Convex Hull Results on Quadratic Programs with Non-Intersecting Constraints

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Abstract

Let $\mathcal{F} \subseteq \mathbb{R}^n$ be a nonempty closed set. Understanding the structure of the closed convex hull $\overline{\mathcal{C}}(\mathcal{F}) := \overline{\text{conv}}\,\{\, (x, xx^T) \mid x \in \mathcal{F} \,\}$ in the lifted space is crucial for solving quadratic programs related to $\mathcal{F}$. A set $\mathcal{G} \subseteq \mathbb{R}^n$ with complicated structure can be constructed by intersecting simple sets. This paper discusses the relationship between $\overline{\mathcal{C}}(\mathcal{F})$ and $\overline{\mathcal{C}}(\mathcal{G})$, where $\mathcal{G}$ results by adding non-intersecting quadratic constraints to $\mathcal{F}$. We prove that $\overline{\mathcal{C}}(\mathcal{G})$ can be represented as the intersection of $\overline{\mathcal{C}}(\mathcal{F})$ and half spaces defined by the added constraints. The proof relies on a complete description of the asymptotic cones of sets defined by a single quadratic equality and a partial characterization of the recession cone of $\overline{\mathcal{C}}(\mathcal{G})$. Our proof generalizes an existing result for bounded quadratically defined $\mathcal{F}$ with non-intersecting hollows and several results on $\overline{\mathcal{C}}(\mathcal{G})$ for $\mathcal{G}$ defined by non-intersecting quadratic constraints. The result also implies a sufficient condition for when the lifted closed convex hull of an intersection equals the intersection of the lifted closed convex hulls.

Keywords: Convex hull, Non-intersecting, Semidefinite programming, Asymptotic cone, Quadratically constrained quadratic programming

1 Introduction

Let $\mathcal{F} \subseteq \mathbb{R}^n$ be a nonempty closed set. In this paper, we are interested in the structure of the closed convex hull

$$\overline{\mathcal{C}}(\mathcal{F}) := \overline{\text{conv}}\left\{\, (x, xx^T) \;\middle|\; x \in \mathcal{F} \,\right\}$$

in the lifted space. The set $\overline{\mathcal{C}}(\mathcal{F})$ is closely related to optimization problems with quadratic objective. Specifically, such a problem minimizes a quadratic function of x (defined by

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$Q \in \mathcal{S}^n$, where $\mathcal{S}^n$ is the set of $n \times n$ real symmetric matrices, and $c \in \mathbb{R}^n$) over $\mathcal{F}$:

$$\begin{aligned} v := \inf \quad & x^T Q x + 2c^T x \\ s.t. \quad & x \in \mathcal{F}. \end{aligned} \tag{1}$$

With a new matrix variable $X = xx^T$, the objective function of (1) can be linearized as $Q \bullet X + 2c^T x$, where $Q \bullet X := \sum_{i,j} Q_{ij} X_{ij}$ is the Frobenius inner product of Q and X. Due to the linearity of the objective function, one can convexify the feasible region and obtain the following well-known convex reformulation (e.g. [22, 16]):

$$\begin{aligned} v = \inf \quad & Q \bullet X + 2c^T x \\ s.t. \quad & (x, X) \in \overline{\mathcal{C}}(\mathcal{F}). \end{aligned} \tag{2}$$

Despite being convex, (2) is computationally intractable due to the lack of explicit expression of $\overline{\mathcal{C}}(\mathcal{F})$. Therefore, understanding the structure of $\overline{\mathcal{C}}(\mathcal{F})$ is crucial for solving problem (1). Moreover, even a partial understanding of $\overline{\mathcal{C}}(\mathcal{F})$ is desired, as valid inequalities for $\overline{\mathcal{C}}(\mathcal{F})$ can help tighten the lower bounds of the convex relaxations of (1).

An important class of problems in the form of (1) is quadratically constrained quadratic programming (QCQP), where $\mathcal{F}$ is defined by quadratic constraints. That is,

$$\mathcal{F} := \{\, x \in \mathbb{R}^n \mid x^T A_i x + 2a_i^T x + \alpha_i \leq 0,\ \ i \in I \,\}, \tag{3}$$

where $I = \{1, \ldots, m\}$, $A_i \in \mathcal{S}^n$, $a_i \in \mathbb{R}^n$, and $\alpha_i \in \mathbb{R}$ for $i \in I$. Here, $\mathcal{F}$ may be nonconvex and/or unbounded. Since QCQP is NP-hard [24], it is unrealistic to expect a complete characterization of $\overline{\mathcal{C}}(\mathcal{F})$ for every quadratically defined set $\mathcal{F}$. On the other hand, $\overline{\mathcal{C}}(\mathcal{F})$ has been proved to be semidefinite representable in special cases. For instance, when $\mathcal{F}$ is defined by a single inequality [18, 26, 17, 27], a single equality [27, 29], or an interval-bounded inequality [25, 28], it is known that $\overline{\mathcal{C}}(\mathcal{F}) = \mathcal{S}(\mathcal{F})$ under mild assumptions, where

$$\mathcal{S}(\mathcal{F}) := \{\, (x, X) \in \mathbb{R}^n \times \mathcal{S}^n \mid A_i \bullet X + 2a_i^T x + \alpha_i \leq 0,\ \ i \in I,\ \ X \succeq xx^T \,\}$$

is the Shor relaxation of $\mathcal{F}$. From a different but insightful perspective, the topic is also related to the discussion of rank-1 generated cones [20, 3]. The Shor relaxation $\mathcal{S}(\mathcal{F})$ can be considered as a slice of a closed convex cone. Instead of focusing on the geometry of $\overline{\mathcal{C}}(\mathcal{F})$ directly, this approach studies when the underlying closed convex cone is rank-1 generated. They show that if the closed convex cone is rank-1 generated, then $\overline{\mathcal{C}}(\mathcal{F}) = \mathcal{S}(\mathcal{F})$. Other semidefinite representable cases, for which $\overline{\mathcal{C}}(\mathcal{F}) \subsetneq \mathcal{S}(\mathcal{F})$, include when $\overline{\mathcal{C}}(\mathcal{F})$ is defined by a convex quadratic and multiple non-intersecting linear constraints [27, 31, 8, 21, 11], and when $\mathcal{F}$ is a low-dimensional polyhedron [2]. See [7] for a survey and the references therein. Even when $\overline{\mathcal{C}}(\mathcal{F})$ is not known to be semidefinite representable, the structure and applications of $\overline{\mathcal{C}}(\mathcal{F})$ are also widely studied. Related concepts in the literature include the cone of nonnegative quadratic functions [27], set-semidefiniteness [15, 16, 13], generalized copositivity [9], completely positivity over sets [5] and set-copositivity [6].

To explore the lifted closed convex hull for more sets, we take a small step and consider a closed set $\mathcal{G} \in \mathbb{R}^n$ resulting by adding one more constraint to an $\mathcal{F}$ with known $\overline{\mathcal{C}}(\mathcal{F})$. Specifically, suppose that $\mathcal{G} := \mathcal{F} \cap \mathcal{H}$ where $\mathcal{H}$ is defined by a single inequality, we hope to derive $\overline{\mathcal{C}}(\mathcal{G})$ based on $\overline{\mathcal{C}}(\mathcal{F})$ and the simple structure of $\mathcal{H}$. If this is successful, we can repeat the process to generate more convex hull results in the lifted space.

We are interested in the relation between $\overline{\mathcal{C}}(\mathcal{G})$, $\overline{\mathcal{C}}(\mathcal{F})$ and $\overline{\mathcal{C}}(\mathcal{H})$. By definition, it is clear that

$$\overline{\mathcal{C}}(\mathcal{G}) = \overline{\mathcal{C}}(\mathcal{F} \cap \mathcal{H}) \subseteq \overline{\mathcal{C}}(\mathcal{F}) \cap \overline{\mathcal{C}}(\mathcal{H}).$$

On the other hand, $\overline{\mathcal{C}}(\mathcal{G})$ can be a proper subset of $\overline{\mathcal{C}}(\mathcal{F}) \cap \overline{\mathcal{C}}(\mathcal{H})$ in general. Such an example can be found even when $\mathcal{F}$ and $\mathcal{H}$ are as simple as two intersecting ellipsoids [8].

In this paper, we propose a sufficient condition for $\overline{\mathcal{C}}(\mathcal{G}) = \overline{\mathcal{C}}(\mathcal{F}) \cap \overline{\mathcal{C}}(\mathcal{H})$. Specifically, we show that the equation holds when $\mathcal{H}$ is defined by a non-intersecting quadratic inequality with nonzero Hessian. For the rest of the paper, unless stated otherwise, we focus on sets with the following structure:

- $\mathcal{F}$ is a nonempty closed set in $\mathbb{R}^n$;
- $\mathcal{H} := \{ x \in \mathbb{R}^n \mid x^T W x + 2w^T x + \omega \le 0 \}$ is a nonempty proper subset of $\mathbb{R}^n$, where $W \in \mathcal{S}^n$, $w \in \mathbb{R}^n$, $\omega \in \mathbb{R}$;
- $\mathcal{G} = \mathcal{F} \cap \mathcal{H}$.

Note that although the study is motivated by and primarily applied to quadratically defined sets, our approach does not rely on the quadratic structure of $\mathcal{F}$ or $\mathcal{G}$. Moreover, we omit the trivial cases when $\mathcal{H} = \emptyset$ or $\mathbb{R}^n$ in the discussion. We show in the paper that $\overline{\mathcal{C}}(\mathcal{G}) = \overline{\mathcal{C}}(\mathcal{F}) \cap \overline{\mathcal{C}}(\mathcal{H})$ under the following assumptions.

Assumption 1 (nonzero)**.** $W \neq 0$.

The nonzero assumption assures that $\mathcal{H}$ is not linearly defined. The necessity of the assumption is demonstrated by Example 1 in Section 3.

Assumption 2 (non-intersecting)**.**

$$x^T W x + 2w^T x + \omega = 0 \quad \Longrightarrow \quad x \in \mathcal{F}. \tag{4}$$

Geometrically, the non-intersecting assumption is satisfied if and only if the boundary of $\mathcal{H}$ is contained in $\mathcal{F}$ (and equivalently contained in $\mathcal{G}$).

When $\mathcal{F}$ is quadratically defined as in (3), the non-intersecting assumption (4) holds if and only if

$$\sup \{ x^T A_i x + 2a_i^T x + \alpha_i \mid h(x) = 0 \} \le 0 \quad \forall\, i \in I. \tag{5}$$

where $h(x) := x^T W x + 2w^T x + \omega$. If there exist $\hat{x}$ and $\bar{x}$ such that $h(\hat{x}) < 0 < h(\bar{x})$, then the optimization problem in (5) enjoys exact Shor relaxations due to the S-Lemma with equality [29]. Therefore, the non-intersecting assumption (4) can be checked by solving semidefinite programs. On the other hand, if $h(x) \ge 0$ for all $x \in \mathbb{R}^n$ and $\mathcal{H} \neq \emptyset$, then $W \succeq 0$ and $w \in \text{Range}(W)$. In this case, $h(x) = (x+b)^T W (x+b)^T + \beta$ for some $b \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$. We observe that $\beta \ge 0$ since $h(x) \ge 0$ for all $x \in \mathbb{R}^n$, and that $\beta \le 0$ since $\mathcal{H} \neq \emptyset$. Therefore, the constraint $h(x) = 0$ is equivalent to the affine equality constraint $W^{1/2}(x+b) = 0$. With suitable substitution, the optimization problem in (5) can be transformed to unconstrained quadratic problems and solved easily.

Concepts similar to Assumption 2 have been mentioned in [30] and [3]. We restate those concepts here to avoid possible confusion. In [30], two linear constraints are called "non-intersecting" if the hyperplanes defined by the constraints do not intersect *inside the unit ball*. In [3], "non-interacting" constraints are explained as that if any of the constraints is active at a certain point x, then all the other constraints are satisfied *strictly* at x.

Under Assumptions 1 and 2, we show in this paper that $\overline{\mathcal{C}}(\mathcal{G}) = \overline{\mathcal{C}}(\mathcal{F}) \cap \mathcal{L}(\mathcal{H})$ (Theorem 1), where

$$\mathcal{L}(\mathcal{H}) := \left\{ (x, X) \;\middle|\; W \bullet X + 2w^T x + \omega \leq 0 \right\}.$$

Since $\mathcal{H}$ is defined by a single quadratic inequality, it is known that $\overline{\mathcal{C}}(\mathcal{H}) = \mathcal{S}(\mathcal{H})$. We then have $\overline{\mathcal{C}}(\mathcal{G}) = \overline{\mathcal{C}}(\mathcal{F}) \cap \mathcal{L}(\mathcal{H}) = \overline{\mathcal{C}}(\mathcal{F}) \cap \mathcal{S}(\mathcal{H}) = \overline{\mathcal{C}}(\mathcal{F}) \cap \overline{\mathcal{C}}(\mathcal{H})$. Our proof approach is motivated by the prior work on bounded quadratic programs with hollows [30]. When $\mathcal{F}$ is bounded, so is $\mathcal{G}$, and $\overline{\mathcal{C}}(\mathcal{G})$ is reduced to

$$\mathcal{C}(\mathcal{G}) := \operatorname{conv} \left\{ (x, xx^T) \;\middle|\; x \in \mathcal{G} \right\}.$$

When $\mathcal{F}$ is bounded and quadratically defined, it is shown in [30] that $\mathcal{C}(\mathcal{G}) = \mathcal{C}(\mathcal{F}) \cap \mathcal{L}(\mathcal{H})$ under the non-intersecting assumption. (The non-intersecting assumption implies the nonzero assumption in the bounded case.) In this paper, we generalize the result to allow general unbounded closed $\mathcal{F}$. This generalization allows more intriguing applications. We provide four quadratically defined[1] examples in Section 4.

We remark here that the proofs in [30] do not generalize to the unbounded case directly. In particular, two proofs are provided in [30]. The first one relies on discussing the locations of optimal solutions of $\min \{ x^T Q x + 2c^T x \mid x \in \mathcal{F} \}$, while the second considers the extreme points of $\mathcal{C}(\mathcal{F}) \cap \mathcal{L}(\mathcal{H})$. For the first proof, when $\mathcal{F}$ is compact, an optimal solution of $\min \{ x^T Q x + 2c^T x \mid x \in \mathcal{F} \}$ always exists due to the Weierstrass extreme value theorem. However, in the unbounded case, an optimal solution may be unattainable even if the optimal value is finite. For the second, when $\mathcal{F}$ is bounded, $\mathcal{C}(\mathcal{G})$ is closed and is generated by its extreme points, which are in the form of (x, xx^T). When $\mathcal{F}$ is unbounded, $\mathcal{C}(\mathcal{G})$ is not necessarily closed, and $\overline{\mathcal{C}}(\mathcal{G})$ is generated by both its extreme points and its extreme directions. However, characterizing the extreme directions of $\overline{\mathcal{C}}(\mathcal{G})$ seems not to be an easy task.

To overcome the difficulty, we tailor a technical lemma by Dickinson et al. [13] to build a connection between $\overline{\mathcal{C}}(\mathcal{G})$ and $\mathcal{C}(\mathcal{G})$. As the connection is related to the asymptotic cone of $\mathcal{G}$, we introduce basic properties of the asymptotic cones and explore the structure of the asymptotic cones of sets defined by a quadratic equality in Section 2. In Section 3, we use the connection to build the proof of $\overline{\mathcal{C}}(\mathcal{G}) = \overline{\mathcal{C}}(\mathcal{F}) \cap \mathcal{L}(\mathcal{H})$, for which two nontrivial pieces of the claim are considered sequentially. To show the necessity of the assumptions, a counterexample is provided after the proof. Three corollaries follow the main result with a generalization of the non-intersecting concept to allow multiple constraints in $\mathcal{H}$. In Section 4, we provide four examples where the theory can be applied. The paper is concluded in Section 5.

[1] The examples are all quadratically defined because little is known about $\overline{\mathcal{C}}(\mathcal{F})$ when $\mathcal{F}$ is non-quadratic.

Notation: For a nonempty set $S \subseteq \mathbb{R}^n$, we denote its boundary, interior, closure by $\mathrm{bd}(S)$, $\mathrm{int}(S)$, and $\overline{S}$, respectively. The (possibly nonconvex) cone generated by S is denoted by $\mathrm{cone}(S)$, and the conic hull of S is represented as $\mathrm{cone}\,\mathrm{conv}(S)$. Their closures are represented as $\overline{\mathrm{cone}}(S)$ and $\overline{\mathrm{cone}}\,\mathrm{conv}(S)$, respectively. The cardinality of S is denoted by $|S|$, and we use $\pm S$ to represent the set $S \cup (-S)$. When S is convex, the recession cone of S is denoted by $\mathrm{Rec}(S)$, and the set of extreme points of S is denoted by $\mathrm{ext}(S)$.

2 Asymptotic cones

In this section, we focus on describing the unboundedness of a nonconvex set. We first introduce the asymptotic cones of nonempty sets.

For a nonempty set $S \subseteq \mathbb{R}^n$, the asymptotic cone of S is defined as

$$S_\infty := \left\{ d \in \mathbb{R}^n \;\middle|\; \exists\, t_k \to \infty, \{x^k\}_k \subseteq S \text{ such that } \lim_{k\to\infty} \frac{x^k}{t_k} = d \right\}.$$

The asymptotic cone S_∞ can be regarded as an extension of the recession cone of S. In particular, when S is convex, S_∞ coincides with the recession cone of S. We refer the readers to [4] for more detailed discussions about S_∞. In the following lemma, we list some properties of S_∞ which relate to our study.

Lemma 1 ([4]). *Let $S \subseteq \mathbb{R}^n$ be nonempty. Then,*

1. *S_∞ is a closed cone;*
2. *S is bounded if and only if $S_\infty = \{0\}$;*
3. *for any $\emptyset \neq T \subseteq S$, $T_\infty \subseteq S_\infty$;*
4. *for any $T \subseteq \mathbb{R}^n$ such that $S \cap T \neq \emptyset$, $(S \cap T)_\infty \subseteq S_\infty \cap T_\infty$;*
5. *if S is a closed convex set that contains no line, then $S = \mathrm{conv}(\mathrm{ext}(S)) + S_\infty$.*

2.1 Sets defined by one quadratic constraint

When S is a polyhedron, it is well known that S_∞ (i.e. $\mathrm{Rec}(S)$) can be described explicitly in a simple form. Unlike in the polyhedral case, we are not aware of any general characterization of S_∞ when S is defined by quadratic constraints. A special case is when S is defined by a single quadratic inequality. The following characterization of S_∞ is provided by Dickinson et al.

Proposition 1 ([13]). *For $S = \{\, x \in \mathbb{R}^n \mid x^T A x + 2a^T x + \alpha \leq 0 \,\} \neq \emptyset$, we have*

$$S_\infty = \begin{cases} \{\, d \in \mathbb{R}^n \mid d^T A d \leq 0 \,\}, & \text{if } A \not\succeq 0 \\ \{\, d \in \mathbb{R}^n \mid d^T A d \leq 0,\ a^T d \leq 0 \,\}, & \text{if } A \succeq 0. \end{cases}$$

To facilitate the discussion in Section 3, we are interested in the form of S_∞ when S is defined by an *equality* constraint. Our result in this section can be regarded as a complement to Proposition 1 to fully characterize S_∞ when S is defined by a single quadratic (equality or inequality) constraint.

Lemma 2. *Suppose that $S = \{ x \in \mathbb{R}^n \mid x^T Ax + 2a^T x + \alpha = 0 \}$ is nonempty and A has a positive eigenvalue. If d satisfies $d^T Ad = 0$ and $a^T d < 0$, then $d \in S_\infty$.*

Proof. Let v be any vector such that $v^T Av > 0$. Consider the following bivariate quadratic function

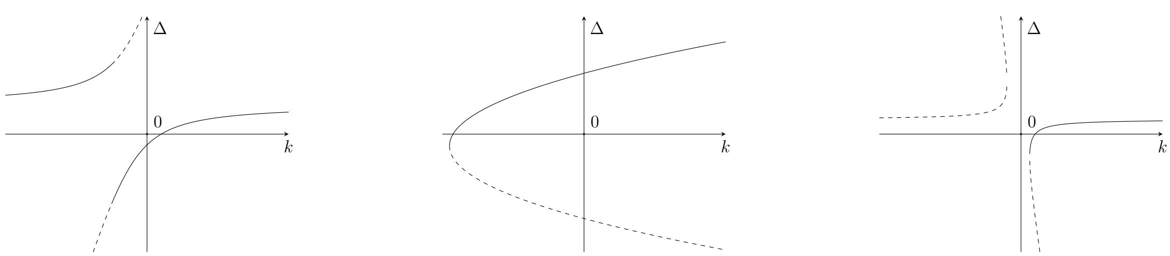
$$\begin{aligned} f(k,\Delta) :&= (kd + \Delta v)^T A(kd + \Delta v) + 2a^T(kd + \Delta v) + \alpha \\ &= (v^T Av)\Delta^2 + 2(kd^T Av + a^T v)\Delta + (2ka^T d + \alpha). \end{aligned}$$

Since $a^T d < 0$, there exists $K \in \mathbb{R}$ such that $f(k, 0) = 2ka^T d + \alpha < 0$ for all $k \geq K$. For each $k \geq K$, let $\delta_k := a^T v + kd^T Av$ and

$$\Delta_k := \begin{cases} \left(-\delta_k + \sqrt{\delta_k^2 - (v^T Av)(2ka^T d + \alpha)}\right) / \left(v^T Av\right), & \text{if } \delta_k \geq 0 \\ \left(-\delta_k - \sqrt{\delta_k^2 - (v^T Av)(2ka^T d + \alpha)}\right) / \left(v^T Av\right), & \text{if } \delta_k < 0. \end{cases} \tag{6}$$

Then $f(k, \Delta_k) = 0$ and $\lim_{k\to\infty} \Delta_k/k = 0$. This implies that $d \in S_\infty$ as $kd + \Delta_k v \in S$ and $\lim_{k\to\infty}(kd+\Delta_k v)/k = d$. (See Figure 1 for the three cases of the level set $\{ (k,\Delta) \mid f(k,\Delta) = 0 \}$.)

□ □



(a) (b) (c)

Figure 1: Plots of possible level sets $\{ (k,\Delta) \mid f(k,\Delta) = 0 \}$ in the proof of Lemma 2. The solid segments represent how Δ_k is defined in (6).

Lemma 3. *Suppose $S = \{ x \in \mathbb{R}^n \mid x^T Ax + 2a^T x + \alpha = 0 \}$ is nonempty. If d satisfies $d^T Ad = 0$ and $a^T d = 0$, then $\pm d \in S_\infty$.*

Proof. The proof is straightforward when $A = 0$. Now suppose that $A \neq 0$ and v is a vector in S. Similar to the proof of Lemma 2, let

$$\begin{aligned} f(k,\Delta) :&= (kd + \Delta v)^T A(kd + \Delta v) + 2a^T(kd + \Delta v) + \alpha \\ &= (v^T Av)\Delta^2 + 2(kd^T Av + a^T v)\Delta + \alpha. \end{aligned}$$

We consider two cases. First, if $d^T Av = 0$, then $f(k, 1) = v^T Av + 2a^T v + \alpha = 0$ for all $k \in \mathbb{R}$. Therefore, $d \in S_\infty$ as $kd + v \in S$ and $\lim_{k\to\infty}(kd + v)/k = d$. Second, if $d^T Av \neq 0$, then, $|kd^T Av + a^T v| \to \infty$ as $k \to \infty$. For sufficiently large k, if $v^T Av \neq 0$, we set Δ_k as defined in (6) (with $a^T d = 0$). If $v^T Av = 0$, we set $\Delta_k := -\alpha/(2kd^T Av + 2a^T v)$. With either definition of Δ_k, $kd + \Delta_k \in S$ and $\lim_{k\to\infty} \Delta_k = 0$. Therefore, $d \in S_\infty$. Replacing d with $-d$ in the above proof, we have $-d \in S_\infty$. □ □

Combining the results above, for $S = \{ x \in \mathbb{R}^n \mid x^T Ax + 2a^T x + \alpha = 0 \}$, we provide an explicit description of S_∞ below.

Proposition 2. *For $S = \{ x \in \mathbb{R}^n \mid x^T Ax + 2a^T x + \alpha = 0 \} \neq \emptyset$, we have*

$$S_\infty = \begin{cases} \{ d \in \mathbb{R}^n \mid d^T Ad = 0,\ a^T d \leq 0 \}, & \text{if } A \succeq 0 \text{ and } A \neq 0 \\ \{ d \in \mathbb{R}^n \mid d^T Ad = 0,\ a^T d \geq 0 \}, & \text{if } A \preceq 0 \text{ and } A \neq 0 \\ \{ d \in \mathbb{R}^n \mid d^T Ad = 0 \}, & \text{if } A \text{ is indefinite} \\ \{ d \in \mathbb{R}^n \mid a^T d = 0 \}, & \text{if } A = 0. \end{cases}$$

Proof. For the forward containment, note that

$$\begin{aligned} S_\infty &= \left(\{ x \in \mathbb{R}^n \mid x^T Ax + 2a^T x + \alpha \leq 0 \} \cap \{ x \in \mathbb{R}^n \mid x^T Ax + 2a^T x + \alpha \geq 0 \}\right)_\infty \\ &\subseteq \{ x \in \mathbb{R}^n \mid x^T Ax + 2a^T x + \alpha \leq 0 \}_\infty \cap \{ x \in \mathbb{R}^n \mid x^T Ax + 2a^T x + \alpha \geq 0 \}_\infty \\ &= \begin{cases} \{ d \in \mathbb{R}^n \mid d^T Ad = 0,\ a^T d \leq 0 \}, & \text{if } A \succeq 0 \text{ and } A \neq 0 \\ \{ d \in \mathbb{R}^n \mid d^T Ad = 0,\ a^T d \geq 0 \}, & \text{if } A \preceq 0 \text{ and } A \neq 0 \\ \{ d \in \mathbb{R}^n \mid d^T Ad = 0 \}, & \text{if } A \text{ is indefinite} \\ \{ d \in \mathbb{R}^n \mid a^T d = 0 \}, & \text{if } A = 0, \end{cases} \end{aligned}$$

where the last equality comes from Proposition 1.

For the reverse containment, note that A has a positive eigenvalue in the first and third cases, and $-A$ has a positive eigenvalue in the second and third cases. Then the reverse containment is a direct consequence of Lemmas 2 and 3.

□ □

3 The closed convex hull result

In this section, we prove $\overline{\mathcal{C}}(\mathcal{G}) = \overline{\mathcal{C}}(\mathcal{F}) \cap \mathcal{L}(\mathcal{H})$ under Assumptions 1 and 2. We start from a simple observation. Let $T := \{ Y \in \mathcal{S}^{n+1} \mid Y_{11} = 1 \}$. By definition, it is easy to check that

$$\operatorname{conv} \left\{ \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}^T \,\middle|\, x \in S \right\} = \operatorname{cone} \operatorname{conv} \left\{ \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}^T \,\middle|\, x \in S \right\} \cap T,$$

where Y_{11} is the top left element of Y. We observe that the same statement also holds for the corresponding closures.

Lemma 4. *Let $S \subseteq \mathbb{R}^n$ be a nonempty closed set. Then*

$$\overline{\mathrm{conv}} \left\{ \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}^T \,\middle|\, x \in S \right\} = \overline{\mathrm{cone}}\, \mathrm{conv} \left\{ \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}^T \,\middle|\, x \in S \right\} \cap T.$$

Proof. The forward containment "$\subseteq$" is straightforward. Now let Y be a matrix of the set on the right side of the equation. Then $Y_{11} = 1$, and there exists a sequence $\{Y_m\}_m$ such that $Y_m \to Y$ as $m \to \infty$ and

$$Y_m = \sum_{i=1}^{k_m} \lambda_{m_i} \begin{pmatrix} 1 \\ x_{m_i} \end{pmatrix} \begin{pmatrix} 1 \\ x_{m_i} \end{pmatrix}^T$$

for some $k_m \geq 0$, λ_{m_i} and $x_{m_i} \in S$. In particular, $\lambda_m := \sum_{i=1}^{k_m} \lambda_{m_i} \to 1$ as $m \to \infty$. Then $\tilde{Y}_m := Y_m / \lambda_m \in \mathrm{conv}\left\{ \binom{1}{x} \binom{1}{x}^T \mid x \in S \right\}$ and $\tilde{Y}_m \to Y$ as $m \to \infty$. Therefore, $Y \in \overline{\mathrm{conv}}\left\{ \binom{1}{x} \binom{1}{x}^T \mid x \in S \right\}$. □ □

The following lemma from [13] is crucial to characterize the closed convex hull.

Lemma 5 ([13])**.** *Let $S \subseteq \mathbb{R}^n$ be a nonempty closed set. Then*

$$\overline{\mathrm{cone}}\, \mathrm{conv} \left\{ yy^T \mid y \in \{1\} \times S \right\} = \mathrm{conv} \left\{ yy^T \mid y \in \mathrm{cone}(\{1\} \times S) \cup (\{0\} \times S_\infty) \right\}.$$

Interpreting Lemma 5 by rewriting the equation in equivalent forms, we have the following lemma to characterize the difference between the convex hull $\mathcal{C}(S)$ and its closure $\overline{\mathcal{C}}(S)$.

Lemma 6. *Let $S \subseteq \mathbb{R}^n$ be a nonempty closed set. Then*

$$\overline{\mathcal{C}}(S) = \mathcal{C}(S) + \mathrm{conv} \left\{ (0, dd^T) \mid d \in S_\infty \right\}.$$

Proof. By Lemma 5,

$$\begin{aligned}
&\overline{\mathrm{cone}}\, \mathrm{conv} \left\{ \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}^T \,\middle|\, x \in S \right\} \\
&= \mathrm{conv} \left\{ yy^T \,\middle|\, y \in \mathrm{cone}(\{1\} \times S) \cup (\{0\} \times S_\infty) \right\} \\
&= \mathrm{conv} \left\{ yy^T \,\middle|\, y \in \mathrm{cone}(\{1\} \times S) \right\} + \mathrm{conv} \left\{ yy^T \,\middle|\, y \in \{0\} \times S_\infty \right\} \\
&= \mathrm{cone}\, \mathrm{conv} \left\{ \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}^T \,\middle|\, x \in S \right\} + \mathrm{conv} \left\{ \begin{pmatrix} 0 \\ d \end{pmatrix} \begin{pmatrix} 0 \\ d \end{pmatrix}^T \,\middle|\, d \in S_\infty \right\}.
\end{aligned}$$

Intersecting both sides of the equation with T, we have

$$\begin{aligned}
\overline{\mathrm{conv}} \left\{ \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}^T \,\middle|\, x \in S \right\} = \mathrm{conv} &\left\{ \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}^T \,\middle|\, x \in S \right\} \\
&+ \mathrm{conv} \left\{ \begin{pmatrix} 0 \\ d \end{pmatrix} \begin{pmatrix} 0 \\ d \end{pmatrix}^T \,\middle|\, d \in S_\infty \right\}
\end{aligned}$$

by Lemma 4. Dropping the first component of the matrices, which is fixed to 1, the above equation is equivalent to $\overline{\mathcal{C}}(S) = \mathcal{C}(S) + \mathrm{conv}\{(0, dd^T) \mid d \in S_\infty\}$. □ □

In fact, Lemma 6 helps build a connection between $\mathrm{Rec}(\overline{\mathcal{C}}(S))$ and S_∞. Applying the description of the asymptotic cone in Section 2 to $\mathrm{bd}(\mathcal{H}) = \{ x \in \mathbb{R}^n \mid x^T W x + 2w^T x + \omega = 0 \}$, we have the following key observation.

Proposition 3. *If $d^T W d = 0$, then $(0, dd^T) \in \mathrm{Rec}(\overline{\mathcal{C}}(\mathcal{G}))$.*

Proof. Since $W \neq 0$ by Assumption 1, Proposition 2 indicates that $\{d \in \mathbb{R}^n \mid d^T W d = 0\} = \pm(\mathrm{bd}(\mathcal{H}))_\infty$. By Assumption 2, $\mathrm{bd}(\mathcal{H})$ is contained in $\mathcal{G}$. Consequently, $\pm(\mathrm{bd}(\mathcal{H}))_\infty \subseteq \pm\mathcal{G}_\infty$ by Lemma 1. Therefore, for any $(x, X) \in \overline{\mathcal{C}}(\mathcal{G}), \lambda \geq 0$ and $d \in \mathbb{R}^n$ such that $d^T W d = 0$,

$$\begin{aligned}(x, X) + \lambda(0, dd^T) &\in \overline{\mathcal{C}}(\mathcal{G}) + \mathrm{conv}\,\{ (0, dd^T) \mid d \in \pm\mathcal{G}_\infty \} \\ &= \overline{\mathcal{C}}(\mathcal{G}) + \mathrm{conv}\,\{ (0, dd^T) \mid d \in \mathcal{G}_\infty \} \\ &= \mathcal{C}(\mathcal{G}) + \mathrm{conv}\,\{ (0, dd^T) \mid d \in \mathcal{G}_\infty \} = \overline{\mathcal{C}}(\mathcal{G}),\end{aligned}$$

where the second equation holds because of Lemma 6. That is, $(0, dd^T) \in \mathrm{Rec}(\overline{\mathcal{C}}(\mathcal{G}))$. □
□

With the help of Assumption 1 and Proposition 2, Proposition 3 establishes a connection between $\{d \in \mathbb{R}^n \mid d^T W d = 0\}$ and $\mathrm{bd}(\mathcal{H})$. Without Assumption 1, $\{d \in \mathbb{R}^n \mid d^T W d = 0\}$ is equal to $\mathbb{R}^n$ and provides no information about $\mathcal{H}$. In addition, the proof of Proposition 3 is the only place where Assumption 1 and Proposition 2 are explicitly used in the proof of the main result (Theorem 1).

As another technical lemma for the main proof, we restate the famous rank-1 decomposition by Sturm and Zhang.

Lemma 7 ([27])**.** *Let V be a symmetric matrix, and suppose $Y \succeq 0$ with $V \bullet Y = 0$ and $\mathrm{rank}(Y) = r$. Then there exists a rank-1 decomposition $Y = \sum_{i=1}^r y^i (y^i)^T$ such that $y^i \neq 0$ and $(y^i)^T V y^i = 0$ for all $i = 1, \ldots, r$.*

The proof of our main theorem is constructed by the following two propositions. In the first proposition, we show that $\overline{\mathcal{C}}(\mathcal{F}) \cap \mathrm{bd}(\mathcal{L}(\mathcal{H})) \subseteq \overline{\mathcal{C}}(\mathcal{G})$ by proving a more general statement.

Proposition 4. *If $X \succeq xx^T$ and $W \bullet X + 2w^T x + \omega = 0$, then $(x, X) \in \overline{\mathcal{C}}(\mathcal{G})$.*

Proof. Since $\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succeq 0$ and $\begin{pmatrix} \omega & w^T \\ w & W \end{pmatrix} \bullet \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} = 0$, by Lemma 7, there exist nonzero $y^j = \begin{pmatrix} z_0^j \\ z^j \end{pmatrix} \in \mathbb{R}^{1+n}$ for $j = 1, \ldots, r$ such that

$$0 = \begin{pmatrix} z_0^j \\ z^j \end{pmatrix}^T \begin{pmatrix} \omega & w^T \\ w & W \end{pmatrix} \begin{pmatrix} z_0^j \\ z^j \end{pmatrix} = (z^j)^T W z^j + 2 z_0^j w^T z^j + \omega (z_0^j)^2 \tag{7}$$

and

$$\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} = \sum_{i=1}^r y^i (y^i)^T = \sum_{j \in J} (z_0^j)^2 \begin{pmatrix} 1 \\ x^j \end{pmatrix} \begin{pmatrix} 1 \\ x^j \end{pmatrix}^T + \sum_{j \notin J} \begin{pmatrix} 0 \\ z^j \end{pmatrix} \begin{pmatrix} 0 \\ z^j \end{pmatrix}^T,$$

where $J := \{ j \mid y_1^j \neq 0 \}$ and $x^j = z^j/(z_0^j)$ for $j \in J$. Equivalently, $\sum_{j\in J}(z_0^j)^2 = 1$ and

$$(x, X) = \sum_{j\in J}(z_0^j)^2(x^j, x^j(x^j)^T) + \sum_{j\notin J}(0, z^j(z^j)^T).$$

By (7), $(z^j)^T W z^j = 0$ for $j \notin J$. Therefore, Proposition 3 indicates that $\sum_{j\notin J}(0, z^j(z^j)^T) \in \mathrm{Rec}(\overline{\mathcal{C}}(\mathcal{G}))$. Also by (7), $x^j \in \mathrm{bd}(\mathcal{H}) \subseteq \mathcal{G}$ for all $j \in J$. Therefore, $(x, X) \in \mathcal{C}(\mathcal{G}) + \mathrm{Rec}(\overline{\mathcal{C}}(\mathcal{G})) \subseteq \overline{\mathcal{C}}(\mathcal{G})$. □ □

We remark here that when W is (positive or negative) definite, $|J| = r$ and the proof of Proposition 4 reduces to the alternative proof of Corollary 1 in [30]. When W is not definite, the term $\sum_{j\notin j}(0, z^j(z^j)^T)$ is related to $\mathrm{Rec}(\overline{\mathcal{C}}(\mathcal{G}))$ in our proof by the prior discussion on the asymptotic cones, which helps generalize the result.

Leveraging Proposition 4, we show in the following proposition that $\mathcal{C}(\mathcal{F}) \cap \mathrm{int}(\mathcal{L}(\mathcal{H})) \subseteq \overline{\mathcal{C}}(\mathcal{G})$. In [30], this case is trivial as it suffices to consider the extreme points of $\mathcal{C}(\mathcal{F}) \cap \mathcal{L}(\mathcal{H})$ when $\mathcal{F}$ is compact. Here, we need to adopt a different approach due to the unboundedness of $\mathcal{F}$. We consider an arbitrary point (x, X) in $\mathcal{C}(\mathcal{F}) \cap \mathrm{int}(\mathcal{L}(\mathcal{H}))$ and decompose it into "rank-1" points in $\mathcal{C}(\mathcal{F})$. If all the "rank-1" points are in $\mathcal{L}(\mathcal{H})$, then (x, X) is in $\mathcal{C}(\mathcal{G})$; if some "rank-1" point is not in $\mathcal{L}(\mathcal{H})$, we construct a convex combination of the point and (x, X) which is in $\mathrm{bd}(\mathcal{L}(\mathcal{H}))$, and consider the convex combination instead.

Proposition 5. *If $(x, X) \in \mathcal{C}(\mathcal{F})$ and $W \bullet X + 2w^T x + \omega < 0$, then $(x, X) \in \overline{\mathcal{C}}(\mathcal{G})$.*

Proof. Since $(x, X) \in \mathcal{C}(\mathcal{F})$, there exist $x^j \in \mathcal{F}$, $\mu_j > 0$ for $j = 1, \ldots, p$, such that $\sum \mu_j = 1$ and

$$(x, X) = \sum_{j=1}^{p} \mu_j(x^j, x^j(x^j)^T).$$

Let $J := \{ j \mid (x^j)^T W x^j + 2w^T x^j + \omega \leq 0 \}$. If $|J| = p$, then $x^j \in \mathcal{F} \cap \mathcal{H} = \mathcal{G}$ for $j = 1, \ldots, p$, and therefore $(x, X) \in \mathcal{C}(\mathcal{G}) \subseteq \overline{\mathcal{C}}(\mathcal{G})$. If $|J| < p$, then for each $j \notin J$, we have $W \bullet (x^j(x^j)^T) + 2w^T x^j + \omega > 0$. Since the hyperplane $\{ (x, X) \mid W \bullet X + 2w^T x + \omega = 0 \}$ separates $(x^j, x^j(x^j)^T)$ and (x, X), there exists $\gamma_j \in (0, 1)$ such that

$$(\hat{x}^j, \hat{X}^j) := \gamma_j(x, X) + (1 - \gamma_j)(x^j, x^j(x^j)^T)$$

satisfies $W \bullet \hat{X}^j + w^T \hat{x}^j + \omega = 0$. Moreover, as a convex combination of two points (x, X) and $(x^j, x^j(x^j)^T)$ in $\mathcal{C}(\mathcal{F})$, $(\hat{x}^j, \hat{X}^j)$ is in $\mathcal{C}(\mathcal{F})$. Since $\hat{X}^j \succeq \hat{x}^j(\hat{x}^j)^T$, Proposition 4 indicates that $(\hat{x}^j, \hat{X}^j) \in \overline{\mathcal{C}}(\mathcal{G})$. By the definition of $(\hat{x}^j, \hat{X}^j)$,

$$\begin{aligned}(x, X) &= \sum_{j\in J} \mu_j(x^j, x^j(x^j)^T) + \sum_{j\notin J} \mu_j(x^j, x^j(x^j)^T) \\ &= \sum_{j\in J} \mu_j(x^j, x^j(x^j)^T) + \sum_{j\notin J} \mu_j \left(\frac{1}{1-\gamma_j}(\hat{x}^j, \hat{X}^j) - \frac{\gamma_j}{1-\gamma_j}(x, X) \right).\end{aligned}$$

Let $\sigma := 1 + \sum_{j \notin J} \frac{\mu_j \gamma_j}{1-\gamma_j}$, then

$$(x, X) = \sum_{j \in J} \frac{\mu_j}{\sigma}(x^j, x^j(x^j)^T) + \sum_{j \notin J} \frac{\mu_j}{\sigma(1-\gamma_j)}(\hat{x}^j, \hat{X}^j),$$

which is a convex combination of points in $\overline{\mathcal{C}}(\mathcal{G})$. Therefore, $(x, X) \in \overline{\mathcal{C}}(\mathcal{G})$. □ □

Using a continuity argument, Proposition 5 can be generalized to $\overline{\mathcal{C}}(\mathcal{F}) \cap \mathrm{int}(\mathcal{L}(\mathcal{H})) \subseteq \overline{\mathcal{C}}(\mathcal{G})$.

Corollary 1. *If $(x, X) \in \overline{\mathcal{C}}(\mathcal{F})$ and $W \bullet X + 2w^T x + \omega < 0$, then $(x, X) \in \overline{\mathcal{C}}(\mathcal{G})$.*

Proof. If $(x, X) \in \overline{\mathcal{C}}(\mathcal{F})$, there exists a sequence $\{ (x^t, X^t) \}_t \subseteq \mathcal{C}(\mathcal{F})$ such that $(x^t, X^t) \to (x, X)$ as $t \to \infty$. Since $W \bullet X + 2w^T x + \omega < 0$, for sufficiently large t, $W \bullet X^t + 2w^T x^t + \omega < 0$. By Proposition 5, $(x^t, X^t) \in \overline{\mathcal{C}}(\mathcal{G})$. The proof is completed by taking $t \to \infty$. □ □

Summarizing the above, we state the main theorem of this section as follows.

Theorem 1. *Under Assumptions 1 and 2, $\overline{\mathcal{C}}(\mathcal{G}) = \overline{\mathcal{C}}(\mathcal{F}) \cap \mathcal{L}(\mathcal{H})$.*

Proof. The forward direction "$\subseteq$" is easy since $\overline{\mathcal{C}}(\mathcal{G}) \subseteq \overline{\mathcal{C}}(\mathcal{F}) \cap \overline{\mathcal{C}}(\mathcal{H}) \subseteq \overline{\mathcal{C}}(\mathcal{F}) \cap \mathcal{L}(\mathcal{H})$. The other direction is given by combining Proposition 4 and Corollary 1. □ □

The non-intersecting assumption (Assumption 2) is essential in Theorem 1. We refer the readers to [30] for counterexamples when $W \succ 0$ and the non-intersecting assumption is missing. The following example shows that the nonzero assumption (Assumption 1) cannot be dropped.

Example 1. *Let $\mathcal{F} = \{ x \in \mathbb{R} \mid -x - 2 \le 0 \} = [-2, \infty)$, $\mathcal{H} = \{x \in \mathbb{R} \mid -x + 1 \ge 0\} = (-\infty, 1]$, and $\mathcal{G} := \mathcal{F} \cap \mathcal{H} = [-2, 1]$. Obviously, the non-intersecting assumption is satisfied as $\mathrm{bd}(\mathcal{H}) = \{1\} \subseteq \mathcal{F}$. However,*

$$\overline{\mathcal{C}}(\mathcal{G}) = \{ (x, X) \in \mathbb{R}^2 \mid X \le 2 - x, X \ge x^2 \},$$

which is a proper subset of

$$\overline{\mathcal{C}}(\mathcal{F}) \cap \mathcal{L}(\mathcal{H}) = \{ (x, X) \in \mathbb{R}^2 \mid -2 \le x \le 1, X \ge x^2 \}.$$

We conclude this section with three corollaries of Theorem 1, which extend our main result to sets defined by multiple quadratic constraints. Let $\mathcal{H}' := \{ x \in \mathbb{R}^n \mid x^T W_k x + 2w_k^T x + \omega_k \le 0, \ k \in K \}$ be a nonempty proper subset of $\mathbb{R}^n$, where $W_k \in \mathcal{S}^n$, $w_k \in \mathbb{R}^n$, $\omega_k \in \mathbb{R}$, and $K = \{1, \ldots, \ell\}$. Correspondingly, we define

$$\mathcal{L}(\mathcal{H}') := \left\{ (x, X) \;\middle|\; W_k \bullet X + 2w_k^T x + \omega_k \le 0, \ \forall\, k \in K \right\}.$$

The first corollary is a direct extension of Theorem 1.

Corollary 2. *For a nonempty closed set $\mathcal{F} \subseteq \mathbb{R}^n$ and $\mathcal{G} = \mathcal{F} \cap \mathcal{H}'$, $\overline{\mathcal{C}}(\mathcal{G}) = \overline{\mathcal{C}}(\mathcal{F}) \cap \mathcal{L}(\mathcal{H}')$ under the following assumptions.*

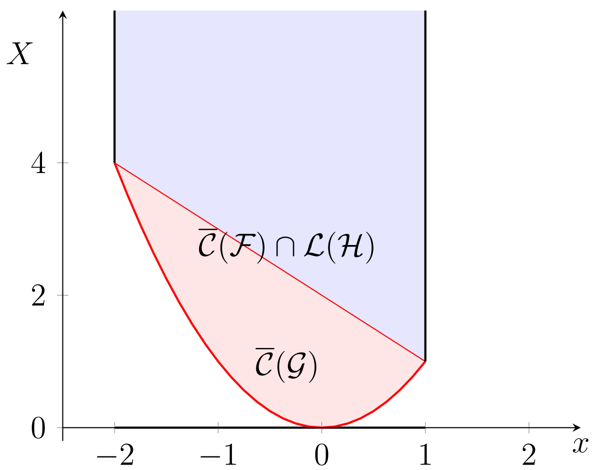


Figure 2: In Example 1, $\overline{\mathcal{C}}(\mathcal{G})$ is a bounded set while $\overline{\mathcal{C}}(\mathcal{F}) \cap \mathcal{L}(\mathcal{H})$ is unbounded.

Assumption 3. *$W_k \neq 0$ for all $k \in K$.*

Assumption 4. *For all $k \in K$,*

$$x^T W_k x + 2w_k^T x + \omega_k = 0 \quad \Longrightarrow \quad \begin{cases} x \in \mathcal{F}, \\ x^T W_j x + 2w_j^T x + \omega_j \leq 0, \quad \forall\, j \in K \setminus \{k\}. \end{cases}$$

Proof. Let $\mathcal{H}_k := \{\, x \in \mathbb{R}^n \mid x^T W_k x + 2w_k^T x + \omega_k \leq 0 \,\}$ for each $k \in K$. We have $\mathcal{H}' = \bigcap_{k\in K} \mathcal{H}_k$. When $\ell = 1$, the statement is reduced to Theorem 1. For $\ell \geq 2$, the corollary can be proved by repeatedly applying Theorem 1 to $\mathcal{H}_k$ and $\mathcal{F} \cap \mathcal{H}_1 \cap \cdots \cap \mathcal{H}_{k-1}$. □ □

The second corollary shows that $\overline{\mathcal{C}}(\mathcal{G}) = \mathcal{S}(\mathcal{G})$ when $\mathcal{G}$ is defined by non-intersecting quadratic constraints with nonzero Hessians. Special cases and variants of the corollary can be spotted in the literature. To name a few: the non-binding constraints in [31], the generalized trust region subproblem in [25], and the non-interacting constraints in [3].

Corollary 3. *Let $\mathcal{G} = \{\, x \in \mathbb{R}^n \mid x^T W_k x + 2w_k^T x + \omega_k \leq 0,\ k \in K \,\}$ be a set defined by non-intersecting quadratic inequalities with nonzero Hessian matrices. That is, for all $k \in K$, $W_k \neq 0$ and*

$$x^T W_k x + 2w_k^T x + \omega_k = 0 \quad \Longrightarrow \quad x^T W_j x + 2w_j^T x + \omega_j \leq 0 \ \ \forall\, j \in K \setminus \{k\}.$$

Then, $\overline{\mathcal{C}}(\mathcal{G}) = \mathcal{S}(\mathcal{G}) = \{\, (x, X) \mid W_k \bullet X + 2w_k^T x + \omega_k \leq 0,\, k \in K,\ X \succeq xx^T \,\}$.

Proof. Note that $\mathcal{G}$ can be decomposed as $\mathcal{G} = \mathcal{F} \cap \mathcal{H}'$, where $\mathcal{F} = \mathbb{R}^n$ and $\mathcal{H}' = \mathcal{G}$. Since Assumptions 3 and 4 are satisfied, Corollary 2 implies

$$\begin{aligned} \overline{\mathcal{C}}(\mathcal{G}) &= \overline{\mathcal{C}}(\mathbb{R}^n) \cap \mathcal{L}(\mathcal{G}) \\ &= \{\, (x, X) \mid W_k \bullet X + 2w_k^T x + \omega_k \leq 0,\, k \in K,\ X \succeq xx^T \,\} \\ &= \mathcal{S}(\mathcal{G}). \end{aligned}$$

□ □

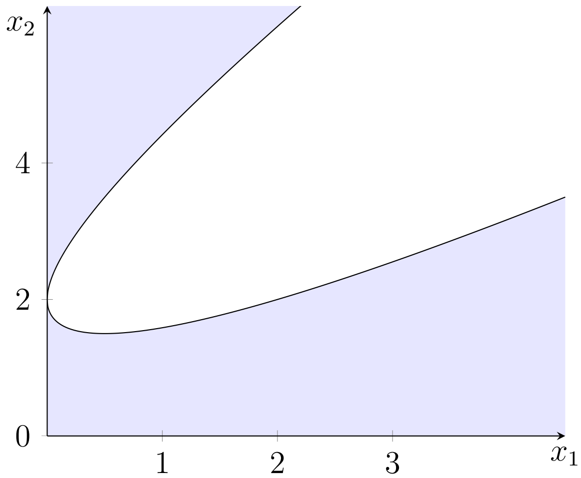


Figure 3: The first quadrant with a parabolic hollow.

The last corollary can be interpreted as a sufficient condition for $\overline{\mathcal{C}}(\mathcal{F}\cap\mathcal{H}') = \overline{\mathcal{C}}(\mathcal{F})\cap\overline{\mathcal{C}}(\mathcal{H}')$. Note that $\overline{\mathcal{C}}(\mathcal{F})\cap\mathcal{L}(\mathcal{H}') = \overline{\mathcal{C}}(\mathcal{F})\cap\mathcal{S}(\mathcal{H}')$. By Theorem 1 and Corollary 3, we have the following statement.

Corollary 4. *For a nonempty closed set $\mathcal{F} \subseteq \mathbb{R}^n$, $\overline{\mathcal{C}}(\mathcal{F} \cap \mathcal{H}') = \overline{\mathcal{C}}(\mathcal{F}) \cap \overline{\mathcal{C}}(\mathcal{H}')$ under Assumptions 3 and 4.*

4 Examples

In this section, we provide four examples to show how the theory in Section 3 can be applied to derive new convex hull results in the lifted space. The first example is a toy example, which is depicted in Figure 3.

Example 2. *Let $\mathcal{F} := \{\, x \in \mathbb{R}^2 \mid x_1 \geq 0, x_2 \geq 0 \,\}$ be the nonnegative quadrant, $\mathcal{H} := \{x \in \mathbb{R}^2 | -(x_1 - x_2)^2 + 2x_2 - 1 \leq 0\}$, and $\mathcal{G} := \mathcal{F} \cap \mathcal{H}$. It is known that $\overline{\mathcal{C}}(\mathcal{F})$ is the doubly nonnegative cone [2], that is,*

$$\overline{\mathcal{C}}(\mathcal{F}) = \{\, (x, X) \mid X \succeq xx^T,\ X \geq 0,\ x \geq 0 \,\}.$$

Since $\mathrm{bd}(\mathcal{H}) \subseteq \mathcal{G}$, the non-intersecting assumption (Assumption 2) is satisfied. Therefore, Theorem 1 indicates that

$$\overline{\mathcal{C}}(\mathcal{G}) = \overline{\mathcal{C}}(\mathcal{F}) \cap \mathcal{L}(\mathcal{H}) = \left\{ (x, X) \;\middle|\; \begin{array}{l} -(X_{11} + X_{22}) + 2X_{12} + 2x_2 - 1 \leq 0, \\ X \succeq xx^T,\ X \geq 0,\ x \geq 0 \end{array} \right\}.$$

The next example is a disjunctive mixed-integer set in $\mathbb{R}^2$. The closed convex hull of mixed-integer sets in the lifted space have been widely studied, e.g. in [14] and [10]. We hope the derivation the basic example can shed some light on future study of the geometry of the lifted closed convex hull for more complicated mixed-integer sets.

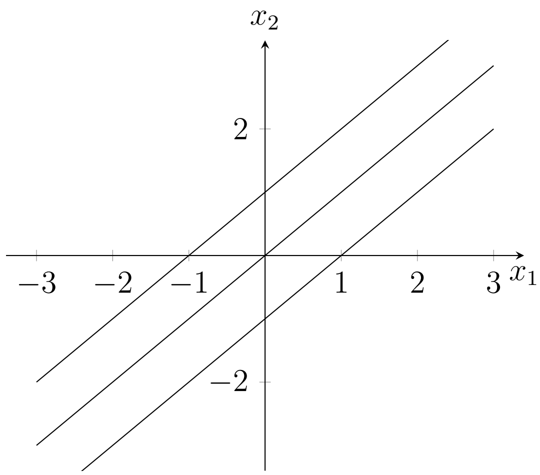


Figure 4: Three parallel lines.

Example 3. *Let $\mathcal{G} := \{ x \in \mathbb{R}^2 \mid x_1 - x_2 \in \{-1, 0, 1\} \}$ be a disjunctive set composed of the union of three parallel lines, see Figure 4. We can rewrite $\mathcal{G}$ as a set defined by non-intersecting quadratic inequalities. That is,*

$$\mathcal{G} = \left\{ x \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} (x_1 - x_2 - 1)(x_1 - x_2 + 1) \leq 0, \\ -(x_1 - x_2 - 1)(x_1 - x_2) \leq 0, \\ -(x_1 - x_2 + 1)(x_1 - x_2) \leq 0 \end{array} \right\}.$$

Since the Hessians of the defining quadratic inequalites are nonzero, Corollary 3 indicates that

$$\overline{\mathcal{C}}(\mathcal{G}) = \mathcal{S}(\mathcal{G}) = \left\{ (x, X) \;\middle|\; \begin{array}{l} X_{11} + X_{22} - 2X_{12} - 1 \leq 0 \\ X_{11} + X_{22} - 2X_{12} - x_1 + x_2 \geq 0 \\ X_{11} + X_{22} - 2X_{12} + x_1 - x_2 \geq 0 \\ X \succeq xx^T \end{array} \right\}.$$

The problem we consider in the next example arises from an extension of the Weber Problem [12] with restricted regions [1, 19].

Example 4. *The Weber problem determines the location of a facility that minimizes the sum of the transportation costs from this facility to n sites. In a traditional Weber problem, the transportation costs are proportional to the Euclidean distance. However, in some realistic situations, it would be more appropriate to assume that the transportation costs were proportional to the squared Euclidean distance [12]. On the other hand, restricted regions have been taken into considerations for the Weber problem and facility location problems [1, 19].*

These restricted regions, while allowing travel through them, prohibit the placement of a facility. Different shapes of the restricted regions have been considered, e.g. polyhedral restricted regions and circular restricted regions [23].

In this example, we consider an extended Weber problem with squared Euclidean distance and disjoint circular restricted regions. Let $a_1, \ldots, a_n \in \mathbb{R}^2$ be the location of n sites. The transportation costs from a facility $x \in \mathbb{R}^2$ to a_i $(i = 1, \ldots, n)$ is assumed to be $w_i \|x - a_i\|_2^2$, *where $w_i > 0$ is a weight. Let b_k and r_k be the center and radius, respectively, of the k-th restricted region ($k = 1, \ldots, K$). We seek for an optimal location x of a facility, so that the total transportation costs are minimized and x is not located in the interior of any of the K disjoint regions. The problem can be formulated as a QCQP:*

$$\begin{aligned} \inf \quad & \sum_{i=1}^{n} w_i \|x - a_i\|_2^2 \\ s.t. \quad & \|x - b_k\|_2^2 \geq r_k^2, \quad k = 1, \ldots, K. \end{aligned}$$

Despite of being nonconvex, the feasible region $\mathcal{G} := \{ x \in \mathbb{R}^2 \mid \|x - b_k\|_2^2 \geq r_k^2 \}$ is defined by non-intersecting quadratic inequalities with nonzero Hessians. Therefore, $\overline{\mathcal{C}}(\mathcal{G}) = \mathcal{S}(\mathcal{G})$ by Corollary 3. The problem is then equivalent to a semidefinite program:

$$\begin{aligned} \inf \quad & \sum_{i=1}^{n} w_i (tr(X) - 2a_i^T x + a_i^T a_i) \\ s.t. \quad & tr(X) - 2b_k^T x + b_k^T b_k \geq r_k^2, \quad k = 1, \ldots, K, \\ & X \succeq xx^T. \end{aligned}$$

In the last example, we regenerate a semidefinite reformulation for a generalized trust-region subproblem with a milder assumption.

Example 5. *The trust-region subproblem (TRS) minimizes a quadratic function over the unit ball. It is well known that the standard semidefinite relaxation of TRS is exact. In this example, we consider a generalized TRS with interval bounds.*

$$\begin{aligned} \inf \quad & x^T Q x + 2q^T x \\ s.t. \quad & \ell \leq x^T A x + 2a^T x \leq u, \end{aligned} \tag{GTRS}$$

where $Q, A \in \mathcal{S}^n$, $q, a \in \mathbb{R}^n$, and $-\infty < \ell \leq u < \infty$. Note that A is not necessarily positive semidefinite.

This problem has been widely studied in the literature, e.g. [29, 25, 28]. It is shown in [28] that the following semidefinite relaxation

$$\begin{aligned} \inf \quad & Q \bullet X + 2q^T x \\ s.t. \quad & \ell \leq A \bullet X + 2a^T x \leq u, \\ & X \succeq xx^T \end{aligned} \tag{SDP-GTRS}$$

is exact under two assumptions:

1. *(nonzero) $A \neq 0$;*
2. *(Slater's condition) There exists $\hat{x}$ such that $\ell < \hat{x}^T A \hat{x} + 2a^T \hat{x} < u$ in the case when $\ell < u$; there exist $\hat{x}$ and $\bar{x}$ such that $\hat{x}^T A \hat{x} + 2a^T \hat{x} < 0 < \bar{x}^T A \bar{x} + 2a^T \bar{x}$ in the case when $\ell = u$.*

With our approach, since $\mathcal{G} := \{ x \in \mathbb{R}^n \mid \ell \leq x^T A x + 2a^T x \leq u \}$ is defined by two non-intersecting quadratic inequalities, we have

$$\overline{\mathcal{C}}(\mathcal{G}) = \mathcal{S}(\mathcal{G}) = \{ (x, X) \mid \ell \leq A \bullet X + 2a^T x \leq u, \ X \succeq xx^T \}$$

if $A \neq 0$. As a result, (SDP-GTRS) is exact as long as $A \neq 0$. Note that the second assumption (Slater's condition) in [28] is not required in our approach.

To illustrate the difference, we denote the Lagrangian dual problem of (GTRS) by (D-GTRS) and denote the conic dual problem of (SDP-GTRS) by (SDD-GTRS). We also use $v(\cdot)$ to represent the optimal value of each problem. The approach in [28] proves that $v(GTRS) = v(D\text{-}GTRS)$ with both the nonzero assumption and the Slater's condition. It is also shown in [28] that the Slater's condition for (GTRS) is equivalent to the one for (SDP-GTRS). Since $v((SDP\text{-}GTRS)) = v(SDD\text{-}GTRS)$ under the latter Slater's condition and (SDD-GTRS) is an equivalent reformulation of (D-GTRS), it is concluded that $v((SDP\text{-}GTRS) = v(SDD\text{-}GTRS) = v(D\text{-}GTRS) = v(GTRS)$. On the other hand, our approach directly connects (GTRS) and (SDP-GTRS) without considering the dual problem. Therefore, the Slater's condition is not required to prove the exactness of (SDP-GTRS). See Figure 5.

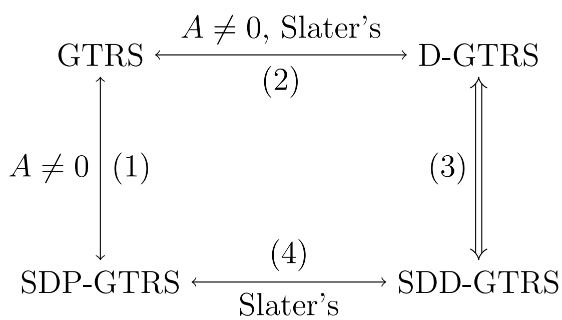


Figure 5: A flowchart showing the two approaches mentioned in Example 5. Our direct approach is only concerned with arc (1) whereas the method in [28] traverses arcs (2),(3) and (4).

5 Conclusion

For closed sets $\mathcal{F}$ and $\mathcal{H}'$, we consider the relation between $\overline{\mathcal{C}}(\mathcal{F} \cap \mathcal{H}')$ and $\overline{\mathcal{C}}(\mathcal{F})$. We show that $\overline{\mathcal{C}}(\mathcal{F} \cap \mathcal{H}') = \overline{\mathcal{C}}(\mathcal{F}) \cap \mathcal{L}(\mathcal{H}') = \overline{\mathcal{C}}(\mathcal{F}) \cap \overline{\mathcal{C}}(\mathcal{H}')$ when $\mathcal{H}'$ is defined by quadratic constraints with nonzero Hessians and the non-intersecting assumption is satisfied. This result generalizes the bounded case in [30] and other non-intersecting cases captured by Corollary 3. To prove the result, we provide a complete characterization of the asymptotic cones of sets defined by a single quadratic equality as a byproduct.

The convex hull result can be applied to any nonempty closed set $\mathcal{F}$ with known lifted closed convex hull $\overline{\mathcal{C}}(\mathcal{F})$ to generate new convex hull results. The result can also be interpreted as a sufficient condition for $\overline{\mathcal{C}}(\mathcal{F} \cap \mathcal{H}') = \overline{\mathcal{C}}(\mathcal{F}) \cap \overline{\mathcal{C}}(\mathcal{H}')$. For future research, it is worthwhile to explore other sufficient conditions and necessary conditions for $\overline{\mathcal{C}}(\mathcal{F} \cap \mathcal{H}') = \overline{\mathcal{C}}(\mathcal{F}) \cap \overline{\mathcal{C}}(\mathcal{H}')$. It would also be interesting to see how the convex hull result can help in the computation of mixed-integer QCQP.

To conclude this paper, we bring to the reader's attention a simple but interesting example, which is beyond the reach of the results in this paper. Let $\mathcal{G} := \{\, x \in \mathbb{R}^2 \mid x_1 x_2 \leq 1, x \geq 0 \,\}$. A natural decomposition of $\mathcal{G}$ is to set $\mathcal{F} := \{\, x \in \mathbb{R}^n \mid x \geq 0 \,\}$ and $\mathcal{H} := \{\, x \in \mathbb{R}^2 \mid x_1 x_2 \leq 1 \,\}$. Note that $\mathcal{H}$ has two branches, and the boundary of the branch in the 3rd quadrant is not contained in $\mathcal{G}$. Therefore, the non-intersecting assumption is violated. Such cases are worth exploring as bilinear terms are fundamental substructures in mixed-integer QCQP. Future research along the path is desired for cases when the non-intersecting assumption is relaxed.

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