VARIATIONAL INEQUALITIES GOVERNED BY STRONGLY PSEUDOMONOTONE VECTOR FIELDS ON HADAMARD MANIFOLDS

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Abstract. We consider variational inequalities governed by strongly pseudomonotone vector fields on Hadamard manifolds. The existence and uniqueness results of the solution, linear convergence, error estimates and finite convegence for sequences generated by a modified projection method for solving variational inequalities are investigated. Some examples and numerical experiments are also given to illustrate our results.

1. Introduction

Recently, many important concepts and methods of optimization theory and nonilnear analysis have been extended from linear spaces to the setting of manifolds. For example, the notion of monotonicity were extended from linear spaces to manifolds and have been studied intensively in [35, 41, 42, 54] and references therein. Subdifferential calculus for nonsmooth functions on Riemannian manifolds is developed in [12, 25, 34] where some constrained optimization problems, nonclassical problems of calculus variations and Hamilton-Jacobi equations on Riemannian manifolds are studied. We refer the reader to [2, 5, 6, 7] and references therein for more results concerning the existence of solutions of other problems on manifolds. Some numerical methods for solving optimization problems on manifolds can be found in [3, 8, 9, 10, 11, 13, 23, 35, 48, 55, 56]. Weak sharp minima for constrained optimization problems on Riemannian manifolds and its applications to the study of finite convergence of proximal point algorithms for solving optimization problems can be found in [36, 38, 14, 56]. It is worth noting that the extension from linear spaces to Riemannian manifolds has some important advantages. For example, non-convex optimization problems can be transformed to convex problems, constrained problems can be reduced to unconstrained problems on Riemannian manifolds and non-monotone vector fields can be transformed into monotone vector fields on manifolds by choosing a suitable Riemannian metric. However, in general, a manifold does not have a linear structure and when we replace linear spaces by Riemannian manifolds (in particular Hadamard manifolds), the line segment is replaced by a geodesic. Many known properties and techniques in the linear space setting do not work in the setting of manifolds. Therefore, the extension of the concepts, techniques and results for various problems from Euclidean spaces to Riemannian manifolds is natural and interesting.

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Variational inequality in finite dimensional spaces was introduced by Hartman and Stampacchia [24] in the early 1960s. It is a useful mathematical model unifying many important concepts in applied mathematics such as complementary problems, network equilibrium problems, obstacle problems, system of nonlinear equations, fixed point problems, optimization problems (see, e.g., [20, 32, 33]). Two basic and important issues for variational inequality problems are the existence of solutions and approximation the solutions. There have been a large number of results concerned with the existence of solutions and methods for solving variational inequalities in the literature (see, e.g., [1, 4, 17, 19, 27, 28, 29, 30, 31, 52] and the references therein).

Variational inequality problems on manifolds were first introduced and established by Németh in [43] for single-valued vector fields on Hadamard manifolds. In that paper, he generalizes some basic existence and uniqueness results of classical theory of variational inequalities on Euclidean spaces to Hadamard manifolds. He also proposed an open problem on how to extend his results on Hadamard manifolds to Riemannian manifolds. By establishing the existence and uniqueness of solutions for variational inequality problems on Riemannian manifolds, Li et al. in [39] solved completely the open problem proposed by Németh. The results then were extended to the set-valued vector fields on Riemannian manifolds in [37] by Li and Yao. Convergence property of the proximal point algorithm for solving the set-valued variational inequality problems on Riemannian manifolds was also investigated. Other results about numerical methods for solving variational inequality problems on manifolds can be found in [9, 16, 22, 35, 37, 47, 49, 50] and references therein. We also refer the reader to [45] for results about the finite convergence property of sequences generated by the (inexact) proximal point algorithm for solving variational inequality on Hadamard manifods under weak sharpness of the solution set. Noting that variational inequalities governed by strongly pseudomonotone mappings in linear spaces have been investigated in many papers and they still attracts many researchers (see, e.g., [15, 21, 27, 31, 26] and references therein). In contrast to the linear setting, we are not aware of any results about both existence and approximation solutions for strongly pseudomonotone variational inequalities on manifolds. So it is natural to study variational inequalities on manifolds under the strong pseudomotonicity of vector fields.

In this paper, we first prove existence and uniqueness theorems for variational inequalities governed by strongly pseudomonotone vector fields on Hadamard manifolds. We then study the modified projection type method for solving strongly pseudomonotone variational inequalities. Linear convergence and error estimates for sequences generated by the modified projection method with suitable step sizes are investigated. These results extend the analogous results from linear space setting to Hadamard manifolds. We also introduce the notion of linear conditioning for variational inequalities on Hadamard manifolds and present a finite convergence result for sequences generated by by the modified projection method under linear conditioning assumption. We also provide some examples and numerical experiments to support our results.

2. Preliminaries

In this section, we recall some fundamental definitions, notations and useful results about Riemannian geometry which will be used throughout this paper. We refer the reader to, for instances, [18, 46, 51] for more details.

Let M be a connected finite dimensional differentiable manifold. The tangent space of M at a point $x \in M$ is denoted by T_xM which is vector space of the same dimension as M. The tangent bundle of M, denoted by TM, is $TM = \cup_{x\in M} T_xM$ which is naturally a manifold. We assume that M is endowed with a Riemannian metric to become a Riemannian manifold. We denote by $\langle \cdot, \cdot \rangle_x$ the inner product on T_xM. The corresponding norm to the inner product $\langle \cdot, \cdot \rangle_x$ on T_xM is denoted by $|| \cdot ||_x$. If no confusion arises, then the subscript is omitted.

For a piecewise smooth curve $\gamma : [a, b] \to M$ joining $p := \gamma(a)$ to $q := \gamma(b)$, the length of γ is defined by

$$L(\gamma) := \int_a^b ||\gamma'(t)||dt.$$

The minimal length of all such curves joining p and q is called the Riemannian distance and is denoted by $d(p, q)$. This distance induces the original topology on M. Let $p \in M$ and $r > 0$. The open metric ball and the closed metric ball at p with radius r are denoted respectively by $B(p, r)$ and $\overline{B}(p, r)$ which are defined respectively as

$$B(p, r) = \{q \in M : d(p, q) < r\} \quad \text{and} \quad \overline{B}(p, r) = \{q \in M : d(p, q) \leq r\}.$$

We denote by $\mathbb{B}_p$ the closed unit ball of T_pM, i.e.,

$$\mathbb{B}_p := \{v \in T_pM : ||v|| \leq 1\}.$$

Let S be a nonempty subset of M. We denote by $\overline{S}$ and ∂S the closure and the boundary of S with respect to tho topology induced by the distance d, respectively. The distance from $p \in M$ to S is defined by

$$d(p, S) := \inf\{d(p, q) : q \in S\}.$$

Let ∇ be the Levi-Civita connection associated with the Riemannian metric and γ be a smooth curve in M. A vector field V is said to be parallel along γ if $\nabla_{\gamma'} V = \mathbf{0}$, where $\mathbf{0}$ is the zero tangent vector. We say that γ is a geodesic if γ' itself is parallel along γ. A geodesic joining p to q is said to be minimal if its length equals $d(p, q)$ and this geodesic is called a minimizing geodesic. A Riemannian manifold is complete if every geodesic is defined for all $-\infty < t < +\infty$. By the Hopf-Rinow theorem, if M is complete then any pair of points in M can be joined by a minimal geodesic. Moreover, (M, d) is a complete metric space and bounded closed subsets are compact.

Let $P_{\gamma,\cdot,\cdot}$ denote the parallel transport on the tangent bundle TM along γ with respect to ∇ which is defined by

$$P_{\gamma,\gamma(b),\gamma(a)}(v) = V(\gamma(b)), \quad \forall a, b \in \mathbb{R}, v \in T_{\gamma(a)}M,$$

where V is the unique vector field satisfying $\nabla_{\gamma'(t)}V = \mathbf{0}$ for all t and $V(\gamma(a)) = v$. For any $a, b, b_1, b_2 \in \mathbb{R}$, it holds that

$$P_{\gamma,\gamma(b_2),\gamma(b_1)} \circ P_{\gamma,\gamma(b_1),\gamma(a)} = P_{\gamma,\gamma(b_2),\gamma(a)}, \quad \text{and} \quad P^{-1}_{\gamma,\gamma(b),\gamma(a)} = P_{\gamma,\gamma(a),\gamma(b)}.$$

If γ is a minimal geodesic joining x to y, then we write $P_{y,x}$ instead of $P_{\gamma,y,x}$. The parallel transport $P_{y,x}$ is an isometry from T_xM to T_yM, that is, the parallel transport preserve the inner product

$$\langle P_{y,x}u, P_{y,x}v\rangle_y = \langle u, v\rangle_x, \quad \forall u, v \in T_xM.$$

From now on, we always assume that M is an m-dimensional Hadamard manifold, i.e. a complete, simply connected Riemannian manifold of non-positive sectional curvature. Let $p \in M$. The exponential map $\exp_p : T_pM \to M$ at p is defined by $\exp_p v := \gamma_v(1, p)$ for each $v \in T_pM$, where $\gamma(\cdot) := \gamma_v(\cdot, p)$ is the geodesic starting from p with velocity v, that is, $\gamma(0) = p$ and $\gamma'(0) = v$. Moreover, we have that $\exp_p tv = \gamma_v(t, p)$ for any real number t and $\exp_p \mathbf{0} = \gamma_v(0, p) = p$. For each $x \in M$, the exponential map $\exp_x : T_xM \to M$ is a diffeomorphism and its inverse is the map $\exp_x^{-1}$ from M to T_xM. For any $x, y \in M$, we have $d(x, y) = ||\exp_x^{-1} y||$.

Let p_1, p_2 and p_3 be three points in M. The set consisting of these three points and three minimal geodesics γ_i joining p_i to p_{i+1}, where $i = 1, 2, 3(\text{mod}3)$, is called a geodesic triangle and is denoted by $\Delta(p_1p_2p_3)$.

Proposition 2.1. [46] *(Comparison result for triangles). Let $\Delta(p_1p_2p_3)$ be a geodesic triangle. For each $i = 1, 2, 3(mod\, 3)$, let $\gamma_i : [0, \ell_i] \to M$ denote the geodesic joining p_i to p_{i+1}, and $\ell_i = L(\gamma_i)$ and α_i be the angle between tangent vectors $\gamma_i'(0)$ and $\gamma_{i-1}'(\ell_{i-1})$. Then*

(i) $\alpha_1 + \alpha_2 + \alpha_3 \le \pi$;
(ii) $\ell_i^2 + \ell_{i+1}^2 - 2\ell_i\ell_{i+1}\cos\alpha_{i+1} \le \ell_{i-1}^2$;
(iii) $\ell_{i+1}\cos\alpha_{i+2} + \ell_i\cos\alpha_i \ge \ell_{i+2}$.

Since

$$\left\langle \exp^{-1}_{p_{i+1}} p_i, \exp^{-1}_{p_{i+1}} p_{i+2} \right\rangle = d(p_i, p_{i+1}) d(p_{i+1}, p_{i+2}) \cos\alpha_{i+1},$$

we can rewrite the inequality (ii) of Proposition 2.1 in terms of the distance and the exponential map as follows

$$d^2(p_i, p_{i+1}) + d^2(p_{i+1}, p_{i+2}) - 2\left\langle \exp^{-1}_{p_{i+1}} p_i, \exp^{-1}_{p_{i+1}} p_{i+2} \right\rangle \le d^2(p_{i-1}, p_i) \tag{2.1}$$

and

$$d^2(p_i, p_{i+1}) \le \left\langle \exp^{-1}_{p_i} p_{i+2}, \exp^{-1}_{p_i} p_{i+1} \right\rangle + \left\langle \exp^{-1}_{p_{i+1}} p_{i+2}, \exp^{-1}_{p_{i+1}} p_i \right\rangle. \tag{2.2}$$

Remark 2.1. [35] If $x, y \in M$ and $v \in T_yM$, then

$$\left\langle v, -\exp_y^{-1} x \right\rangle = \left\langle v, P_{y,x}\exp_x^{-1} y \right\rangle = \left\langle P_{x,y}v, \exp_x^{-1} y \right\rangle. \tag{2.3}$$

Remark 2.2. [16] Let $x, y, z \in M$ and $v \in T_xM$. By using (2.2) and Remark 2.1,

$$\left\langle v, \exp_x^{-1} y \right\rangle \le \left\langle v, \exp_x^{-1} z \right\rangle + \left\langle v, P_{x,z}\exp_z^{-1} y \right\rangle. \tag{2.4}$$

Lemma 2.1. [35] *Let $x_0 \in M$ and $\{x_n\} \subset M$ with $x_n \to x_0$. Then the following assertions hold.*

(i) *For any $y \in M$, we have $\exp_{x_n}^{-1} y \to \exp_{x_0}^{-1} y$ and $\exp_y^{-1} x_n \to \exp_y^{-1} x_0$.*
(ii) *If $v_n \in T_{x_n}M$ and $v_n \to v_0$, then $v_0 \in T_{x_0}M$.*
(iii) *Given $u_n, v_n \in T_{x_n}M$ and $u_0, v_0 \in T_{x_0}M$, if $u_n \to u_0$ and $v_n \to v_0$, then $\langle u_n, v_n \rangle \to \langle u_0, v_0 \rangle$.*
(iv) *For any $u \in T_{x_0}M$, the function $F : M \to TM$, defined by $F(x) = P_{x,x_0}u$ for each $x \in M$ is continuous on M.*

Definition 2.1. A subset $K \subset M$ is said to be (geodesic) convex if for any two point p and q in K, the geodesic joining p to q is contained in K, that is, if $\gamma : [a, b] \to M$ is a geodesic such that $\gamma(a) = p$ and $\gamma(b) = q$, then $\gamma(ta + (1 - t)b) \in K$ for all $t \in [0, 1]$.

The projection of a point $x \in M$ onto a subset K of a Hadamard manifold M is defined by

$$P(x, K) := \{p \in K : d(x, p) = d(x, K)\}.$$

Proposition 2.2. [53] *Let K be a closed convex subset of a Hadamard manifold M. Then, for any $x \in M$, $P(x, K)$ is a singleton set. Also, for any $p \in M$, the following assertions are equivalent:*

(i) *$y = P(p, K)$;*
(ii) *$\langle \exp_y^{-1} p, \exp_y^{-1} q \rangle \le 0$ for all $q \in K$.*

We now recall the definitions of the normal cone and the tangent cone to a closed convex subset in Hadamard manifolds; for more details see [25]. Let K be a closed convex subset of M and let $x \in K$. The normal cone to K at a point $x \in K$ is the set

$$N_K(x) := \{v \in T_xM : \langle v, \exp_x^{-1} y \rangle \le 0 \quad \text{for all} \quad y \in K\}.$$

The tangent cone to K at $x \in K$ is the set

$$T_K(x) = \{v \in T_xM : \langle v, w \rangle \le 0 \text{ for all } \ w \in N_K(x)\}.$$

That is, $T_K(x) = [N_K(x)]^\circ$. Note that if H is a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and $C \subset H$, then the polar C° of C is defined by $C^\circ = \{v \in H : \langle v, w \rangle \le 0 \ \forall w \in C\}$.

Let X be a subset of M. A vector field V on X is a mapping $V : X \to TM$ such that $V(x) \in T_xM$ for each $x \in M$.

Definition 2.2. [54, 47] Let M be a Hadamard manifold and X be a convex subset of M. A vector field $V : X \to TM$ is said to be

(i) monotone on X if

$$\langle V(x), \exp_x^{-1} y \rangle + \langle V(y), \exp_y^{-1} x \rangle \le 0, \quad \forall x, y \in X,$$

(ii) strongly monotone on X with modulus μ if there exists $\mu > 0$ such that

$$\langle V(x), \exp_x^{-1} y \rangle + \langle V(y), \exp_y^{-1} x \rangle \le -\mu d^2(x, y), \quad \forall x, y \in X,$$

(iii) pseudomonotone on X if

$$\langle V(x), \exp_x^{-1} y\rangle \geq 0 \quad \Longrightarrow \quad \langle V(y), \exp_y^{-1} x\rangle \leq 0, \quad \forall x, y \in X,$$

(iv) strongly psedomonotone on X with modulus μ if there exists $\mu > 0$ such that

$$\langle V(x), \exp_x^{-1} y\rangle \geq 0 \quad \Longrightarrow \quad \langle V(y), \exp_y^{-1} x\rangle \leq -\mu d^2(x, y), \quad \forall x, y \in X,$$

(v) Lipschitz continuous on X with Lipschitz constant L if there exists $L > 0$ such that

$$||P_{y,x}V(x) - V(y)|| \leq Ld(x, y), \quad \forall x, y \in X.$$

Remark 2.3. It is obvious that $(ii) \Rightarrow (i)$, $(i) \Rightarrow (iii)$, $(ii) \Rightarrow (iv)$ and $(iv) \Rightarrow (iii)$. However, inverse implications are not true in general.

Let $\mathcal{D} \subset M$ be an open set and $f : \mathcal{D} \to \mathbb{R}$ be a function. If f is differentiable, then we denote its gradient by $\text{grad}f$. If f is twice differentiable, its Hessian is denoted by $\text{Hess}f$. The norm of the Hessian of f at $p \in M$ is given by

$$||\text{Hess}f(p)|| := \sup\{||\text{Hess}f(p)v|| : v \in T_pM, ||v|| = 1\}.$$

The following result presents a characterization for twice continuously differentiable functions with Lipschitz continuous gradient vector field.

Lemma 2.2. [23] *Let $f : \mathcal{D} \to \mathbb{R}$ be a twice continuously differentiable function. The gradient vector field of f is Lipschitz with constant $L \geq 0$ if and only if there exists $L \geq 0$ such that $||\text{Hess}f(p)|| \leq L$ for all $p \in \mathcal{D}$.*

Let X be a closed convex subset of M and $V : X \to TM$ be a vector field. Németh [43] introduced the variational inequality problem (in short, VIP) on Hadamard manifolds: *Find $x \in X$ such that*

$$\langle V(x), \exp_x^{-1} y\rangle \geq 0 \quad \textit{for all} \;\; y \in X. \tag{2.5}$$

The variational inequality problem on Hadamard manifolds is an extension of classical variational inequality problems on linear spaces. More precisely, if $M = \mathbb{R}^n$, then VIP (2.5) reduces to

$$\textit{Find} \;\; x \in X \;\; \textit{such that} \quad \langle V(x), y - x\rangle \geq 0 \quad \textit{for all} \;\; y \in X. \tag{2.6}$$

Concerning the existence of solutions of variational inequality problems on manifolds under different assumptions, we refer the reader to [43, 39, 37]. We recall the following result (see, [39, Corollary 4.1] or [37, Corollary 3.7]) which will be useful in the sequel.

Theorem 2.1. *Let $X \subset M$ be a compact convex set and V is continuous. Then, VIP (2.5) has at least one solution.*

3. Existence results of solutions for VIP

This section is devoted to the study of the existence of solutions of the variational inequality problems. The following result extends [20, Proposition 2.2.3] and [20, Theorem 2.3.4] from Euclidean spaces to Hadamard manifolds. Note that [20, Proposition 2.2.3] was proved by using degree-theoretic approach. We refer the reader to [28] for an alternative proof of [20, Proposition 2.2.3] using the regularization approach that we borrow the idea for proving the first part of Theorem 3.1.

Theorem 3.1. *Let X be a closed convex subset of M and $V : X \to TM$ is continuous. Consider the following statements:*

(a) *There exists $x_0 \in X$ such that the set*

$$X_0 := \{x \in X : \left\langle V(x), \exp_x^{-1} x_0 \right\rangle > 0\}$$

is bounded (possibly empty).

(b) *There exist a bounded convex open set Ω and $x_0 \in X \cap \Omega$ such that*

$$\left\langle V(x), \exp_x^{-1} x_0 \right\rangle \le 0, \quad \forall x \in X \cap \partial\Omega. \tag{3.1}$$

(c) *VIP (2.5) has a solution.*

It holds that (a) $\Rightarrow$ (b) $\Rightarrow$ (c). Moreover, if V is pseudomonotone on X, then (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c).

Proof. $(a) \Rightarrow (b)$. Assume that (a) holds. Since X_0 is bounded, there exists $r > 0$ such that $\Omega := B(x_0, r)$ contains X_0 and x_0. It is obvious that Ω is bounded, convex and open. Moreover, $\partial\Omega \cap X_0 = \partial B(x_0, r) \cap X_0 = \emptyset$. Thus, (3.1) holds and (b) follows.

We now show that $(b) \Rightarrow (c)$. Assume that (b) holds. Set $X_\Omega = X \cap \overline{\Omega}$. Then, X_Ω is nonempty, compact and convex. For each $k \in \mathbb{N}$, we define the vector field $V_k : X \to TM$ by

$$V_k(x) = V(x) - \frac{1}{k} \exp_x^{-1} x_0, \quad x \in X.$$

Then, by Lemma 2.1, V_k is continuous on X for every $k \in \mathbb{N}$. By Theorem 2.1, the variational inequality problem: *Find $x^* \in X_\Omega$ such that*

$$\left\langle V_k(x^*), \exp_{x^*}^{-1} x \right\rangle \ge 0, \quad \forall x \in X_\Omega,$$

has a solution $x_k \in X_\Omega$ for each $k \in \mathbb{N}$. That is,

$$\left\langle V_k(x_k), \exp_{x_k}^{-1} x \right\rangle \ge 0, \quad \forall x \in X_\Omega. \tag{3.2}$$

Equivalently,

$$\left\langle V(x_k) - \frac{1}{k} \exp_{x_k}^{-1} x_0, \exp_{x_k}^{-1} x \right\rangle \ge 0, \tag{3.3}$$

for all $x \in X_\Omega$.

We claim that $x_k \in \Omega$ for all $k \in \mathbb{N}$. Assume to the contrary that there exists an index k such that $x_k \notin \Omega$. This means that $x_k \in \partial\Omega$. By (3.1), one has that

$$\left\langle V(x_k), \exp_{x_k}^{-1} x_0 \right\rangle \le 0. \tag{3.4}$$

Since $x_0 \in X_\Omega$, substituting x by x_0 in (3.3), we have

$$\left\langle V(x_k) - \frac{1}{k}\exp_{x_k}^{-1}x_0, \exp_{x_k}^{-1}x_0 \right\rangle \geq 0.$$

This yields

$$\langle V(x_k), \exp_{x_k}^{-1}x_0\rangle \geq \frac{1}{k}\langle \exp_{x_k}^{-1}x_0, \exp_{x_k}^{-1}x_0\rangle = \frac{1}{k}||\exp_{x_k}^{-1}x_0||^2 = \frac{1}{k}d^2(x_k, x_0).$$

Combining with (3.4), we get

$$\frac{1}{k}d^2(x_k, x_0) \leq \langle V(x_k), \exp_{x_k}^{-1}x_0\rangle \leq 0.$$

This contradicts the fact that $x_k \neq x_0$. So x_k must belong to Ω for all $k \in \mathbb{N}$.

We shall show that x_k is a solution of the variational inequality problem: *Find* $x^* \in X$ *such that*

(3.5) $$\langle V_k(x^*), \exp_{x^*}^{-1}x\rangle \geq 0, \quad \forall x \in X.$$

Indeed, for each $k \in \mathbb{N}$, since Ω is open and $x_k \in \Omega$, there exists $r_k > 0$ such that $B(x_k, r_k) \subset \Omega$. We now fix $k \in \mathbb{N}$, for each $z \in X$ with $z \neq x_k$, we choose a positive number δ_k such that

$$\delta_k < \min\left\{1, \frac{r_k}{d(x_k, z)}\right\}.$$

Set $u_k = \exp_{x_k}^{-1}z \in T_{x_k}M$. We have

$$d(x_k, \exp_{x_k}\delta_k u_k) = \delta_k||u_k|| = \delta_k||\exp_{x_k}^{-1}z|| = \delta_k d(x_k, z) < r_k.$$

Thus, $\exp_{x_k}\delta_k u_k \in B(x_k, r_k) \subset \Omega$. Moreover, since $\delta_k < 1$, $x, x_k \in X$ and X is convex, we have $\exp_{x_k}\delta_k u_k \in X$. It follows that $\exp_{x_k}\delta_k u_k \in X \cap \Omega = X_\Omega$. It follows from (3.2) that

$$\left\langle V_k(x_k), \exp_{x_k}^{-1}\exp_{x_k}\delta_k u_k\right\rangle \geq 0,$$

which implies that

$$\left\langle V_k(x_k), \exp_{x_k}^{-1}z\right\rangle \geq 0.$$

Since z is taken arbitrary in X, the latter inequality holds for all $z \in X$. It means that x_k is a solution of the variational inequality (3.5).

We next show that the problem VIP (2.5) has a solution. Since $\{x_k\}$ is a bounded sequence in the closed X, there exists a subsequence $\{x_{k_i}\}$ of $\{x_k\}$ converging to some point $x^* \in X$. On the other hand, we have for all $i \in \mathbb{N}$ and $x \in X$ that

$$\left\langle V_{k_i}(x_{k_i}), \exp_{x_{k_i}}^{-1}x\right\rangle \geq 0,$$

or, equivalently,

$$\left\langle V(x_{k_i}) - \frac{1}{k_i}\exp_{x_{k_i}}^{-1}x_0, \exp_{x_{k_i}}^{-1}x\right\rangle \geq 0$$

for all $i \in \mathbb{N}$ and $x \in X$. By the continuity of V and by Lemma 2.1, letting $i \to \infty$ in the latter inequality, one gets

$$\left\langle V(x^*), \exp_{x^*}^{-1}x\right\rangle \geq 0, \quad \forall x \in X.$$

Thus, x^* is a solution of VIP (2.5) and this ends the proof of the implication $(b) \Rightarrow (c)$.

Assume now that V is pseudomonotone on X. It is sufficient to show that $(c) \Rightarrow (a)$. Assume that (c) holds. Let x_0 be a solution of VIP (2.5). Then,

$$\left\langle V(x_0), \exp_{x_0}^{-1} x \right\rangle \geq 0, \quad \forall x \in X.$$

By the pseudomonotonicity of V, we have

$$\left\langle V(x), \exp_{x}^{-1} x_0 \right\rangle \leq 0, \quad \forall x \in X.$$

This implies that the set X_0 is empty. The proof is complete. □

Remark 3.1. In the proof of Theorem 3.1, we use the property that any metric ball in Hadamard manifolds is convex. This property is not true for Riemannian manifolds in general. Therefore, it is interesting to extend the result in Theorem 3.1 to the Riemannian manifolds.

We next establish the solution existence result for variational inequalities governed by strongly pseudomonotone vector fields.

Theorem 3.2. *Let $X \subset M$ be a nonempty closed convex set and $V : X \to TM$ be a continuous vector field. If V is strongly pseudomonotone on X, then the problem VIP (2.5) has a unique solution.*

Proof. Let $\mu > 0$ be the modulus of the strong pseudomonotonicity of V. We first show that the problem VIP (2.5) has a solution. By Theorem 3.1, it is enough to find a point $\tilde{x} \in X$ such that the set

$$\widetilde{X} := \{x \in X : \langle V(x), \exp_x^{-1} \tilde{x} \geq 0\}$$

is bounded. We choose $\tilde{x}$ to be any point in X. For any $x \in \widetilde{X}$, we have

$$\left\langle V(x), \exp_x^{-1} \tilde{x} \right\rangle \geq 0.$$

It follows from the strong pseudomonotonicity of V that

$$\langle V(\tilde{x}), \exp_{\tilde{x}}^{-1} x \rangle \leq -\mu d^2(x, \tilde{x}).$$

Then, by the Cauchy - Schwarz inequality, one has

$$\begin{aligned} \mu d^2(x, \tilde{x}) &\leq \left\langle -V(\tilde{x}), \exp_{\tilde{x}}^{-1} x \right\rangle \\ &\leq ||V(\tilde{x})||.||\exp_{\tilde{x}}^{-1} x|| = ||V(\tilde{x})||.d(x, \tilde{x}). \end{aligned}$$

This implies that

$$d(x, \tilde{x}) \leq \frac{||V(\tilde{x})||}{\mu}$$

and that $\widetilde{X}$ is contained in the closed metric ball at center $\tilde{x}$ with radius $||V(\tilde{x})||/\mu$, i.e., $\widetilde{X}$ is bounded. Therefore, VIP (2.5) has at least a solution.

Assume now that VIP (2.5) has two solutions x^* and y^*. Then,

$$\left\langle V(x^*), \exp_{x^*}^{-1} y^* \right\rangle \geq 0 \tag{3.6}$$

and

$$\left\langle V(y^*), \exp_{y^*}^{-1} x^* \right\rangle \geq 0 \tag{3.7}$$

The strong pseudomonotonicity of V and (3.7) imply that

$$\left\langle V(x^*), \exp_{x^*}^{-1} y^* \right\rangle \leq -\mu d^2(x^*, y^*). \tag{3.8}$$

From (3.6) and (3.8), we have

$$\mu d^2(x^*, y^*) \leq -\left\langle V(x^*), \exp_{x^*}^{-1} y^* \right\rangle \leq 0$$

which leads to $x^* = y^*$. This ends the proof. □

We now provide examples to illustrate the validity of Theorem 3.2.

Example 3.1. Let $\mathbb{R}_{++} = \{x \in \mathbb{R} : x > 0\}$ and $M = (\mathbb{R}_{++}, \langle \cdot, \cdot \rangle)$ be the Riemannian manifold with the Riemannian metric

$$\langle u, v \rangle := \frac{1}{x^2} uv \quad \text{for } x \in M, \text{and} \quad u, v \in T_x M$$

The Riemannian distance $d : M \times M \to \mathbb{R}_+$ is given by

$$d(x, y) = \left| \ln \left(\frac{x}{y} \right) \right|$$

for all $x, y \in M$, see, for example [13]. The sectional curvature of M is zero and it holds that M is a Hadamard manifold. For each $x \in M$, the tangent plane $T_x M$ at x equals to $\mathbb{R}$. The unique geodesic γ starting from $x = \gamma(0) \in M$ with velocity $v = \gamma'(0) \in T_x M$ is defined by

$$\gamma(t) = x e^{\left(\frac{v}{x}\right) t}.$$

Thus,

$$\exp_x tv = x e^{\left(\frac{v}{x}\right) t}.$$

Moreover, for any $x, y \in M$, we have

$$y = \exp_x \left(d(x, y) \frac{\exp_x^{-1} y}{d(x, y)} \right) = x e^{\frac{\exp_x^{-1} y}{x d(x,y)} d(x,y)} = x e^{\frac{\exp_x^{-1} y}{x}}.$$

Hence, the inverse of exponential map is defined as

$$\exp_x^{-1} y = x \ln \left(\frac{y}{x} \right).$$

Let $X = [1, 2]$. Then, X is a closed convex subset of M. We consider the vector field $V : X \to TM$ defined by

$$V(x) = (1 - \ln x) x, \quad \forall x \in X.$$

We first see that V is not monotone on X. Indeed, for all $x, y \in X$ with $x \neq y$, we have

$$\begin{aligned}
\langle V(x), \exp_x^{-1} y \rangle + \langle V(y), \exp_y^{-1} x \rangle &= \frac{1}{x^2}(1 - \ln x) x.x. \ln \left(\frac{y}{x} \right) + \frac{1}{y^2}(1 - \ln y) y.y. \ln \left(\frac{x}{y} \right) \\
&= (1 - \ln x)(\ln y - \ln x) + (1 - \ln y)(\ln x - \ln y) \\
&= \ln^2 \left(\frac{x}{y} \right) = d^2(x, y) > 0.
\end{aligned}$$

We now show that V is strongly pseudomonotone on X. Indeed, let $x, y \in X$ be such that $\langle V(x), \exp_x^{-1} y\rangle \geq 0$. Then,

$$0 \leq \frac{1}{x^2}(1-\ln x)x.x.\ln\left(\frac{y}{x}\right) = (1-\ln x)\ln\left(\frac{y}{x}\right).$$

Since $1-\ln x > 0$ for all $x \in X$, the latter inequality implies that $x \leq y$. As $x, y \in [1,2]$ and $x \leq y$, we have

$$0 \leq \ln\left(\frac{y}{x}\right) < 1.$$

Thus,

$$\ln^2\left(\frac{y}{x}\right) \leq \ln\left(\frac{y}{x}\right).$$

Now,

$$\begin{aligned}\langle V(y), \exp_y^{-1} x\rangle &= \frac{1}{y^2}(1-\ln y)y.y.\ln\left(\frac{x}{y}\right) = (1-\ln y)\ln\left(\frac{x}{y}\right)\\ &\leq (1-\ln 2)\ln\left(\frac{x}{y}\right) = -(1-\ln 2)\ln\left(\frac{y}{x}\right)\\ &\leq -(1-\ln 2)\ln^2\left(\frac{y}{x}\right) = -(1-\ln 2)d^2(x,y).\end{aligned}$$

Thus, V is strongly pseudomonotone on X with modulus $\mu = 1-\ln 2$. It is evident that V is continuous. Applying Theorem 3.2, we conclude that the problem VIP (2.5) has a unique solution. Let x_* be the unique solution of VIP (2.5). Then, for all $y \in X$, we have

$$\begin{aligned}0 &\leq \left\langle V(x_*), \exp_{x_*}^{-1} y\right\rangle\\ &= \frac{1}{x_*^2}(1-\ln x_*)x_*.x_*\ln\left(\frac{y}{x_*}\right)\\ &= (1-\ln x_*)\ln\left(\frac{y}{x_*}\right)\end{aligned}$$

Equivalently,

$$\ln\left(\frac{y}{x_*}\right) \geq 0 \quad \text{for all } y \in X.$$

This implies that $x_* = 1$.

In Example 3.1, the set X is compact. The existence of solutions of VIP (2.5) can be obtained by applying Theorem 2.1. However, to obtain the uniqueness, we apply Theorem 3.2. We next provide an example in which X is not compact.

Example 3.2. Let M, $\langle\cdot,\cdot\rangle$ and d be as in Example 3.1. Let $X = [1, \infty)$ and $V : X \to TM$ be defined by

$$V(x) = \left(\frac{1}{32}+\frac{1}{x}\right)x\ln x \quad \text{for } x \in X.$$

Then, V is strongly pseudomonotone. Indeed, let $x, y \in X$ be such that $\langle V(x), \exp_x^{-1} y\rangle \geq 0$. That is,

$$\frac{1}{x^2}\left(\frac{1}{32}+\frac{1}{x}\right)x\ln x.x\ln\left(\frac{y}{x}\right) \geq 0$$

or, equivalently, $y \geq x$. In this case, we have

$$
\begin{aligned}
\langle V(y), \exp_y^{-1}x\rangle &= \frac{1}{y^2}\left(\frac{1}{32}+\frac{1}{y}\right) y\ln y.y\ln\left(\frac{x}{y}\right)\\
&= \left(\frac{1}{32}+\frac{1}{y}\right)\ln y\ln\left(\frac{x}{y}\right)\\
&\leq \left(\frac{1}{32}+\frac{1}{y}\right)\ln y\ln\left(\frac{x}{y}\right)+\left(\frac{1}{32}+\frac{1}{y}\right)\ln x\ln\left(\frac{y}{x}\right)\\
&= \left(\frac{1}{32}+\frac{1}{y}\right)(\ln y-\ln x)\ln\left(\frac{x}{y}\right)\\
&= -\left(\frac{1}{32}+\frac{1}{y}\right)\ln^2\left(\frac{x}{y}\right)\\
&\leq -\frac{1}{32}d^2(x,y)
\end{aligned}
$$

Thus, V is strongly pseudomonotone with modulus $\frac{1}{32}$. On the other hand, let $x = e^4$ and $y = e^5$, we have

$$
\begin{aligned}
\langle V(x), \exp_x^{-1}y\rangle + \langle V(y), \exp_y^{-1}x\rangle &= \left(\frac{1}{32}+\frac{1}{x}\right)\ln x.\ln\left(\frac{y}{x}\right)+\left(\frac{1}{32}+\frac{1}{y}\right)\ln y\ln\left(\frac{x}{y}\right)\\
&= \left[\left(\frac{1}{32}+\frac{1}{x}\right)\ln x-\left(\frac{1}{32}+\frac{1}{y}\right)\ln y\right]\ln\left(\frac{y}{x}\right)\\
&= \left[\left(\frac{1}{32}+\frac{1}{e^4}\right)\ln e^4-\left(\frac{1}{32}+\frac{1}{e^5}\right)\ln e^5\right]\ln\left(\frac{e^5}{e^4}\right)\\
&= 0.00832 > 0.
\end{aligned}
$$

Hence, V is not monotone on X.

It is obvious that V is continuous. Applying Theorem 3.2, VIP (2.5) has a unique solution. Let $\bar{x}$ be the solution of VIP (2.5). Then, $\langle V(\bar{x}), \exp_{\bar{x}}^{-1}y\rangle \geq 0$ for all $y \in X$. That is,

$$
\frac{1}{\bar{x}^2}\left(\frac{1}{32}+\frac{1}{\bar{x}}\right)\bar{x}\ln\bar{x}.\bar{x}\ln\left(\frac{y}{\bar{x}}\right) \geq 0, \quad \forall y \in X,
$$

or,

$$
\ln\bar{x}.\ln\left(\frac{y}{\bar{x}}\right) \geq 0, \quad \forall y \in X.
$$

This is equivalent to $\bar{x} = 1$. Therefore, VIP (2.5) has a unique solution $\bar{x} = 1$.

4. Modified projection method: linear and finite convergence

In this section, we study the modified projection method for solving the variational inequality problem VIP (2.5) when the vector filed V is strongly pseudomonotone and is Lipschitz continuous on X.

Algorithm 4.1. (Modified Projection Method)

- **Initialization**: Choose an initial point $x_0 \in X$. Set $k = 0$ and let $\{\lambda_k\} \subset (0, +\infty)$ be a sequence of real numbers.

- **Iterative step**: At stage k, given $x_k \in X$, compute x_{k+1} such that

$$\left\langle P_{x_{k+1},x_k}V(x_k) - \frac{1}{\lambda_k}\exp^{-1}_{x_{k+1}}x_k, \exp^{-1}_{x_{k+1}}y\right\rangle \geq 0, \quad \text{for all } y \in X. \tag{4.1}$$

Lemma 4.1. *The Algorithm 4.1 is well defined. More presicsely, given $x_k \in X$, there exists a unique point $x_{k+1} \in X$ satisfying (4.1).*

Proof. For each $k \in \mathbb{N}$, consider the vector field $V_k : X \to TM$ defined by

$$V_k(x) = P_{x,x_k}V(x_k) - \frac{1}{\lambda_k}\exp^{-1}_x x_k, \quad x \in X.$$

For $x, y \in X$, we have

$$\begin{aligned}
&\left\langle V_k(x), \exp^{-1}_x y\right\rangle + \left\langle V_k(y), \exp^{-1}_y x\right\rangle \\
=\ &\left\langle P_{x,x_k}V(x_k) - \frac{1}{\lambda_k}\exp^{-1}_x x_k, \exp^{-1}_x y\right\rangle + \left\langle P_{y,x_k}V(x_k) - \frac{1}{\lambda_k}\exp^{-1}_y x_k, \exp^{-1}_y x\right\rangle \\
=\ &\left\langle P_{x,x_k}V(x_k), \exp^{-1}_x y\right\rangle + \left\langle P_{y,x_k}V(x_k), \exp^{-1}_y x\right\rangle \\
&-\frac{1}{\lambda_k}\left[\left\langle \exp^{-1}_x x_k, \exp^{-1}_x y\right\rangle + \left\langle \exp^{-1}_y x_k, \exp^{-1}_y x\right\rangle\right].
\end{aligned} \tag{4.2}$$

By Remark 2.1 and Remark 2.2, we have

$$\begin{aligned}
\left\langle P_{x,x_k}V(x_k), \exp^{-1}_x y\right\rangle &= \left\langle P_{x_k,x}\circ P_{x,x_k}V(x_k), P_{x_k,x}\exp^{-1}_x y\right\rangle \\
&= \left\langle V(x_k), P_{x_k,x}\exp^{-1}_x y\right\rangle \\
&\leq \left\langle V(x_k), P_{x_k,x}\exp^{-1}_x x_k\right\rangle + \left\langle V(x_k), P_{x,x_k}\circ P_{x_k,x}\exp^{-1}_{x_k} y\right\rangle \\
&= \left\langle V(x_k), -\exp^{-1}_{x_k} x\right\rangle + \left\langle V(x_k), \exp^{-1}_{x_k} y\right\rangle
\end{aligned} \tag{4.3}$$

Similarly, one has

$$\left\langle P_{y,x_k}V(x_k), \exp^{-1}_y x\right\rangle \leq \left\langle V(x_k), -\exp^{-1}_{x_k} y\right\rangle + \left\langle V(x_k), \exp^{-1}_{x_k} x\right\rangle. \tag{4.4}$$

Moreover, by (2.1), we have

$$\left\langle \exp^{-1}_x x_k, \exp^{-1}_x y\right\rangle \geq d^2(x, x_k) + d^2(x, y) - d^2(y, x_k) \tag{4.5}$$

and

$$\left\langle \exp^{-1}_y x_k, \exp^{-1}_y x\right\rangle \geq d^2(y, x_k) + d^2(x, y) - d^2(x, x_k) \tag{4.6}$$

Combining (4.2) - (4.6), we obtain

$$\left\langle V_k(x), \exp^{-1}_x y\right\rangle + \left\langle V_k(y), \exp^{-1}_y x\right\rangle \leq -\frac{1}{\lambda_k}d^2(x, y).$$

Hence, V_k is strongly monotone with modulus $\frac{1}{\lambda_k}$ and then V_k is strongly pseudomonotone with modulus $\frac{1}{\lambda_k}$. Moreover, by Lemma 2.1, V_k is continuous. Applying Theorem 3.2, there exists a unique $x_{k+1} \in X$ satisfying (4.1). □

Remark 4.1. When M is an Euclidean space, Algorithm 4.1 reduces to the well-known projection method for solving variational inequality problems (see, e.g., [27] and references therein). Algorithm 4.1 is also considered in [9] for the case when the vector fields are strongly monotone. Lemma 4.1 is a special case of [9, Proposition 5]. However, in the proof of [9, Proposition 5] the authors use the relation $P_{a,b} \circ P_{b,c}v = P_{a,c}v$ for $a, b, c \in M$ and $v \in T_cM$ which is not true in general. Note that in [9], the authors study the method when the step sizes $\{\lambda_k\}$ is contained in an interval $[a, b]$ for suitable positive constants $a < b$.

4.1. **Linear convergence of the modified projection method.** In this subsection, we study convergence property for sequences generated by Algorithm 4.1 with different step size rules. We first need the following technical lemma.

Lemma 4.2. *Assume that $X \subset M$ is nonempty closed convex and $V : X \to TM$ is strongly pseudomonotone on X with a modulus μ and Lipschitz continuous with a constant L. If $\{x_k\}$ is a sequence generated by Algorithm 4.1 and x^* is the unique solution of VIP (2.5), then*

$$(1 + 2\lambda_k\mu - \lambda_k^2 L^2)d^2(x_{k+1}, x^*) \le d^2(x_k, x^*), \quad \forall k \in \mathbb{N}. \tag{4.7}$$

Proof. Let x^* be the solution of VIP (2.5). From (4.1), we have

$$\left\langle P_{x_{k+1},x_k}V(x_k) - \frac{1}{\lambda_k}\exp^{-1}_{x_{k+1}}x_k, \exp^{-1}_{x_{k+1}}x^* \right\rangle \ge 0.$$

It follows that

$$\langle \exp^{-1}_{x_{k+1}}x_k, \exp^{-1}_{x_{k+1}}x^*\rangle \le \lambda_k\langle P_{x_{k+1},x_k}V(x_k), \exp^{-1}_{y_{k+1}}x^*\rangle. \tag{4.8}$$

Since x^* is the solution of VIP (2.5), $\langle V(x^*), \exp^{-1}_{x^*}x_{k+1}\rangle \ge 0$. By the strong pseudomonotonicity of V, one has

$$\langle V(x_{k+1}), \exp^{-1}_{x_{k+1}}x^*\rangle \le -\mu d^2(x_{k+1}, x^*).$$

Then, by the Lipschitz continuity of V,

$$\begin{aligned}\langle P_{x_{k+1},x_k}V(x_k), \exp^{-1}_{x_{k+1}}x^*\rangle &= \langle V(x_{k+1}), \exp^{-1}_{x_{k+1}}x^*\rangle - \langle V(x_{k+1}) - P_{x_{k+1},x_k}V(x_k), \exp^{-1}_{x_{k+1}}x^*\rangle \\ &\le -\mu d^2(x_{k+1}, x^*) + ||V(x_{k+1}) - P_{x_{k+1},x_k}V(x_k)||.||\exp^{-1}_{x_{k+1}}x^*|| \\ &\le -\mu d^2(x_{k+1}, x^*) + Ld(x_k, x_{k+1}).d(x_{k+1}, x^*).\end{aligned}$$

Combining with (4.8) and using the Cauchy-Schwarz inequality, we get

$$\begin{aligned}2\langle \exp^{-1}_{x_{k+1}}x_k, \exp^{-1}_{x_{k+1}}x^*\rangle &\le -2\lambda_k\mu d^2(x_{k+1}, x^*) + 2\lambda_k Ld(x_k, x_{k+1}).d(x_{k+1}, x^*) \\ &\le -2\lambda_k\mu d^2(x_{k+1}, x^*) + d^2(x_k, x_{k+1}) + \lambda_k^2L^2d^2(x_{k+1}, x^*) \\ &= (\lambda_k^2L^2 - 2\lambda_k\mu)d^2(x_{k+1}, x^*) + d^2(x_k, x_{k+1})\end{aligned} \tag{4.9}$$

On the other hand, by (2.1),

$$d^2(x_k, x_{k+1}) + d^2(x_{k+1}, x^*) - d^2(x_k, x^*) \le 2\langle \exp^{-1}_{x_{k+1}}x_k, \exp^{-1}_{x_{k+1}}x^*\rangle.$$

This and (4.9) give

$$d^2(x_k, x_{k+1}) + d^2(x_{k+1}, x^*) - d^2(x_k, x^*) \le (\lambda_k^2L^2 - 2\lambda_k\mu)d^2(x_{k+1}, x^*) + d^2(x_k, x_{k+1})$$

which leads to (4.7). The proof is complete. ☐

Theorem 4.1. *Assume that $X \subset M$ is nonempty closed convex and $V : X \to TM$ is strongly pseudomonotone on X with a modulus μ and Lipschitz continuous with a constant L. Let $\{x_k\}$ be the sequence generated by Algorithm 4.1 with*

$$0 < \bar{\lambda} \leq \lambda_k \leq \lambda < \frac{2\mu}{L^2}, \quad \forall k \in \mathbb{N}, \tag{4.10}$$

where $\bar{\lambda}$ and λ are some positive constants. Then, $\{x_k\}$ converges linearly to the unique solution x^* of VIP (2.5). Moreover,

$$d(x_{k+1}, x^*) \leq \frac{\alpha^{k+1}}{1-\alpha} d(x_1, x_0)$$

and

$$d(x_{k+1}, x^*) \leq \frac{\alpha}{1-\alpha}$$

for all $k \in \mathbb{N}$. Here

$$\alpha = \frac{1}{\sqrt{1 + \bar{\lambda}(2\mu - \lambda L^2)}} \in (0, 1).$$

Proof. By (4.10), one has $1 + 2\lambda_k\mu - \lambda_k^2 L^2 \geq 1 + \bar{\lambda}(2\mu - \lambda L^2) > 1$ for all $k \in \mathbb{N}$. Thus,

$$\alpha = \frac{1}{\sqrt{1 + \bar{\lambda}(2\mu - \lambda L^2)}} \in (0, 1).$$

From (4.7), we have for all $k \in \mathbb{N}$ that

$$d(x_{k+1}, x^*) \leq \frac{1}{\sqrt{1 + 2\lambda_k\mu - \lambda_k^2 L^2}} d(x_k, x^*) \leq \alpha d(x_k, x^*).$$

This means that $\{x_k\}$ converges linearly to x^*. It follows from the latter inequality that

$$d(x_{k+1}, x^*) \leq \alpha d(x_k, x^*) \leq \cdots \leq \alpha^{k+1} d(x_0, x^*).$$

Moreover,

$$d(x_k, x^*) \leq d(x_k, x_{k+1}) + d(x_{k+1}, x^*) \leq d(x_k, x_{k+1}) + \alpha d(x_k, x^*).$$

Hence,

$$d(x_k, x^*) \leq \frac{1}{1-\alpha} d(x_k, x_{k+1}).$$

Therefore,

$$d(x_{k+1}, x^*) \leq \alpha^{k+1} d(x_0, x^*) \leq \frac{\alpha^{k+1}}{1-\alpha} d(x_1, x_0)$$

and

$$d(x_{k+1}, x^*) \leq \alpha d(x_k, x^*) \leq \frac{\alpha}{1-\alpha} d(x_k, x^*).$$

The proof is complete. ☐

We now consider Algorithm 4.1 with diminishing step size rules.

Theorem 4.2. *Assume that $X \subset M$ is nonempty closed convex and $V : X \to TM$ is strongly pseudomonotone on X with a modulus μ and Lipschitz continuous with a constant L. Let $\{x_k\}$ be the sequence generated by Algorithm 4.1 with*

$$\sum_{k=0}^{\infty} \lambda_k = \infty, \quad \text{and} \quad \lim_{k\to\infty} \lambda_k = 0. \tag{4.11}$$

Then, $\{x_k\}$ converges to the unique solution of VIP (2.5). Moreover, there exists $k_0 \in \mathbb{N}$ such that $\lambda_k(2\mu - \lambda_k L^2) > 0$ and

$$d(x_{k+1}, x^*) \leq \frac{1}{\sqrt{\Pi_{i=k_0}^{k}[1 + \lambda_i(2\mu - \lambda_i L^2)]}} d(x_{k_0}, x^*) \tag{4.12}$$

for all $k \geq k_0$.

Proof. Since $\lim_{k\to\infty} \lambda_k = 0$, there exists $k_0 \in \mathbb{N}$ such that $\lambda_k L^2 < \mu$ for all $k \geq k_0$. That is, $\lambda_k(2\mu - \lambda_k L^2) > \lambda_k \mu > 0$ for all $k \geq k_0$.

From (4.7), we have for $k \geq k_0$ that

$$\begin{aligned} d(x_{k+1}, x^*) &\leq \frac{1}{\sqrt{1 + \lambda_k(2\mu - \lambda_k L^2)}} d(x_k, x^*) \\ &\leq \frac{1}{\sqrt{1 + \lambda_k(2\mu - \lambda_k L^2)}} . \frac{1}{\sqrt{1 + \lambda_{k-1}(2\mu - \lambda_{k-1} L^2)}} d(x_{k-1}, x^*) \\ &\leq \cdots \\ &\leq \frac{1}{\sqrt{\Pi_{i=k_0}^{k}[1 + \lambda_i(2\mu - \lambda_i L^2)]}} d(x_{k_0}, x^*). \end{aligned}$$

Thus, we have proved (4.12).

For each $k \in \mathbb{N}$, set $\delta_k = \lambda_k(2\mu - \lambda_k L^2)$. Since $\delta_k > \lambda_k \mu$ for all $k \geq k_0$, by (4.11), one has

$$\sum_{k=k_0}^{\infty} \delta_k = \infty.$$

Thus,

$$\frac{1}{\Pi_{i=k_0}^{k}(1 + \delta_i)} \leq \frac{1}{1 + \sum_{i=k_0}^{k} \delta_i} \to 0 \quad \text{as} \quad k \to \infty.$$

Therefore,

$$d(x_{k+1}, x^*) \leq \frac{1}{\sqrt{\Pi_{i=k_0}^{k}[1 + \lambda_i(2\mu - \lambda_i L^2)]}} d(x_{k_0}, x^*) = \frac{1}{\Pi_{i=k_0}^{k}(1 + \delta_i)} d(x_{k_0}, x^*) \to 0$$

as $k \to \infty$. This implies that $\{x_k\}$ converges to x^*. The proof is complete. □

4.2. **Finite convergence of the modified projection method.** In this subsection, we provide condition for which the modified projection method is terminated after a finite iterations. For our aim, we introduce the following concept of linear conditioning for the solution set of a variational inequality problem on Hadamard manifolds. For the definition of linear conditioning, we assume that the solution set X^* of VIP (2.5) is convex. Concerning the convexity of the solution set, we recall the following result which can be deduced from [37, Corollary 4.7] .

Proposition 4.1. *If the vector field V is monotone on X, then the solution set X^* of VIP (2.5) is convex.*

Definition 4.1. The solution set X^* of VIP (2.5) is said to be linearly conditioned if there exists $\alpha > 0$ such that

$$-\left\langle V(x), \exp_x^{-1} P(x, X^*)\right\rangle \geq \alpha d(x, X^*), \quad \forall x \in X. \tag{4.13}$$

The constant α is called a modulus of the linear conditioning.

Remark 4.2. When $M = \mathbb{R}^n$, then (4.13) reduces to

$$-\langle V(x), P(x, X^*) - x\rangle \geq \alpha d(x, X^*), \quad \forall x \in X$$

which is a special case of linear conditioning for bifunctions (see, [40, 44]).

We recall the following concept for solution set of variational inequalities on manifolds which was introduced in [45].

Definition 4.2. [45] The solution set X^* of VIP (2.5) is said to be weakly sharp if there is a constant $\alpha > 0$ such that

$$\alpha \mathbb{B}_z \subset V(z) + [T_X(z) \cap N_{X^*}(z)]^\circ, \quad \text{for each } z \in X^*. \tag{4.14}$$

The constant α is called a modulus of the weak sharpness of X^*.

Remark 4.3. Notice that if the solution set X^* is weakly sharp and V is monotone, then X^* is linearly conditioned (see, [45, Proposition 3.2]).

Remark 4.4. If the solution set X^* is singleton, say $X^* = \{x^*\}$, then (4.13) is rewritten as

$$\alpha d(x, X^*) \leq -\left\langle V(x), \exp_x^{-1} x^*\right\rangle, \quad \forall x \in X.$$

Example 4.1. Let M, X and V as in Example 3.1. The solution set of VIP (2.5) is $X^* = \{1\}$. For all $x \in X$, we have

$$\begin{aligned} -\left\langle V(x), \exp_x^{-1} x^*\right\rangle &= -\frac{1}{x^2}(1 - \ln x)x.x \ln\left(\frac{x^*}{x}\right) \\ &= (1 - \ln x)\ln\left(\frac{x}{x^*}\right) \\ &\geq (1 - \ln 2) d(x, x^*). \end{aligned}$$

Thus, X^* is linearly conditioned with modulus $\alpha = 1 - \ln 2$. Note that, in this example, the solution set X^* is also weakly sharp. Indeed, one can compute that $N_{X^*}(1) = \mathbb{R}$ and $T_X(1) = \mathbb{R}_+$ and

$$V(1) + [N_{X^*}(1) \cap T_X(1)]^\circ = 1 + (-\infty, 0] = (-\infty, 1].$$

Thus,

$$\mathbb{B}_1 \subset V(1) + [N_{X^*}(1) \cap T_X(1)]^\circ,$$

i.e., X^* is weakly sharp with modulus 1.

Example 4.2. Let M, X and V as in Example 3.2. If x^* is the solution of VIP (2.5), then for $x \in X$, we have

$$-\left\langle V(x), \exp_x^{-1} x^* \right\rangle = -\left(\frac{1}{32} + \frac{1}{x}\right) \ln x \ln\left(\frac{x^*}{x}\right) = \left(\frac{1}{32} + \frac{1}{x}\right) \ln x d(x, x^*).$$

Hence, X^* is not linearly conditioned. We can also see that

$$V(1) + [N_{X^*}(1) \cap T_X(1)]^\circ = (-\infty, 0]$$

and there is no $\alpha > 0$ such that

$$\alpha \mathbb{B}_1 \subset V(1) + [N_{X^*}(1) \cap T_X(1)]^\circ.$$

Thus, X^* is not weakly sharp.

Remark 4.5. It is still open whether the linear conditioning of the solution set X^* implies the weak sharpness of X^* (see the open question in [45, Remark 3.2]).

We next establish the finite convergence of the modified projection method under the linear conditioning of the solution set.

Theorem 4.3. *Let V be strongly pseudomonotone with modulus μ and Lipschitz with constant L. Let $\{x_k\}$ be a sequence generated by Algorithm 4.1 with*

$$0 < a \le \lambda_k \le b < 2\mu/L^2, \quad \forall k \in \mathbb{N},$$

where a and b are some positive constants. If the solution set X^* is linearly conditioned, then $x_k \in X^*$ for all k sufficiently large.

Proof. Suppose that $X^* = \{x^*\}$ is linearly conditioned with modulus $\alpha > 0$. That is,

$$\alpha d(x, x^*) \le -\left\langle V(x), \exp_x^{-1} x^* \right\rangle, \quad \forall x \in X.$$

It follows that

$$\alpha d(x_{k+1}, x^*) \le -\left\langle V(x_{k+1}), \exp_{x_{k+1}}^{-1} x^* \right\rangle, \quad \forall k \in \mathbb{N}. \tag{4.15}$$

For all $k \in \mathbb{N}$, one has

$$\begin{aligned}
-\left\langle V(x_{k+1}), \exp_{x_{k+1}}^{-1} x^* \right\rangle &= \left\langle P_{x_{k+1},x_k} V(x_k) - V(x_{k+1}), \exp_{x_{k+1}}^{-1} x^* \right\rangle \\
&\quad + \left\langle -P_{x_{k+1},x_k} V(x_k), \exp_{x_{k+1}}^{-1} x^* \right\rangle \\
&\le ||P_{x_{k+1},x_k} V(x_k) - V(x_{k+1})||.||\exp_{x_{k+1}}^{-1} x^*|| \\
&\quad - \frac{1}{\lambda_k} \left\langle \exp_{x_{k+1}}^{-1} x_k, \exp_{x_{k+1}}^{-1} x^* \right\rangle \\
&\le L d(x_{k+1}, x_k).d(x_{k+1}, x^*) + \frac{1}{\lambda_k} ||\exp_{x_{k+1}}^{-1} x_k||.||\exp_{x_{k+1}}^{-1} x^*|| \\
&= \left(L + \frac{1}{\lambda_k}\right) d(x_{k+1}, x_k).d(x_{k+1}, x^*).
\end{aligned}$$

Combining with (4.15), we have

$$\alpha d(x_{k+1}, x^*) \leq \left(L + \frac{1}{\lambda_k}\right) d(x_{k+1}, x_k).d(x_{k+1}, x^*), \quad \forall k \in \mathbb{N}. \tag{4.16}$$

If the conclusion of the theorem is false, then there exists a subsequence of $\{x_k\}$ which, without loss of generality, is still denoted by $\{x_k\}$ such that $x_k \neq x^*$ for all k. Hence, by (4.16), we get

$$\alpha \leq \left(L + \frac{1}{\lambda_k}\right) d(x_{k+1}, x_k) \leq \left(L + \frac{1}{a}\right) [d(x_k, x^*) + d(x_{k+1}, x^*)],$$

for all $k \in \mathbb{N}$. Letting $k \to \infty$ and having in mind that $\lim_{k\to\infty} x_k = x^*$, we obtain $\alpha \leq 0$ which is a contradiction. Therefore, $x_k \in X^*$ for all k sufficiently large. □

4.3. **Numerical examples.** In this subsection, we provide some numerical experiments for the modified projection method.

Let the Hadamard manifold M be as in Example 3.1. As in [13], if $f : \mathbb{R}_{++} \to \mathbb{R}$ is a twice differentiable function, the gradient and the Hessian of f are given by

$$\text{grad} f(x) = x^2 f'(x) \quad \text{and} \quad \text{Hess} f(x) = f''(x) + x^{-1} f'(x), \tag{4.17}$$

where f' and f'' denote the first and second derivatives of f in the Euclidean sense.

Example 4.3. Let the Hadamard manifold M, the set X and the vector field V be as in Example 3.1. Let $f : M \to \mathbb{R}$ be defined by

$$f(x) = \ln x - \frac{1}{2}\ln^2 x, \quad \forall x \in M.$$

Then, f is twice continuously differentiable on M in the Euclidean sense. By (4.17), one has

$$\text{grad} f(x) = (1 - \ln x)x = V(x) \quad \text{and} \quad \text{Hess} f(x) = -\frac{1}{x^2}, \quad \forall x \in X.$$

It is easy to see that V is Lipschitz continuous on X with constant $L = 1$. The relation (4.1) is equivalent to

$$\frac{1}{x_{k+1}^2}\left(V(x_k) - \frac{1}{\lambda_k}\exp^{-1}_{x_{k+1}} x_k\right) \exp^{-1}_{x_{k+1}} y \geq 0, \quad \forall y \in [1, 2],$$

or

$$\left[(1 - \ln x_k)x_k - \frac{1}{\lambda_k} x_{k+1} \ln\left(\frac{x_k}{x_{k+1}}\right)\right] \ln\left(\frac{y}{x_{k+1}}\right) \geq 0, \quad \forall y \in [1, 2].$$

The finite convergence results with different step sizes $\{\lambda_k\}$ and different initial points x_0 are given in Table 1. We see that the Algorithm 4.1 is terminated faster when the step sizes are larger. Note that in this example $2\mu/L^2 = 2 - \ln 2 \approx 1.30685$.

Example 4.4. Let the Hadamard manifold M, the set X and the vector field V be as in Example 3.2. Now, let $f : \mathbb{R}_{++} \to \mathbb{R}$ be defined by

$$f(x) = \frac{1}{64}\ln^2 x - \frac{1 + \ln x}{x}, \quad \forall x \in M.$$

k	$\lambda_k = 1/10$		$\lambda_k = (k+1)/(2k+10)$		$\lambda_k = 1$	
0	1.5	2	1.5	2	1.5	2
1	1.407935	1.937647	1.407935	1.937647	1	1.198537
2	1.311962	1.870889	1.243569	1.824983	1	1
3	1.212521	1.799621	1.012176	1.661630	1	1
4	1.110190	1.723779	1	1.441960	1	1
5	1.005696	1.643358	1	1.158021	1	1
6	1	1.558421	1	1	1	1
7	1	1.469113	1	1	1	1
8	1	1.375676	1	1	1	1
9	1	1.278466	1	1	1	1
10	1	1.177969	1	1	1	1
11	1	1.074811	1	1	1	1
12	1	1	1	1	1	1

TABLE 1. Finite convergence for Algorithm 4.1

Then, f is twice continuously differentiable on M in the Euclidean sense and by (4.17), we have

$$\text{grad} f(x) = \frac{1}{32} x \ln x + \ln x = V(x), \quad \text{and} \quad \text{Hess} f(x) = \frac{1}{32x^2} + \frac{1 - \ln x}{x^3}, \quad \forall x \in X.$$

One can see that V is Lipschitz continuous on X with constant $L = 33/32$. In this case, $2\mu/L^2 = 64/1089 \approx 0.05878$.

Table 2 and Figure 1 show the numerical behavior of Algorithm 4.1 with two different initial points x_0 and with a constant step size $\lambda_k = 1/18$. Table 3 and Figure 2 show the numerical behavior of Algorithm 4.1 with two different initial points x_0 and with diminishing step sizes $\lambda_k = 1/\sqrt{k+2}$. We see that in all case the iteration points converge to 1, the unique solution of the variational inequality.

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Iterate k	x_k with $x_0 = 5$	x_k with $x_0 = 8$
0	5	8
1	4.895517	7.854259
10	4.029121	6.622970
20	3.219436	5.420137
40	2.057784	3.514055
60	1.423815	2.237478
80	1.147336	1.512444
100	1.047405	1.182220
120	1.014789	1.059240
140	1.004566	1.018551
160	1.001405	1.005735
180	1.000432	1.001766
190	1.000239	1.000979
200	1.000133	1.000543

TABLE 2. The numerical result for Example 4.4 with $\lambda_k = 1/18$

Iterate k	x_k with $x_0 = 5$	x_k with $x_0 = 8$
0	5	8
1	3.950846	6.562728
5	1.858389	3.267718
10	1.177621	1.671944
15	1.042302	1.186732
20	1.011958	1.055270
25	1.003872	1.018138
30	1.001391	1.006542
35	1.000542	1.002551
40	1.000225	1.001060
45	1.000099	1.000465

TABLE 3. The numerical result for Example 4.4 with $\lambda_k = 1/\sqrt{k+2}$

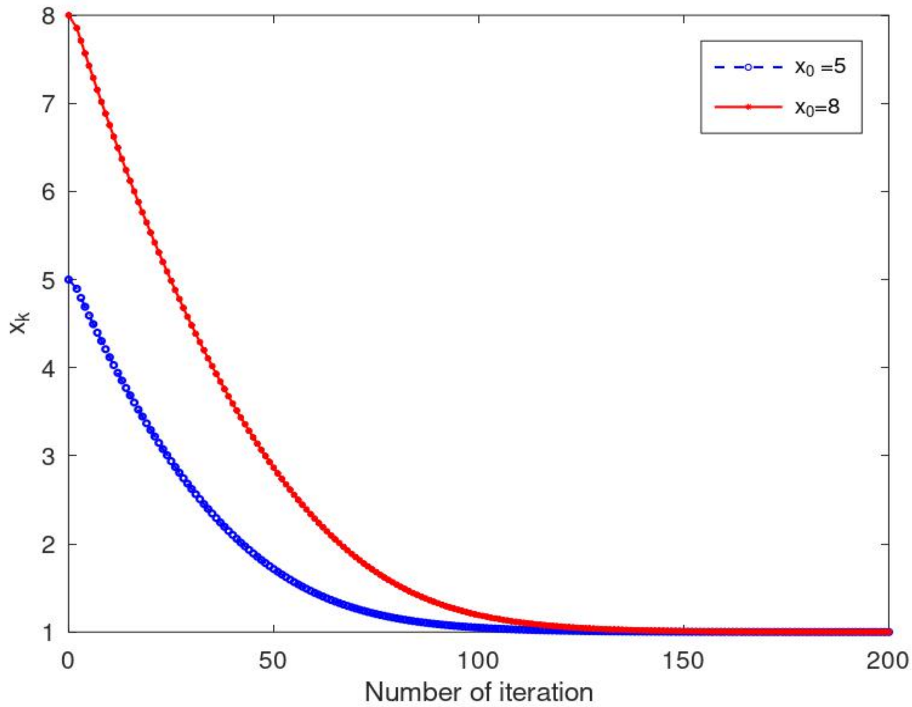

Figure 1. Iterative process of Example 4.4 with different initial points and $\lambda_k = 1/18$

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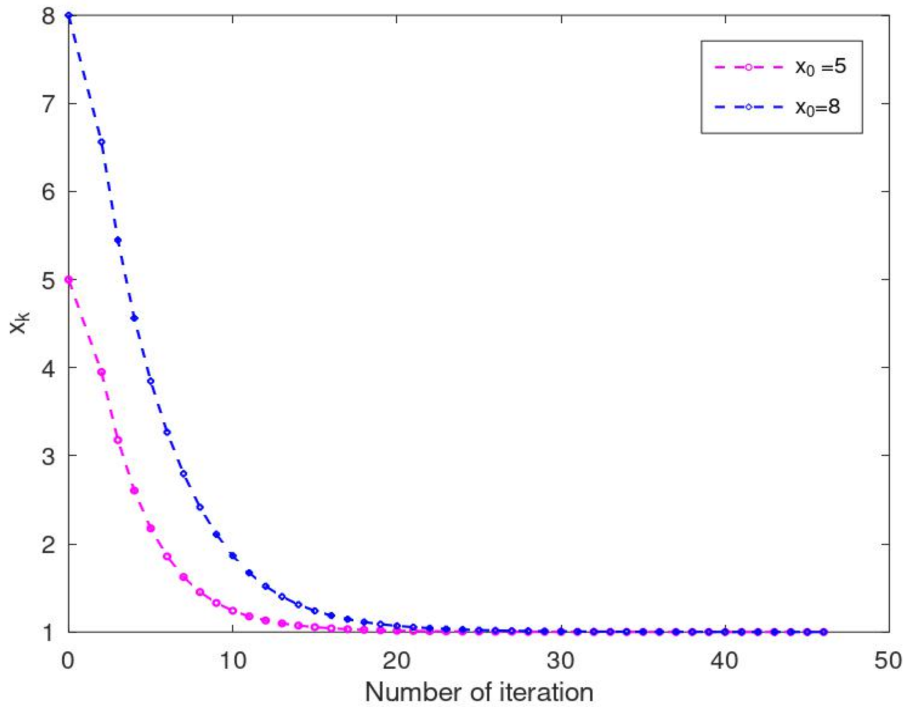

FIGURE 2. Iterative process of Example 4.4 with different initial points and $\lambda_k = 1/\sqrt{k+2}$

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