*, where the follower structure set Ω consists of some type of network structures composed of vertices and edges, like shortest paths [24], spanning trees [43], cliques [19, 29, 30], connected components [2, 17], matchings [45], dominating sets [33], and vertex covers [8], among others. For these problems, a leader's strategy $U \in \Pi$ comprises a subset of vertices or edges that are blocked/removed from the network by the leader in order to prevent the follower from selecting any structure $T \in \Omega$ that contains elements in U.

Other types of problems outside network interdiction can be modeled by these binary settings as well. Notice that the realizations of the strategies U's and structures T's are not specified by Definition 1, and, depending on the contextual problem, they could represent multiple things. For example, they could be subsets of some ground set (e.g., the set of arcs as in network interdiction), sequences of actions (e.g., the tasks of a scheduling interdiction problem [11]), or structures with a stochastic nature (e.g., the (s, t)-connected subgraphs with random capacities in stochastic network interdiction [15]). In what follows, we treat Π and Ω as abstract sets in order to discuss some general results, but, along the way, we provide multiple examples of problems with specific realizations of Π and Ω; particularly, in Sections 3 and 4.

Most interdiction games including the ones that we classify as binary admit two different modeling perspectives depending on the way in which the leader's objective is defined. Perhaps the most commonly studied perspective corresponds to the case where the leader's strategies in Π are restricted by a budget b—with respect to cost function c—and the leader's objective is set to maximize the minimum weight of the residual structures that can then be selected by the follower. We will refer to this variation as the *budget* (BDG) version of the problem. Alternatively, a perspective that has received considerable less attention corresponds to the case in which the leader minimizes the cost of selecting an interdiction strategy that forces the minimum weight among the structures that can be selected by the follower to reach at least a predefined target threshold r. We will refer to this variation as the *target* (TGT) version of the problem.

Given the binary interdiction function $\mathcal{I}$ provided before, the following definitions can be used to provide a general representation of these two BIG variations.

Definition 2. Let $\mathcal{T} \subseteq \Omega$ be any subset of the follower's structures and $U \in \Pi$ be a leader's strategy. We then define

1. $\mathcal{I}(U, \mathcal{T}) = 1$ if the strategy U blocks all the structures in $\mathcal{T}$,
2. $\Omega_U = \{T \in \Omega \mid \mathcal{I}(U, T) = 1\}$ to be the set of follower's structures that are blocked by U.

Then, both the BDG and TGT versions of BIG can be modeled as follows.

$$(\text{BDG}) \quad \max_{U\in\Pi} \quad \min_{T\in\Omega\setminus\Omega_U} w(T) \qquad (\text{TGT}) \quad \min_{U\in\Pi} \quad c(U)$$
$$\text{s.t.} \quad c(U) \le b; \qquad\qquad \text{s.t.} \quad \min_{T\in\Omega\setminus\Omega_U} w(T) \ge r.$$

It is not difficult to see that these optimization problems are two versions of the same decision problem and as such are polynomial-time equivalent in their general complexity. In fact, under some mild assumptions, a solution approach for one of them can be interweaved with a binary search procedure to solve instances of the other [7, 43]. However, noting that for the most part this type of interdiction problems are NP-hard, such type of approach is unpractical as progress may stall at each step of the binary search. Therefore, most developments found in the literature have been specifically tailored to solve either one or the other, depending on the context of the problem at hand.

In this paper we study modeling strategies to solve the TGT version of BIG, and show that under minor considerations, the proposed methodology can also be used seamlessly to model and solve problems stated from the BDG perspective. The main contributions of this paper are listed below:

1. We study a general type binary interdiction game that characterizes a wide range of interdiction games as instance problems. We provide high-level solution approaches that can be applied to the problems that fall into this category.

2. We analyze a set-covering type formulation that can be used for solving binary interdiction games whose elements are defined over one or two ground sets. The technical tools developed along with this formulation can be used for identifying what we call critical structures in various interdiction problems.

3. We develop a class of valid inequalities and use them to strengthen the set-covering formulation. This class of inequalities unifies and extends others that have been developed for specific interdiction problems. We also provide conditions for them to be facet-defining and conclude with a general discussion about their separation.

The rest of the paper is organized as follows. In section 2, we provide a general literature review and discuss several high-level solution approaches commonly used in the literature. In section 3, we define the ground sets based binary interdiction game and study an integer programming formulation that can be used to solve it. We also provide a transformation theorem that can convert many two ground sets interdiction games into an equivalent one ground set counterpart. In section 4, we develop a set of valid inequalities to strengthen the integer programming formulation. We study necessary and sufficient conditions for those inequalities to be facet-defining and discuss some separation approaches. Finally, in the last section, we conclude the paper and provide some guidelines for future work.

2 Background and Previous Work

Given the adversarial nature of these games, it is not surprising that the practical motivation behind most early developments stemmed from military operations [10, 11, 12, 20, 21, 32, 39]; however, the number of novel applications in other contexts has consistently grown over the years. Nowadays is common to find interdiction problems being solved in a wide variety of areas such as transportation [6, 23], communications [44], cyber security [18], power networks [1], immunization strategies [40], human trafficking prevention [26], conservation planing [36], and evacuation planning [31], among others.

One of the main reasons for the recent popularity of these models is that they can be used to identify critical vulnerabilities in operational and infrastructure networks that may need to be protected amidst extreme events. Even in cases where there is not a clear adversarial conflict, someone interested in identifying the worst possible damage that some infrastructure may endure during a critical event can design an interdiction model by taking the stance of an intelligent opponent whose aim is to cause the maximum disruption. From this perspective—as previously noted in [38]—interdiction games are in a way analogous to some traditional robust optimization problems, but with the leader and follower roles reversed.

In addition to the practical motivation, many studies have been devoted to develop general solution approaches. We concentrate our discussion on exact methods, particularly those based on mathematical programming. We provide a general overview of three solution approaches commonly found in the literature. For a more comprehensive discussion, we refer the interested reader to a recent survey [38]. Furthermore, since most interdiction problems have a natural bilevel structure, several techniques developed for bilevel optimization can be applied to solve interdiction problems as well. We recommend the following studies [5, 16, 25] for a general overview on recent solution methods for bilevel optimization.

2.1 Dualize-and-Combine Methods

One of the most common solution approaches for solving interdiction problems is the so-called "dualize-and-combine" method [38]. This type of strategy has become ubiquitous whenever the inner problem has a dual representation that admits strong duality [15, 24]. The main advantage of this approach is that the bilevel structure of the problem is transformed into a single level reformulation by fixing the leader's strategy U and then dualizing the follower's problem.

For the specific case of BIG, if the follower's problem can be formulated as a convex program and the set of available structures $\Omega \setminus \Omega_U$ could be represented by a family of indicator vectors from a feasible set $Y^p_{\Omega\setminus\Omega_U}$, then the following reformulations are valid, as long as any constraint qualification holds. Here, we use $Y^d_{\Omega\setminus\Omega_U}$ and w^d to denote the dual feasible space and the dual objective function.

$$\max_{U\in\Pi} \quad \max_{\mathbf{y}\in Y^d_{\Omega\setminus\Omega_U}} w^d(\mathbf{y}) \qquad\qquad \min_{U\in\Pi} \quad c(U)$$
$$\text{s.t.} \quad c(U) \le b; \qquad\qquad \text{s.t.} \quad \max_{\mathbf{y}\in Y^d_{\Omega\setminus\Omega_U}} w^d(\mathbf{y}) \ge r.$$

Furthermore, because of strong duality, the maximization of the inner problem can then be omitted, as any feasible dual solution $\mathbf{y} \in Y^d_{\Omega\setminus\Omega_U}$ yields a lower bound on the follower's primal problem.

In many instances of BIG, the follower's problem admits a linear programming formulation, thus the dualized inner problem has a simple structure. This type of solution method can be often found in the literature for interdicting network structures such as shortest paths [24], maximum flows [15], minimum cost multicommodity flows [27], and minimum spanning trees [9, 43].

2.1.1 Sampling and Enumerative Methods

In many instances of BIG, it is possible to identify the subclass of follower's structures that the leader must block by analyzing the solution space of both players. Usually, the cardinality of this subclass of structures is quite large, so this type of methods use formulations that begin by targeting only a small portion of those structures, and then progressively identify others to be blocked by sampling the follower's reaction to the leader's strategies being found. Several studies have shown that this type of formulations often produce better results in practice when compared to the dualize-and-combine reformulations [19, 24, 28, 43]. We use the interaction function $\mathcal{I}$ to illustrate the general idea behind this methods for both the TGT and BDG versions of BIG. The general formulations we study in Section 3 can be placed in this category.

By the definition of the TGT version, after the leader's strategy $U \in \Pi$ is chosen, the weight of any structure left available for the follower must be greater than or equal to the target level r. Thus, any structure $T \in \Omega$ with a weight $w(T) < r$ must be blocked by the leader's strategy U. This observation leads to the following definition of critical structures.

Definition 3 (Critical Structures)**.** Given the follower's structure set Ω along with a interdiction target r, we define $\hat{\Omega} := \{T \in \Omega \mid w(T) < r\}$ to be the set of *critical* structures.

Importantly, we note that the term *critical* has been used for multiple other purposes in the literature; particularly, in reference to some connectivity properties of graphs [2, 42]. Given the contextual differences, we believe there should be little room for confusion; however, we will provide any further clarification when needed (see Section 3.2).

Proposition 1. *A strategy $U \in \Pi$ is feasible if and only if $\mathcal{I}(U, \hat{\Omega}) = 1$.*

Based on this proposition, we can formulate any instance of TGT as follows,

$$\min_{U\in\Pi} \quad c(U) \tag{1a}$$
$$\text{s.t.} \quad \mathcal{I}(U,T) = 1 \quad \forall T \in \hat{\Omega}. \tag{1b}$$

That is, any feasible strategy for the leader has to block all the critical structures of the follower. For the BDG version, it is common to use variations of the following formulation,

$$
\begin{aligned}
\max_{U \in \Pi : c(U) \le b} \quad & z && \text{(2a)}\\
\text{s.t.} \quad & z \le w(T) + M_T \cdot \mathcal{I}(U,T) \quad \forall T \in \Omega, && \text{(2b)}
\end{aligned}
$$

where M_T is a suitable big-M type of constant that ensures the corresponding constraint is satisfied when a structure T is blocked by a strategy U.

In both reformulations, the cardinality of the constraint set $\hat{\Omega}$ and Ω could be quite large in practice. Thus, a branch-and-cut implementation built upon the interaction between two problems—a master and a subproblem—is generally used to solve these formulations [19, 24, 28, 43]. At each iteration, the master problem optimizes the leader's objective over a subset of the constraint set, finding a candidate solution $U \in \Pi$. Then, the algorithm parameterizes the subproblem with solution U, which is used to potentially reveal additional structures that can be used to complement the leader's constraint set in subsequent iterations. This iterative procedure terminates when the objective of the subproblem is either greater than r for TGT, or equal to the underestimated objective value provided by the master problem for BDG.

2.1.2 Supervalid Constraints

Supervalid constraints are often used within the branch-and-cut framework described before for cutting off feasible solutions that are dominated by a given incumbent solution. This type of cuts share some similarities with other types of inequalities developed in other contexts [22], like during *probing* [3]. Their usage to solve interdiction problems can be traced back to the seminal paper by Israeli and Wood [24], where those are specifically tailored to solve shortest path interdiction problems.

Definition 4 (Supervalid Constraint [24])**.** In the process of solving a mathematical programming formulation, a constraint $f(x) \le 0$ (or $f(x) = 0$) is said to be supervalid if adding it into the formulation will not remove all the optimal solutions or an incumbent solution of the problem is already optimal.

By this definition, any valid constraint of the problem is also supervalid; however, some supervalid constraint are expected to cut off feasible (even optimal) solutions, as long as the formulation retains at least one optimal solution. This technique has been used for solving instances of both TGT [43] and BDG [24]. Interestingly, the use of supervalid inequalities for solving instances of BIG reveals an interesting connection between both the BDG and TGT versions. We start by introducing the following family of constraints that is naturally supervalid for formulation (2). These constraints are a direct generalization of the ones proposed in [24].

Proposition 2. *When solving Formulation* (2)*, given an incumbent solution whose objective value is $\underline{z}$, any constraint that describes the set of strategies*

$$\mathcal{R} = \{U \in \Pi \mid \mathcal{I}(U, \hat{\Omega}) = 1\}, \ \textit{where } \hat{\Omega} = \{T \in \Omega \mid w(T) \le \underline{z}\}$$

is supervalid.

Proof. Let z^* be the optimal objective value of the optimization problem. If the incumbent $\underline{z}$ is already optimal, then we are done. Assume $\underline{z} < z^*$ where z^* is the optimal objective value, then an optimal solution U^* has to block any structure T with a weight that is less or equal to $\underline{z}$. That is, $\mathcal{I}(U^*, T) = 1$ for all T having a weight $w(T) \le \underline{z}$, which is the definition of $\mathcal{R}$. A valid constraint that describes the region $\mathcal{R}$ will not remove any elements in $\mathcal{R}$, thus is supervalid since it retains the optimal solutions unless the incumbent is already optimal. □

In [24], instead of considering all the valid constraints that describe the space $\mathcal{R}$, the authors proved that a specific class of inequalities is supervalid. The computational experiments in [24] also provide evidence that adding these supervalid constraints improves the runtime performance of the proposed formulation. Adding supervalid constraints is often beneficial because in the process of removing integer feasible solutions, other fractional solutions might be cut from the master problem's feasible region as well, thus resulting in the algorithm finding tighter upper bounds.

Notice that the set $\mathcal{R}$ is just a special case of the feasible region of TGT. Indeed, if we set the target value r equal to $\underline{z} + \epsilon$ for some small ϵ, then $\mathcal{R}$ is simply the set of strategies that can block all the critical structures whose weights are less or equal to r. This offers another way to solve the BDG version as follows,

$$\begin{aligned} \min_{U \in \Pi : c(U) \le b} \quad & c(U) \\ \text{s.t.} \quad & \mathcal{I}(U, T) = 1 \quad \forall T \in \hat{\Omega}_{z^*}, \end{aligned}$$

where z^* represents the optimal objective value of the BDG version problem. This formulation is very similar to the TGT formulation (1). Both can be solved via branch-and-cut, where the master problem produces candidate strategies for the leader and the subproblem identifies any critical structures that remains available for the follower and generates the corresponding constraints. However, there are several differences between them. First, the space of feasible leader strategies is limited in the above formulation by the budget constraint $c(U) \le b$. Second, the critical structure set $\hat{\Omega}_{z^*}$ depends on the optimal objective value z^*—which is not known by the leader in advance. However, the elements in this set can be identified sequentially using any incumbent solution $\underline{z}$. In fact, we have $\hat{\Omega}_{\underline{z}} \subseteq \hat{\Omega}_{\underline{z}'}$ for $\underline{z}' \ge \underline{z}$, where $\underline{z}'$ is an incumbent solution identified after $\underline{z}$. Thus, in the process of solving the above formulation, the set $\hat{\Omega}_{\underline{z}}$ will converge to $\hat{\Omega}_{z^*}$ eventually, as the algorithm keeps finding better incumbent solutions. Finally, the TGT formulation (1) halts when an optimal solution is obtained, while the above formulation for the BDG version only stops when the master problem becomes infeasible. Then, the algorithm chooses the incumbent solution as the optimal solution.

Despite these minor implementation differences, the above formulation derived from Proposition 2 allows us to consider BDG as a special case of TGT. For this reason, we will henceforth focus on the TGT version of BIG. We now proceed to describe the proposed formulations.

3 Mathematical Formulations

In this section, we study a general framework for solving binary interdiction games. We set up the stage by introducing two well-known examples of problems that fit the proposed binary classification, which—along with others that will be introduced later in this section—will be used throughout the rest of the paper for illustrative purposes. Since our focus is on the TGT version of the problem, we present them from such perspective. Then, we will continue by defining the interdiction game's *ground sets*, which can be seen as the building blocks for the leader's strategies and follower's structures, and will allow us to study a general context-free formulation.

Example 1 (Shortest Path Interdiction [24].)**.** Given a graph $G = (V, E)$, with edge costs $\{c_e\}_{e \in E}$ and edge weights $\{w_e\}_{e \in E}$, two terminal vertices $s, t \in V$, and an interdiction target r, the shortest path interdiction problem seeks to identify a set of edges $U \subseteq E$ of minimum cost $\sum_{e \in U} c_e$, so that when removed from G, the length of the shortest (s, t)-paths that are left, measured with respect to the edge weights, is at least r. ■

Example 2 (Minimum Dominating Set Edge Interdiction [33].)**.** Given an undirected graph $G = (V, E)$, with vertex weights $\{w_i\}_{i \in V}$ and edge costs $\{c_e\}_{e \in E}$, a dominating set consists of a subset of vertices $T \subseteq V$ such that every vertex in V is either in T or adjacent to some other vertex in T (e.g., $\{v_4, v_5\}$ in Figure 1). Let r be an interdiction target value; then, the minimum dominating set edge interdiction problem seeks to identify a subset of edges $U \subseteq E$ of minimum cost $\sum_{e \in U} c_e$, so that when removed from G, the minimum weight of any dominating set left in the graph is at least r. ■

In general, the *ground sets* of a binary interdiction game are the sets of indivisible elements that compose the leader's strategies and the follower's structures (i.e., the building blocks or atoms of the game). A binary interdiction game is said to be played over a *single* ground set Δ, if both the leader's strategies and the follower's structures are composed of elements from Δ, i.e., $\Pi \subseteq 2^\Delta$ and $\Omega \subseteq 2^\Delta$. Many interdiction games fit this particular setup. For instance, in the shortest path interdiction problem described before, each feasible strategy for the leader is an edge set $U \subseteq E$, while each follower's feasible structure is an (s,t)-path—which can also be represented directly as an edge set $T \subseteq E$. Thus, for this game $\Delta = E$ and, consequently, $\Pi = 2^E$ and $\Omega = \{T \subseteq E \mid T \text{ forms an } (s,t)\text{-path}\}$. Many other interdiction games (see Examples 3 and 4) fall into this single ground set category.

Similarly, an interdiction game is said to be played over *two* ground sets if the leader's strategies and the follower's structures are composed of elements from two different ground sets, denoted Δ_l and Δ_f, respectively. The minimum dominating set edge interdiction problem described before is a problem of this type because any feasible strategy for the leader is an edge set $U \subseteq E$, while the follower's structures are dominating sets, which are defined over the set of vertices V. Therefore, $\Delta_l = E$, $\Delta_f = V$ and, consequently $\Pi = 2^E$ and $\Omega = \{T \subseteq V \mid T \text{ forms an dominating set}\}$. Most network interdiction games are typical examples of BIG instances defined over one or two ground sets, for that both the strategies and structures are often defined in terms of either the vertices or the edges of the networks.

In this section, we study a general mathematical formulation that can be used to solve both types of games. Some instances of this formulation have appeared in the literature in recent years derived to solve some specific network interdiction problems. Here, we provide a general characterization of it. We will first describe such a formulation to model games played over a single ground set, and then we will provide a transformation theorem that, under some basic assumptions, can be used to reduce any problem played over two ground sets into a game defined over the ground set of the leader.

3.1 Binary Interdiction Games with One Ground Set

We define this class of problems as follows.

Definition 5 (BIG-1)**.** A binary interdiction game defined over a single ground set is a BIG where both the strategy set Π and the structure set Ω consist of subsets of the same ground set Δ. That is, $\Pi \subseteq 2^\Delta$ and $\Omega \subseteq 2^\Delta$.

In any instance of BIG-1, the leader aims to block all the follower's critical structures in $\hat{\Omega}$. Then, based on the definition of BIGs provided in Section 1, the following assumption defines the interaction between any strategy-structure pair $(U, T) \in \Pi \times \Omega$.

Assumption 1. Given an instance of BIG-1, we will assume that $\mathcal{I}(U, T) = 1$ if and only if $T \setminus U \notin \hat{\Omega}$, for any strategy pair $(U, T) \in \Pi \times \Omega$.

By representing the members of Π and Ω as subsets of the game's ground set Δ, the leader's strategies and the follower's structures can be ordered by the inclusion relation ($\subseteq$). That is, both Π and Ω can be defined as partially ordered sets (posets) under $\subseteq$. Some poset related notions that will be used hereafter follow.

Definition 6. Given a poset $(P, \subseteq)$,

1. let $m(P)$ be the set of all minimal elements in P.
2. Q is a subposet of P if Q is a subset of P with the inherited ordering.
3. Q is a lower set in P if Q is a subposet of P and, for any $u_1, u_2 \in P$ such that $u_1 \subseteq u_2$, $u_2 \in Q$ implies that $u_1 \in Q$. The dual notion of a lower set is called an upper set in P.
4. Given a subset U of a poset P, $\uparrow U = \{u' \in P \mid u' \supseteq u \text{ for some } u \in U\}$ is the upper closure of U in P. The dual version $\downarrow U$ is the lower closure of U in P.

Additionally, given an instance of BIG-1 with the corresponding strategy and structure sets Π and Ω defined over ground set Δ, we will also assume that the weight w and cost c functions of the game are monotone and modular over the elements in Δ. That is, $c(U) = \sum_{a \in U} c_a$ and $w(T) = \sum_{a \in T} w_a$, for any $U \in \Pi$ and $T \in \Omega$, respectively. This assumption is not a strong requirement, but will help simplify our exposition.

3.1.1 Solution Characterization

In this section, we will characterize the solution space of BIG-1 using an integer programming formulation. We start with the following lemma.

Lemma 1. *For any instance of BIG-1 that satisfies Assumption 1, given a strategy $U \subseteq \Delta$ and a family of critical structures $\hat{\Omega} \subseteq \Omega \subseteq 2^{\Delta}$, then $T \setminus U \notin \hat{\Omega}$ for all $T \in \hat{\Omega}$ if and only if $|U \cap T| \geq 1$ for all $T \in m(\hat{\Omega})$.*

Proof. First, note that $|U \cap T| \geq 1$ for all $T \in m(\hat{\Omega})$ implies that $T \setminus U \notin \hat{\Omega}$ because of the minimality of T. Thus, we are left to show that blocking the minimal structures $m(\hat{\Omega})$ is equivalent to blocking the whole set $\hat{\Omega}$. The necessity is trivial so we focus on proving sufficiency. To this end, we conduct structural induction on the poset $\hat{\Omega}$. The base case contains all the minimal elements in $\hat{\Omega}$, which are covered by the premise. Now, we consider an arbitrary non-minimal $T \in \hat{\Omega}$. We have $T \cap U \neq \emptyset$; otherwise, there must exist some minimal $T' \subsetneq T$ such that $T' \cap U = \emptyset$, which contradicts the base case. Hence, we have $(T \setminus U) \subsetneq T$. Then, either $T \setminus U \notin \hat{\Omega}$, in which case we are done, or $T \setminus U \in \hat{\Omega}$, in which case we can apply the induction assumption to derive $T \setminus U = (T \setminus U) \setminus U \notin \hat{\Omega}$, a contradiction. This proves the induction step. □

According to this lemma, blocking all the structures in $\hat{\Omega}$ by a strategy U, given Assumption 1, occurs if and only if U blocks all the minimal structures in $\hat{\Omega}$. Based on this result, we have the following theorem that characterizes the solution space of BIG-1.

Theorem 1. *For any instance of BIG-1 that satisfies Assumption 1, a leader's strategy $U \in \Pi$ is feasible for the TGT version of the problem if and only if $|U \cap T| \geq 1$ for all $T \in m(\hat{\Omega})$.*

Proof. By the definition of TGT, a strategy U is feasible if and only if $\mathcal{I}(U, T) = 1$ for all $T \in \hat{\Omega}$, which is equivalent to say that $T \setminus U \notin \hat{\Omega}$, for all $T \in \hat{\Omega}$ under Assumption 1. Then, by Lemma 1, this is the same as blocking all the minimal structures in $\hat{\Omega}$, i.e., $|U \cap T| \geq 1$ for all $T \in m(\hat{\Omega})$. □

Theorem 1 directly yields the following valid integer programming formulation for any instance of BIG-1.

$$
\begin{aligned}
\text{(MCS)} \quad \min \quad & \sum_{a \in \Delta} c_a x_a && (3) \\
\text{s.t.} \quad & \sum_{a \in T} x_a \geq 1 \quad \forall T \in m(\hat{\Omega}) && (4) \\
& \mathbf{x} \in X_\Pi, && (5)
\end{aligned}
$$

where $\mathbf{x} = \{x_a\}_{a \in \Delta}$ is the indicator vector of the leader's strategy $U \in \Pi$ and X_Π is the space of all such vectors. We call this formulation the *Minimal Critical Structures (MCS) formulation*.

Clearly, the MCS formulation shows that most instances of BIG-1 under Assumption 1 can be modeled as set covering problems, where the set of elements to be covered is composed of all the minimal critical structures in $\hat{\Omega}$. An immediate consequence of this is that previous results about the facial structure of the set covering polytope [4, 34, 35] directly apply to MCS, and therefore any family of valid inequalities developed for set covering can be used to strengthen the MCS formulation too. We will study some other general valid inequalities for this formulation in Section 4.

3.1.2 Examples of BIG-1

Besides the shortest path interdiction problem described before, many other problems fit into this category. We now list some of them for later reference. The following graph will be used to describe instances of these examples.

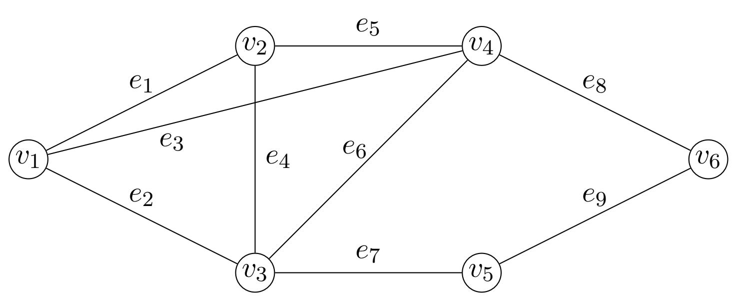

Figure 1: Example of a graph $G = (V, E)$ with $V = \{v_1, \ldots, v_6\}$ and $E = \{e_1, \ldots, e_9\}$.

Example 1 (Shortest Path Interdiction, Cont.)**.** For the graph given in Figure 1, let $s = v_1$ and $t = v_6$, and assume that paths $T_1 = \{e_2, e_6, e_8\}$ and $T_2 = \{e_3, e_8\}$ are the only two paths having a weight that is less than the predefined target value r. Then, $\hat{\Omega} = \{T_1, T_2\}$. Notice that in this case $m(\hat{\Omega}) = \hat{\Omega}$ because all (s,t)-paths are minimal. Based on Theorem 1, the following formulation can be used to solve the TGT version of this shortest path interdiction problem.

$$
\begin{aligned}
\min \quad & \mathbf{c}^T \mathbf{x} \\
\text{s.t.} \quad & x_2 + x_6 + x_8 \geq 1 \\
& x_3 + x_8 \geq 1 \\
& \mathbf{x} \in \{0,1\}^E.
\end{aligned}
$$

By the relationship between BDG and TGT explained in Section 2.1.2, these inequalities can also be used as supervalid inequalities to strengthen other formulations that solve the BDG version. Indeed, in [24], these inequalities have been implemented for this purpose. ■

Example 3 (Spanning Tree Interdiction [43])**.** Given an undirected graph $G = (V, E)$, with edge weights $\{w_e\}_{e\in E}$ and edge costs $\{c_e\}_{e\in E}$, a *spanning tree* is a connected acyclic subgraph that spans the vertex set V (e.g., $\{e_1, e_2, e_5, e_7, e_8\}$ in Figure 1). Let r be an interdiction target value; then, the spanning tree interdiction problem seeks to identify a subset of edges $U \subseteq E$ of minimum cost $\sum_{e\in U} c_e$, so that when removed from G, the minimum weight of any spanning tree left in the graph is at least r.

For the graph given in Figure 1, suppose that $T_1 = \{e_1, e_2, e_3, e_7, e_8\}$ and $T_2 = \{e_3, e_5, e_6, e_7, e_8\}$ are the only two spanning trees having a weight less than the target value r, then the minimal critical structure set is $m(\hat{\Omega}) = \{T_1, T_2\}$. In this case, we also have $m(\hat{\Omega}) = \hat{\Omega}$ due to the minimality of spanning trees. According to Theorem 1, the following formulation is valid to solve the TGT version spanning tree interdiction.

$$
\begin{aligned}
\min \quad & \mathbf{c}^T\mathbf{x} \\
\text{s.t.} \quad & x_1 + x_2 + x_3 + x_7 + x_8 \geq 1 \\
& x_3 + x_5 + x_6 + x_7 + x_8 \geq 1 \\
& \mathbf{x} \in \{0, 1\}^E.
\end{aligned}
$$

This is formulation was called the *Critical Spanning Tree* formulation in [43], and, as for the shortest path interdiction case, these constraints can also be used as supervalid cuts for solving the BGT version. ■

Example 4 (Maximum Clique Vertex Interdiction [30])**.** Given an undirected graph $G = (V, E)$, with vertex weights $\{w_i\}_{i\in V}$ and vertex costs $\{c_i\}_{i\in V}$, a clique consists of a subset of vertices $T \subseteq V$ whose induced graph is complete (e.g., $\{v_1, v_2, v_3, v_4\}$ in Figure 1). Let r be an interdiction target value; then, the maximum clique interdiction problem seeks to identify a subset of vertices $U \subseteq V$ of minimum cost $\sum_{i\in V} c_i$, so that when removed from G, the maximum weight of any clique left in the graph is at the most r. The critical structures in this problem consist of all the cliques with a weight greater than r. Contrary to the previous examples, not all the critical cliques are necessarily minimal because any subset of a critical clique whose weight is greater than r is also a critical clique.

In this example, suppose that $w_i = 1$, for all $i \in V$ and let $r = 2$. Then, for the graph given in Figure 1, $T_1 = \{v_1, v_2, v_3\}$, $T_2 = \{v_1, v_3, v_4\}$, $T_3 = \{v_1, v_2, v_4\}$, $T_4 = \{v_2, v_3, v_4\}$, and $T_5 = \{v_1, v_2, v_3, v_4\}$ are the critical cliques that compose $\hat{\Omega}$. Then, $m(\hat{\Omega}) = \{T_1, T_2, T_3, T_4\}$, since T_5 is not minimal. Theorem 1 allows us to use the following formulation,

$$
\begin{aligned}
\min \quad & \mathbf{c}^T\mathbf{x} \\
\text{s.t.} \quad & x_1 + x_2 + x_3 \geq 1 \\
& x_1 + x_3 + x_4 \geq 1 \\
& x_1 + x_2 + x_4 \geq 1 \\
& x_2 + x_3 + x_4 \geq 1 \\
& \mathbf{x} \in \{0, 1\}^V.
\end{aligned}
$$

In [30], a similar formulation was developed; however, one major difference is that some of the inequalities associated with non-minimal critical cliques having right hand sides greater than 1 are used in that paper. Later in Section 4, we will show that such inequalities are part of a larger family of valid inequalities. ■

Other graph structures such as, independent sets, Hamiltonian cycles, vertex covers, and connected components can also be used to define similar binary interdiction games. Moreover, each structure may have two potential problems depending on whether the leader's strategies are defined over the vertices or edges. In the next section, we will also show that problems where the ground sets of the leader and follower are different can also be modeled with a variation of the MCS formulation.

3.2 Binary Interdiction Games with Two Ground Sets

As before, we define this class of problems as follows.

Definition 7 (BIG-2)**.** A binary interdiction game defined over two ground sets is a BIG where the strategy set Π and the structure set Ω consist of subsets of two different ground sets Δ_l and Δ_f, respectively. That is, $\Pi \subseteq 2^{\Delta_l}$ and $\Omega \subseteq 2^{\Delta_f}$.

We now introduce the counterpart of Assumption 1 for this type of games. Before stating this assumption, we derive some intuition behind this assumption using the minimum dominating set edge interdiction example that we introduced earlier.

Example 2 (Minimum Dominating Set Edge Interdiction, Cont.)**.** As discussed before, here $\Delta_l = E$ and $\Delta_f = V$. Given a dominating set $T \in \Omega$, let $\overline{T} = V \setminus T$ be its complement, and $(T, \overline{T}) = \{\{i,j\} \in E \mid i \in T, j \in \overline{T}\}$ be the cut-set between T and $\overline{T}$. A *whip* of dominating set T is a set of edges $S \subseteq (T, \overline{T})$ so that exactly one edge in S is incident to each vertex $i \in \overline{T}$ [14, 33].[1]

It is easy to see that the vertices in T compose a dominating set if and only if there exists a whip $S \in (T, \overline{T})$; otherwise, there must exist a vertex $i \in \overline{T}$ that is not incident to any vertex in T, thus contradicting the fact that T is a dominating set.[2] Furthermore, one can observe that it is possible for a dominating set to have multiple whips; for instance, in the graph depicted in Figure 1, the sets $S_1 = \{e_1, e_6, e_9\}$ and $S_2 = \{e_4, e_6, e_9\}$ are two of the several whips associated with dominating set $T = \{v_1, v_3, v_5\}$.

An important consequence of this observation, is that for any vertex set $T \subseteq V$, if there is a whip S in cut-set $\in (T, \overline{T})$, then such a whip S is a *witness*—defined over Δ_l—to the statement $T \in \Omega$.[3] Furthermore, from the perspective of the interdiction game, given a critical dominating set $T \in \hat{\Omega}$, if the leader wants to block the follower from selecting T by interdicting edges, then the leader must attack all the whips associated with T; otherwise, T would remain a dominating set. For example, for the graph in Figure 1, if the leader's strategy U contains edge e_9, then $T = \{v_1, v_3, v_5\}$ is no longer a critical dominating set in $G(V, E \setminus U)$, as there is no whip of T that remains in the cut-set $(T, \overline{T})$ of the residual graph $G(V, E \setminus U)$. ■

Intuitively, if all the critical structures in $\hat{\Omega}$ have this type of witnesses, then those can be seen as a translation of the elements in $\hat{\Omega}$ into a set of critical structures defined over the leader's ground set Δ_l. In this section, we are interested in interdiction games where such a translation exist. We now proceed to formalize these ideas.

[1] In [14], whips are defined as the subgraph $G(V, S)$. We use the edge definition however for the sake of convenience.

[2] Note that $S = \emptyset$ is a valid whip for T if and only if $T = V$, as in such a case $\overline{T} = \emptyset$.

[3] While witness is mostly used in formal logic, the analogous term *certificate* is also commonly used for this purpose, particularly in computational complexity.

Definition 8 (Structure's witnesses)**.** Consider a BIG-2 defined over the ground sets Δ_l and Δ_f, where $\Omega \subseteq 2^{\Delta_f}$ is the set of follower's structures. We will say that a set $S \subseteq \Delta_l$ is a *witness* of structure $T \subseteq \Delta_f$ if and only if S proves that $T \in \Omega$. The set of witnesses of structure $T \in \Omega$ is denoted $D_T = \{S \in \Delta_l \mid S \text{ certifies that } T \in \Omega\}$ and, for any subset $\bar{\Omega} \subseteq \Omega$,

$$\phi(\bar{\Omega}) = \bigcup_{T \in \bar{\Omega}} m(D_T) \tag{6}$$

denotes the set of minimal witnesses of the structures in $\bar{\Omega}$. Any instance of BIG-2 with this type of *structure/witness* relationship is said to be *leader-certifiable.*

Note that each set of witnesses D_T, for any $T \in \Omega$, can also be ordered by inclusion, so we can use the minimal operator $m(\cdot)$ for posets introduced in Section 3.1.

We are now in position to state the counterpart of Assumption 1 for instances of BIG-2.

Assumption 2. Given a leader-certifiable instance of BIG-2, we will assume that for any strategy pair $(U, T) \in \Pi \times \Omega$, function $\mathcal{I}(U, T) = 1$ if and only if $U \cap S \neq \emptyset$, for every witness $S \in D_T$.

Importantly, under Assumptions 1 and 2, it is not difficult to see that in general BIG-1 is a subclass of BIG-2. In such a case, because $\Delta_l = \Delta_f = \Delta$, each $T \in \Omega$ acts as its own witness. In what follows, we will show that, under Assumption 2, any leader-certifiable instance of BIG-2 can be transformed into an instance of BIG-1.

Theorem 2 (BIG-2 Transformation Theorem)**.** *In any BIG-2 that satisfies Assumption 2, a leader's strategy U is feasible to the TGT version of the problem if and only if $|U \cap S| \geq 1$ for all $S \in m(\phi(\hat{\Omega}))$, where $\hat{\Omega}$ is the set of critical structures.*

Proof. By the definition of the TGT version of BIG, a strategy U is feasible to the leader if and only if $\mathcal{I}(U, T) = 1$ for all $T \in \hat{\Omega}$. Then, we have the following equivalence chain.

$$\begin{aligned}
&\forall T \in \hat{\Omega},\ \mathcal{I}(U, T) = 1 \\
\iff &\forall T \in \hat{\Omega}, \forall S \in D_T,\ |U \cap S| \geq 1 \\
\iff &\forall T \in \hat{\Omega}, \forall S \in m(D_T),\ |U \cap S| \geq 1 \\
\iff &\forall S \in \phi(\hat{\Omega}),\ |U \cap S| \geq 1 \\
\iff &\forall S \in m(\phi(\hat{\Omega})),\ |U \cap S| \geq 1,
\end{aligned}$$

where the first equivalence is due to Assumption 2, the second and the last equivalences are by Lemma 1, and the third one is due to the definition of the witnesses for the structures in $\hat{\Omega}$. □

A consequence of Theorem 2 is that any instance of BIG-2 that satisfies Assumption 2 can be solved using the MCS formulation, after replacing the minimal critical structures with the set of their minimal witnesses $m(\phi(\hat{\Omega}))$ as follows,

$$\begin{aligned}
\text{(MCS)} \quad \min \quad & \sum_{a \in \Delta_l} c_a x_a \\
\text{s.t.} \quad & \sum_{a \in S} x_a \geq 1 \quad \forall S \in m(\phi(\hat{\Omega})) \\
& \mathbf{x} \in X_\Pi,
\end{aligned}$$

where $\mathbf{x} = \{x_a\}_{a \in \Delta_l}$ is an indicator vector of the leader's strategy $U \in \Pi$ and X_Π is the space of all such vectors.

3.2.1 Examples of BIG-2

Besides the minimum dominating set edge interdiction problem described before, many other problems fit into this category as well. We now discuss some examples.

Example 2 (Minimum Dominating Set Edge Interdiction, Cont.)**.** For the graph given in Figure 1, assume $T_1 = \{v_1, v_5\}$ and $T_2 = \{v_1, v_6\}$ are the only minimal critical dominating sets. As we discussed previously, the witnesses of the dominating sets are their whips. Thus, $m(D_{T_1}) = \{\{e_1, e_2, e_3, e_9\}, \{e_1, e_3, e_7, e_9\}\}$, $m(D_{T_2}) = \{\{e_1, e_2, e_3, e_9\}, \{e_1, e_2, e_8, e_9\}\}$, and consequently we have the following set of witnesses $\phi(\hat{\Omega}) = \{\{e_1, e_2, e_3, e_9\}, \{e_1, e_3, e_7, e_9\}, \{e_1, e_2, e_8, e_9\}\}$ for set $\hat{\Omega}$. Based on Theorem 2, the following formulation can be used to solve the TGT version of this minimum dominating set edge interdiction problem.

$$\begin{aligned}
\min \quad & \mathbf{c}^T\mathbf{x} \\
\text{s.t.} \quad & x_1 + x_2 + x_3 + x_9 \geq 1 \\
& x_1 + x_2 + x_8 + x_9 \geq 1 \\
& x_1 + x_3 + x_7 + x_9 \geq 1 \\
& \mathbf{x} \in \{0, 1\}^E.
\end{aligned}$$

Importantly, [33] derived a similar formulation, but they also added some additional inequalities for non-minimal whips. In Section 4, we will show that such inequalities are part of a larger family of valid inequalities. ■

As one may expect, identifying set $\phi(\hat{\Omega})$ is crucial when solving any instance of BIG-2. In some cases, as for the following example, the set of witnesses of each critical structure $T \in \hat{\Omega}$ is rather trivial to derive, while for others like for the minimum dominating set edge interdiction problem such a set is more difficult to identify.

Example 5 (Maximum Clique Edge Interdiction [29])**.** This example is similar to the one given in Example 4, but here the leader interdict edges instead of vertices; hence, the cost function c is defined over E. As for Example 4, the critical structures consist of all the cliques with a weight greater than r. Given a vertex set $T \subseteq V$, it is easy to see that $T \in \Omega$ if and only if the edge set $S = \binom{T}{2}$ is a subset of E. Contrary to the dominating set example, where each dominating set may have multiple witnesses (i.e., whips), here each clique $T \in \Omega$ has a unique witness. For the graph given in Figure 1, assume $\hat{\Omega} = \{T_1, T_2, T_3, T_4, T_5\}$, where $T_1 = \{v_1, v_2, v_3\}$, $T_2 = \{v_1, v_3, v_4\}$, $T_3 = \{v_1, v_2, v_4\}$, $T_4 = \{v_2, v_3, v_4\}$, and $T_5 = \{v_1, v_2, v_3, v_4\}$. Then, $m(D_{T_1}) = \{e_1, e_2, e_4\}$, $m(D_{T_2}) = \{e_2, e_3, e_6\}$, $m(D_{T_3}) = \{e_1, e_3, e_5\}$, $m(D_{T_4}) = \{e_4, e_5, e_6\}$, and $m(D_{T_5}) = \{e_1, e_2, e_3, e_4, e_5, e_6\}$. Based on Theorem 2, the following formulation can be used to solve the TGT version of this maximum clique edge interdiction problem.

$$\begin{aligned}
\min \quad & \mathbf{c}^T\mathbf{x} \\
\text{s.t.} \quad & x_1 + x_2 + x_4 \geq 1 \\
& x_2 + x_3 + x_6 \geq 1 \\
& x_1 + x_3 + x_5 \geq 1 \\
& x_4 + x_5 + x_6 \geq 1 \\
& \mathbf{x} \in \{0, 1\}^E.
\end{aligned}$$

As one may expect, this formulation is quite similar to the one provided in Example 4, where the leader interdict the vertices of the graph. ■

Example 6 (Critical Edge Detection Problems on Graphs [2, 37, 42])**.** Most critical element detection problems consist of identifying a collection of vertices or edges of a graph whose removal optimally deteriorates a connectivity metric of the graph. One oft-used metric to measure connectivity is the weight of the largest connected component of the residual graph [37]. An example of this problem modeled as a BIG defined from the TGT perspective follows.[4]

Given a graph $G = (V, E)$ with vertex weights $\{w_i\}_{i\in V}$ and edge costs $\{c_e\}_{e\in E}$, let r be the interdiction target value. The critical edge detection problem then seeks to identify a subset of edges $U \subseteq E$ of minimum cost $\sum_{i\in V} c_i$, so that when removed from G, the weight of the largest connected component left in the graph is at the most r. The critical structures in this problem consist of all the vertex subsets that induce connected subgraphs in G, whose total weight is greater than r. Given a vertex set $T \subseteq V$, it is easy to see that $T \in \Omega$ if and only if there is a tree in G that spans T. Then, the edge set $S \subseteq E$ of such a tree is a witness for T. As for the dominating set example, each connected vertex set T may have multiple witnesses (i.e., the edges of any tree that spans T). For the graph given in Figure 1, assuming that $m(\hat{\Omega}) = \{T_1, T_2\}$ where $T_1 = \{v_1, v_2, v_3\}$ and $T_2 = \{v_3, v_4, v_5\}$, we have $m(D_{T_1}) = \{\{e_1, e_2\}, \{e_1, e_4\}\}, \{e_2, e_4\}\}$ and $m(D_{T_2}) = \{\{e_6, e_8\}\}$. Based on Theorem 2, the following formulation can be used to solve this problem.

$$
\begin{aligned}
\min \quad & \mathbf{c}^T\mathbf{x} \\
\text{s.t.} \quad & x_1 + x_2 \geq 1 \\
& x_1 + x_4 \geq 1 \\
& x_2 + x_4 \geq 1 \\
& x_6 + x_8 \geq 1 \\
& \mathbf{x} \in \{0, 1\}^E.
\end{aligned}
$$

Other variations of critical element detection problems can also be modeled as BIGs. ■

4 Polyhedral Analysis

In this section, we study the convex hull of the feasible solutions for BIGs. For the general discussion, we consider the single ground set version of the problem, but provide a few examples of the two ground set version along the way. Furthermore, in order to conduct our facial analysis, we will assume that $\Pi = 2^\Delta$; however, all the inequalities we study in this section are also valid for the case where $\Pi \subseteq 2^\Delta$.

4.1 Elementary Facets

Given a BIG defined over ground set Δ, where $\Pi = 2^\Delta$ and $\Omega \subseteq 2^\Delta$ are the sets of leader's strategies and follower's structures, respectively, let X denote the set of incidence vectors of the feasible solutions for the leader. That is,

$$X := \left\{ \mathbf{x} \in \{0, 1\}^{|\Delta|} \mid \sum_{a\in T} x_a \geq 1, \ \forall T \in m(\hat{\Omega}) \right\}.$$

Furthermore, let $\mathscr{P} := \text{conv}(X)$ be the convex hull of the elements in X. The following proposition provides some basic properties of $\mathscr{P}$.

[4]In this example, the term critical edges is used in reference to the edges that are removed, while for us the term critical structures is used in reference to the leader's targets.

Proposition 3. *Assuming that $|T| \geq 2$, for all $T \in \hat{\Omega}$, then the following statements about $\mathscr{P}$ are true,*

1. *$\mathscr{P}$ is full dimensional.*
2. *Given an element $a \in \Delta$, the inequality $x_a \geq 0$ induces a facet of $\mathscr{P}$ if and only if $\{a, a'\} \notin \hat{\Omega}$, for all $a' \in \Delta \setminus \{a\}$.*
3. *Given an element $a \in \Delta$, the inequality $x_a \leq 1$ induces a facet of $\mathscr{P}$.*
4. *Given a critical structure $T \in \hat{\Omega}$, the inequality $\sum_{a \in T} x_a \geq 1$ induces a facet of $\mathscr{P}$ if and only if for all $a \in \Delta \setminus T$, there exists some $a' \in T$ such that $T \setminus \{a'\} \cup \{a\}$ is not a critical structure.*

These statements are easy to verify and follow directly from the polyhedral analysis of the set covering polytope given in [4], so we omit the proof here. Similarly, all the facet-defining inequalities for set covering that have been discovered previously [4, 34, 35] can also be applied directly to strengthen the MCS formulation. In the following sections we will study other general families of valid inequalities for $\mathscr{P}$ that are different from the ones derived therein. In what follows, we will assume that $|T| \geq 2$, for all $T \in \hat{\Omega}$.

4.2 Vitality Inequalities

We begin our discussion with the following definition.

Definition 9 (Vital Component/Vitality Index)**.** Given a set $S \subseteq \Delta$, the elements of the following set are called the *vital components* of S:

$$\mathcal{V}_S = \{S' \subseteq S \mid T \not\subseteq S \setminus S',\ \forall T \in m(\hat{\Omega})\}.$$

Furthermore, the *vitality index* of set S is defined as $\gamma_S = \min_{S' \in \mathcal{V}_S} |S'|$.

From the definition, it should be clear that the set of vital components $\mathcal{V}_S$ of any set $S \subseteq \Delta$ is an upper set under inclusion; therefore, interdicting any minimal vital component of S (i.e., any $S' \in m(\mathcal{V}_S)$) suffices to block all the critical structures contained in S.

We have the following basic facts about the vitality index. They can be trivially verified, so we omit the proof.

Lemma 2. *Given the set of critical structures $\hat{\Omega} \subseteq 2^{\Delta}$, and a set $S \subseteq \Delta$, the vitality index of set S satisfies the following properties:*

1. *$\gamma_S = 1$ for all $S \in m(\hat{\Omega})$.*
2. *$\gamma_S \geq 1$ if and only if $S \in \uparrow \hat{\Omega}$,*
3. *$\gamma_{S \cup \{a\}} - \gamma_S = 0$ or 1 for all $a \in \Delta \setminus S$.*

Using the concept of vital components, we can directly identify a necessary condition for an inequality to be valid for $\mathscr{P}$.

Theorem 3. *Let $\sum_{a \in S} \alpha_a x_a \geq b$ with $\alpha_a \geq 0, \forall a \in S$ and $b > 0$ be a valid inequality for $\mathscr{P}$, then $S \in \uparrow \hat{\Omega}$.*

Proof. Assume that such a valid inequality exists for a set $S \notin \uparrow \hat{\Omega}$. First, if $m(\hat{\Omega}) = \emptyset$, note that the solution $x_a = 0, \forall a \in \Delta$ belongs to $\mathscr{P}$, but violates the inequality, as $b > 0$. Then, if $m(\hat{\Omega}) \neq \emptyset$, the assumption $S \notin \uparrow \hat{\Omega}$ implies that $T \setminus S \neq \emptyset, \forall T \in m(\hat{\Omega})$. Therefore, one can construct a set $Q \subseteq \Delta$ so that $Q \cap (T \setminus S) \neq \emptyset, \forall T \in m(\hat{\Omega})$. It is easy to see that the solution $x_a = 1, \forall a \in Q$ and $x_a = 0, \forall a \in \Delta \setminus Q$ belongs to $\mathscr{P}$ because it interdicts at least one element of all the minimal critical structures. However, since $Q \cap S = \emptyset$, such solution does not satisfy the inequality, as $b > 0$. □

Intuitively, any vital component of a set $S \subseteq \Delta$ is a subset $S' \subseteq S$ such that, if the leader interdicts all the elements in S', then all the critical structures contained in S would be blocked. The vitality index of S then is the minimum number of elements in S that the leader must interdict in order to block all the critical structures contained in S. If set S does not contain any critical structure, then the empty set is a vital component of S and therefore S's vitality index would be 0. Also, for $S = \Delta$, the set $\mathcal{V}_\Delta$ is then composed of all the feasible strategies for the leader.

From the definition of the vitality indices, we can directly obtain the following set of valid inequalities for $\mathscr{P}$.

$$\sum_{a \in S} x_a \geq \gamma_S, \quad S \in \uparrow \hat{\Omega}. \tag{8}$$

From Lemma 2, when $S \in m(\hat{\Omega})$, we obtain a constraint from the MCS formulation. In other words, constraints (4) are the particular case of these inequalities with $\gamma_S = 1$. Also, as discussed before in Examples 2 and 4, some particular cases of this type of inequalities have appeared in [29, 30, 33]. Here, (8) can be seen as a generalization of them.

An important observation is that, depending on the selection of set S, these inequalities can be rather loose; particularly, when S contains multiple vital components whose sizes are significantly larger than the vitality index γ_S. Intuitively, the reason for their weakness is that these inequalities only capture the information of the smallest vital components of S. In what follows, we will study a stronger family of valid inequalities that consider information derived from all the other vital components as well.

Definition 10 (Disjoint Decomposition)**.** Given a family of subsets $\mathcal{V} \subseteq 2^\Delta$ of a ground set Δ, a *disjoint decomposition* of $\mathcal{V}$ is a partition $\{\mathcal{V}_i\}_{i \in I}$ of $\mathcal{V}$ such that all the parts are also pairwise disjoint with respect to their elements; i.e., $(\bigcup \mathcal{V}_i) \cap (\bigcup \mathcal{V}_j) = \emptyset$ for any distinct i and j in I, where $\bigcup \mathcal{V}_i$ is the union of all the sets contained in $\mathcal{V}_i$, i.e., $\bigcup \mathcal{V}_i = \bigcup_{S \in \mathcal{V}_i} S$. A disjoint decomposition of $\mathcal{V}$ is called the *finest* if it cannot be decomposed further. Otherwise, we say the disjoint decomposition is *coarse*.

There are many different ways to construct a disjoint decomposition from a given family of sets $\mathcal{V}$. In fact, given any disjoint decomposition $\{\mathcal{V}_i\}_{i \in I}$ of a set $\mathcal{V}$, if any two of its parts are combined, the result is a coarser disjoint decomposition. That is, for any $i_1, i_2 \in I$, the new partition $\{\mathcal{V}_i\}_{i \in I \setminus \{i_1, i_2\}} \cup \{\mathcal{V}_{i_1} \cup \mathcal{V}_{i_2}\}$ is also a disjoint decomposition. Consequently, the coarsest disjoint decomposition contains only one part, set $\mathcal{V}$ itself. On the other hand, the finest decomposition is often non-trivial and is unique by the following proposition.

Proposition 4. *Given a family of subsets $\mathcal{V} \subseteq 2^\Delta$, there is a unique finest disjoint decomposition of $\mathcal{V}$ up to relabeling.*

Proof. Towards a contradiction, we assume $\{\mathcal{V}_i\}_{i \in I}$ and $\{\mathcal{U}_j\}_{j \in J}$ are two non-isomorphic disjoint decompositions of $\mathcal{V}$. This implies that there exists a set $S \in \mathcal{V}$ for which the parts that contain it in both decompositions are different. That is, for the parts $\mathcal{V}_0$ and $\mathcal{U}_0$, where $S \in \mathcal{V}_0 \in \{\mathcal{V}_i\}_{i \in I}$ and $S \in \mathcal{U}_0 \in \{\mathcal{U}_j\}_{j \in J}$, $\mathcal{V}_0 \triangle \mathcal{U}_0 \neq \emptyset$.[5] Notice that the choice of $\mathcal{V}_0$ and $\mathcal{U}_0$ is unique since both

[5]Here, $\triangle$ stands for the symmetric difference between sets.

decompositions are partitions of $\mathcal{V}$. Without loss of generality, we assume $\mathcal{V}_0 \setminus \mathcal{U}_0 \neq \emptyset$. Now, because sets $\mathcal{V}_0 \setminus \mathcal{U}_0$ and $\mathcal{V}_0 \cap \mathcal{U}_0$ belong to different parts in $\{\mathcal{U}_j\}_{j\in J}$, then $(\bigcup(\mathcal{V}_0 \setminus \mathcal{U}_0)) \cap (\bigcup(\mathcal{V}_0 \cap \mathcal{U}_0)) = \emptyset$. Furthermore, both sets $\mathcal{V}_0 \setminus \mathcal{U}_0$ and $\mathcal{V}_0 \cap \mathcal{U}_0$ are nonempty, thus, splitting $\mathcal{V}_0$ into $\mathcal{V}_0 \setminus \mathcal{U}_0$ and $\mathcal{V}_0 \cap \mathcal{U}_0$ produces a strictly finer disjoint decomposition than $\{\mathcal{V}_i\}_{i\in I}$, which contradicts the assumption that $\{\mathcal{V}_i\}_{i\in I}$ is a finest disjoint decomposition. □

Based on the finest disjoint decomposition of a set $S \in \uparrow \hat{\Omega}$, we define the vitality inequalities as follows,

Definition 11 (Vitality Inequalities)**.** Given $S \in \uparrow \hat{\Omega}$, let $\{\mathcal{V}_i\}_{i\in I}$ be the finest disjoint decomposition of $m(\mathcal{V}_S)$ with $\gamma_i = \min_{S' \in \mathcal{V}_i} |S'|$, the *Vitality Inequality* of S is given by

$$\sum_{i\in I} \left(\prod_{j\in I: j\neq i} \gamma_j \right) \sum_{a\in \bigcup \mathcal{V}_i} x_a \geq \prod_{i\in I} \gamma_i. \tag{9}$$

Here, we take the convention that the product defined over an empty set is equal to one.

Importantly, this inequality is not necessarily in the simplest form since all the coefficients and the RHS may share a common divisor. Moreover, after transforming it into the simplest form, there will be no unit coefficients unless $\max_{i\in I} \gamma_i = lcm\{\gamma_i\}_{i\in I}$, where lcm stands for the least common multiple. In this case, stronger inequalities could be produced via the Chvátal-Gomory rounding procedure [13]. The following is an example of vitality inequalities for the spanning tree interdiction problem.

Example 4 (Spanning Tree Interdiction, Cont.)**.** Given an set $S = \{e_1, \dots, e_8\}$ that induces the spanning subgraph in Figure 2. Suppose all spanning trees contained in S are critical, then a vital component is any subset of S that disconnects the graph. Hence, we have

$$\begin{aligned} m(\mathcal{V}_S) = &\{\{e_1, e_2\}, \{e_1, e_3\}, \{e_2, e_3\}, \{e_4\}, \{e_5, e_6\}, \\ &\{e_5, e_7\}, \{e_5, e_8\}, \{e_6, e_7\}, \{e_6, e_8\}, \{e_7, e_8\}\}, \end{aligned}$$

where $\gamma_S = 1$. The inequality derived from (8) is

$$\sum_{a\in S} x_a \geq 1. \tag{10}$$

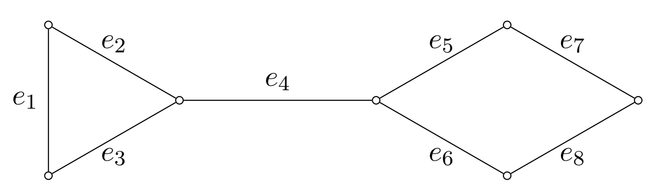

Figure 2: Example of a set $S \subseteq E$ for the spanning tree interdiction problem.

It is easy to see that this inequality is in fact dominated by the inequality associated with any subset of S that induces a spanning tree. On the other hand, given the finest disjoint decomposition of $m(\mathcal{V}_S)$:

- $\mathcal{V}_1 = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$, where $\gamma_1 = 2$;
- $\mathcal{V}_2 = \{\{4\}\}$, where $\gamma_2 = 1$;

- $\mathcal{V}_3 = \{\{5,6\},\{5,7\},\{5,8\},\{6,7\},\{6,8\},\{7,8\}\}$, where $\gamma_3 = 2$,

the corresponding vitality inequality is,

$$2\sum_{a\in\mathcal{V}_1} x_a + 4a_4 + 2\sum_{a\in\mathcal{V}_3} x_a \geq 4,$$

which after simplification becomes,

$$\sum_{a\in S: a\neq 4} x_a + 2x_4 \geq 2. \tag{11}$$

This inequality clearly dominates (10) and is not dominated by any of the inequalities associated with subsets of S that induce spanning trees. This particular vitality inequality is called a cactus inequality in [43]. We will show later that the vitality inequalities in fact comprise a larger family of valid inequalities. ■

The inequality (11) can be produced directly from (9) using a special disjoint decomposition of $\mathcal{V}_S$, which we call the *disjoint vitality decomposition*. In the above example, we can form this decomposition by merging the parts $\mathcal{V}_1$ and $\mathcal{V}_3$. In general, the disjoint vitality decomposition can be obtained from the finest decomposition by merging parts that have the same value of γ_i.

Definition 12 (Disjoint Vitality Decomposition)**.** Given $S \subseteq \Delta$, let $m(\mathcal{V}_S)$ be the minimal vital components of S, then the disjoint vitality decomposition of $m(\mathcal{V}_S)$ is the finest disjoint decomposition such that the minimal cardinality of the elements in each part is different. That is, let $\gamma_i = \min_{S'\in\mathcal{V}_i} |S'|$, then $\gamma_i \neq \gamma_j$ for every two different parts $\mathcal{V}_i$ and $\mathcal{V}_j$.

Clearly, the disjoint vitality decomposition of a given $m(\mathcal{V}_S)$ is unique and can be generated from the finest disjoint decomposition by merging any two parts $\mathcal{V}_i$ and $\mathcal{V}_j$ for which $\gamma_i = \gamma_j$. Moreover, the inequalities produced from the finest disjoint decomposition and the disjoint vitality decomposition are equivalent because in the inequality derived from the finest disjoint decomposition, all the γs that are equal are multiplied an equal number of times on both sides of the inequalities (9).

Importantly, we define the disjoint vitality decompositions not only because they produce a simplified form of the corresponding vitality inequalities. Also, constructing the disjoint vitality decomposition is more direct and intuitive than the finest disjoint decomposition for certain choices of $S \in\uparrow \hat{\Omega}$. Some examples are provided in Section 4.4.

Theorem 4. *Given a set $S \in\uparrow \hat{\Omega}$, the corresponding vitality inequality of S is valid for $\mathscr{P}$.*

Proof. Let $U \subseteq \Delta$ be any feasible strategy for the leader and $\bar{\mathbf{x}}$ be its indicator vector. Since U must block all critical structures contained in S, then, by Definition 9, there must be a set $S' \in m(\mathcal{V}_S)$ so that $S' \subseteq U$. Assume that S' belongs to some part $\mathcal{V}_i$ of the finest disjoint decomposition of $m(\mathcal{V}_S)$, then $\sum_{a\in\bigcup\mathcal{V}_i} \bar{x}_a \geq \gamma_i$ because $|S'| \geq \gamma_i$. After multiplying both sides of this expression by the constant $\prod_{j\in I: j\neq k} \gamma_j$, we obtain

$$\prod_{j\in I: j\neq i} \gamma_j \sum_{a\in\bigcup\mathcal{V}_i} \bar{x}_a \geq \prod_{i\in I_s} \gamma_i,$$

proving that $\bar{\mathbf{x}}$ satisfies (9) for the given set S. □

4.3 Facial Analysis of Vitality Inequalities

The following theorem characterizes necessary and sufficient conditions required for the vitality inequalities to be facet-defining.

Theorem 5. *Given $S \in\uparrow \hat{\Omega}$, let $\{\mathcal{V}_i\}_{i\in I}$ be the corresponding finest disjoint decomposition with $\gamma_i = \min_{S'\in\mathcal{V}_i} |S'|$. Then, the vitality inequality* (9) *induces a facet of $\mathscr{P}$ if both of the following conditions are satisfied,*

1. *Each part $\mathcal{V}_i$ contains at least $|\bigcup \mathcal{V}_i|$ elements of size γ_i such that their indicator vectors are affinely independent.*
2. *For every $a \in \Delta \setminus S$, there exists a set S' in some part $\mathcal{V}_i$ of the disjoint decomposition such that $|S'| = \gamma_i$ and $S \setminus S' \cup \{a\} \notin\uparrow \hat{\Omega}$.*

Moreover, condition 1 *is also necessary, and condition* 2 *is necessary if $\gamma_i = \gamma_S$, for all $i \in I$.*

Proof. First, notice that the indicator vector of any feasible leader strategy $U \subseteq \Delta$ satisfies (9) at equality if and only if $U \cap S = S'$, where S' is a set that belongs to some part $\mathcal{V}_i$ and $|S'| = \gamma_i$.

To prove sufficiency, we will show that under conditions 1 and 2 it is possible to construct $|\Delta|$ affinely independent feasible solutions that lie on the face defined by the vitality inequality (9) associated with set S. First, from condition 1, we can pick $|\bigcup \mathcal{V}_i|$ sets of size γ_i from each part $\mathcal{V}_i$, such that their indicator vectors are affinely independent. Let S' be any of these sets. It is easy to see that $U = S' \cup (\Delta \setminus S)$ is a feasible strategy of the leader, and since we can create a solution like this for all the $|\bigcup \mathcal{V}_i|$ sets selected from each part $\mathcal{V}_i$, this procedure yields $\sum_{i\in I} |\bigcup \mathcal{V}_i| = |S|$ affinely independent feasible solutions that lie on the face induced by (9). Second, for a given $a \in \Delta \setminus S$, let S' be the set provided by condition 2. We can then construct the solution $U = S' \cup (\Delta \setminus S) \cup \{a\}$, which is feasible since $S \setminus S' \cup \{a\}$ contains no critical structures. Repeating this procedure for each $a \in \Delta \setminus S$ yields $|\Delta \setminus S|$ additional feasible solutions that are also on the face induced by (9). From the construction, it is easy to check all these solutions are also affinely independent.

For the necessity of condition 1, we assume some part $\mathcal{V}_i$ violates condition 1 and show that it is impossible to construct enough affinely independent solutions lying on the face induced by the inequality. Notice that affine independence and linear independence are equivalent in this case since the dimension of any face is strictly less than n and the zero solution is not feasible due to $\gamma_i > 0$. Let M be any $|\Delta|$ by $|\Delta|$ matrix that is composed by $|\Delta|$ feasible solutions (as columns) that satisfy (9) at equality, then we partition the rows of M into two submatrices M_1 and M_2 where the former corresponds to the elements in $\bigcup \mathcal{V}_i$. Clearly, we have $rank(M) \leq rank(M_1) + rank(M_2)$ as rank function is submodular. By assumption, $\bigcup \mathcal{V}_i$ violates condition 1, which implies $rank(M_1) < |\bigcup \mathcal{V}_i|$. Then, we have $rank(M) < |\bigcup \mathcal{V}_i| + |\Delta \setminus \bigcup \mathcal{V}_i| = |\Delta|$, proving the desired result.

Finally, we use the contrapositive to show that condition 2 is necessary if $\gamma_i = \gamma_S$, for all $i \in I$. Assume that the vitality inequality of S induces a facet of $\mathscr{P}$ and also that we can pick an element $a' \in \Delta \setminus S$ that violates condition 2. Then, it is easy to see that all the vital components of set $\bar{S} = S \cup \{a'\}$ must contain a'. This implies that the finest disjoint decomposition of $\mathcal{V}_{\bar{S}}$ is composed of only one part and also $\gamma_{\bar{S}} = \gamma_S + 1$ (as for any vital component of $S' \in \mathcal{V}_S$, the set $S' \cup \{a'\}$ is a vital component for $\bar{S}$). Therefore, the corresponding mixed vitality inequality for $\bar{S}$ is $\sum_{a\in\bar{S}} x_a \geq \gamma_S + 1$. This inequality along with $-x_{a'} \geq -1$ can be combined to generate the vitality inequality of S, which implies that the inequality does not induce a facet. □

We now discuss a procedure to verify in practice whether condition 1 is satisfied by any instance under some special considerations.

Proposition 5. *Given a set $S \subseteq \Delta$, its set of minimal vital components $m(\mathcal{V}_S)$, and the corresponding finest disjoint decomposition $\{\mathcal{V}_i\}_{i\in I}$, let $n_i = |\bigcup \mathcal{V}_i|$ and k_i be the number of elements in $\mathcal{V}_i$ of size γ_i. Furthermore, let M_i be $n_i \times k_i$ comprising the indicator vectors of the elements in $\mathcal{V}_i$. Then, if one of the following requirements is satisfied, then condition 1 in Theorem 5 is also satisfied.*

1. *Every subset of $\bigcup \mathcal{V}_i$ of size γ_i is in $\mathcal{V}_i$.*
2. *When $\gamma_i = 2$ and M_i is the incidence matrix of a connected non-bipartite graph (connected graph without even cycles). This is also a necessary condition when $\gamma_i = 2$.*
3. *When $\gamma_i = 2$ and the number of elements in $\mathcal{V}_i$ of size γ_i is strictly greater than*

$$\begin{cases} n_i^2/4, & \text{if } n_i \text{ is even,} \\ (n_i^2-1)/4, & \text{if } n_i \text{ is odd.} \end{cases}$$

Proof. Case 1 is trivial since we have enough elements to form a linearly independent square matrix. For Case 2, when $\gamma_i = 2$, M_i is the incidence matrix of a graph with n_i vertices and k_i edges. This graph is also connected since it is in the finest disjoint decomposition. Then, Case 2 is a direct result of Theorem 1 in [41], which states that the rows of the incidence matrix of a graph are linearly independent over the reals if and only if no connected component is bipartite. To prove Case 3, we calculate the maximum number of edges in a bipartite graph with n_i vertices, since having more edges implies the requirement of the aforementioned theorem from [41] will be satisfied. With n_i vertices, a bipartition will split the vertices into a set of cardinality x and another of cardinality $n_i - x$. Then, the maximum number of edges across two sets is $x(n-x)$. The maximizer of this function is equal to $n_i^2/4$ and $(n_i^2-1)/4$ for an even or odd n_i, respectively. ☐

4.4 Examples of Vitality Inequalities

In this section we provide several examples of vitality inequalities that can be used to strengthen the MCS formulation to solve some of the problems presented in previous sections. We will also show that the vitality inequalities either dominate or generalize other families that have been proposed in recent years [29, 30, 33, 43].

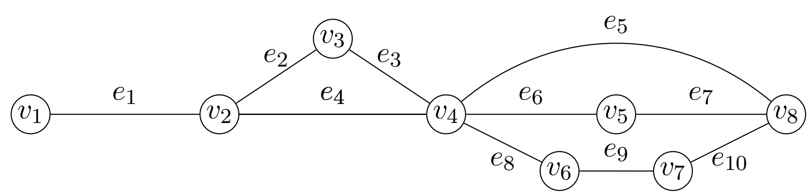

Figure 3: Example of a set $S \subseteq E$ to derive vitality inequalities for shortest path interdiction.

Example 1 (Shortest Path Interdiction, Cont.)**.** Figure 3 depicts a subgraph H of some graph G, where $s = v_1$ and $t = v_8$. Suppose all the paths in H are critical (6 paths in total), and let S be the edge set of H, then $m(\mathcal{V}_S)$ contains all the minimal (s,t)-cuts in H. That is, $m(\mathcal{V}_S) = \{\{e_1\}, \{e_2, e_4\}, \{e_3, e_4\}, \{e_5, e_6, e_8\}, \{e_5, e_6, e_9\}, \{e_5, e_6, e_{10}\}, \{e_5, e_7, e_8\}, \{e_5, e_7, e_9\}, \{e_5, e_6, e_{10}\}\}$. So, we have $\gamma_S = 1$ since $\{a_1\}$ is a minimal-sized vital component of S. As discussed in Section 4.2, the corresponding inequality (8) is $\sum_{e\in S} x_e \geq 1$, which is quite weak since it is dominated by any of the constraints (4) associated with a critical path in H.

Now, by definition, the disjoint vitality decomposition of $m(\mathcal{V}_S)$ comprises sets $\mathcal{V}_1 = \{\{e_1\}\}$, $\mathcal{V}_2 = \{\{e_2, e_4\}, \{e_3, e_4\}\}$, and $\mathcal{V}_3 = \{\{e_5, e_6, e_8\}, \{e_5, e_6, e_9\}, \{e_5, e_6, e_{10}\}, \{e_5, e_7, e_8\}, \{e_5, e_7, e_9\}, \{e_5, e_6, e_{10}\}\}$, where $\bigcup \mathcal{V}_1 = \{e_1\}$, with $\gamma_1 = 1$, $\bigcup \mathcal{V}_2 = \{e_2, e_3, e_4\}$ with $\gamma_2 = 2$, and $\bigcup \mathcal{V}_3 = \{e_5, e_6, \cdots, e_{10}\}$ with $\gamma_3 = 3$. This gives the following vitality inequality,

$$6x_1 + \sum_{e=2}^{4} 3x_e + \sum_{e=5}^{10} 2x_e \geq 6.$$

Now, suppose all the paths except $\{e_1, e_4, e_5\}$ and $\{e_1, e_2, e_3, e_5\}$ are critical, then $m(\mathcal{V}_S)$ is similar to the one of the previous example, with the exception that all sets in $\mathcal{V}_3$ do not contain edge e_5. Then, the corresponding mixed vitality inequality is,

$$2x_1 + \sum_{e \in \{2,\ldots,10\} \setminus \{5\}} x_e \geq 2.$$

These inequalities can be heuristically separated. ■

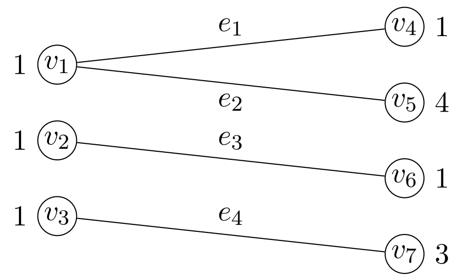

Figure 4: Example of a set $S \subseteq E$ to derive vitality inequalities for the minimum dominating set edge interdiction.

Example 2 (Minimum Dominating Set Edge Interdiction, Cont.)**.** Figure 4 depicts a subgraph $H(V, S)$ of some graph $G = (V, E)$ with $S \subseteq E$. Furthermore, assume that the number next to each vertex corresponds to its weight w_i, and that the target weight for the interdiction problem is $r = 7$. In this example, we will use edge set S to generate a vitality inequality.

There are multiple critical dominating sets in H. For example, the sets $T_1 = \{v_1, v_2, v_3\}$ with weight $w(T_1) = 3$, $T_2 = \{v_1, v_3, v_6\}$ with weight $w(T_2) = 3$, and $T_3 = \{v_1, v_3, v_6, v_7\}$ with weight $w(T_3) = 6$, among many others. It is easy to verify that the minimal vital components of S are $m(\mathcal{V}_S) = \{\{2\}, \{1, 4\}, \{3, 4\}\}$, hence $\gamma_S = 1$. Notice that, for example, set $\{e_4\}$ is not a vital component of S because after removing such an edge, the critical dominating set T_3 would remain in H. Since $\gamma_S = 1$, the corresponding inequality (8) is $\sum_{e \in S} x_e \geq 1$, which is weak as it is dominated by the constraint (4) associated with critical whip $\{e_1, e_2, e_4\}$.[6] Now, the disjoint vitality decomposition of $m(\mathcal{V}_T)$ is composed of the following parts $\{\mathcal{V}_1, \mathcal{V}_2\}$, where $\mathcal{V}_1 = \{\{2\}\}$ and $\mathcal{V}_2 = \{\{1, 3\}, \{3, 4\}\}$. Hence, the corresponding vitality inequality is,

$$x_1 + 2x_2 + x_3 + x_4 \geq 2. \tag{12}$$

These inequalities can be heuristically separated as well. ■

[6] This type of inequalities is used in [33].

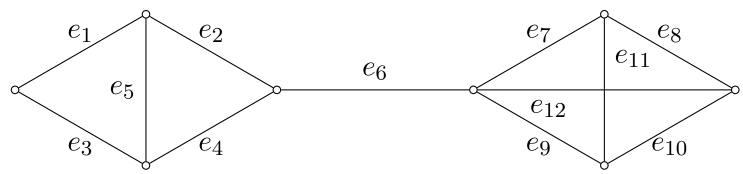

Figure 5: Example of a set $S \subseteq E$ to derive vitality inequalities for minimum spanning tree interdiction.

Example 3 (Minimum Spanning Tree Interdiction, Cont.)**.** In [43], the authors developed a class of inequalities induced by spanning cacti that are a special case of the vitality inequalities.[7] Here, we provide an example of a vitality inequality that is not associated with a cactus. Figure 5 depicts a subgraph $H(V, S)$ of some graph $G = (V, S)$ with $S \subseteq E$. Suppose all spanning trees in H are critical, then $m(\mathcal{V}_s) = \{\{e_1, e_3\}, \{e_2, e_4\}, \{e_1, e_2, e_5\}, \{e_3, e_4, e_5\}, \{e_1, e_4, e_5\}, \{e_2, e_3, e_5\}, \{e_6\}, \{e_7, e_8, e_{11}\}, \{e_9, e_{10}, e_{11}\}, \{e_7, e_9, e_{12}\}, \{e_8, e_{10}, e_{12}\}, \{e_7, e_{10}, e_{11}, e_{12}\}, \{e_8, e_9, e_{11}, e_{12}\}\}$. The disjoint vitality decomposition of $\mathcal{V}_S$ has three parts, where $\bigcup \mathcal{V}_1 = \{1, 2, \ldots, 5\}$, $\bigcup \mathcal{V}_2 = \{6\}$, and $\bigcup \mathcal{V}_3 = \{7, 8, \ldots, 12\}$, with $\gamma_1 = 2$, $\gamma_2 = 1$, and $\gamma_3 = 3$, respectively. This produces the following vitality inequality,

$$\sum_{a=1}^{5} 3x_a + 6x_6 + \sum_{a=7}^{12} 2x_a \geq 6.$$

These inequalities can be separated in as similar fashion as in [43]. ■

We now conclude this section with a brief discussion about the separation process of the vitality inequalities as well as some basic properties of those for the shortest path interdiction problem.

4.5 Further Discussion

The process of separating vitality inequalities depends on the instance being solved, so instead of providing a generic separation method, we discuss an idea that seems to work well in practice for several problems. There are three steps to separate a vitality inequality from any given subset S: (1) determine the set of vital components $\mathcal{V}_S$; (2) identify the finest disjoint decomposition $\{\mathcal{V}_i\}_{i \in I}$ of $\mathcal{V}_S$; (3) construct the vitality inequality. The last two steps are computationally easier than the first since (2) can be done by sequentially merging parts in $\mathcal{V}_S$ and (3) is a direct application of Definition 11. Thus, the main difficulty of the separation routine resides in Step 1, and consequently on the selection of set S.

The vitality inequalities can be generated for any choice of $S \in \uparrow \hat{\Omega}$. However, the larger the set S is, the more difficult the separation process becomes. For example, at the extreme case where $S = \Delta$, generating the corresponding inequality requires knowing all the feasible strategies for the leader—including the optimal one—which renders the process of identifying such an inequality rather useless. Furthermore, as S grows, it becomes less likely for the inequality to induce a facet of $\mathscr{P}$, as condition 1 of Theorem 4 becomes more difficult to be satisfied (see Proposition 5). Therefore, a good practice seems to be to select a set S that is relatively small. This not only makes the separation process easier, but also increases the chances of it making an impact on the algorithm's performance.

One possible strategy that often works well is to begin with a minimal critical structure $S = T \in \hat{\Omega}$ and then progressively add to S a few extra elements from $\Delta \setminus T$ until the minimum vitality

[7] A cactus is defined as is a connected graph in which every edge belongs to at most one cycle. The graph from Figure 2 is an example of a cactus.

decomposition has at least one part $\mathcal{V}_i$ with $\gamma_i > 1$. If successful, one may try to find an additional inequality by adding a few more elements until there is a new part with a bigger γ_i. It is easy to see that when $S = T$, the set of minimum vital components $m(\mathcal{V}_S)$ is composed of all $a \in S$, as by the definition of the critical structures, removing any of those directly blocks structure T (see Definition 9). Additionally, since all the vital components are of size one, the resulting vitality inequality is exactly the same constraint (4) associated with T. Now, as pointed out before, adding a few extra elements, $S = T \cup \{a_1, \dots, a_k\}$, for some small k may lead to some minimum vital component in $m(\mathcal{V}_S)$ whose size is two or more, thus yielding a non-trivial vitality inequality.

Consider for instance the example depicted in Figure 2 for the minimum spanning tree interdiction problem. There, from subgraph H, we can extract set $T = \{e_1, e_2, e_4, e_5, e_6, e_6, e_7\}$, which corresponds to the edges a critical spanning tree. Notice that by selecting instead set $S = T \cup \{e_3, e_8\}$ after adding only two edges, it was possible to identify a more complex vitality inequality (11), which is in fact facet inducing according to Theorem 4. This type of basic exploration procedure is relatively easy to implement.

A similar idea can be observed from the example depicted in Figure 4 associated with the minimum dominating set edge interdiction problem. There, we can extract set $T = \{e_1, e_2, e_4\}$, which corresponds to a minimal critical whip associated with dominating set $\{v_1, v_2, v_3, v_6\}$. Notice that by selecting instead $S = T \cup \{e_3\}$, after adding only one edge, it was possible to identify a more complex vitality inequality (12).

For the case of the shortest path interdiction, one might be inclined to apply a similar procedure and begin with a critical path $S = T \in \hat{\Omega}$ and keep adding edges to S until $m(\mathcal{V}_S)$ contains some vital component of size two. Despite being a valid procedure to identify vitality inequalities for this problem, the resulting inequality may not be as strong as one may expect.

Proposition 6. *For the shortest path interdiction problem described in Example 1, let $S \in \uparrow \hat{\Omega}$ and $\mathcal{V}_S$ be its set of vital components. If the finest disjoint decomposition $\{\mathcal{V}_i\}_{i \in I}$ of $\mathcal{V}_S$ satisfies $\gamma_i \leq 2$ for all $i \in I$, then the corresponding vitality inequality* (9) *is facet-defining if and only if there is no part $\mathcal{V}_i$ with $\gamma_i = 2$.*

Proof. The sufficiency should be clear because if $\gamma_i = 1$ for all $i \in I$, then by Lemma 2, S is composed of a unique critical path in $\hat{\Omega}$. Thus, the corresponding vitality inequality (9) of S reduces to a critical structures inequality (4) and conditions 1 and 2 in Theorem 5 is trivially satisfied.

To proof necessity, if there is at least one part $\mathcal{V}_i$ of the finest disjoint decomposition for which $\gamma_i = 2$, then S contains a substructure with two parallel subpaths, say P_1 and P_2, each contained in at least one different critical path. Clearly, this substructure $P_1 \cup P_2$ will induce a part $\tilde{\mathcal{V}}_i$ in the finest disjoint decomposition. By definition, a minimal vital component contained in $\tilde{\mathcal{V}}_i$ needs to block all the critical paths contained in S, thus, the members of $\tilde{\mathcal{V}}_i$ are always of the form $\{e_1, e_2\}$ for some $e_1 \in P_1$ and some $e_2 \in P_2$. It can be easily verified that $\tilde{\mathcal{V}}_i$ does not satisfy the requirements of Proposition 5. Thus, Condition 1 of Theorem 5 cannot be satisfied. □

We note that this theorem does not necessarily imply that separating vitality inequalities for the shortest path interdiction problem is useless. Inequalities of this type may not induce a facet, however they may still beneficial to cut off fractional solutions. Moreover, the theorem does not rule out the case of a set S for which there are parts in the finest disjoint decomposition of $\mathcal{V}_S$ with $\gamma_i > 2$.

This concludes our developments related to the vitality inequalities.

5 Conclusion

In this paper we studied a general type binary interdiction games that characterizes a wide range of adversarial optimization problems. We studied a set-covering type formulation that can be used for solving binary interdiction games whose elements are defined over one or two ground sets. We analyzed some properties of the convex hull of feasible solutions for the problem and developed a class of valid inequalities that generalizes other families that have appeared in the literature in recent years. We also provided conditions for them to be facet-defining and concluded with general a discussion about their separation. Several examples of problems in the context of network interdiction were presented to help with the exposition.

Some interesting possibilities for future work include identifying other families of valid and supervalid inequalities to strengthen the proposed formulation. Also, one may be inclined to generalize our results for interdiction problems that do not necessarily fit the binary classification introduced in this paper. For example, cases where the leader strategies affect the follower's objective function, instead of the set of available structures.

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