

# Total Coloring and Total Matching: Polyhedra and Facets

Luca Ferrarini, Stefano Gualandi

*Università di Pavia, Dipartimento di Matematica “F. Casorati”*

---

## Abstract

A total coloring of a graph  $G = (V, E)$  is an assignment of colors to vertices and edges such that neither two adjacent vertices nor two incident edges get the same color, and, for each edge, the end-points and the edge itself receive different colors. Any valid total coloring induces a partition of the elements of  $G$  into total matchings, which are defined as subsets of vertices and edges that can take the same color. In this paper, we propose Integer Linear Programming models for both the Total Coloring and the Total Matching problems, and we study the strength of the corresponding Linear Programming relaxations. The total coloring is formulated as the problem of finding the minimum number of total matchings that cover all the graph elements, and we prove that this relaxation is tighter than a natural assignment model. This covering formulation can be solved by a column generation algorithm, where the pricing subproblem corresponds to the Weighted Total Matching Problem. Hence, we study the Total Matching Polytope. We introduce two families of nontrivial valid inequalities: congruent- $2k3$  cycle inequalities based on the parity of the vertex set induced by the cycle, and clique inequalities induced by complete subgraphs of even order. We prove that congruent- $2k3$  cycle inequalities are facet-defining only when  $k = 4$ , while the even cliques are always facet-defining. Since the separation problem of the clique inequalities of even order is NP-hard, we get a polyhedral proof of the NP-hardness of the Weighted Total Matching Problem.

*Keywords:* Integer Programming, Combinatorial Optimization, Total Coloring, Total Matching

---

## 1. Introduction

Let us consider a simple and undirected graph  $G = (V, E)$  and let  $D = V \cup E$  be the set of its elements. We say that a pair of elements  $a, b \in D$  are *adjacent* if  $a$  and  $b$  are adjacent vertices, or if they are incident edges, or if  $a$  is an edge incident to a vertex  $b$ . If two elements  $a, b \in D$  are not adjacent, they are *independent*. A *matching* is a subset of edges  $M \subseteq E$  such that  $e \cap f = \emptyset$  for all  $e, f \in M$  with  $e \neq f$ . A matching is called *perfect* if it covers all vertices, that is, has size  $\frac{1}{2}|V|$ .

---

*Email addresses:* [l.ferrarini3@campus.unimib.it](mailto:l.ferrarini3@campus.unimib.it) (Luca Ferrarini), [stefano.gualandi@unipv.it](mailto:stefano.gualandi@unipv.it) (Stefano Gualandi)

We define  $\nu(G) := \max\{|M| : M \text{ is a matching}\}$ . Given a set of colors  $K = \{1, \dots, k\}$ , a  $k$ -total coloring of  $G$  is an assignment  $\phi : D \rightarrow K$  such that  $\phi(a) \neq \phi(b)$  for every pair of adjacent elements  $a, b \in D$ . Each subset of elements assigned to the same color by  $\phi$  defines a *total matching*, that is, a subset  $T \subseteq D$  where the elements are pairwise independent. Hence, a  $k$ -total coloring induces a partition of the elements in  $D$  into  $k$  disjoint total matchings. The minimum value of  $k$  such that  $G$  admits a  $k$ -total coloring is called the total chromatic number, and it is denoted by  $\chi_T(G)$ . A total matching of maximum cardinality is denoted by  $\nu_T(G)$ .

The Total Coloring Problem consists of finding  $\chi_T(G)$ . It is an NP-hard problem [38], which is studied mainly in graph theory [43] for the conjecture attributed independently to Vizing [42] and Behzad [3] that relates  $\chi_T(G)$  to the maximum degree  $\Delta(G)$  of the nodes in  $G$ . The conjecture states that  $\chi_T(G) \leq \Delta(G) + 2$ . Note that  $\Delta(G) + 1$  is a trivial lower bound on  $\chi_T(G)$ . Hence, if the conjecture were true, we would be left with the NP-Complete problem of deciding whether  $\chi_T(G) = \Delta(G) + 1$ . While the conjecture is valid for specific classes of graphs (e.g., see [41]), the conjecture is still open for general graphs<sup>1</sup>. The Total Coloring Problem generalizes both the Vertex Coloring Problem, where we have to color only the vertices of  $G$ , and the Edge Coloring Problem, where instead we have to color only the edges. The Vertex Coloring Problem belongs to the list of 21 NP-hard problems introduced by E. Karp in [18], and it was tackled in the literature by many exact polyhedral approaches (for a recent survey, see [28]).

The most efficient exact approaches to the Vertex Coloring Problem are based on set covering formulations [30, 10, 27, 14], where every single set of the covering represents a subset of vertices taking the same color, corresponding hence to a (maximal) stable set of  $G$ . Similarly, the best polyhedral approach to the Edge Coloring Problem is based on a set covering formulation, where the edges are covered by (maximal) matchings of  $G$  [33, 19]. An interesting alternative formulation for the Edge Coloring Problem is based on a different ILP model based on a binary encoding of the problem variables [20]. While in the literature there are other interesting approaches to graph coloring problems (e.g., branch-and-cut [31], semidefinite programming [17], decision diagrams [40], constraint satisfiability [13], memetic algorithms [24]), and other interesting types of coloring problems (e.g., equitable coloring [22, 4], graph multicoloring [11], sum coloring [5], selective graph coloring [6]), in this work, we focus on a polyhedral approach to the Total Coloring Problem.

A problem related to the total coloring of a graph is the Total Matching Problem, where we look for a set of vertices and edges which together yield an independent set. The Total Matching generalizes both the Matching Problem, where we look for an independent set of edges [8], and the Stable Set Problem, where instead we look for an independent set of vertices [36, 35]. The first work on the Total Matching Problem appeared in [1], where the authors derive lower and upper bounds on the size of a maximum total matching. Despite the strong connection with the Matching Problem, the Total Matching Problem is less studied in the operations research literature.

---

<sup>1</sup>When submitting this paper, we have found a paper on arxiv claiming a proof for Vizing's conjecture [32].

In particular, significant results are obtained only for structured graphs, as cycles, paths, full binary trees, hypercubes, and complete graphs, see [21]. This work presents a first polyhedral study of the Total Matching Problem deriving several facet-defining inequalities for its polytope.

The Total Coloring Problem has several practical applications, for instance, in Match Scheduling [16], Network Task Efficiency, and Math Art [21]. As an example of match scheduling, consider the martial art tournament problem, which can be modeled using a *tournament graph*  $G = (V, E)$  and a set of colors  $K$  defined as follows. We add a vertex  $i$  to  $V$  for each player, and an edge  $\{i, j\}$  to  $E$  for each match. Then, for each time period of the tournament, we introduce a color in  $K$ . The assignment of a color  $k \in K$  to an edge  $\{i, j\}$  represents the scheduled time period of the match between players  $i$  and  $j$ . The assignment of a color  $k \in K$  to a vertex  $i$  represents a rest time for player  $i$  during the time period associated with color  $k$ . Given this graph formulation, no pair of incident edges get the same color because no player can be in two matches at once; no vertex can be incident to an edge with the same color as the vertex because no player should have a match during his rest time; no pair of adjacent vertices should get the same color because no two matched players can leave the stage simultaneously. Hence, a proper total coloring of the tournament graph represents a feasible scheduling of the tournament, and the total chromatic number represents the minimum number of time periods to schedule the tournament.

*Our contributions.* The main results of this paper are:

- A set covering formulation of the Total Coloring Problem based on maximal total matchings, which can be solved by column generation. Motivated by the pricing subproblem, we introduce the Maximum Weighted Total Matching Problem.
- The definition of two families of nontrivial valid inequalities that we call the *congruent-2k3 cycle inequality*, which are based on the parity of the cycle, and the *even clique inequality*, which are based on complete graphs of even cardinality. We prove that congruent-2k3 cycle inequalities are facet-defining only when  $k = 4$  (see Proposition 13), while the even cliques (but not the odd clique) are always facet-defining (see Theorem 1).
- A polyhedral proof of the NP-hardness of the Weighted Total Matching Problem.

*Outline.* The outline of this paper is as follows. In the next paragraph, we fix the notation. In Section 2, we present two Integer Linear Programming (ILP) formulations of the Total Coloring Problem, and we introduce the Maximum Weighted Total Matching Problem. In Section 3, we study the Total Matching Polytope, proposing several facet-defining inequalities. In Section 4, we present two families of nontrivial valid inequalities: cycle inequalities based on the parity of the vertex set induced by the cycle, and clique inequalities induced by complete subgraphs of even order. Using the complexity of the separation problem for even clique inequalities, we provide a polyhedral proof of the NP-hardness of the Weighted Total Matching Problem. In Section 5, we conclude the paper with a discussion on future works.

*Notation.* The graphs considered in this paper are simple and undirected. Given a graph  $G = (V, E)$ , we define  $n = |V|$  and  $m = |E|$ . For a vertex  $v \in V$ , we denote by  $\delta(v)$  the set of edges incident to  $v$  and by  $N_G(v)$  the set of vertices adjacent to  $v$ . The degree of a vertex is  $|\delta(v)|$ , in particular, we denote by  $\Delta(G) = \max\{|\delta(v)| \mid v \in V\}$ . For a subset of vertices  $U \subseteq V$ , let  $G[U]$  be the subgraph induced by  $U$  on  $G$ . We define  $\delta(U) := \{e \in E \mid e = \{u, v\}, u \in U, v \in V \setminus U\}$ .

## 2. Total Coloring and Total Matching Models: ILP models

In this section, we first present an assignment Integer Linear Programming (ILP) model for the Total Coloring Problem. Second, we introduce a stronger set covering formulation based on the idea of covering the elements of the graph by the minimum number of maximal total matchings. Third, we introduce the Weighted Total Matching Problem.

### 2.1. Total Coloring: Assignment model

Let  $G = (V, E)$  be a graph and let  $K$  be the set of available colors, with  $|K| \geq \Delta(G) + 1$ . We introduce binary variables  $x_{vk} \in \{0, 1\}$  for every vertex  $v$  and binary variables  $y_{ek} \in \{0, 1\}$  for every edge  $e$  to denote whether they get assigned color  $k$ . Besides, we introduce the binary variables  $z_k$  to indicate whether any element uses color  $k$ . Using these variables, our assignment ILP model for the Total Coloring Problem is as follows.

$$\chi_T(G) := z_{IP}^{(A)} = \min \sum_{k \in K} z_k \quad (1)$$

$$\text{s.t.} \quad \sum_{k \in K} x_{vk} = 1 \quad \forall v \in V \quad (2)$$

$$\sum_{k \in K} y_{ek} = 1 \quad \forall e \in E \quad (3)$$

$$x_{vk} + \sum_{e \in \delta(v)} y_{ek} \leq z_k \quad \forall v \in V, \forall k \in K \quad (4)$$

$$x_{vk} + x_{wk} + y_{ek} \leq z_k \quad \forall e \in E, \forall k \in K \quad (5)$$

$$x_{vk} \in \{0, 1\} \quad \forall v \in V, \forall k \in K \quad (6)$$

$$y_{ek} \in \{0, 1\} \quad \forall e \in E, \forall k \in K. \quad (7)$$

The objective function (1) minimizes the number of used colors. Constraints (2)–(3) ensure that every vertex and every edge get assigned a color. Constraint (4) enforces that all edges  $e$  incident to a vertex  $v$ , and the vertex  $v$  itself, take a different color; at the same time, the constraints guarantee that the corresponding variable  $z_k$  is set to 1 whenever color  $k$  is used by at least an element of  $G$ . Constraint (5) imposes that for each edge  $e = \{i, j\}$  at most one element among  $\{e, i, j\}$  can take color  $k$ , and it sets the corresponding variable  $z_k$  accordingly. If we relax the integrality constraints

(6) and (7), we get a Linear Programming relaxation. We denote the optimal value of the LP relaxation by  $z_{LP}^{(A)}$ .

The LP relaxation of model (1)–(7) yields the following lower bound.

**Proposition 1.** *Let  $G = (V, E)$  be a graph. Then, we have  $\chi_T(G) \geq z_{LP}^{(A)} \geq \Delta + 1$ .*

**Proof:** Let  $x_{vk} = y_{ek} = \frac{1}{\Delta+1}$  for  $k = 1, \dots, \Delta + 1$  and  $z_k = 1$  for  $k = 1, 2, \dots, \Delta + 1$ . Notice that this assignment gives a feasible solution for the LP relaxation of (1)–(7). Since  $|K| \geq \Delta + 1$ , the assertion follows immediately.  $\square$

## 2.2. Total Coloring: Set Covering model

The assignment model (1)–(7) is easy to write, but it suffers from symmetry issues: any permutation of the color classes indexed by  $k$  generates the same optimal solution [29, 15]. To overcome this issue and to get a stronger LP lower bound, we introduce a set covering formulation based on maximal total matchings. A total matching is (inclusion-wise) maximal if it is not a subset of any other total matching. Note that the number of maximal total matchings in a graph is strictly less than the number of total matchings.

Let  $\mathcal{T}$  be the set of all maximal total matchings of  $G$ . Let  $\lambda_t$  be a binary decision variable indicating if the matching  $t \subset \mathcal{T}$  is selected (or not) for representing a color class. The 0–1 parameter  $a_{vt}$  indicates if vertex  $v$  is contained in the total matching  $t$ . Similarly, the 0–1 parameter  $b_{et} = 1$  indicates if edge  $e$  is contained in the total matching  $t$ . The following set covering model is a valid formulation for the Total Coloring Problem.

$$\chi_T(G) = z_{IP}^{(C)} := \min \sum_{t \in \mathcal{T}} \lambda_t \quad (8)$$

$$\text{s.t.} \quad \sum_{t \in \mathcal{T}} a_{vt} \lambda_t \geq 1 \quad \forall v \in V \quad (9)$$

$$\sum_{t \in \mathcal{T}} b_{et} \lambda_t \geq 1 \quad \forall e \in E \quad (10)$$

$$\lambda_t \in \{0, 1\} \quad \forall t \in \mathcal{T}. \quad (11)$$

Given an optimal solution of the previous problem, whenever an element of  $G$  appears in  $t > 1$  maximal total matchings, it is always possible to recover a proper total coloring by removing the element from  $t - 1$  of those total matchings. Note that the covering model has an exponential number of variables, one for each maximal total matching in  $G$ . We denote by  $z_{LP}^{(C)}$  the optimum value of the LP relaxation of problem (8)–(11).

If we introduce the dual variables  $\alpha_v$  for constraints (9) and the variables  $\beta_e$  for constraints

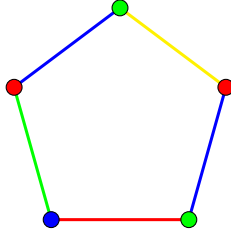


Figure 1: A total coloring of a cycle of length 5 with  $k = 4 = \Delta(G) + 2$  colors. The optimal value of the LP relaxation of the assignment model (1)–(7) is equal to  $z_{LP}^{(A)} = 3$ , while the optimal value of the LP relaxation of the set covering model (8)–(11) is equal to  $z_{LP}^{(C)} = \frac{10}{3}$ .

(10), we can write the dual of the set covering LP relaxation as follows.

$$z_{LP}^{(C)} := \min \sum_{t \in \mathcal{T}} \lambda_t \quad \text{s.t. (9)–(10), } \lambda_t \geq 0, \forall t \in \mathcal{T} \quad (\text{Primal}) \quad (12)$$

$$= \max \sum_{v \in V} \alpha_v + \sum_{e \in E} \beta_e \quad (\text{Dual}) \quad (13)$$

$$\text{s.t. } \sum_{v \in V} a_{vt} \alpha_v + \sum_{e \in E} b_{et} \beta_e \leq 1 \quad \forall t \in \mathcal{T} \quad (14)$$

$$\alpha_v, \beta_e \geq 0 \quad \forall v \in V, \forall e \in E. \quad (15)$$

For this LP covering relaxation, the following proposition holds.

**Proposition 2.** *Let  $G = (V, E)$  be a graph. Then, we have  $\chi_T(G) \geq z_{LP}^{(C)} \geq \Delta(G) + 1$ .*

**Proof:** Consider a vertex  $v$  of maximum degree, and let  $\Delta(G) = k$ , where  $N_G(v) := \{v_1, \dots, v_k\}$  and  $\delta(v) := \{e_1, \dots, e_k\}$ . Consider the total matching  $T_0 := \{v\}$ , and the additional  $k$  distinct total matchings  $T_i := \{v_i, e_{i+1}\}$  for all  $i = 1, \dots, k - 1$  and  $T_k := \{v_k, e_1\}$ . Hence, we have  $k + 1$  total matchings, which can be used to define a feasible dual solution: we set  $\alpha_v = 1$ ,  $\alpha_{v_i} = \beta_{e_{i+1}} = \frac{1}{2}$  for all  $i = 1, \dots, k - 1$  and  $\alpha_{v_k} = \beta_{e_1} = \frac{1}{2}$ . Thus, summing up all these dual values in the dual objective function, we get the valid lower bound result  $z_{LP}^{(S)} \geq \Delta(G) + 1$ .  $\square$

The example in Figure 1 shows that the optimal value of the LP relaxation of the set covering model can be tighter than the value of the LP assignment relaxation. Next, we prove that the LP covering relaxation always provides a lower bound at least as strong as that of the LP assignment relaxation. Our proof uses the equivalence of the set covering relaxation  $z_{LP}^{(C)}$  with a set partitioning relaxation, where the inequality constraints (9)–(10) are replaced with equality constraints. Herein, we denote by  $z_{LP}^{(P)}$  the optimal value of the LP partitioning relaxation. The proof of the following result is straightforward.

**Lemma 1.**  $z_{LP}^{(C)} = z_{LP}^{(P)}$ .

We are now ready to prove the following proposition.

**Proposition 3.** *Let  $G = (V, E)$  be a graph. Then, we have  $\chi_T(G) \geq z_{LP}^{(C)} \geq z_{LP}^{(A)} \geq \Delta + 1$ .*

**Proof:** It suffices to prove that the set covering model (8)–(11) can be obtained by applying the Dantzig-Wolfe reformulation of the assignment model (1)–(7). We exploit the block structure of the constraint matrix. First, we group the variables by a fixed color  $k \in K$ . Let  $\mathbf{u}_k := (x_{v_1k}, x_{v_2k}, \dots, x_{v_nk}, y_{e_1k}, y_{e_2k}, \dots, y_{e_mk}, z_k)$  be the vector associated to the decision variable of the assignment model and let  $\mathbf{z} := (z_1, \dots, z_k)$ . We notice that the constraint matrix has the following block structure:

$$\begin{aligned}
\min \quad & \mathbf{1}^T \mathbf{z} \\
& A_1 \mathbf{u}_1 + A_2 \mathbf{u}_2 + \dots + A_{|K|} \mathbf{u}_{|K|} = \mathbf{1} \\
& B_1 \mathbf{u}_1 \leq \mathbf{0} \\
\text{s.t.} \quad & B_2 \mathbf{u}_2 \leq \mathbf{0} \\
& \vdots \\
& B_{|K|} \mathbf{u}_{|K|} \leq \mathbf{0} \\
& \mathbf{u}_k \in \{0, 1\}^{n+m+1}, \forall k = 1, 2, \dots, |K|
\end{aligned}$$

The corresponding sub-block matrices can be written as:

$$A_j = \left[ \begin{array}{c|c} I_{v,j} & \mathbf{0}_{n \times (m+1)} \\ \hline \mathbf{0}_{m \times n} & I_{e,j} \end{array} \right], \quad B_j = \left[ \begin{array}{c|c} I_v & B_{v,j} \\ \hline B_{e,j} & I_e \end{array} \right].$$

The blocks  $A_j$  for  $j = 1, \dots, |K|$  correspond to the constraint matrices of (2)–(3), where  $I_{v,j}$  is the identity matrix relative to the vertex components, and  $I_{e,j}$  is the identity matrix relative to the edge components with one more column with all zeros. The blocks  $B_j$  for  $j = 1, \dots, |K|$  correspond to constraints (4)–(5), where  $B_{v,j}$  and  $B_{e,j}$  are the edge-vertex incidence matrix and the vertex-edge incidence matrix respectively, both with one more column of all minus ones indicating the color  $j$ . Notice that the blocks  $A_1 = A_2 = \dots = A_{|K|}$  and  $B_1 = B_2 = \dots = B_{|K|}$  are identical, since they are incidence matrices of the same graph. Now, define  $P_t := \{\mathbf{w}_t \in \{0, 1\}^{n+m+1} \mid B_t \mathbf{w}_t \leq \mathbf{0}\}$  for  $t = 1, \dots, |K|$ . Thus, for a fixed  $k \in K$ , we can express  $\mathbf{u}_t = \sum_{j \in P_k} \lambda_j^t \mathbf{w}_j$  such that  $\sum_{j \in P_t} \lambda_j^t = 1$ , where the variables  $\lambda_j^t$  for  $j = 1, \dots, |P_t|$  correspond to the convexity coefficients with respect to the points of  $\text{conv}(P_t)$ . In order to guarantee the integrality of the solution and to select exactly one of the feasible solution, we impose that  $\lambda_j^t \in \{0, 1\}$  for  $j = 1, \dots, |K|$ . Since we cannot distinguish between colors and the blocks  $B_j$  are the same, we have the same feasible regions, and thus, we can define  $P := P_1 = P_2 = \dots = P_{|K|}$ . Let  $A_{v,j}$  be the upper block matrix corresponding to the vertex components of  $A_j$  and  $A_{e,j}$  be the below block matrix corresponding to the edge components, then

we can rewrite the model as:

$$\min \sum_{j=1}^{|K|} \sum_{t \in P} \lambda_t^j \mathbf{w}_t \quad (16)$$

$$\text{s.t.} \quad \sum_{j=1}^{|K|} \sum_{t \in P} (A_{v,j} \mathbf{w}_t) \lambda_t^j = \mathbf{1} \quad (17)$$

$$\sum_{j=1}^{|K|} \sum_{t \in P} (A_{e,j} \mathbf{w}_t) \lambda_t^j = \mathbf{1} \quad (18)$$

$$\sum_{t \in P} \lambda_t^j = 1 \quad \forall j \in K \quad (19)$$

$$\lambda_t^j \in \{0, 1\} \quad \forall t \in P, \forall j \in K. \quad (20)$$

Notice that  $A_{v,j} \mathbf{w}_t = A_{v,t}$  and  $A_{e,j} \mathbf{w}_t = A_{e,t}$ , where  $A_{v,t} = (a_{vt}, \mathbf{0}_m)_{v \in V}$  and  $A_{e,t} = (\mathbf{0}_n, a_{et})_{e \in E}$  are characteristic vectors of a total matching  $t$ , since each element belonging to the same color class corresponds to a total matching. From the previous observation, we can replace for every  $t \in P$ ,  $\lambda_t := \sum_{j=1}^{|K|} \lambda_t^j$ . In addition, since we can select at most one total matching  $t \in P$  for every color class  $j \in K$ , we replace constraint (19) with  $\lambda_t \in \{0, 1\}$ . The final integer program with the Dantzig-Wolfe reformulation becomes:

$$\min \sum_{t \in P} \lambda_t \quad (21)$$

$$\text{s.t.} \quad \sum_{t \in P} a_{vt} \lambda_t = 1 \quad \forall v \in V \quad (22)$$

$$\sum_{t \in P} a_{et} \lambda_t = 1 \quad \forall e \in E \quad (23)$$

$$\lambda_t \in \{0, 1\} \quad \forall t \in P, \quad (24)$$

where  $P$  represents the set of all possible total matchings. □

### 2.3. Column generation and Weighted Total Matchings

We can solve problem (12), or equivalently its dual (13)–(15), by considering a subset  $\bar{\mathcal{T}} \subset \mathcal{T}$ , and by applying a column generation algorithm, where looking for a primal negative reduced cost variables corresponds to look for a violated dual constraint [25, 2, 7, 12]. Given a dual feasible solution  $\bar{\alpha}$  and  $\bar{\beta}$ , the separation problem of the dual constraints (14) reduces to the following



Maximum Weighted Total Matching.

$$\nu_T(G, \bar{\alpha}, \bar{\beta}) := \max \sum_{v \in V} \bar{\alpha}_v x_v + \sum_{e \in E} \bar{\beta}_e y_e \quad (25)$$

$$\text{s.t.} \quad \sum_{e \in \delta(v)} y_e \leq 1 - x_v \quad \forall v \in V \quad (26)$$

$$x_v + x_w \leq 1 - y_e \quad \forall e = \{v, w\} \in E \quad (27)$$

$$x_v, y_e \in \{0, 1\} \quad \forall v \in V, \forall e \in E. \quad (28)$$

Note that constraints (26) and (27) together define the valid constraints for total matchings of  $G$ . In addition, whenever the optimal value  $\nu_T(G, \bar{\alpha}, \bar{\beta}) > 1$ , the corresponding total matching gives a violated constraint (14). That is, problem (25)–(28) is the pricing subproblem for solving our set covering model by column generation.

Motivated by the solution of the pricing subproblem (25)–(28), in the next section, we study valid (facet) inequalities of the Total Matching Polytope.

### 3. Facet inequalities for the Total Matching Polytope

In this section, we study the feasible region of the Maximum Weighted Total Matching Problem (25)–(28), and we provide facet-defining inequalities for the corresponding polytope. The most important original contribution of this paper is given in Theorem 1, where we prove that the even clique inequalities are facet-defining for the Total Matching Polytope.

#### 3.1. Total Matching Polytope

The Total Matching Polytope is defined as the *convex hull* of characteristic vectors of total matchings. Hence, given a total matching  $T$ , the corresponding characteristic vector is defined as follows.

$$\chi[T] = \begin{cases} z_a = 1 & \text{if } a \in T \subseteq D = V \cup E, \\ z_a = 0 & \text{otherwise.} \end{cases}$$

where  $z = (x, y) \in \{0, 1\}^{n+m}$ ,  $x$  corresponds to the vertex variables and  $y$  to the edges variables.

**Definition 1.** *The Total Matching Polytope of a graph  $G = (V, E)$  is defined as:*

$$P_T(G) := \text{conv}\{\chi[T] \subseteq \mathbb{R}^{n+m} \mid T \subseteq D = V \cup E \text{ is a total matching}\}.$$

**Proposition 4.**  *$P_T(G)$  has the following valid inequalities:*

$$\sum_{e \in \delta(v)} y_e \leq 1 - x_v \quad \forall v \in V \quad (29)$$

$$x_v + x_w \leq 1 - y_e \quad \forall e = \{v, w\} \in E \quad (30)$$

$$x_v, y_e \geq 0 \quad \forall v \in V, \forall e \in E. \quad (31)$$

**Proof:** In a total matching, by definition, we can take for each vertex  $v$  at most one edge incident to  $v$  or the vertex itself (constraints (29)). For every edge  $e = \{v, w\}$ , a total matching contains at most one element among  $e, v, w$  (constraints (30)). Clearly, the variables must be nonnegative.  $\square$

The following proposition implies that the valid inequalities that are facet-defining are nonredundant, and, hence, they represent a minimal system defining  $P_T(G)$ .

**Proposition 5.**  $P_T(G)$  is full-dimensional, that is,  $\dim(P_T(G)) = n + m$ .

**Proof:** We have that the origin, the unit vectors  $\chi[\{v\}]$  for every  $v \in V$  and  $\chi[\{e\}]$  for every  $e \in E$  belong to  $P_T(G)$ , and clearly they are linearly independent. Thus, we have  $n + m + 1$  affinely independent points.  $\square$

We establish now an important connection between total matchings of a graph and the stable sets of the *adjoint graph*, defined as follows. Consider the graph  $G$  and its corresponding line graph  $L(G)$ , that is, the graph obtained from  $G$  having one vertex for each edge  $e \in E(G)$ , and where two vertices are linked by an edge if the corresponding edges in  $G$  are incident to the same vertex in  $G$ . Starting from the line graph, we construct a new graph  $H = (V \cup V(L(G)), E(L(G)) \cup E')$ , that we will call the *line-full graph*, where  $E'$  is the set of edges connecting the vertices of  $L(G)$  to vertices of  $G$ , if and only if  $v \in V(L(G))$  is an edge of  $G$ . We call a *doubling* of an edge the operation that adds an edge between a pair of vertices.

**Definition 2.** Let  $G$  be a graph and  $H$  its corresponding line-full graph. The graph  $W$  obtained from  $H$  applying a doubling of an edge for every pair of vertices  $\{v, w\} \in V(H) \setminus V(L(G))$  such that  $e = \{v, w\} \in E(G)$ , is called the **adjoint graph** of  $G$ .

Note that in the line graph, if  $|\delta(v)| = l$ , then we have a corresponding clique  $K_l$ . In addition, by doubling the edges, we can create triangles in the adjoint graph. Hence, as shown in Figure 2, the adjoint graph can be described as the union of cliques  $K_3$  and general cliques. We can prove that total matchings of  $G$  correspond to stable sets of its adjoint graph  $W$ . In the following, we denote as  $P_{stable}$  the Stable Set Polytope.

**Proposition 6.** Let  $G$  be a graph and  $W$  its adjoint. Then,  $P_T(G) = P_{stable}(W)$ .

**Proof:** The characteristic vectors of the stable sets of  $W$  correspond to the characteristic vectors of total matchings of  $G$ , and, hence, the vertices of  $P_{stable}(W)$  are the vertices of  $P_T(G)$ .  $\square$

### 3.2. Perfect Total Matchings

A total matching is *perfect* if every vertex of the graph is covered by a total matching, that is, every vertex is either in the total matching or one of its incident edges belongs to the total matching. We prove next, that for any graph  $G$ , we can always find a perfect total matching.

**Proposition 7.** Every graph  $G$  has a perfect total matching.

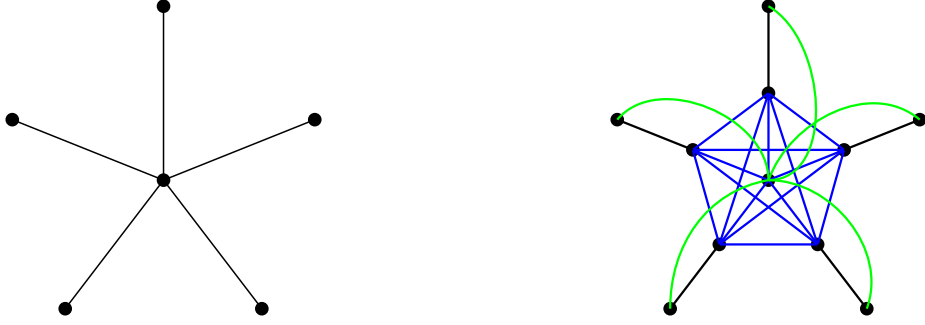


Figure 2: A star graph on the left and the corresponding adjoint graph on the right.

**Proof:** If  $G$  has a perfect matching, it is trivial. Otherwise, let us suppose that  $G$  has no perfect matching. Given a subset of vertices  $S \subseteq V$ , let  $k$  be the number of odd components of  $G$  that is, the number of maximal connected components of odd order. We denote the odd components as  $O_1, O_2, \dots, O_k$ . By applying the Tutte's theorem [39, 23], we have  $k > |S|$ . Notice that, since the maximum size of a matching in an odd component is  $\frac{|V(O_i)|-1}{2}$  for  $i = 1, 2, \dots, k$ , there is a vertex that is not covered by a matching, we call it a left-out vertex. Instead, we have a perfect matching  $N$  that covers all the vertices in the even components. Now, let  $T$  be a total matching of  $G$ . We show how to construct  $T$  so that all the vertices of  $G$  are covered by  $T$ . First, for each odd component we can construct a maximum matching  $M_i$  of size  $\frac{|V(O_i)|-1}{2}$  for every  $i = 1, 2, \dots, k$ , in which we choose as a left-out vertex one of the vertices connecting an odd component to  $S$ . Let  $v_i$  be the left-out vertex by  $M_i$  of the component  $O_i$  for  $i = 1, 2, \dots, k$ . Now, take one edge of  $|S|$  odd components connecting  $v_i$  to the set  $S$  and consider the matching  $S_O := \{e = \{v_i, s_i\} \mid s_i \in V(S)\}$  for  $i = 1, 2, \dots, |S|\}$ . Since  $k > |S|$ , for each of the remaining components, we have a left-out vertex that cannot be covered by a matching  $M_i$  and in particular, in order to form an independent set of elements, we cannot choose an edge connecting  $S$  to the odd component. Consider the set  $L$  of these vertices and define  $T := M_1 \cup M_2 \cdots \cup M_k \cup S_O \cup L \cup N$ . Since every vertex is covered by  $T$  by construction, the assertion follows.  $\square$

The previous proposition allows us to define the Perfect Total Matching Polytope. Let  $P_{PT}(G)$  be the convex hull of all perfect total matchings of  $G$ .

**Proposition 8.** *Let  $G$  be a graph. The following inequalities are valid for  $P_{PT}(G)$ .*

$$\sum_{e \in \delta(v)} y_e = 1 - x_v \quad \forall v \in V \quad (32)$$

$$x_v + x_w = 1 - y_e \quad \forall e = \{v, w\} \in E \quad (33)$$

$$x_v, y_e \geq 0 \quad \forall v \in V, \forall e \in E. \quad (34)$$

In practice, for any perfect total matching, the inequalities describing the feasible region of total matchings are all tight.

### 3.3. Facet-defining inequalities

In the following paragraphs, we prove that the valid inequalities describing the feasible region of the Total Matching Polytope are all facet-defining.

**Proposition 9.** *Let  $G$  be a graph. Then, the inequality*

$$\sum_{e \in \delta(v)} y_e \leq 1 - x_v \quad \forall v \in V, \quad (35)$$

*is facet-defining for the Total Matching Polytope  $P_T(G)$ .*

**Proof:** Fix a vertex  $v \in V$  with  $|\delta(v)| = k$ , and let  $\delta(v) := \{e_1, \dots, e_k\}$  and  $N_G(v) := \{v_1, \dots, v_k\}$ . Let  $F := \{z \in P_T(G) \mid \lambda^T z = \lambda_0\}$  be the face corresponding to the inequality (35). We have to exhibit  $n + m$  affinely independent points satisfying (35) at equality. Consider the unit vector  $x$  with  $x_v = 1$ , and zero the other entries, and the unit vectors with  $y_{e_i} = 1$  for  $i \in \{1, \dots, k\}$ , and zero the other components. These  $k + 1$  vectors belong to  $F$  and they are linearly independent. Thus, they are also affinely independent. Now, pick the vertex  $v$  together with an edge  $e \notin \delta(v)$ . This is a total matching, and, in addition, we can form  $m - k$  distinct total matchings, whose characteristic vectors are linearly independent and so affinely independent. Indeed, consider the set  $S := \{\chi[\{v\}] + \chi[\{e\}] \mid \forall e \notin \delta(v)\}$ . The matrix having as columns these vectors has rank  $m - k$ , and together with the previous vectors, we have  $m + 1$  affinely independent points. Then, if we take the vertex  $v$  together with  $w \notin N_G(v)$ , we can build  $n - (k + 1)$  distinct total matchings, and the corresponding points are linearly independent. Finally, note that  $T_i = \{v_i, e_{i+1}\}$  for  $i = 1, \dots, k - 1$  and  $T_k = \{v_k, e_1\}$  are distinct total matchings, and the corresponding vectors  $\chi[T_i]$  for  $i = 1, \dots, k$  are linearly independent, since the matrix having as columns these vectors forms the identity matrix relative to the vertex components of  $N_G(v)$ . This completes the proof, since we have found  $n + m$  affinely independent points.  $\square$

**Proposition 10.** *The inequality*

$$x_v + x_w \leq 1 - y_e \quad \forall e = \{v, w\} \in E, \quad (36)$$

*is facet-defining for the Total Matching Polytope  $P_T(G)$ .*

**Proof:** Consider an edge  $e = \{v, w\}$ . Let  $|\delta(v)| = k$  and  $|\delta(w)| = l$  with  $N_G(v) := \{v_1, \dots, v_k\}$  and  $N_G(w) := \{w_1, \dots, w_l\}$ . Denote also  $\delta(v) := \{e_{v_1}, \dots, e_{v_k}\}$  and  $\delta(w) := \{e_{w_1}, \dots, e_{w_l}\}$ . We have to exhibit  $n + m$  affinely independent points belonging to the face  $F := \{z \in P_T(G) \mid x_v + x_w = 1 - y_e\}$ . Consider the set  $S_u := \{z \in P_T(G) \mid y_e = 1, x_u = 1, u \notin e, z_i = 0, \forall i \in (V \setminus \{u\}) \cup (E \setminus \{e\})\}$ . Notice that  $S_u \subseteq F$ . Thus, we can construct  $n - 2$  distinct total matchings and together with the two unit vectors  $\chi[\{v\}]$  and  $\chi[\{w\}]$ , we can show  $n$  affinely independent points. Since the matrix with columns the characteristic vectors of these total matchings are linearly independent, we have the

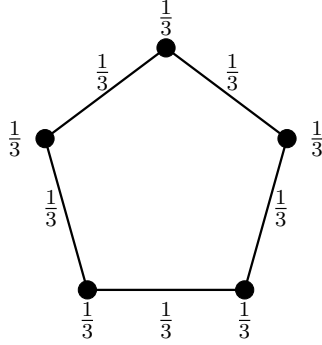


Figure 3: A vertex  $\mathbf{x} = \frac{1}{3}\mathbf{1}$  of the cycle  $C_5$ .

identity matrix  $I_v$  relative to the vertex components, so they are also affinely independent. Now, consider the vector  $z \in \mathbb{R}^{n+m}$  with  $y_e = 1$ , and zero the other components. Clearly,  $z$  belongs to  $F$ . Then, take the vertex  $w$  and consider an edge  $f \notin \delta(w)$ , and note that  $T_f = \{w, f\}$  is a total matching. In this way, we can build  $m - l$  distinct total matchings. In addition, take the vertex  $v$  and consider an edge  $f' \in \delta(w)$ ,  $f' \neq e$ , also in this case  $T_{f'} = \{v, f'\}$  is a total matching. Note that we can form  $l - 1$  distinct total matchings of this type excluding the edge  $e$ . Since the matrix having as columns these  $m$  vectors is the identity matrix  $I_e$  relative to the edge components, the corresponding  $m$  characteristic vectors are linearly independent. Hence, we have found  $n + m$  affinely independent points satisfying the inequality at equality.  $\square$

**Proposition 11.** *The inequalities*

$$x_v \geq 0, \quad \forall v \in V, \quad \text{and} \quad y_e \geq 0, \quad \forall e \in E,$$

are facet-defining for the Total Matching Polytope  $P_T(G)$ .

**Proof:** Fix an edge  $e \in E$  and let  $F := \{z \in P_T(G) \mid y_e = 0\}$ . Consider the unit vectors  $\chi[\{v\}]$  for every  $v \in V$  and  $\chi[\{f\}]$  for every  $f \neq e$ . The matrix  $A$  with columns these vectors is the identity matrix  $I$  of order  $(n \times m) \times (n \times m - 1)$ , thus it has  $\text{rank}(A) = n + m - 1$  and so  $\dim(F) = n + m - 1$ . Similarly, fix a vertex  $v \in V$  and consider  $\hat{F} := \{z \in P_T(G) \mid x_v = 0\}$ . The vectors  $\chi[\{u\}]$  for  $u \in V, u \neq v$  and  $\chi[\{e\}]$  for  $e \in E$  are  $n + m - 1$  linearly independent vectors and are contained in  $\hat{F}$ . This completes the proof.  $\square$

In the following section, we introduce nontrivial facet-defining inequalities for the Total Matching Polytope.

#### 4. Congruent- $2k3$ cycle and even clique inequalities

In the previous section, we have proved that all the inequalities defining the feasible region of the Total Matching Problem are facet-defining. In this section, we introduce two families of nontrivial valid inequalities.

#### 4.1. Congruent-2k3 cycle inequalities

The inequalities (29)–(31) define the feasible region of total matchings and they are facet-defining, but they do not describe the complete convex hull of the Total Matching Polytope. For instance, Figure 3 shows that using only those inequalities, we have that for a cycle  $C$  of length 5, the point  $z_a = \frac{1}{3}$  for all  $a \in V(C) \cup E(C)$  belongs to  $P_T(C)$  and it is a vertex. However, in [21], the authors show that the cardinality of a maximum total matching in a cycle of cardinality  $k \in \mathbb{N}$  is equal to  $\lfloor \frac{2k}{3} \rfloor$ . Thus, we introduce an inequality that cuts off these nonintegral solutions for cycles, which we call the **congruent-2k3 cycle inequality**.

**Proposition 12.** *Let  $C_k$  be an induced cycle. Then, if  $k \equiv 1 \pmod{3}$  or  $k \equiv 2 \pmod{3}$ , the congruent-2k3 cycle inequality defined as*

$$\sum_{v \in V(C_k)} x_v + \sum_{e \in E(C_k)} y_e \leq \left\lfloor \frac{2k}{3} \right\rfloor \quad (37)$$

is facet-defining for  $P_T(C_k)$ .

**Proof:** Let  $F := \{z \in P_T(C_k) \mid \lambda^T z = \lambda_0\}$  be a facet of  $P_T(C_k)$  such that  $\tilde{F} := \{z \in P_T(C_k) \mid \tilde{\lambda}^T z = \tilde{\lambda}_0\} \subseteq F$  where the inequality  $\tilde{\lambda}^T z \leq \tilde{\lambda}_0$  corresponds to the inequality (37). We want to prove that there exists  $a \in \mathbb{R}$  such that  $\lambda = a\tilde{\lambda}$  and  $\lambda_0 = a\tilde{\lambda}_0$ . We distinguish two cases based on the parity of the cycle. We label the vertices  $V(C_k) := \{v_0, \dots, v_{k-1}\}$ , so that  $v_i$  is adjacent to  $v_{i-1}$  for  $i = 0, 1, \dots, k-1 \pmod{k}$ , and the edges  $E(C_k) := \{e_0, \dots, e_{k-1}\}$ , so that  $e_i = \{v_i, v_{i+1}\}$  for  $i = 0, 1, \dots, k-1 \pmod{k}$ .

**Case 1:** ( $k \equiv 1 \pmod{3}$ ). Consider the total matching  $T_0 := \{v_i, e_{i+1} \mid 0 \leq i \leq k-4, \text{ for } i \equiv 0 \pmod{3}\}$ . This is a maximal total matching, since every element in  $T_0$  is mutually nonadjacent. The number of elements of  $T_0$  is twice the numbers of integers  $i$  satisfying the condition, that is,  $|T_0| = \frac{2(k-1)}{3}$ , and, hence,  $\chi[T_0] \in \tilde{F}$  and, in particular,  $\chi[T_0] \in F$ . Note that the set  $\{v_{k-2}, e_{k-2}\}$  is not contained in  $T_0$ , because of our description of  $T_0$ . Now, consider the total matchings  $T_0^- := (T_0 \setminus \{e_{k-3}\}) \cup \{v_{k-2}\}$  and  $T_0^+ := (T_0 \setminus \{e_{k-3}\}) \cup \{e_{k-2}\}$ . In this way, we obtain two distinct total matchings with the same cardinality, whose characteristic vectors belong to  $\tilde{F}$ . Since  $\chi[T_0^+] \in F$  and  $\chi[T_0^-] \in F$ , then  $\lambda^T \chi[T_0] = \lambda^T \chi[T_0^+]$  and  $\lambda^T \chi[T_0] = \lambda^T \chi[T_0^-]$ , thus  $\lambda_{e_{k-3}} = \lambda_{v_{k-2}} = \lambda_{e_{k-2}}$ , where  $\lambda_{v_i}$  is the cost coefficient corresponding to the vertex  $v_i$  and  $\lambda_{e_i}$  is the coefficient relative to the edge  $e_i = \{v_i, v_{i+1}\}$ . Now, consider the function  $\sigma : C \rightarrow C$  such that  $\sigma(v_i) = v_{i+1}$  and  $\sigma(e_i) = e_{i+1}$ . Indeed,  $\sigma$  shifts every element to the next position with respect to the ordering of the vertices and the edges. Composing  $k-1$  times the shifting function on  $T_0$ , we obtain the following total matchings  $\sigma(T_0), \sigma^2(T_0), \dots, \sigma^{k-1}(T_0)$ . For a fixed  $i$ , denote  $\sigma^i(T_0) := T_i$ . These are still total matchings and each characteristic vector  $\chi[T_i] \in \tilde{F}$ . Notice also that  $T_i$  does not contain  $\{v_{i-2}, e_{i-2}\}$ , for  $i = 1$  the corresponding set is  $\{v_{k-1}, e_{k-1}\}$ . So, following the same previous procedure, we deduce that  $\lambda_{e_{i-3}} = \lambda_{v_{i-2}} = \lambda_{e_{i-2}}$  for all  $i = 1, \dots, k-1 \pmod{k}$ . This implies that there exists  $a \in \mathbb{R}$  such that  $\lambda = a\mathbf{1}$ . Then, since  $\chi[T_i] \in \tilde{F}$ , we have that  $\lambda^T \chi[T_i] = a(\mathbf{1}^T \chi[T_i]) = a\tilde{\lambda}_0$ . We conclude that, since

$(\lambda, \lambda_0) = a(\mathbf{1}, \tilde{\lambda}_0)$ ,  $\lambda^T z \leq \lambda_0$  is a scalar multiple of the cycle inequality.

**Case 2:** ( $k \equiv 2 \pmod{3}$ ). Consider the total matching  $T_0 := \{v_i, e_{i+1} \mid 0 \leq i \leq k-5, \text{ for } i \equiv 0 \pmod{3}\} \cup \{v_{k-2}\}$ . Notice that now  $e_{k-2} \notin T_0$ . Also in this case  $\chi[T_0] \in \tilde{F}$ , since  $|T_0| = \frac{2(k-2)}{3} + 1 = \lfloor \frac{2k}{3} \rfloor$ . We can construct other two total matchings with the same cardinality  $\hat{T}_0 := (T_0 \setminus \{v_{k-2}\}) \cup \{e_{k-2}\}$  and  $\tilde{T}_0 := (\hat{T}_0 \setminus \{e_{k-4}\}) \cup \{v_{k-3}\}$ . Note that  $\chi[\hat{T}_0], \chi[\tilde{T}_0] \in \tilde{F}$ , and so they also lie in  $F$ . Thus,  $\lambda^T \chi[\hat{T}_0] = \lambda^T \chi[T_0]$  and  $\lambda^T \chi[\tilde{T}_0] = \lambda^T \chi[T_0]$ . From the first equality, we deduce that  $\lambda_{v_{k-2}} = \lambda_{e_{k-2}}$  and for the second one,  $\lambda_{v_{k-3}} = \lambda_{e_{k-4}}$ . We conclude as in the Case 1 by applying the shifting function  $\sigma$ , so we have a scalar multiple of the cycle inequality.  $\square$

In particular, we notice that when the induced cycle is  $C_4$ , then the corresponding cycle inequality is facet-defining for the Total Matching Polytope of the entire graph.

**Proposition 13.** *Let  $G$  be a graph and let  $C_4$  be the induced cycle of four vertices. Then, the inequality:*

$$\sum_{v \in V(C_4)} x_v + \sum_{e \in E(C_4)} y_e \leq 2 \quad (38)$$

*is facet-defining for  $P_T(G)$ .*

**Proof:** Denote by  $\tilde{F}$  the face induced by the inequality (38). Suppose by contradiction that  $\tilde{F}$  is contained in  $F := \{z \in P_T(G) \mid \lambda^T z = \lambda_0\}$ . By proposition (12), the corresponding inequality inducing the face  $F$  has the form  $a(\sum_{v \in V(C_4)} x_v + \sum_{e \in E(C_4)} y_e) + \sum_{l \notin C_4} \lambda_l^T z_l \leq 2a$  for  $a \in \mathbb{R}$ . Denote as  $V(C_4) := \{v_0, v_1, v_2, v_3\}$  and  $E(C_4) := \{e_{i,i+1} = \{v_i, v_{i+1}\} \mid i = 0, 1, 2, 3 \pmod{4}\}$ . Consider the matching  $M := \{e_{0,1}, e_{2,3}\}$ , then the corresponding characteristic vector lies on  $\tilde{F}$ . Since  $M \cap \{u\} = \emptyset$  for every  $u \notin V(C_4)$ ,  $T_u := M \cup \{u\}$  is a total matching whose characteristic vector lies on  $\tilde{F}$ . This implies that  $\lambda_u = 0$  for every  $u \notin V(C_4)$ . Similarly,  $M \cap \{e\} = \emptyset$  for every  $e \notin \delta(V(C_4)) \cup E(C_4)$ , so  $T_e := M \cup \{e\}$  is a total matching whose characteristic vector lies on  $\tilde{F}$ . This implies that  $\lambda_e = 0$  for every  $e \notin \delta(V(C_4)) \cup E(C_4)$ . Now, let  $S := \{v \in V(C_4) \mid \delta(V(C_4)) \neq \emptyset\}$ . Fix a vertex  $v_i \in S$ , and consider the total matching  $T_{v_i} := \{e_{i+1,i+2}, v_{i+3}\} \cup \{e_{v_i}\}$ , where  $i$  is taken modulo 4 and  $e_{v_i} \in \delta(V(K_h)) \cap \delta(v_i)$ . It is easy to see that the characteristic vector of  $T_{v_i}$  lies on  $\tilde{F}$ , in particular exactly one edge  $e \in \delta(v_i) \cap \delta(V(C_4))$  is chosen, so  $\lambda_{e_{v_i}} = 0$  for every  $e_{v_i} \in \delta(v_i) \cap \delta(V(C_4))$ . In this way, repeating the same argument for all  $v \in S$ , we obtain that  $\lambda_{e_{v_k}} = 0$  for every  $e_{v_k} \in \delta(V(K_h))$ . This completes the proof since we have proved that  $\lambda_l = 0$  for all  $l \notin C_4$ .  $\square$

*Separation of congruent- $2k3$  cycle inequalities.* In this paragraph, we deal with the problem of separating the facet-defining inequalities given by the class of the congruent- $2k3$  cycle inequalities. Given a fractional optimal solution of the LP relaxation of the pricing subproblem, the separation for the congruent- $2k3$  cycle inequalities consists of either finding an inequality in this class that is violated by a cycle inequality or proving that all inequalities are satisfied. To this end, we propose an Integer Linear Programming formulation for solving this separation problem.

Let  $(c_v, w_e)$  denote the fractional optimal solution to the current LP problem, and let  $x_v$  and  $y_e$  denote the decision variables of the problem of finding a congruent- $2k3$  cycle in a graph  $G$ . The separation problem consists of maximizing the following value

$$\alpha := \sum_{v \in V} c_v x_v + \sum_{e \in E} w_e y_e - \left\lfloor \frac{2k}{3} \right\rfloor, \quad (39)$$

where  $k$  is the cardinality of the cycle induced by the variables  $x_v$  and  $y_e$ . Thus, we want to detect a maximum weighted cycle, where node and edge weights are  $(c_v, w_e)$ , and the cycle contains a number of nodes that is not a multiple of three. Whenever  $\alpha > 0$ , we have a violated cycle. Otherwise all the congruent- $2k3$  cycle inequalities are satisfied. Since  $k \equiv 1, 2 \pmod{3}$ , we can express  $k = 3z + t$  where  $z \in \mathbb{Z}$  and  $t \in \{1, 2\}$ , and we can rewrite the floor expression in (39) as follows

$$\left\lfloor \frac{2k}{3} \right\rfloor = \left\lfloor \frac{2(3z + t)}{3} \right\rfloor = \begin{cases} 2z & \text{if } t = 1 \\ 2z + 1 & \text{if } t = 2, \end{cases}$$

and, hence, we get

$$\left\lfloor \frac{2k}{3} \right\rfloor = 2z + t - 1. \quad (40)$$

Another important element of our ILP model for the separation of congruent- $2k3$  cycle inequalities is the connectivity constraints, which we formulate exploiting the ideas presented in [26], by setting a network flow model. Given the original graph  $G = (V, E)$  the flow networks is defined as  $H = (V, A)$ , where  $A := \bigcup_{\{i,j\} \in E} \{(i, j), (j, i)\}$ . The network  $H$  has a single source node that introduces all the flow, while every node that belongs to the cycle is the sink of a single unit of flow. However, we do not fix in advance the source node, and we let variables  $s_i \in \{0, 1\}$  for  $i = 1, \dots, n$  to indicate which node of  $H$  is the source. Then, we introduce the variables  $u_i \in \mathbb{Z}_+$  for every vertex  $v_i \in V$  to indicate the overall amount of flow originated at the only source node  $i$  having  $s_i = 1$ . Indeed, we have that  $u_i > 0$  only for the sink node. The complete ILP model for the separation of congruent- $2k3$  cycle inequalities is the following:



$$\max \sum_{v \in V} c_v x_v + \sum_{e \in E} w_e y_e - (2z + t - 1) \quad (41)$$

$$\text{s.t.} \quad \sum_{e \in \delta(v)} y_e = 2x_v \quad \forall v \in V \quad (42)$$

$$\sum_{v \in V} x_v = 3z + t \quad \forall v \in V, \forall e \in E \quad (43)$$

$$x_i + \sum_{(i,j) \in A} f_{ij} = u_i + \sum_{(j,i) \in A} f_{ji} \quad \forall i \in V \quad (44)$$

$$\sum_{i=1}^n s_i = 1 \quad \forall i \in V \quad (45)$$

$$u_i \leq n \cdot s_i \quad \forall i \in V \quad (46)$$

$$f_{ij} \leq n \cdot y_e \quad \forall (i,j) \in A \quad (47)$$

$$y_e, x_v \in \{0, 1\} \quad \forall v \in V, \forall e \in E \quad (48)$$

$$u_i \in \mathbb{Z}_+ \quad \forall i \in V \quad (49)$$

$$z \in \mathbb{Z}_+, t \in \{1, 2\}. \quad (50)$$

The objective function (41) includes the relation specified in (40). Constraints (42) ensure that the subgraph induced by the variables  $x_v$  and  $y_e$  is a union of disjoint cycles, since every node has either degree zero or two. Constraints (43) impose the congruence on the length of the cycle, which cannot be a multiple of three. Constraints (44) impose the flow conservation at every node, and constraints (45) impose that a single vertex is the origin of the flow. Constraints (46) impose that all the vertices but the source have  $u_i = 0$ , that is, they do not originate any unit of flow. For every flow variable  $f_{ij}$ , constraints (47) set the capacity of the flow variables to zero whenever  $y_e = 0$ , that is, whenever arc  $e$  is not included in the cycle.

The ILP model (41)–(50) permits us to look for the most violated congruent- $2k3$  cycle inequality by solving a single problem. Alternatively, we could solve a simplified version of the separation problem by fixing in advance both the source node  $s_i$  and the value of variable  $t$ . In this way, to find the most violated inequality, we have to solve two (easier) subproblems for every node, for a total of  $6n$  subproblems. However, each subproblem reduces to a Shortest Path Problem defined on an auxiliary directed graph having nonnegative weights, as shown in the proof of the following proposition.

**Proposition 14.** *The separation problem of the congruent- $2k3$  cycle inequality is in  $P$ .*

**Proof:** The separation problem consists of a sequence of  $2n$  Minimum Weighted  $s, t$ -Path problems from a source node  $s$  to the target node  $t$  of an auxiliary graph. Let  $G = (V, E)$  be a weighted graph where  $(c_v, w_e)$  are the optimal values of the current LP relaxation. Starting from  $G = (V, E)$ , we

construct a weighted directed graph  $H = (N, A)$  in the following way. For every vertex  $v \in V$ , we introduce three nodes labelled as  $v_0, v_1, v_2$  in  $N$ . Now, for each edge  $e = \{v, w\} \in E$ , we introduce three arcs  $a_i \in A$  with respect to the permutation  $\sigma = (012)$ , that is,  $a_i = (v_i, w_{\sigma(i)})$ , with  $i = 0, 1, 2$ . Observe that a path from  $v_0$  to  $v_1$  gives a path  $P_k$  of size  $k \equiv 1 \pmod{3}$ , and a path from  $v_0$  to  $v_2$  gives a path  $P_k$  of size  $k \equiv 2 \pmod{3}$ . Next, we distinguish the two cases.

**Case 1:** ( $k \equiv 1 \pmod{3}$ ). In this case, we have  $\lfloor \frac{2k}{3} \rfloor = \frac{2(k-1)}{3}$ , and the separation problem reads as follows:

$$\exists C_k : \frac{2}{3}|C_k| - \sum_{v \in V(C_k)} c_v x_v - \sum_{e \in E(C_k)} w_e x_e < \frac{2}{3}.$$

Since we look for the most violated inequality, for each node  $v \in V$ , the separation problem is equivalent to a Minimum Weighted  $s, t$ -Path Problem where the source is  $v_0$  and the target is  $v_1$ . Now, we define the costs on the arcs as  $l_{a=(i,j)} := \frac{2}{3} - c_i - w_{e=\{i,j\}} + 1$ , for every  $a = (i, j) \in A$ . We know that  $c_i + c_j + w_{e=\{i,j\}} \leq 1$  due to feasibility of constraints (27) and, hence, the costs are positive. Let  $P_1$  be a minimum weighted path in  $H$  from  $v_0$  to  $v_1$ . By construction, the path  $P_1$  in  $H$  corresponds to a cycle  $C_k$  in  $G$  of length  $k \equiv 1 \pmod{3}$ , where for each node  $v_i \in N$  we consider the corresponding node  $v \in V$ . If we sum up all the costs on the path  $P_1$ , we obtain:

$$l(P_1) := \sum_{(i,j) \in A(P_1)} l_{(i,j)} = \frac{2}{3}|P_1| - \sum_{i \in V(C_k)} c_i - \sum_{e \in E(C_k)} w_e + |P_1|. \quad (51)$$

Hence, the path  $P_1$  yields a violated congruent- $2k3$  cycle  $C_k$  in  $G$  if and only if  $l(P_1) - |P_1| < \frac{2}{3}$ .

**Case 2:** ( $k \equiv 2 \pmod{3}$ ). In this case, we have  $\lfloor \frac{2k}{3} \rfloor = \frac{2(k-2)}{3}$ , and the separation problem reads as follows:

$$\exists C_k : \frac{2}{3}|C_k| - \sum_{v \in V(C_k)} c_v x_v - \sum_{e \in E(C_k)} w_e x_e < \frac{4}{3}.$$

Hence, we have to find a minimum weighted path  $P_2$  from  $v_0$  to  $v_2$  for each node  $v$  in  $V$ . We define the arc costs  $l_a$  as before, and we get a maximum violated cycle if and only if  $l(P_2) - |P_2| < \frac{4}{3}$ .

In conclusion, by solving  $2n$  shortest path problems on a directed graph with positive weights, we get the most violated congruent- $2k3$  cycle inequalities in polynomial time.  $\square$

#### 4.2. Even and odd clique inequalities

At this point, we focus on valid inequalities that can be derived by complete subgraphs  $K_h$  of  $G$ , with  $h \leq n$ . This leads to consider the following valid inequality.

**Proposition 15.** *Let  $G$  be a graph, and let  $K_h$  a clique of order  $h \leq n$  of  $G$ . Then,*

$$\sum_{v \in V(K_h)} x_v + \sum_{e \in E(K_h)} y_e \leq \left\lceil \frac{h}{2} \right\rceil \quad (52)$$

*is a valid inequality for  $P_T(G)$ .*

First, it is useful to notice that the result obtained by Padberg in [34] for a maximal clique can be extended also for the Total Matching Polytope.

**Proposition 16.** *Let  $G$  be a graph and let  $K_h$  be a maximal clique of  $G$ . Then, the inequality*

$$\sum_{v \in V(K_h)} x_v \leq 1 \quad (53)$$

*is facet-defining for  $P_T(G)$ .*

**Proof:** Let  $G$  be a graph and let  $K_h \subseteq G$  be a maximal clique. We have to exhibit  $n + m$  affinely independent points which belong to the face  $F := \{z \in P_T(G) \mid \sum_{v \in V(K_h)} x_v = 1\}$ . We know that, since  $K_h$  is maximal, by Theorem 2.4 in [34], we can easily construct  $n$  of such points belonging to  $F$ . Now, fix a vertex  $v \in V(K_h)$  and denote  $N_G(v) \cap V(K_h) := \{v_0, v_1, \dots, v_{h-2}\}$  and  $\delta(v) \cap E(K_h) := \{e_i = \{v, v_i\} \mid i = 0, 1, \dots, h-2\}$ . Define the total matching  $T_v^{\bar{e}} := \{v, \bar{e}\}$ , where  $\bar{e} \notin \delta(v)$  and notice that  $\chi[T_v^{\bar{e}}] \in F$ . Thus, we can construct the set of 0-1 vectors  $\{\chi[T_v^{\bar{e}}] \mid \forall e \notin \delta(v)\} \subseteq F$ . It is easy to see that the corresponding characteristic vectors are affinely independent, so up to now we have found  $|E \setminus \delta(v)|$ . Then, fix a vertex  $w \in V(K_h)$  with  $w \neq v$ , and consider the set of total matchings  $T_w^e := \{\{w, e\} \mid \forall e \in \delta(V(K_h)) \cap \delta(v)\}$ . By construction, the set of the corresponding characteristic vectors of  $T_w^e$  is contained in  $F$ . Finally, the vectors  $\chi[T_{v_i, e_{i+1}}]$  for  $i = 0, 1, \dots, h-2 \pmod{h-1}$  with one in entry  $x_{v_i}$  and  $y_{e_{i+1}}$  and zero the other components, are characteristic vectors lying on  $F$  and are affinely independent, thus we have  $|E|$  affinely independent points. We have found in total  $n + m$  affinely independent points, since the matrix having the columns the characteristic vectors found assumes the following form:

$$\left[ \begin{array}{c|c} A_v & B \\ \hline \mathbf{0} & I_e \end{array} \right],$$

where  $A_v$  represents the vertex components of the  $n$  points and  $B$  the characteristic vectors of total matchings relative to a fixed vertex in the clique and exactly one edge not belonging to the clique itself. This completes the proof.  $\square$

In particular, when the subgraph  $K_h$  has even cardinality, we get the following result.

**Proposition 17.** *Let  $K_h$  be a complete graph, where  $h \in \mathbb{N}$  is an even number. Then, the **even clique inequality** defined as*

$$\sum_{v \in V(K_h)} x_v + \sum_{e \in E(K_h)} y_e \leq \frac{h}{2} \quad (54)$$

is facet-defining for the Total Matching Polytope  $P_T(K_h)$ .

**Proof:** Let  $G = K_h$  be a complete graph, where  $h = 2l$  for  $l \in \mathbb{N}$  and let  $V(K_h) := \{v_1, v_2, \dots, v_{2l}\}$  and  $E(K_h) := \{e_{i,j} = \{v_i, v_j\} \mid \forall i, j \in \{1, 2, \dots, 2l\}, i \neq j\}$ . First, we show that the even clique inequalities are valid for  $P_T(G)$ . Since  $K_h$  is a complete graph of even order, it admits a perfect matching  $M$ . Notice that any stable set  $S$  intersects  $K_h$  in at most one vertex, thus a maximum total matching  $T$  can be obtained by a perfect matching, or by deleting from a perfect matching an edge  $e = \{i, j\}$  and adding one of its endpoints. This implies that  $|T| \leq l$ . Next, we prove that the face induced by an even clique inequality is facet-defining. To this end, consider a face  $F := \{z \in P_T(G) \mid \lambda^T z = \lambda_0\}$  and let  $F' := \{z \in P_T(G) \mid \tilde{\lambda}^T z = \tilde{\lambda}_0\}$ , where  $\tilde{\lambda}^T z \leq \tilde{\lambda}_0$  corresponds to the even clique inequality. Suppose that  $F' \subseteq F$ , we want to show that every inequality of  $F$  is a scalar multiple of the even clique inequality. Place the vertices  $v_1, v_2, \dots, v_{2l-1}$  at equal distances on a circle and place  $v_{2l}$  in the center. Starting from this configuration, we show a decomposition of  $K_h$  into disjoint union of perfect matchings, such that  $E(K_h) = M_1 \cup M_2 \cup \dots \cup M_{h-1}$ . Notice also that a perfect matching  $M$  can be naturally identified as a total matching. Now, fix an index  $i$  and consider the edge that connects a vertex  $v_i$  to the center  $v_{2l}$  of the circle, we call  $c_i = \{v_i, v_{2l}\}$  the *central edge*, and consider the set of edges  $E_i := \{e_{i+j, i-j} = \{v_{i+j}, v_{i-j}\} \mid \forall j \in \{1, \dots, \frac{h}{2} - 1\}\}$ , where the indexes run modulo  $h - 1$ . It turns out that  $M_i := E_i \cup \{c_i\}$  is a perfect matching. In this way, repeating the same construction we can form  $h - 1$  distinct perfect matchings  $M_i$ , with  $\chi[M_i] \in F'$ , for all  $i \in \{1, 2, \dots, 2l - 1\}$ . Now, we can construct a total matching with the same cardinality of the perfect matchings just constructed. Consider an edge  $e = \{v_j, v_k\} \in M_i$  of a fixed perfect matching  $M_i$ . Then,  $T_k := (M_i \setminus \{e_{j,k}\}) \cup \{v_k\}$  and  $T_j := (M_i \setminus \{e_{j,k}\}) \cup \{v_j\}$  are total matchings. Observe that  $\chi[T_j] \in F'$  and  $\chi[T_k] \in F'$ , in particular these characteristic vectors lie on  $F$ . This implies that  $\lambda_{v_j} = \lambda_{v_k} = \lambda_{e_{j,k}}$ , since  $\lambda^T \chi[T_j] = \lambda^T \chi[T_k] = \lambda^T \chi[M_i]$ , where we denote as  $\lambda_a$  the cost coefficient for the element  $a \in D = V \cup E$ . In particular, we apply this construction for all the edges of the same perfect matching  $M_i$ . Repeating the same argument for all the perfect matchings in the decomposition, we obtain that  $\lambda_v = \lambda_e$  for  $e \in \delta(v)$ ,  $\forall v \in V$ , and since the cost coefficients for the endpoints of each edge are the same by construction, and we consider only perfect matchings (we can touch each vertex), we deduce that there exists  $a \in \mathbb{R}$  such that  $\lambda = a\mathbf{1}$ . Thus, this implies that  $\lambda_0 = a \frac{h}{2}$ . We conclude that  $\lambda^T z \leq \lambda_0$  is a scalar multiple of the even clique inequality since  $(\lambda, \lambda_0) = a(\mathbf{1}, \frac{h}{2})$ . This completes the proof.  $\square$

Now, we are ready to prove the main theorem of this section.

**Theorem 1.** *Let  $G$  be a graph, and let  $K_h$  be a complete graph, where  $h$  is even. Then, the even clique inequality defined as*

$$\sum_{v \in V(K_h)} x_v + \sum_{e \in E(K_h)} y_e \leq \frac{h}{2}$$

*is facet-defining for the Total Matching Polytope  $P_T(G)$ .*

**Proof:** Let  $K_h$  be a complete subgraph of even order of  $G$ . We denote as  $F$  the face induced by

the even clique inequality. By Proposition 17, we can find  $|V(K_h)| + |E(K_h)|$  affinely independent points satisfying at equality the even clique inequality. Now, fix a perfect matching  $M$  of  $G[V(K_h)]$ . Since  $M \cap \{u\} = \emptyset$  for every  $u \notin V(K_h)$ ,  $T_u := M \cup \{u\}$  is a total matching. Observe that,  $\chi[T_u] \in F$ . Thus, the set of characteristic vectors  $\{\chi[T_u] \mid \forall u \notin V(K_h)\}$  is contained in  $F$  and the corresponding  $|V \setminus V(K_h)|$  points are affinely independent. Clearly, it is easy to see that they are still affinely independent with respect to the previous points, so we have  $n$  points up to now. Similarly,  $T_e := M \cup \{e\}$  for every  $e \notin \delta(V(K_h)) \cup E(K_h)$  is a total matching, since  $M \cap \{e\} = \emptyset$ . Consequently, also the set of vectors  $\{\chi[T_e] \mid \forall e \notin \delta(V(K_h)) \cup E(K_h)\}$  is contained in  $F$ , and the corresponding points are affinely independent. Now, let  $S := \{v \in V(K_h) \mid \delta(V(K_h)) \neq \emptyset\}$ . We can construct a total matching  $T_{\bar{s}} := (M_s \setminus \{e\}) \cup \{\bar{s}\}$ , where  $e = \{s, \bar{s}\} \in E(K_h)$ ,  $s \in S$  and  $M_s$  is a perfect matching of  $G[V(K_h)]$  with one end-point in  $s$ . Then,  $T_{\bar{e}_s} := T_{\bar{s}} \cup \{\bar{e}_s\}$  for every  $\bar{e}_s \in \delta(V(K_h)) \cap \delta(s)$ , is a total matching whose characteristic vector lies on  $F$ . Repeating the same construction for all the edges  $e_s \in \delta(V(K_h))$ , we can obtain distinct total matchings for every  $s \in S$  whose characteristic vectors belong to  $F$ , where the corresponding points are affinely independent. In this way, we have found  $n+m$  affinely independent points belonging to  $F$ , since we can rearrange the rows of the matrix having as columns these points in such a way that we get the following form:

$$\left[ \begin{array}{c|c|c} A_{K_h} & B_{K_h} & C_{K_h} \\ \hline \mathbf{0} & \tilde{I}_v & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \tilde{I}_e \end{array} \right],$$

where the matrices  $A_{K_h}, B_{K_h}, C_{K_h}$  have dimension  $|V(K_h)| \times |E(K_h)|$  and correspond to the vertex and edge components of  $K_h$ . The rest of the blocks are the zero and identity matrices of the remaining vertex and edge components. This completes the proof.  $\square$

**Proposition 18.** *Let  $G$  be a graph, and let  $K_h$  be a complete subgraph of  $G$ , where  $h$  is odd. Then, the **odd clique inequality** defined as*

$$\sum_{v \in V(K_h)} x_v + \sum_{e \in E(K_h)} y_e \leq \frac{h+1}{2} \quad (55)$$

*is valid for the Total Matching Polytope  $P_T(G)$ , but it is not facet-defining.*

**Proof:** Let  $K_h$  be a clique of odd order. Since in a total matching of  $K_h$  we can pick at most one vertex, and the size of the largest matching is  $\frac{h-1}{2}$ , we can take at most  $\frac{h-1}{2} + 1 = \frac{h+1}{2}$  elements of a total matching, as shown in Figure 5. Therefore, this implies that the odd clique inequality is valid for  $P_T(G)$ . Now, we prove that it is not facet-defining. Adding a vertex  $u$  to the clique  $K_h$ , we can form a clique of even order  $K_{h+1} := (V(K_{h+1}), E(K_{h+1}))$ , where  $V(K_{h+1}) := V(K_h) \cup \{u\}$  and  $E(K_{h+1}) := E(K_h) \cup \{e = \{u, v\} \mid v \in V(K_h)\}$ . Then, the inequality

$$\sum_{v \in V(K_h)} x_v + \sum_{e \in E(K_h)} y_e \leq \frac{h+1}{2}$$

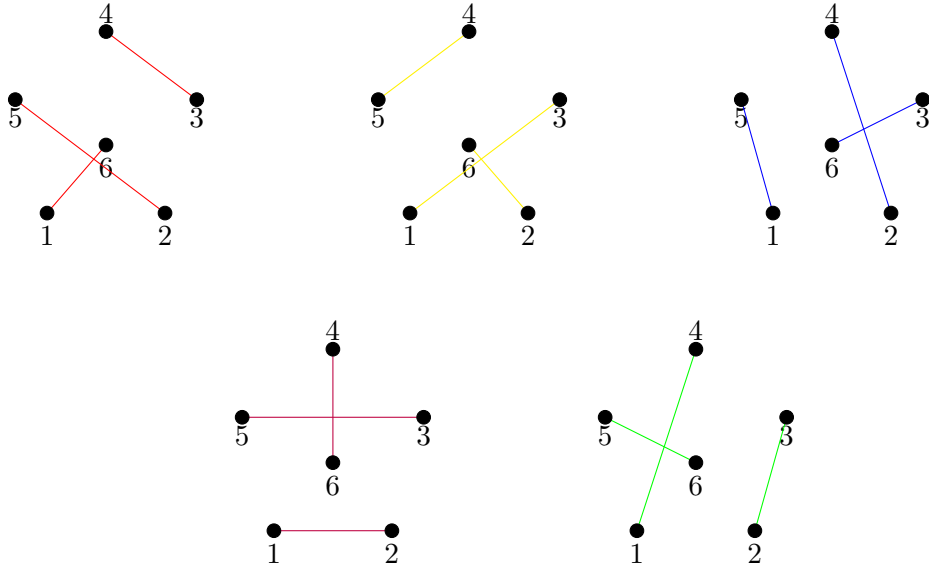


Figure 4: Five perfect matchings of  $K_6$

is dominated by

$$\sum_{v \in V(K_{h+1})} x_v + \sum_{e \in E(K_{h+1})} y_e \leq \frac{h+1}{2}$$

This completes the proof.  $\square$

We stressed out the fact that, even if the odd clique inequality is maximal it remains not facet-defining. Indeed, suppose that  $K_h$  is a maximal clique of odd order. We know that

$$\sum_{v \in V(K_h)} x_v \leq 1$$

is facet-defining, and it is easy to notice that the following is a valid inequality for the Total Matching Polytope

$$\sum_{e \in E(K_h)} y_e \leq \frac{h-1}{2}.$$

Thus, the sum of the two inequalities gives the odd-clique inequality.

*Separation for the even clique inequalities.* We propose the following ILP model to detect a maximum violated even clique, which is based on the maximum edge weighted clique model discussed

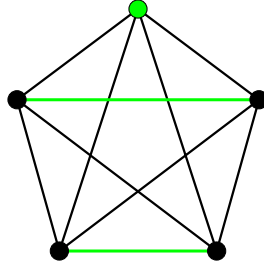


Figure 5: A complete  $K_5$  graph. In green, a possible maximal total matching.

in [37]:

$$\max \sum_{v \in V} \left( c_v - \frac{1}{2} \right) x_v + \sum_{e \in E} \left( w_e - \frac{1}{2} \right) y_e \quad (56)$$

$$\text{s.t.} \quad \sum_{v \in I} x_v \leq 1 \quad \forall I \in \mathcal{I} \quad (57)$$

$$\sum_{v \in V} x_v + \sum_{e \in E} y_e = 2z \quad (58)$$

$$x_v \leq y_e \quad \forall e = \{u, v\} \in E \quad (59)$$

$$x_u \leq y_e \quad \forall e = \{u, v\} \in E \quad (60)$$

$$x_v, y_e \in \{0, 1\} \quad \forall v \in V, \forall e \in E \quad (61)$$

$$z \in \mathbb{Z}, \quad (62)$$

where  $\mathcal{I}$  represents the set of all maximal stable sets of  $G$ . Since we want to detect a clique of even order, we introduce the integer variable  $z \in \mathbb{Z}$ . The constraints (57) impose that we can select at most one vertex from a stable set in the clique found. Notice that there are exponentially (in the size of the graph) many constraints of this type, and their separation is NP-hard. As a consequence, the problem (56)–(62) is NP-hard in general, and it contains the maximum edge weighted clique as a special case [37]. The NP-hardness of problem (56)–(62) allows us to give a polyhedral proof of the NP-hardness also of the Weighted Total Matching Problem.

**Theorem 2.** *The Weighted Maximum Total Matching Problem is NP-hard.*

**Proof:** Consider the optimization problem  $\nu(G, \alpha, \beta)$  for  $P_T(G)$ . Since the Total Matching Polytope is full-dimensional (Proposition 5) and the separation problem for the even clique inequality is NP-hard (Theorem 1), by applying the Equivalence Theorem between Optimization and Separation, (see [9], Chap. 6.4, pp. 174–181), we conclude that solving problem  $\nu(G, \alpha, \beta)$  is NP-hard.  $\square$

## 5. Conclusion and future works

In this paper, we have proposed polyhedral approaches to the Total Coloring and Total Matching Problem. We have introduced two ILP models for the Total Coloring Problem: the assignment

model and a set covering model by maximal total matchings. We can obtain the second model by applying a Dantzig-Wolfe reformulation to the first model. For the Total Matching Problem, our main contributions include the characterization of the feasible region of the Total Matching Polytope and the introduction of two families of nontrivial valid inequalities: congruent- $2k3$  cycle inequalities that are facet-defining when  $k = 4$ , and the facet-defining even clique inequalities. Curiously, we have also shown that the odd clique inequalities are valid, but they are not facet-defining. Finally, our polyhedral study of the Total Matching Problem has permitted an alternative proof for the NP-hardness of the Weighted Total Matching Problem.

As future work, we plan to give a complete description of the Total Matching Polytope for certain classes of graphs. For instance, since we have a complete description of the Stable Set Polytope for bipartite graphs, our research direction goes towards identifying of new facet-defining inequalities that will completely describe the Total Matching Polytope for bipartite graphs and, likely, for complete graphs.

## Acknowledgments

The research was partially supported by the Italian Ministry of Education, University and Research (MIUR): Dipartimenti di Eccellenza Program (2018–2022) - Dept. of Mathematics “F. Casorati”, University of Pavia.

We are deeply indebted to Stefano Coniglio for discussions on the equivalence between optimization and separation as presented in [9].

## References

- [1] Alavi, Y., Behzad, M., Lesniak-Foster, L. M., and Nordhaus, E. A. (1977). Total matchings and total coverings of graphs. *Journal of Graph Theory*, 1:135–140.
- [2] Barnhart, C., Johnson, E. L., Nemhauser, G. L., Savelsbergh, M. W., and Vance, P. H. (1998). Branch-and-price: Column generation for solving huge integer programs. *Operations research*, 46(3):316–329.
- [3] Behzad, M. (1965). *Graphs and their chromatic numbers*. PhD thesis, Michigan State University.
- [4] Dantas, S., de Figueiredo, C. M. H., Mazzuocolo, G., Preissmann, M., dos Santos, V. F., and Sasaki, D. (2016). On the equitable total chromatic number of cubic graphs. *Discrete Applied Math.*, 209:84–91.
- [5] Delle Donne, D., Furini, F., Malaguti, E., and Calvo, R. W. (2020). A branch-and-price algorithm for the minimum sum coloring problem. *Discrete Applied Math.*
- [6] Demange, M., Ekim, T., Ries, B., and Tanasescu, C. (2015). On some applications of the selective graph coloring problem. *European Journal of Operational Research*, 240(2):307–314.
- [7] Desaulniers, G., Desrosiers, J., and Solomon, M. M. (2006). *Column generation*. Springer Science & Business Media.
- [8] Edmonds, J. (1965). Maximum matching and a polyhedron with 0, 1-vertices. *Journal of research of the National Bureau of Standards B*, 69(125-130):55–56.
- [9] Grötschel, M., Lovász, L., and Schrijver, A. (1988). *Geometric Algorithms and Combinatorial Optimization*, volume 2. Springer-Verlag.



- [10] Gualandi, S. and Malucelli, F. (2012a). Exact solution of graph coloring problems via constraint programming and column generation. *INFORMS Journal on Computing*, 24(1):81–100.
- [11] Gualandi, S. and Malucelli, F. (2012b). A simple branching scheme for vertex coloring problems. *Discrete applied mathematics*, 160(1-2):192–196.
- [12] Gualandi, S. and Malucelli, F. (2013). Constraint programming-based column generation. *Annals of Operations Research*, 204(1):11–32.
- [13] Hébrard, E. and Katsirelos, G. (2020). Constraint and satisfiability reasoning for graph coloring. *Journal of Artificial Intelligence Research*, 69:33–65.
- [14] Held, S., Cook, W., and Sewell, E. C. (2012). Maximum-weight stable sets and safe lower bounds for graph coloring. *Math. Progr. Computation*, 4(4):363–381.
- [15] Jabrayilov, A. and Mutzel, P. (2018). New integer linear programming models for the vertex coloring problem. In *Latin American Symposium on Theoretical Informatics*, pages 640–652. Springer.
- [16] Januario, T., Urrutia, S., Ribeiro, C., and de Werra, D. (2016). Edge coloring: A natural model for sports scheduling. *European Journal of Operational Research*, 254(1):1–8.
- [17] Karger, D., Motwani, R., and Sudan, M. (1998). Approximate graph coloring by semidefinite programming. *Journal of the ACM (JACM)*, 45(2):246–265.
- [18] Karp, R. M. (1975). On the computational complexity of combinatorial problems. *Networks*, 5(1):45–68.
- [19] Lee, J. and Leung, J. (1993). A comparison of two edge-coloring formulations. *Operations Research Letters*, 13(4):215–223.
- [20] Lee, J., Leung, J., and de Vries, S. (2005). Separating type-I odd-cycle inequalities for a binary-encoded edge-coloring formulation. *Journal of Combinatorial Optimization*, 9(1):59–67.
- [21] Leidner, M. E. (2012). *A study of the total coloring of graphs*. PhD thesis, University of Louisville.
- [22] Lih, K.-W. (1998). The equitable coloring of graphs. In *Handbook of Combinatorial Optimization*, pages 2015–2038. Springer.
- [23] Lovász, L. and Plummer, M. D. (2009). *Matching theory*, volume 367. American Mathematical Soc.
- [24] Lü, Z. and Hao, J.-K. (2010). A memetic algorithm for graph coloring. *European Journal of Operational Research*, 203(1):241–250.
- [25] Lübbecke, M. E. and Desrosiers, J. (2005). Selected topics in column generation. *Operations research*, 53(6):1007–1023.
- [26] Maksimović, Z. (2016). A new integer mixed programming formulation for the maximum degree bounded connected subgraph. *International Transactions in Operational Research*, 99(1):99–108.
- [27] Malaguti, E., Monaci, M., and Toth, P. (2011). An exact approach for the vertex coloring problem. *Discrete Optimization*, 8(2):174–190.
- [28] Malaguti, E. and Toth, P. (2010). A survey on vertex coloring problems. *International Transactions in Operational Research*, 17(1):1–34.
- [29] Margot, F. (2007). Symmetric ilp: Coloring and small integers. *Discrete Optimization*, 4(1):40–62.
- [30] Mehrotra, A. and Trick, M. A. (1996). A column generation approach for graph coloring. *INFORMS Journal on Computing*, 8(4):344–354.
- [31] Méndez-Díaz, I. and Zabala, P. (2006). A branch-and-cut algorithm for graph coloring. *Discrete Applied Mathematics*, 154(5):826–847.
- [32] Murthy, T. S. (2020). A proof of the total coloring conjecture. *arXiv preprint arXiv:2003.09658*.
- [33] Nemhauser, G. L. and Park, S. (1991). A polyhedral approach to edge coloring. *Operations Research Letters*, 10(6):315–322.
- [34] Padberg, M. (1973). On the facial structure of the set packing polyhedra. *Mathematical Programming*, 5:199–215.
- [35] Rebennack, S., Oswald, M., Theis, D. O., Seitz, H., Reinelt, G., and Pardalos, P. M. (2011). A branch and cut solver for the maximum stable set problem. *Journal of combinatorial optimization*, 21(4):434–457.

- [36] Rossi, F. and Smriglio, S. (2001). A branch-and-cut algorithm for the maximum cardinality stable set problem. *Operations Research Letters*, 28(2):63–74.
- [37] San Segundo, P., Coniglio, S., Furini, F., and Ljubić, I. (2019). A new branch-and-bound algorithm for the maximum edge-weighted clique problem. *European Journal of Operational Research*, 278(1):76–90.
- [38] Sánchez-Arroyo, A. (1989). Determining the total colouring number is np-hard. *Discrete Mathematics*, 78(3):315–319.
- [39] Tutte, W. T. (1954). A short proof of the factor theorem for finite graphs. *Canadian Journal of Mathematics*, 6:347–352.
- [40] van Hoeve, W.-J. (2020). Graph coloring lower bounds from decision diagrams. In *International Conference on Integer Programming and Combinatorial Optimization*, pages 405–418. Springer.
- [41] Vignesh, R., Geetha, J., and Somasundaram, K. (2018). Total coloring conjecture for certain classes of graphs. *Algorithms*, 11(10):161.
- [42] Vizing, V. G. (1964). On an estimate of the chromatic class of a p-graph. *Discret Analiz*, 3:25–30.
- [43] Yap, H. P. (2006). *Total colourings of graphs*. Springer.